# Varieties of Negation and Contra-classicality in View of Dunn Semantics

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Abstract. In this paper, we discuss J. Michael Dunn's foundational work on the semantics for First Degree Entailment logic (**FDE**), also known as Belnap–Dunn logic (or Sanjaya–Belnap–Smiley–Dunn Four-valued Logic, as suggested by Dunn himself). More specifically, by building on the framework due to Dunn, we sketch a broad picture towards a systematic understanding of *contra-classicality*. Our focus will be on a simple propositional language with negation, conjunction, and disjunction, and we will systematically explore variants of **FDE**, **K3**, and **LP** by tweaking the falsity condition for negation.

*Keywords.* Bi-lateral natural deduction, Contraposition, Contra-classicality, Dunn semantics, Negation, Uni-lateral natural deduction, Variable sharing property

## 1. Introduction

Let us begin with a brief explanation of the three key notions included in the title of our paper, namely, *Dunn semantics*, *contra-classicality* and *negation*.

**Dunn semantics.** The logic of first-degree entailment **FDE**, also known as Belnap–Dunn logic (or Sanjaya–Belnap–Smiley–Dunn Four-valued Logic, as suggested by Dunn himself in [17, p. 95]), is a basic paraconsistent and paracomplete logic that has found many applications in philosophy and different areas of computer science, including the semantics of logic programs and inconsistency-tolerant description logics. The seminal papers [12; 4; 5] on **FDE** from the 1970s have been re-printed in [33], together with some recent essays devoted to Belnap–Dunn Logic.

The system **FDE** has various equivalent semantical presentations, cf. [31]. There exists a four-valued semantics, a so-called "star" semantics, an algebraic semantics, and a two-valued relational semantics due to Dunn [12]. (Note that the results published in [12] were already established and included in [11].) This semantics not only justifies the intuitive reading of the four truth values in the four-valued semantics but also enables a tweaking of the falsity condition of negation so as to obtain certain variants of **FDE**, the paracomplete three-valued strong Kleene logic **K3**, and the paraconsistent three-valued logic of paradox, **LP**. The four-valued semantics and the relational Dunn semantics are very closely related, and there exists a mechanical procedure to turn the many-valued truth tables into pairs of truth and falsity conditions, and vice versa, see [30].

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**Contra-classicality.** The notion of a contra-classical logic has been coined by Lloyd Humberstone [26]. The most prominent non-classical logics such as, for example, minimal logic, intuitionistic logic, and the relevance logics **E** and **R** are subclassical. If they are presented in the vocabulary of classical logic, their consequence relations are subsets of the consequence relation of classical logic. In contrast to this, a contra-classical logic validates consequences that are not valid in classical logic. Various contra-classical logics have been studied in the literature. Examples include Abelian logic (cf. [29], [34]), systems with demi-negation (cf. [25; 26; 35]), certain systems of connexive logic (cf. [47], [49]), and the second-order Logic of Paradox (cf. [23]).

Some of the known contra-classical logic are contra-classical in a way that radically differs from logical orthodoxy insofar as they are non-trivial but negation inconsistent. These logics contain provable contradictions, i.e., they contain formulas $A$ such that both $A$ and the negation $\sim A$ of $A$ are theorems. Whilst **FDE**, **K3**, and **LP** are subclassical logics, we will see that a tweaking of the falsity condition for negation in these logics can give rise to contra-classical systems. Some of the contra-classical variants of **FDE**, **K3**, and **LP** turn out to be negation inconsistent and some are negation incomplete.

**Negation.** There exists an extensive literature on the notion of negation and on which properties a genuine negation connective minimally ought to possess, see, for example, [22; 24; 50; 46; 7; 8; 10]. Although Michael Dunn has made substantial contributions to the study of negation as a modal operator of impossibility or "unnecessity" [13; 14; 15; 18], he clearly had a broader understanding of the concept of negation and even voiced the conviction that negation flip-flops between truth and falsity. Here is a quote from [15, p. 49] (notation adjusted):

> Tim Smiley once good-naturedly accused me of being a kind of lawyer for various non-classical logics. He flattered me with his suggestion that I could make a case for anyone of them, and in particular provide it with a semantics, no matter what the merits of the case [...] But I must say that my own favourite is the 4-valued semantics. I am persuaded that '$\sim A$ is true iff $A$ is false', and that '$\sim A$ is false iff $A$ is true'. And now to paraphrase Pontius Pilate, we need to know more about 'What are truth and falsity?'. It is of course the common view that they divide up the states into two exclusive kingdoms. But there are lots of reasons, motivated by applications, for thinking that this is too simple-minded.

In the present paper, we will study variants of logics in which negation flip-flops between truth and falsity, namely, variants of **FDE**, **K3**, and **LP**. A very weak requirement imposed on a unary connective in a logical system to deserve the classification as a negation connective is that for some formulas $A$ and $B$, neither $A \vdash \sim A$ nor $\sim B \vdash B$, cf. [2; 28]. We will consider one-place connectives that not only satisfy this weak condition but also share the above truth condition for negation: $\sim A$ is true (under a given interpretation) iff $A$ is false (under that interpretation). *Classically* falsity means untruth, so that the truth condition already fixes the falsity condition, but this is not the case in general, and in particular, it is not the case in **FDE**, **K3**, and **LP**, where truth and falsity are two primitive concepts that are on a par. A discussion of semantical opposition understood as an opposition between on the one hand truth and falsity, and on the other hand between truth and untruth can be found in [32], where it is observed that in the four-valued setting of **FDE**, the above truth condition for $\sim A$ together with the understanding of falsity of $\sim A$ as untruth of $A$ results in the "demi-negation" of the system **CP** from [27].

According to Arnon Avron, the requirement that $\sim A$ is true iff $A$ is false represents "the idea of falsehood within the language" [1, p. 160]. We shall keep this truth condition for negated formulas but abandon the classical understanding of falsity as untruth and instead treat truth and falsity as two separate primitive semantical notions of equal importance. There is thus a clear sense in which the unary connectives in this paper written as $\sim$, sometimes with a subscript, can be seen as negations. However, there is now room for tweaking the falsity condition for negation. We will consider all combinations that are possible for **FDE**, **K3**, and **LP** in a classical metatheory. This gives us sixteen variants of **FDE**, four variants of **K3**, and four variants of **LP**. By considering these logics, we are applying what Luis Estrada-González [19; 20] has called "the Bochum Plan."[1]

The themes dealt with in the present paper are among the topics addressed in nine questions we had posed to Prof. J. Michael Dunn in March 2021 together with Grigory Olkhovikov (the notion of negation, the tweaking of falsity conditions, negation inconsistency, bilateralism, contraposition), see [51]. Unfortunately, Mike was no longer able to answer these questions. He passed away on 5 April 2021, a few weeks after he informed us that he is willing to answer our questions.

Before moving further, let us recall some well known results related to **FDE**, **K3**, and **LP**. The language $\mathcal{L}$ consists of a set $\{\sim, \wedge, \vee\}$ of propositional connectives and a countable set $\mathsf{Prop}$ of propositional variables which we denote by $p, q, \ldots$. Furthermore, we denote by $\mathsf{Form}$ the set of formulas defined as usual in $\mathcal{L}$. We denote a formula of $\mathcal{L}$ by $A, B, C, \ldots$ and a set of formulas of $\mathcal{L}$ by $\Gamma, \Delta, \Sigma, \ldots$.

We begin with the many-valued representations of **FDE**, **K3** and **LP**.

**Definition 1.** A four-valued **FDE**-interpretation of $\mathcal{L}$ is a function $v_4 \colon \mathsf{Prop} \longrightarrow \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$. Given a four-valued interpretation $v_4$, this is extended to a function $I_4$ that assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:

| | $\sim$ |
|---|---|
| **t** | **f** |
| **b** | **b** |
| **n** | **n** |
| **f** | **t** |

| $\wedge$ | **t** | **b** | **n** | **f** |
|---|---|---|---|---|
| **t** | **t** | **b** | **n** | **f** |
| **b** | **b** | **b** | **f** | **f** |
| **n** | **n** | **f** | **n** | **f** |
| **f** | **f** | **f** | **f** | **f** |

| $\vee$ | **t** | **b** | **n** | **f** |
|---|---|---|---|---|
| **t** | **t** | **t** | **t** | **t** |
| **b** | **t** | **b** | **t** | **b** |
| **n** | **t** | **t** | **n** | **n** |
| **f** | **t** | **b** | **n** | **f** |

Then, the semantic consequence relation for **FDE** ($\models_{\mathbf{FDE}}$) is defined as follows.

**Definition 2.** For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{FDE}} A$ iff for all four-valued **FDE**-interpretations $v_4$, $I_4(A) \in \mathcal{D}$ if $I_4(B) \in \mathcal{D}$ for all $B \in \Gamma$, where $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$.

Now, if we eliminate the value **b** from the semantics for **FDE**, then we obtain the three-valued semantics for **K3**, as follows.

**Definition 3.** A three-valued **K3**-interpretation of $\mathcal{L}$ is a function $v_3 \colon \mathsf{Prop} \longrightarrow \{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$. Given a three-valued interpretation $v_3$, this is extended to a function $I_3$ that

[1] Note that the Bochum Plan in general does not privilege truth over falsity, so that we could also keep the standard falsity condition for negation and systematically tweak the truth condition.

assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:

| | $\sim$ |
|---|---|
| **t** | **f** |
| **n** | **n** |
| **f** | **t** |

| $\wedge$ | **t** | **n** | **f** |
|---|---|---|---|
| **t** | **t** | **n** | **f** |
| **n** | **n** | **n** | **f** |
| **f** | **f** | **f** | **f** |

| $\vee$ | **t** | **n** | **f** |
|---|---|---|---|
| **t** | **t** | **t** | **t** |
| **n** | **t** | **n** | **n** |
| **f** | **t** | **n** | **f** |

Then, the semantic consequence relation for **K3** ($\models_{\mathbf{K3}}$) is defined as follows.

**Definition 4.** For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{K3}} A$ iff for all three-valued interpretations $v_3$, $I_3(A) \in \mathcal{D}$ if $I_3(B) \in \mathcal{D}$ for all $B \in \Gamma$, where $\mathcal{D} = \{\mathbf{t}\}$.

Moreover, if we eliminate the value **n** from the semantics for **FDE**, then we obtain the three-valued semantics for **LP**, as follows.

**Definition 5.** A three-valued **LP**-interpretation of $\mathcal{L}$ is a function $v_3 : \mathsf{Prop} \longrightarrow \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$. Given a three-valued interpretation $v_3$, this is extended to a function $I_3$ that assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:

| | $\sim$ |
|---|---|
| **t** | **f** |
| **b** | **b** |
| **f** | **t** |

| $\wedge$ | **t** | **b** | **f** |
|---|---|---|---|
| **t** | **t** | **b** | **f** |
| **b** | **b** | **b** | **f** |
| **f** | **f** | **f** | **f** |

| $\vee$ | **t** | **b** | **f** |
|---|---|---|---|
| **t** | **t** | **t** | **t** |
| **b** | **t** | **b** | **b** |
| **f** | **t** | **b** | **f** |

Then, the semantic consequence relation for **LP** ($\models_{\mathbf{LP}}$) is defined as follows.

**Definition 6.** For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{LP}} A$ iff for all three-valued interpretations $v_3$, $I_3(A) \in \mathcal{D}$ if $I_3(B) \in \mathcal{D}$ for all $B \in \Gamma$, where $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$.

Finally, let us recall the Dunn semantics for **FDE**.

**Definition 7.** A *Dunn-interpretation* of $\mathcal{L}$ is a relation, $r$, between propositional variables and the values 1 and 0, namely, $r \subseteq \mathsf{Prop} \times \{1, 0\}$. Given an interpretation, $r$, this is extended to a relation between all formulas and truth values by the following clauses:

$$\begin{array}{lll} {\sim}Ar1 \text{ iff } Ar0, & A \wedge Br1 \text{ iff } Ar1 \text{ and } Br1, & A \vee Br1 \text{ iff } Ar1 \text{ or } Br1, \\ {\sim}Ar0 \text{ iff } Ar1, & A \wedge Br0 \text{ iff } Ar0 \text{ or } Br0, & A \vee Br0 \text{ iff } Ar0 \text{ and } Br0. \end{array}$$

**Definition 8.** A formula $A$ is a *two-valued semantic consequence* of $\Gamma$ ($\Gamma \models_2 A$) iff for all Dunn-interpretations $r$, if $Br1$ for all $B \in \Gamma$ then $Ar1$.

**Remark 9.** We obtain the Dunn semantics for **K3** and **LP** by adding the following constraints, respectively, to $r$: (no-gap) for no $p$, $pr1$ and $pr0$; (no-glut) for all $p$, $pr1$ or $pr0$. Of course, if we add both constraints, then we obtain the semantics for classical logic.

Given our assumption concerning negation, we will systematically consider the variants of **FDE**, **K3** and **LP** by changing the falsity condition for negation, and explore their basic properties.[2]

[2]Note that in a recent article [21], Estrada-González considers the Bochum plan and suggests systematic changes in the evaluation conditions not only for negation, but also for other connectives. By doing so, he emphasized the tweaking of the evaluation clauses as a source of contra-classicality.

## 2. Semantics

Let us now present the semantics for the variations of **FDE**, **K3** and **LP** we consider in the rest of paper. We begin with the variations of **FDE**.

By simple combinatorial considerations, the following sixteen operations exhaust the space of possible connectives that share the truth condition for negation.

| $A$ | $\sim_1 A$ | $\sim_2 A$ | $\sim_3 A$ | $\sim_4 A$ | $\sim_5 A$ | $\sim_6 A$ | $\sim_7 A$ | $\sim_8 A$ |
|---|---|---|---|---|---|---|---|---|
| **t** | **f** | **f** | **f** | **f** | **f** | **f** | **f** | **f** |
| **b** | **b** | **b** | **b** | **b** | **t** | **t** | **t** | **t** |
| **n** | **n** | **n** | **f** | **f** | **n** | **n** | **f** | **f** |
| **f** | **t** | **b** | **t** | **b** | **t** | **b** | **t** | **b** |

| $A$ | $\sim_9 A$ | $\sim_{10} A$ | $\sim_{11} A$ | $\sim_{12} A$ | $\sim_{13} A$ | $\sim_{14} A$ | $\sim_{15} A$ | $\sim_{16} A$ |
|---|---|---|---|---|---|---|---|---|
| **t** | **n** | **n** | **n** | **n** | **n** | **n** | **n** | **n** |
| **b** | **b** | **b** | **b** | **b** | **t** | **t** | **t** | **t** |
| **n** | **n** | **n** | **f** | **f** | **n** | **n** | **f** | **f** |
| **f** | **t** | **b** | **t** | **b** | **t** | **b** | **t** | **b** |

In view of the mechanical procedure described in [30, §2], we obtain falsity conditions for the above operators. We leave the details to interested readers as an easy exercise (the same applies to the variants of **K3** and **LP**, introduced below). Then, we define the semantic consequence relations for the variants with $\sim_i$ instead of $\sim$ (notation: $\models^i_{\mathbf{FDE}}$) as in Definition 1.

**Remark 10.** As one may easily observe, $\sim_1$ is the original negation included in **FDE**. Moreover, $\sim_{16}$ is the connective we discussed in [32]. The other fourteen operations are, to the best of our knowledge, not discussed in the literature.[3] Note that only three of the fourteen operations are *subclassical*. Further details of the operations will be explored in §5.

We now turn to variations of **K3**. By another simple combinatorial consideration, or by eliminating some cases starting from the above considerations for **FDE**, the following four operations exhaust the space of possible connectives that share the truth condition for negation.

| $A$ | $\sim_1 A$ | $\sim_2 A$ | $\sim_3 A$ | $\sim_4 A$ |
|---|---|---|---|---|
| **t** | **f** | **f** | **n** | **n** |
| **n** | **n** | **f** | **n** | **f** |
| **f** | **t** | **t** | **t** | **t** |

Note here that $\sim_2$ is the connective discussed in [41]. Then, we define the semantic consequence relations for the variants with $\sim_i$ instead of $\sim$ (notation: $\models^i_{\mathbf{K3}}$) as in Definition 3.

Finally, we consider the variations of **LP**. By another simple combinatorial consideration, or again by eliminating some cases starting from the above considerations for **FDE**, the following four operations exhaust the space of possible connectives that share the truth condition for negation.

[3] A referee directed our attention to [36] as a reference that covers the connectives that we are discussing in this paper. This, however, is not the case. Note also that there is a crucial difference between [36] and the present paper insofar as we are *not* expanding the language of **FDE**, but only changing the interpretation of negation.

| $A$ | $\sim_1 A$ | $\sim_2 A$ | $\sim_3 A$ | $\sim_4 A$ |
|---|---|---|---|---|
| **t** | **f** | **f** | **f** | **f** |
| **b** | **b** | **b** | **t** | **t** |
| **f** | **t** | **b** | **t** | **b** |

Note here that $\sim_2$ is the connective discussed in [40]. We define the semantic consequence relations for the variants with $\sim_i$ instead of $\sim$ (notation: $\models^i_{\mathbf{LP}}$) as in Definition 5.

## 3. Proof Systems

### 3.1. Unilateral Natural Deduction.

Let us first recall the natural deduction system for **FDE**, **K3** and **LP**. Our presentation below follows the one due to Dag Prawitz in [37, Appendix B], where he considers a certain expansion of **FDE** suggested by David Nelson, namely a logic that can be seen as an expansion of intuitionistic logic by a "strong" negation.[4]

**Definition 11.** The natural deduction rules $\mathcal{R}_{\mathbf{FDE}}$ for **FDE** are all the following rules:

$$\frac{A \quad B}{A \wedge B} \qquad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B} \qquad \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

$$\frac{\sim\sim A}{A}\ (\sim\sim 1) \qquad \frac{A}{\sim\sim A}\ (\sim\sim 2)$$

$$\frac{\sim(A \wedge B) \quad \begin{array}{c}[\sim A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[\sim B]\\ \vdots \\ C\end{array}}{C} \qquad \frac{\sim B}{\sim(A \wedge B)} \qquad \frac{\sim A}{\sim(A \wedge B)} \qquad \frac{\sim(A \vee B)}{\sim B} \qquad \frac{\sim(A \vee B)}{\sim A} \qquad \frac{\sim A \quad \sim B}{\sim(A \vee B)}$$

Moreover, for the natural deduction rules $\mathcal{R}_{\mathbf{K3}}$ and $\mathcal{R}_{\mathbf{LP}}$ for **K3** and **LP**, respectively, we add the ECQ and the Law of the Excluded Middle, respectively:

$$\frac{A \quad \sim A}{B}\ (\mathrm{ECQ}) \qquad \frac{}{A \vee \sim A}\ (\mathrm{LEM}).$$

Then, given any set $\Sigma \cup \{A\}$ of formulas, $\Sigma \vdash_{\mathbf{FDE}} A$ iff for some finite $\Sigma' \subseteq \Sigma$, there is a derivation of $A$ from $\Sigma'$ in the calculus whose rule set is $\mathcal{R}_{\mathbf{FDE}}$. In the same way, we define $\vdash_{\mathbf{K3}}$ and $\vdash_{\mathbf{LP}}$.

We now turn to introduce the natural deduction systems for the variants of our basic systems.

**Definition 12.** The natural deduction rules $\mathcal{R}^i_{\mathbf{FDE}}$ for $\mathbf{FDE}^i$ are all the rules $\mathcal{R}_{\mathbf{FDE}}$ for **FDE** except that we replace $(\sim\sim 1)$ and $(\sim\sim 2)$ by the following rules.

$$\frac{\sim_1\sim_1 A}{A}\ (\sim_1\sim_1 1) \qquad \frac{A}{\sim_1\sim_1 A}\ (\sim_1\sim_1 2)$$

$$\frac{\sim_2\sim_2 A}{A \vee \sim_2 A}\ (\sim_2\sim_2 1) \qquad \frac{A}{\sim_2\sim_2 A}\ (\sim_2\sim_2 2) \qquad \frac{\sim_2 A}{\sim_2\sim_2 A}\ (\sim_2\sim_2 3)$$

$$\frac{\sim_3\sim_3 A \quad \sim_3 A}{A}\ (\sim_3\sim_3 1) \qquad \frac{A}{\sim_3\sim_3 A}\ (\sim_3\sim_3 2) \qquad \frac{}{\sim_3 A \vee \sim_3\sim_3 A}\ (\sim_3\sim_3 3)$$

$$\frac{}{\sim_4\sim_4 A}\ (\sim_4\sim_4)$$

[4] One can also present the system as in [38, p. 304] using two-way rules with double lines. However, for the purpose of making the connection more smooth to bilateral natural deduction systems, we will adopt the presentation by Prawitz.

$$\frac{\sim_5 A \quad \sim_5\sim_5 A}{B}\ (\sim_5\sim_5 1) \quad \frac{\sim_5\sim_5 A}{A}\ (\sim_5\sim_5 2) \quad \frac{A}{\sim_5 A \vee \sim_5\sim_5 A}\ (\sim_5\sim_5 3)$$

$$\frac{A \quad \sim_6 A \quad \sim_6\sim_6 A}{B}\ (\sim_6\sim_6 1) \quad \frac{\sim_6\sim_6 A}{A \vee \sim_6 A}\ (\sim_6\sim_6 2) \quad \frac{A}{\sim_6 A \vee \sim_6\sim_6 A}\ (\sim_6\sim_6 3)$$

$$\frac{\sim_6 A}{A \vee \sim_6\sim_6 A}\ (\sim_6\sim_6 4) \quad \frac{\sim_7 A \quad \sim_7\sim_7 A}{B}\ (\sim_7\sim_7 1) \quad \frac{}{\sim_7 A \vee \sim_7\sim_7 A}\ (\sim_7\sim_7 2)$$

$$\frac{A \quad \sim_8 A \quad \sim_8\sim_8 A}{B}\ (\sim_8\sim_8 1) \quad \frac{}{A \vee \sim_8\sim_8 A}\ (\sim_8\sim_8 2) \quad \frac{}{\sim_8 A \vee \sim_8\sim_8 A}\ (\sim_8\sim_8 3)$$

$$\frac{\sim_9\sim_9 A}{A}\ (\sim_9\sim_9 1) \quad \frac{\sim_9\sim_9 A}{\sim_9 A}\ (\sim_9\sim_9 2) \quad \frac{A \quad \sim_9 A}{\sim_9\sim_9 A}\ (\sim_9\sim_9 3)$$

$$\frac{\sim_{10}\sim_{10} A}{\sim_{10} A}\ (\sim_{10}\sim_{10} 1) \quad \frac{\sim_{10} A}{\sim_{10}\sim_{10} A}\ (\sim_{10}\sim_{10} 2)$$

$$\frac{A \quad \sim_{11}\sim_{11} A}{\sim_{11} A}\ (\sim_{11}\sim_{11} 1) \quad \frac{\sim_{11} A \quad \sim_{11}\sim_{11} A}{A}\ (\sim_{11}\sim_{11} 2)$$

$$\frac{}{A \vee \sim_{11} A \vee \sim_{11}\sim_{11} A}\ (\sim_{11}\sim_{11} 3) \quad \frac{A \quad \sim_{11} A}{\sim_{11}\sim_{11} A}\ (\sim_{11}\sim_{11} 4)$$

$$\frac{A \quad \sim_{12}\sim_{12} A}{\sim_{12} A}\ (\sim_{12}\sim_{12} 1) \quad \frac{}{A \vee \sim_{12}\sim_{12} A}\ (\sim_{12}\sim_{12} 2) \quad \frac{\sim_{12} A}{\sim_{12}\sim_{12} A}\ (\sim_{12}\sim_{12} 3)$$

$$\frac{\sim_{13}\sim_{13} A}{B}\ (\sim_{13}\sim_{13})$$

$$\frac{A \quad \sim_{14}\sim_{14} A}{B}\ (\sim_{14}\sim_{14} 1) \quad \frac{\sim_{14}\sim_{14} A}{\sim_{14} A}\ (\sim_{14}\sim_{14} 2) \quad \frac{\sim_{14} A}{A \vee \sim_{14}\sim_{14} A}\ (\sim_{14}\sim_{14} 3)$$

$$\frac{A \quad \sim_{15}\sim_{15} A}{B}\ (\sim_{15}\sim_{15} 1) \quad \frac{\sim_{15} A \quad \sim_{15}\sim_{15} A}{B}\ (\sim_{15}\sim_{15} 2)$$

$$\frac{}{A \vee \sim_{15} A \vee \sim_{15}\sim_{15} A}\ (\sim_{15}\sim_{15} 3) \quad \frac{A \quad \sim_{16}\sim_{16} A}{B}\ (\sim_{16}\sim_{16} 1) \quad \frac{}{A \vee \sim_{16}\sim_{16} A}\ (\sim_{16}\sim_{16} 2)$$

Based on these, given any set $\Sigma \cup \{A\}$ of formulas, $\Sigma \vdash^i_{\mathbf{FDE}} A$ iff for some finite $\Sigma' \subseteq \Sigma$, there is a derivation of $A$ from $\Sigma'$ in the calculus whose rule set is $\mathcal{R}_{\mathbf{FDE}^i}$.

**Definition 13.** The natural deduction rules $\mathcal{R}^i_{\mathbf{K3}}$ for $\mathbf{K3}^i$ are all the rules $\mathcal{R}_{\mathbf{K3}}$ for **K3** but replacing $(\sim\sim 1)$ and $(\sim\sim 2)$ by the following rules.

$$\frac{\sim_1\sim_1 A}{A}\ (\sim_1\sim_1 1) \quad \frac{A}{\sim_1\sim_1 A}\ (\sim_1\sim_1 2) \quad \frac{}{\sim_2 A \vee \sim_2\sim_2 A}\ (\sim_2\sim_2)$$

$$\frac{\sim_3\sim_3 A}{B}\ (\sim_3\sim_3) \quad \frac{A \quad \sim_4\sim_4 A}{B}\ (\sim_4\sim_4 1) \quad \frac{}{A \vee \sim_4 A \vee \sim_4\sim_4 A}\ (\sim_4\sim_4 2)$$

Based on these, given any set $\Sigma \cup \{A\}$ of formulas, $\Sigma \vdash^i_{\mathbf{K3}} A$ iff for some finite $\Sigma' \subseteq \Sigma$, there is a derivation of $A$ from $\Sigma'$ in the calculus whose rule set is $\mathcal{R}_{\mathbf{K3}^i}$.

**Definition 14.** The natural deduction rules $\mathcal{R}^i_{\mathbf{LP}}$ for $\mathbf{LP}^i$ are all the rules $\mathcal{R}_{\mathbf{LP}}$ for **LP** but replacing $(\sim\sim 1)$ and $(\sim\sim 2)$ by the following rules.

$$\frac{\sim_1\sim_1 A}{A}\ (\sim_1\sim_1 1) \quad \frac{A}{\sim_1\sim_1 A}\ (\sim_1\sim_1 2) \quad \frac{}{\sim_2\sim_2 A}\ (\sim_2\sim_2) \quad \frac{\sim_3 A \quad \sim_3\sim_3 A}{B}\ (\sim_3\sim_3)$$

$$\frac{A \quad \sim_4 A \quad \sim_4\sim_4 A}{B}\ (\sim_4\sim_4 1) \quad \frac{}{A \vee \sim_4\sim_4 A}\ (\sim_4\sim_4 2)$$

Based on these, given any set $\Sigma \cup \{A\}$ of formulas, $\Sigma \vdash^i_{\mathbf{LP}} A$ iff for some finite $\Sigma' \subseteq \Sigma$, there is a derivation of $A$ from $\Sigma'$ in the calculus whose rule set is $\mathcal{R}_{\mathbf{LP}^i}$.

3.2. **Bilateral Natural Deduction.** We will present *bilateral* natural deduction systems for the consequence relations $\models^{i}_{\mathbf{FDE}}$ ($i \in \{1,\ldots,16\}$), $\models^{i}_{\mathbf{K3}}$ ($i \in \{1,\ldots,4\}$), and $\models^{i}_{\mathbf{LP}}$ ($i \in \{1,\ldots,4\}$) along the lines of [48]. These calculi make use of pure (separated) introduction and elimination rules, i.e., rules that introduce into the conclusion or eliminate from the premises only a single connective as the main connective of a compound formula. The systems are, therefore, interesting from the point of view of proof-theoretic semantics, because their rules can be seen as laying down the meaning of the connectives inferentially. We will present the bilateral rules in the style of the natural deduction rules from §3.1, but now with a distinction drawn between proofs and disproofs (refutations) from assumptions that are taken to be true and counterassumptions that are taken to be definitely false. We use single lines in the notation for proofs and double lines in the notation for refutations. Thus, in this section, double lines indicate disproofs. In particular, we write $\overline{A}$ to denote a proof of $A$ from $A$ as an assumption, and $\overline{\overline{A}}$ to denote a refutation of $A$ from $A$ as a counterassumption. This gives the inductive base for a definition of the set of proofs and refutations in any of the systems we will consider. A permitted discharge of assumptions is indicated by square brackets, $[\ ]$, and a permitted discharge of counterassumptions is indicated by double square brackets, $[\![\ ]\!]$. We will simplify the notation by writing $[A]$ instead of $[\overline{A}]$ and $[\![A]\!]$ instead of $[\![\overline{\overline{A}}]\!]$. Moreover, if $\Sigma$ is a set of formulas, then $\Sigma^{+}$ is defined as the set $\{\overline{A} : A \in \Sigma\}$ and $\Sigma^{-}$ as $\{\overline{\overline{A}} : A \in \Sigma\}$.

The introduction and elimination rules for conjunctions and disjunctions from §3.1 then take the following form:[5]

$$\frac{\overline{A} \quad \overline{B}}{A \wedge B} \qquad \frac{\overline{A \wedge B}}{A} \qquad \frac{\overline{A \wedge B}}{B} \qquad \frac{\overline{A}}{A \vee B} \qquad \frac{\overline{B}}{A \vee B} \qquad \frac{\overline{A \vee B} \quad \begin{array}{c}[A]\\ \vdots \\ \overline{C}\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ \overline{C}\end{array}}{C}$$

In the present setup, the dotted lines indicate derivations that may be built up from *both* refutations and proofs. Instead of rules for introducing and eliminating negated conjunctions, disjunctions, and negations into and from proofs, we have rules for introducing and removing disjunctions, conjunctions, and negations into and from disproofs.

**Definition 15.** The set of natural deduction rules $\mathfrak{R}_{\mathbf{FDE}}$ for **FDE** consists of the above rules for $\wedge$ and $\vee$ together with:

$$\begin{array}{c}\overline{\overline{A}} \quad \overline{\overline{B}}\\ \hline\hline A \vee B\end{array} \qquad \begin{array}{c}\overline{\overline{A \vee B}}\\ \hline\hline A\end{array} \qquad \begin{array}{c}\overline{\overline{A \vee B}}\\ \hline\hline B\end{array} \qquad \begin{array}{c}\overline{\overline{A}}\\ \hline\hline A \wedge B\end{array} \qquad \begin{array}{c}\overline{\overline{B}}\\ \hline\hline A \wedge B\end{array} \qquad \begin{array}{c}\overline{\overline{A \wedge B}} \quad \begin{array}{c}[\![A]\!]\\ \vdots \\ \overline{\overline{C}}\end{array} \quad \begin{array}{c}[\![B]\!]\\ \vdots \\ \overline{\overline{C}}\end{array}\\ \hline\hline C\end{array}$$

and the following rules for introducing and eliminating negations into and from proofs and refutations:

$$\frac{\overline{\overline{A}}}{\sim A} \qquad \begin{array}{c}\overline{\sim A}\\ \hline\hline A\end{array} \qquad \begin{array}{c}\overline{A}\\ \hline\hline \sim A\end{array}\ (\sim\sim 1) \qquad \frac{\overline{\overline{\sim A}}}{A}\ (\sim\sim 2)$$

[5] This is the way how these rules are presented in [44], though without abbreviating $[\overline{A}]$ as $[A]$.

Moreover, for the sets of natural deduction rules $\mathfrak{R}_{\mathbf{K3}}$ and $\mathfrak{R}_{\mathbf{LP}}$ for **K3** and **LP**, respectively, we add the rule ECQ and the dilemma rule DIL, respectively, which express a certain interaction between proofs and disproofs:

$$\dfrac{\overline{A} \quad \overline{\overline{A}}}{B}\ (\mathsf{ECQ}) \qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B}\ (\mathsf{DIL})$$

Let $\Sigma \cup \Gamma \cup \{A\}$ be a set of formulas. Then $\Sigma^+ \cup \Gamma^- \vdash^+_{\mathfrak{R}\mathbf{FDE}} A$ ($\Sigma^+ \cup \Gamma^- \vdash^-_{\mathfrak{R}\mathbf{FDE}} A$) iff for some finite $\Sigma' \subseteq \Sigma$ and finite $\Gamma' \subseteq \Gamma$, there is a proof (disproof) of $A$ from $\Sigma'^+ \cup \Gamma'^-$ in the calculus whose rule set is $\mathfrak{R}_{\mathbf{FDE}}$. In the same way, we define the relations $\vdash^+_{\mathfrak{R}\mathbf{K3}}$, $\vdash^-_{\mathfrak{R}\mathbf{K3}}$, $\vdash^+_{\mathfrak{R}\mathbf{LP}}$, and $\vdash^-_{\mathfrak{R}\mathbf{LP}}$.

**Definition 16.** The set of rules $\mathfrak{R}^i_{\mathbf{FDE}}$ for $\mathbf{FDE}^i$, with $i \in \{1, \dots, 16\}$, consists of all the rules of $\mathfrak{R}_{\mathbf{FDE}}$ for **FDE**, but the rules for $\sim$ are replaced by the following introduction and elimination rules:

$$\dfrac{\overline{\overline{A}}}{\sim_i A} \qquad \dfrac{\overline{\sim_i A}}{\overline{A}} \qquad\qquad \dfrac{\overline{A}}{\overline{\sim_1 A}} \qquad \dfrac{\overline{\overline{\sim_1 A}}}{A}$$

$$\dfrac{\overline{\overline{\sim_2 A}} \quad \begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B} \qquad \dfrac{\overline{A}}{\overline{\sim_2 A}} \qquad \dfrac{\overline{\overline{A}}}{\overline{\sim_2 A}} \qquad \dfrac{\overline{\overline{\sim_3 A}} \quad \overline{\overline{A}}}{A} \qquad \dfrac{\overline{A}}{\overline{\sim_3 A}} \qquad \dfrac{\begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket \sim_3 A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B}$$

$$\overline{\overline{\sim_4 A}} \qquad \dfrac{\overline{\overline{A}} \quad \overline{\overline{\sim_5 A}}}{B} \qquad \dfrac{\overline{\overline{\sim_5 A}}}{A} \qquad \dfrac{\overline{A} \quad \begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket \sim_5 A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B}$$

$$\dfrac{\overline{A} \quad \overline{\overline{A}} \quad \overline{\overline{\sim_6 A}}}{B} \qquad \dfrac{\overline{\overline{\sim_6 A}} \quad \begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B} \qquad \dfrac{\overline{A} \quad \begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket \sim_6 A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B} \qquad \dfrac{\overline{\overline{A}} \quad \begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket \sim_6 A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B}$$

$$\dfrac{\overline{\overline{A}} \quad \overline{\overline{\sim_7 A}}}{B} \qquad \dfrac{\begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket \sim_7 A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B} \qquad \dfrac{\overline{A} \quad \overline{\overline{A}} \quad \overline{\overline{\sim_8 A}}}{B} \qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket \sim_8 A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B} \qquad \dfrac{\begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket \sim_8 A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B}$$

$$\dfrac{\overline{\overline{\sim_9 A}}}{A} \qquad \dfrac{\overline{\overline{\sim_9 A}}}{\overline{A}} \qquad \dfrac{\overline{A} \quad \overline{\overline{A}}}{\overline{\sim_9 A}} \qquad \dfrac{\overline{\overline{A}}}{\overline{\sim_{10} A}} \qquad \dfrac{\overline{\overline{\sim_{10} A}}}{\overline{A}}$$

$$\dfrac{\overline{A} \quad \overline{\overline{\sim_{11} A}}}{\overline{A}} \qquad \dfrac{\overline{\overline{A}} \quad \overline{\overline{\sim_{11} A}}}{A} \qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket \sim_{11} A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B} \qquad \dfrac{\overline{A} \quad \overline{\overline{A}}}{\overline{\sim_{11} A}}$$

$$\dfrac{\overline{A} \quad \overline{\overline{\sim_{12} A}}}{\overline{A}} \qquad \dfrac{\begin{array}{c}\llbracket A\rrbracket\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}\llbracket \sim_{12} A\rrbracket\\ \vdots\\ \overline{B}\end{array}}{B} \qquad \dfrac{\overline{\overline{A}}}{\overline{\sim_{12} A}} \qquad \dfrac{\overline{\overline{\sim_{13} A}}}{B}$$

$$\dfrac{\overline{A}\quad\overline{\overline{\sim_{14}A}}}{B}\qquad \dfrac{\overline{\overline{\sim_{14}A}}}{\overline{\overline{A}}}\qquad \dfrac{\overline{\overline{A}}\quad\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array}\quad\begin{array}{c}[\![\sim_{14}A]\!]\\ \vdots\\ \overline{B}\end{array}}{B}$$

$$\dfrac{\overline{A}\quad\overline{\overline{\sim_{15}A}}}{B}\qquad \dfrac{\overline{\overline{A}}\quad\overline{\overline{\sim_{15}A}}}{B}\qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array}\quad\begin{array}{c}[\![A]\!]\\ \vdots\\ \overline{B}\end{array}\quad\begin{array}{c}[\![\sim_{15}A]\!]\\ \vdots\\ \overline{B}\end{array}}{B}\qquad \dfrac{\overline{A}\quad\overline{\overline{\sim_{16}A}}}{B}\qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array}\quad\begin{array}{c}[\![\sim_{16}A]\!]\\ \vdots\\ \overline{B}\end{array}}{B}$$

Let $\Sigma\cup\Gamma\cup\{A\}$ be a set of formulas. Then $\Sigma^{+}\cup\Gamma^{-}\vdash^{i+}_{\mathfrak{R}\mathbf{FDE}}A$ ($\Sigma^{+}\cup\Gamma^{-}\vdash^{i-}_{\mathfrak{R}\mathbf{FDE}}A$) iff for some finite $\Sigma'\subseteq\Sigma$ and finite $\Gamma'\subseteq\Gamma$, there is a proof (disproof) of $A$ from $\Sigma'^{+}$ $\cup\,\Gamma'^{-}$ in the calculus whose rule set is $\mathfrak{R}^{i}_{\mathbf{FDE}}$.

**Definition 17.** For $i\in\{1,2,3,4\}$, the set of natural deduction rules $\mathfrak{R}^{i}_{\mathbf{K3}}$ for $\mathbf{K3}^{i}$ consists all the rules $\mathfrak{R}_{\mathbf{K3}}$ for **K3**, but the rules for $\sim$ are replaced by the following rules:

$$\dfrac{\overline{\overline{A}}}{\sim_i A}\quad \dfrac{\overline{\overline{\sim_i A}}}{\overline{A}}\qquad \dfrac{\overline{A}}{\overline{\sim_1 A}}\quad \dfrac{\overline{\overline{\sim_1 A}}}{A}\qquad \dfrac{\begin{array}{c}[\![A]\!]\\ \vdots\\ \overline{B}\end{array}\quad\begin{array}{c}[\![\sim_{2}A]\!]\\ \vdots\\ \overline{B}\end{array}}{B}$$

$$\dfrac{\overline{\overline{\sim_3 A}}}{B}\qquad \dfrac{\overline{A}\quad\overline{\overline{\sim_4 A}}}{B}\qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array}\quad\begin{array}{c}[\![A]\!]\\ \vdots\\ \overline{B}\end{array}\quad\begin{array}{c}[\![\sim_{4}A]\!]\\ \vdots\\ \overline{B}\end{array}}{B}$$

Let $\Sigma\cup\Gamma\cup\{A\}$ be a set of formulas. Then $\Sigma^{+}\cup\Gamma^{-}\vdash^{i+}_{\mathfrak{R}\mathbf{K3}}A$ ($\Sigma^{+}\cup\Gamma^{-}\vdash^{i-}_{\mathfrak{R}\mathbf{K3}}A$) iff for some finite $\Sigma'\subseteq\Sigma$ and finite $\Gamma'\subseteq\Gamma$, there is a proof (disproof) of $A$ from $\Sigma'^{+}\cup$ $\Gamma'^{-}$ in the calculus whose rule set is $\mathfrak{R}^{i}_{\mathbf{K3}}$.

**Definition 18.** For $i\in\{1,2,3,4\}$, the set of rules $\mathfrak{R}^{i}_{\mathbf{LP}}$ for $\mathbf{LP}^{i}$ comprises all the rules $\mathfrak{R}_{\mathbf{LP}}$ for **LP**, but the rules for $\sim$ are replaced by the following rules:

$$\dfrac{\overline{\overline{A}}}{\sim_i A}\quad \dfrac{\overline{\overline{\sim_i A}}}{\overline{A}}\qquad \dfrac{\overline{A}}{\overline{\sim_1 A}}\quad \dfrac{\overline{\overline{\sim_1 A}}}{A}\qquad \overline{\overline{\sim_2 A}}\qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array}\quad\begin{array}{c}[\![\sim_{4}A]\!]\\ \vdots\\ \overline{B}\end{array}}{B}$$

$$\dfrac{\overline{\overline{A}}\quad\overline{\overline{\sim_3 A}}}{B}\qquad \dfrac{\overline{A}\quad\overline{\overline{A}}\quad\overline{\overline{\sim_4 A}}}{B}$$

Let $\Sigma\cup\Gamma\cup\{A\}$ be a set of formulas. Then $\Sigma^{+}\cup\Gamma^{-}\vdash^{i+}_{\mathfrak{R}\mathbf{LP}}A$ ($\Sigma^{+}\cup\Gamma^{-}\vdash^{i-}_{\mathfrak{R}\mathbf{LP}}A$) iff for some finite $\Sigma'\subseteq\Sigma$ and finite $\Gamma'\subseteq\Gamma$, there is a proof (disproof) of $A$ from $\Sigma'^{+}\cup\Gamma'^{-}$ in the calculus whose rule set is $\mathfrak{R}^{i}_{\mathbf{LP}}$.

We show the bilateral systems to be equivalent with their unilateral counterparts.

**Theorem 19.** *Let $\tau(\Delta^{-})=\{\sim A : \overline{\overline{A}}\in\Delta^{-}\}$. Then (1) $\Sigma^{+}\cup\Gamma^{-}\vdash^{i+}_{\mathfrak{R}FDE}A$ iff $\Sigma\cup\tau(\Gamma^{-})\vdash^{i}_{\mathcal{R}FDE}A$ and (2) $\Sigma^{+}\cup\Gamma^{-}\vdash^{i-}_{\mathfrak{R}FDE}A$ iff $\Sigma\cup\tau(\Gamma^{-})\vdash^{i}_{\mathcal{R}FDE}\sim A$.*

*Proof.* By induction on derivations in $\mathfrak{R}^{i}_{\mathbf{FDE}}$ and $\mathcal{R}^{i}_{\mathbf{FDE}}$. The cases of the rules for introducing and eliminating conjunctions and disjunctions into and from proofs are obvious. We present some of the remaining cases. Direction from left to right, claim (1). By applying the definition of derivations, the induction hypothesis, and rules of $\mathcal{R}^{i}_{\mathbf{FDE}}$, for the derivations on the left we obtain the derivations on the right:

$$\overline{A} \qquad A \qquad \dfrac{\dfrac{}{\sim_9 A}}{A} \qquad \dfrac{\dfrac{}{\sim_2 A} \quad \begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}[\![A]\!]\\ \vdots\\ \overline{B}\end{array}}{B} \qquad \dfrac{\dfrac{\sim_2\sim_2 A}{A\vee\sim_2 A} \quad \begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}[\sim_2 A]\\ \vdots\\ \overline{B}\end{array}}{B} \qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}[\![A]\!]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}[\![\sim_{11} A]\!]\\ \vdots\\ \overline{B}\end{array}}{B}$$

$$\dfrac{\sim_9\sim_9 A}{A} \qquad \dfrac{A\vee(\sim_{11}A\vee\sim_{11}\sim_{11}A) \quad \begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \dfrac{[\sim_{11}A\vee\sim_{11}\sim_{11}A] \quad \begin{array}{c}[\sim_{11} A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}[\sim_{11}\sim_{11} A]\\ \vdots\\ \overline{B}\end{array}}{B}}{B}$$

Direction from left to right, claim (2). By applying the definition of derivations, the induction hypothesis, and rules of $\mathcal{R}^i_{\mathbf{FDE}}$, for the first derivations we obtain the second derivations with the same subscript:

$$\overline{\overline{A}} \qquad \sim A \qquad \dfrac{\overline{A} \quad \overline{\overline{A}}}{\sim_9 A} \qquad \dfrac{A \quad \sim_9 A}{\sim_9\sim_9 A} \qquad \dfrac{\dfrac{}{A\wedge B} \quad \begin{array}{c}[\![A]\!]\\ \vdots\\ \overline{\overline{C}}\end{array} \quad \begin{array}{c}[\![B]\!]\\ \vdots\\ \overline{\overline{C}}\end{array}}{C} \qquad \dfrac{\dfrac{\sim_i(A\wedge B)}{\sim_i A\vee\sim_i B} \quad \begin{array}{c}[\sim_i A]\\ \vdots\\ \overline{\sim_i C}\end{array} \quad \begin{array}{c}[\sim_i B]\\ \vdots\\ \overline{\sim_i C}\end{array}}{\sim_i C}$$

Direction from right to left, claim (1). By applying the definition of derivations, the induction hypothesis, and rules of $\mathfrak{R}^i_{\mathbf{FDE}}$, for the first derivations we obtain the second derivations with the same subscript:

$$A \qquad \overline{A} \qquad \dfrac{\sim_9\sim_9 A}{A} \qquad \dfrac{\dfrac{}{\sim_9 A}}{A} \qquad \dfrac{\sim_{13}\sim_{13} A}{B} \qquad \dfrac{\dfrac{}{\sim_{13} A}}{B}$$

Direction from right to left, claim (2). By applying the definition of derivations, the induction hypothesis, and rules of $\mathfrak{R}^i_{\mathbf{FDE}}$, for the first derivations we obtain the second derivations with the same subscript:

$$\sim A \qquad \overline{\overline{A}} \qquad \dfrac{\sim_i A\wedge\sim_i B}{\sim_i(A\vee B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\sim_i A\wedge\sim_i B}}{\sim_i A}}{A} \quad \dfrac{\dfrac{\dfrac{}{\sim_i A\wedge\sim_i B}}{\sim_i B}}{B}}{A\vee B} \qquad \dfrac{A \quad \sim_9 A}{\sim_9\sim_9 A} \qquad \dfrac{\overline{A} \quad \overline{\overline{A}}}{\sim_9 A} \quad \triangleleft$$

**Theorem 20.** *Let* $\tau(\Delta^-) = \{\sim A : \overline{\overline{A}} \in \Delta^-\}$. *Then (1)* $\Sigma^+\cup\Gamma^- \vdash^{i+}_{\mathfrak{R}\mathbf{K3}} A$ *iff* $\Sigma\cup\tau(\Gamma^-)\vdash^i_{\mathcal{R}\mathbf{K3}} A$ *and (2)* $\Sigma^+\cup\Gamma^- \vdash^{i-}_{\mathfrak{R}\mathbf{K3}} A$ *iff* $\Sigma\cup\tau(\Gamma^-)\vdash^i_{\mathcal{R}\mathbf{K3}} \sim A$.

*Proof.* By induction on derivations in $\mathfrak{R}^i_{\mathbf{K3}}$ and $\mathcal{R}^i_{\mathbf{K3}}$. We present only one more interesting case for the direction from right to left, claim (1). By applying the definition of derivations, the induction hypothesis, and a rule of $\mathfrak{R}^4_{\mathbf{K3}}$, for the first derivations we obtain the second derivations with the same subscript:

$$\dfrac{}{A\vee(\sim_4 A\vee\sim_4\sim_4 A)}$$

$$\dfrac{\dfrac{[A]}{A\vee(\sim_4 A\vee\sim_4\sim_4 A)} \quad \dfrac{\dfrac{\dfrac{[\![A]\!]}{\sim_4 A}}{\sim_4 A\vee\sim_4\sim_4 A}}{A\vee(\sim_4 A\vee\sim_4\sim_4 A)} \quad \dfrac{\dfrac{\dfrac{[\![\sim_4 A]\!]}{\sim_4\sim_4 A}}{\sim_4 A\vee\sim_4\sim_4 A}}{A\vee(\sim_4 A\vee\sim_4\sim_4 A)}}{A\vee(\sim_4 A\vee\sim_4\sim_4 A)} \quad \triangleleft$$

**Theorem 21.** *Let* $\tau(\Delta^-) = \{\sim A : \overline{\overline{A}} \in \Delta^-\}$. *Then (1)* $\Sigma^+ \cup \Gamma^- \vdash^{i+}_{\mathfrak{N}LP} A$ *iff* $\Sigma \cup \tau(\Gamma^-) \vdash^{i}_{\mathcal{R}LP} A$ *and (2)* $\Sigma^+ \cup \Gamma^- \vdash^{i-}_{\mathfrak{N}LP} A$ *iff* $\Sigma \cup \tau(\Gamma^-) \vdash^{i}_{\mathcal{R}LP} \sim A$.

*Proof.* By induction on derivations in $\mathfrak{N}^i_{\mathbf{LP}}$ and $\mathcal{R}^i_{\mathbf{LP}}$. We present only the more interesting cases. Direction from left to right, claim (1). By applying the induction hypothesis and the rule for eliminating disjunctions from proofs in $\mathcal{R}^i_{\mathbf{LP}}$, for the first derivations we obtain the second derivations with the same subscript:

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ \overline{B}\end{matrix} \qquad \begin{matrix}[\![A]\!] \\ \vdots \\ \overline{B}\end{matrix}}{B}\ (\mathsf{DIL}) \qquad\qquad \dfrac{\overline{A \vee \sim_i A} \qquad \begin{matrix}[A] \\ \vdots \\ \overline{B}\end{matrix} \qquad \begin{matrix}[\sim_i A] \\ \vdots \\ \overline{B}\end{matrix}}{B}$$

Direction from right to left, claim (1). By applying rules from $\mathfrak{N}^i_{\mathbf{LP}}$, for the first derivations we obtain the second derivations with the same subscript:

$$\dfrac{}{A \vee \sim_i A} \qquad\qquad \dfrac{\dfrac{[A]}{A \vee \sim_i A} \qquad \dfrac{\dfrac{[\![A]\!]}{\sim_i A}}{A \vee \sim_i A}}{A \vee \sim_i A}\ (\mathsf{DIL})$$

4. Soundness and Completeness

**Theorem 22** (Soundness)**.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *(1)* $\Gamma \vdash^i_{\mathbf{FDE}} A$ *only if* $\Gamma \models^i_{\mathbf{FDE}} A$, *(2)* $\Gamma \vdash^i_{\mathbf{K3}} A$ *only if* $\Gamma \models^i_{\mathbf{K3}} A$, *and (3)* $\Gamma \vdash^i_{\mathbf{LP}} A$ *only if* $\Gamma \models^i_{\mathbf{LP}} A$.

*Proof.* Tedious, but standard. ◁

For the completeness direction, we prepare some well known notions and lemmas.

**Definition 23.** Let $\Sigma$ be a set of formulas. Then, $\Sigma$ is a *theory* iff $\Sigma \vdash A$ implies $A \in \Sigma$, and $\Sigma$ is *prime* iff $A \vee B \in \Sigma$ implies $A \in \Sigma$ or $B \in \Sigma$.

**Lemma 24** (Lindenbaum)**.** *If* $\Sigma \nvdash A$, *then there is* $\Sigma' \supseteq \Sigma$ *such that* $\Sigma' \nvdash A$ *and* $\Sigma'$ *is a prime theory.*

We now define the canonical valuation in the usual manner.

**Definition 25.** For any $\Sigma \subseteq \mathsf{Form}$, let $v^i_\Sigma$ from $\mathsf{Prop}$ to $\{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ be defined as follows:

$$v^i_\Sigma(p) := \begin{cases} \mathbf{t} \text{ iff } \Sigma \vdash^i_{\mathbf{FDE}} p \text{ and } \Sigma \nvdash^i_{\mathbf{FDE}} \sim p; \\ \mathbf{b} \text{ iff } \Sigma \vdash^i_{\mathbf{FDE}} p \text{ and } \Sigma \vdash^i_{\mathbf{FDE}} \sim p; \\ \mathbf{n} \text{ iff } \Sigma \nvdash^i_{\mathbf{FDE}} p \text{ and } \Sigma \nvdash^i_{\mathbf{FDE}} \sim p; \\ \mathbf{f} \text{ iff } \Sigma \nvdash^i_{\mathbf{FDE}} p \text{ and } \Sigma \vdash^i_{\mathbf{FDE}} \sim p. \end{cases}$$

The following lemma is the key for the completeness result.

**Lemma 26.** *If* $\Sigma$ *is a prime theory, then the following hold for all* $B \in \mathsf{Form}$.

$$v^i_\Sigma(B) = \begin{cases} \mathbf{t} \textit{ iff } \Sigma \vdash^i_{\mathbf{FDE}} B \textit{ and } \Sigma \nvdash^i_{\mathbf{FDE}} \sim B; \\ \mathbf{b} \textit{ iff } \Sigma \vdash^i_{\mathbf{FDE}} B \textit{ and } \Sigma \vdash^i_{\mathbf{FDE}} \sim B; \\ \mathbf{n} \textit{ iff } \Sigma \nvdash^i_{\mathbf{FDE}} B \textit{ and } \Sigma \nvdash^i_{\mathbf{FDE}} \sim B; \\ \mathbf{f} \textit{ iff } \Sigma \nvdash^i_{\mathbf{FDE}} B \textit{ and } \Sigma \vdash^i_{\mathbf{FDE}} \sim B. \end{cases}$$

*Proof.* Note first that it is obvious that $v_\Sigma$ well defined. Then the desired result is proved by induction on the construction of $B$. The base case, for atomic formulas, is obvious by the definition. For the induction step, the cases are split based on the connectives. We will here only deal with the case for negation of $\mathbf{FDE}^{16}$.

$v^{16}_\Sigma(\sim_{16} B) = \mathbf{t}$ iff $v^{16}_\Sigma(B) = \mathbf{b}$ (by the definition of $v^{16}_\Sigma$) iff $\Sigma \vdash^{16}_{\mathbf{FDE}} B$ and $\Sigma \vdash^{16}_{\mathbf{FDE}} \sim_{16} B$ (by IH) iff $\Sigma \vdash^{16}_{\mathbf{FDE}} \sim_{16} B$ and $\Sigma \nvdash^{16}_{\mathbf{FDE}} \sim_{16}\sim_{16} B$ (by $(\sim_{15}\sim_{15} 1)$ for the left-to-right direction and $(\sim_{16}\sim_{16} 2)$ for the other direction).

$v^{16}_\Sigma(\sim_{16} B) = \mathbf{b}$ iff $v^{16}_\Sigma(B) = \mathbf{f}$ (by the definition of $v^{16}_\Sigma$) iff $\Sigma \nvdash^{16}_{\mathbf{FDE}} B$ and $\Sigma \vdash^{16}_{\mathbf{FDE}} \sim_{16} B$ (by IH) iff $\Sigma \vdash^{16}_{\mathbf{FDE}} \sim_{16} B$ and $\Sigma \vdash^{16}_{\mathbf{FDE}} \sim_{16}\sim_{16} B$ (by $(\sim_{15}\sim_{15} 2)$ for the left-to-right direction and $(\sim_{16}\sim_{16} 1)$ for the other direction).

$v^{16}_\Sigma(\sim_{16} B) = \mathbf{n}$ iff $v^{16}_\Sigma(B) = \mathbf{t}$ (by the definition of $v^{16}_\Sigma$) iff $\Sigma \vdash^{16}_{\mathbf{FDE}} B$ and $\Sigma \nvdash^{16}_{\mathbf{FDE}} \sim_{16} B$ (by IH) iff $\Sigma \nvdash^{16}_{\mathbf{FDE}} \sim_{16} B$ and $\Sigma \nvdash^{16}_{\mathbf{FDE}} \sim_{16}\sim_{16} B$ (by $(\sim_{16}\sim_{16} 1)$ for the left-to-right direction and $(\sim_{16}\sim_{16} 2)$ for the other direction).

$v^{16}_\Sigma(\sim_{16} B) = \mathbf{f}$ iff $v^{16}_\Sigma(B) = \mathbf{n}$ (by the definition of $v^{16}_\Sigma$) iff $\Sigma \nvdash^{16}_{\mathbf{FDE}} B$ and $\Sigma \nvdash^{16}_{\mathbf{FDE}} \sim_{16} B$ (by IH) iff $\Sigma \nvdash^{16}_{\mathbf{FDE}} \sim_{16} B$ and $\Sigma \vdash^{16}_{\mathbf{FDE}} \sim_{16}\sim_{16} B$ (by $(\sim_{16}\sim_{16} 2)$ for the left-to-right direction and $(\sim_{16}\sim_{16} 1)$ for the other direction).

The other cases are left to the interested readers to be written out in detail. ◁

For the variations of **K3** and **LP**, we need to eliminate the values **b** and **n**, respectively.

We are now ready to prove the completeness result.

**Theorem 27** (Completeness). *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, (1) $\Gamma \vDash^i_{\boldsymbol{FDE}} A$ only if $\Gamma \vdash^i_{\boldsymbol{FDE}} A$, (2) $\Gamma \vDash^i_{\boldsymbol{K3}} A$ only if $\Gamma \vdash^i_{\boldsymbol{K3}} A$, and (3) $\Gamma \vDash^i_{\boldsymbol{LP}} A$ only if $\Gamma \vdash^i_{\boldsymbol{LP}} A$.*

*Proof.* We only deal with the case for $\mathbf{FDE}^i$ since other cases can be established in the same manner. Assume $\Gamma \nvdash^i_{\mathbf{FDE}} A$. Then, by Lemma 24, there is a $\Sigma \supseteq \Gamma$ such that $\Sigma$ is a prime theory and $A \notin \Sigma$, and by Lemma 26, a four-valued valuation $v_\Sigma$ can be defined with $I_\Sigma(B) \in \mathcal{D}$ for every $B \in \Gamma$ and $I_\Sigma(A) \notin \mathcal{D}$. Thus it follows that $\Gamma \nvDash^i_{\mathbf{FDE}} A$, as desired. ◁

## 5. Basic Results

### 5.1. Negation Inconsistency and Negation Incompleteness.

As one may easily observe, all variants of **FDE** are both paraconsistent and paracomplete, all variants of **K3** are paracomplete, but not paraconsistent, and all variants of **LP** are paraconsistent, but not paracomplete. However, for some of the variants, some stronger properties than paraconsistency and paracompleteness hold. The stronger properties we have in mind are the following.

**Definition 28.** A logic **L** is *negation inconsistent* if for some $A$, we have both $B \vDash_{\mathbf{L}} A$ and $B \vDash_{\mathbf{L}} \sim A$ for all $B$. Moreover, a logic **L** is *negation incomplete* if for some $A$, both $A \vDash_{\mathbf{L}} B$ and $\sim A \vDash_{\mathbf{L}} B$ for all $B$.

Then, we obtain the following results.

**Theorem 29.** *$\boldsymbol{LP}^2$, $\boldsymbol{LP}^4$, $\boldsymbol{FDE}^4$, $\boldsymbol{FDE}^8$, $\boldsymbol{FDE}^{12}$ and $\boldsymbol{FDE}^{16}$ are negation inconsistent.*

*Proof.* We prove the result by showing the specific instances of inconsistency. For $\mathbf{LP}^2$, we have $B \vDash^2_{\mathbf{LP}} \sim_2\sim_2 A$ and $B \vDash^2_{\mathbf{LP}} \sim_2\sim_2\sim_2 A$. For $\mathbf{LP}^4$, we have $B \vDash^4_{\mathbf{LP}} \sim_4(A \wedge$

$\sim_4 A \wedge \sim_4 \sim_4 A)$ and $B \models^4_{\mathbf{LP}} \sim_4 \sim_4 (A \wedge \sim_4 A \wedge \sim_4 \sim_4 A)$. For $\mathbf{FDE}^4$, we have $B \models^4_{\mathbf{FDE}} \sim_4 \sim_4 A$ and $B \models^4_{\mathbf{FDE}} \sim_4 \sim_4 \sim_4 A$. For $\mathbf{FDE}^8$, we have $B \models^8_{\mathbf{FDE}} \sim_8 (A \wedge \sim_8 A \wedge \sim_8 \sim_8 A)$ and $B \models^8_{\mathbf{FDE}} \sim_8 \sim_8 (A \wedge \sim_8 A \wedge \sim_8 \sim_8 A)$. For $\mathbf{FDE}^{12}$, we have $B \models^{12}_{\mathbf{FDE}} \sim_{12} \sim_{12} \sim_{12} A$ and $B \models^{12}_{\mathbf{FDE}} \sim_{12} \sim_{12} \sim_{12} \sim_{12} A$. Finally, for $\mathbf{FDE}^{16}$, we have $B \models^{16}_{\mathbf{FDE}} \sim_{16} (A \wedge \sim_{16} \sim_{16} A)$ and $B \models^{16}_{\mathbf{FDE}} \sim_{16} \sim_{16} (A \wedge \sim_{16} \sim_{16} A)$. ◁

**Remark 30.** The other variants of **FDE**, **K3** and **LP** are not negation inconsistent. Indeed, for the variants of **K3**, this is obvious since they are all explosive. For the other variants of **FDE** and **LP**, note that negation inconsistency implies that there is a formula that receives the value **b** for all interpretations. But it is easy to see that this cannot be the case with these variants. For example, the subclassical variants have the set $\{\mathbf{t}, \mathbf{f}\}$ being closed under all three connectives. Similar arguments by looking at sets $\{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$ or $\{\mathbf{n}\}$ will establish the desired results.

**Theorem 31.** ***K3***$^3$*,* ***K3***$^4$*,* ***FDE***$^{13}$*,* ***FDE***$^{14}$*,* ***FDE***$^{15}$ *and* ***FDE***$^{16}$ *are negation incomplete.*

*Proof.* We prove the result by showing the specific instances of incompleteness. For $\mathbf{K3}^3$, we have $\sim_3 \sim_3 A \not\models^3_{\mathbf{K3}} B$ and $\sim_3 \sim_3 \sim_3 A \not\models^3_{\mathbf{K3}} B$. For $\mathbf{K3}^4$, we have $\sim_4 \sim_4 (A \wedge \sim_4 A \wedge \sim_4 \sim_4 A) \not\models^3_{\mathbf{K3}} B$ and $\sim_4 \sim_4 \sim_4 (A \wedge \sim_4 A \wedge \sim_4 \sim_4 A) \not\models^4_{\mathbf{K3}} B$. For $\mathbf{FDE}^{13}$, we have $\sim_{13} \sim_{13} A \not\models^{13}_{\mathbf{FDE}} B$ and $\sim_{13} \sim_{13} \sim_{13} A \not\models^{13}_{\mathbf{FDE}} B$. For $\mathbf{FDE}^{14}$, we have $\sim_{14} \sim_{14} \sim_{14} A \not\models^{14}_{\mathbf{FDE}} B$ and $\sim_{14} \sim_{14} \sim_{14} \sim_{14} A \not\models^{14}_{\mathbf{FDE}} B$. For $\mathbf{FDE}^{15}$, we have $\sim_{15} \sim_{15} (A \wedge \sim_{15} A \wedge \sim_{15} \sim_{15} A) \not\models^{15}_{\mathbf{FDE}} B$ and $\sim_{15} \sim_{15} \sim_{15} (A \wedge \sim_{15} A \wedge \sim_{15} \sim_{15} A) \not\models^{15}_{\mathbf{FDE}} B$. Finally, for $\mathbf{FDE}^{16}$, we have $\sim_{16} (A \vee \sim_{16} \sim_{16} A) \not\models^{16}_{\mathbf{FDE}} B$ and $\sim_{16} \sim_{16} (A \vee \sim_{16} \sim_{16} A) \not\models^{16}_{\mathbf{FDE}} B$. ◁

**Remark 32.** The other variants of **FDE**, **K3** and **LP** are not negation incomplete. Indeed, for the variants of **LP**, this is obvious since they all have (LEM). For the other variants of **FDE** and **K3**, note that negation incompleteness implies that there is a formula that receives the value **n** for all interpretations. But it is easy so see that this cannot be the case by similar considerations we sketched above for the cases with negation inconsistency.

5.2. **Functional Completeness.** We now turn to show that the matrices that characterize some of the contra-classical variants of **FDE**, **K3** and **LP** are functionally complete as a corollary of a general characterization of functional completeness. To this end, we first introduce some related notions.

**Definition 33** (Functional completeness)**.** An algebra $\mathfrak{A} = \langle A, f_1, \ldots, f_n \rangle$, is said to be *functionally complete* provided that every finitary function $f \colon A^m \to A$ is definable by compositions of the functions $f_1, \ldots, f_n$ alone. A matrix $\langle \mathfrak{A}, \mathcal{D} \rangle$ is *functionally complete* if $\mathfrak{A}$ is functionally complete.

**Definition 34** (Definitional completeness)**.** A logic **L** is *definitionally complete* if there exists a functionally complete matrix that is strongly adequate for *L*.

For the characterization of the functional completeness, the following theorem of Jerzy Słupecki is elegant and useful. In order to state the result, we need the following definition.

**Definition 35.** Let $\mathfrak{A} = \langle A, f_1, \ldots, f_n \rangle$ be an algebra, and $f$ be a binary operation defined in $\mathfrak{A}$. Then, $f$ is *unary reducible* iff for some unary operation $g$ definable in

$\mathfrak{A}$, $f(x,y) = g(x)$ for all $x,y \in A$ or $f(x,y) = g(y)$ for all $x,y \in A$. And $f$ is *essentially binary* if $f$ is not unary reducible.

**Theorem 36** (Słupecki, [42]). $\mathfrak{A} = \langle\langle\mathcal{V}, f_1,\ldots,f_n\rangle, \mathcal{D}\rangle$ ($|\mathcal{V}| \geq 3$) *is functionally complete iff in* $\langle\mathcal{V}, f_1,\ldots,f_n\rangle$ *(1) all unary functions on* $\mathcal{V}$ *are definable, and (2) at least one surjective and essentially binary function on* $\mathcal{V}$ *is definable.*

Based on this elegant characterization by Słupecki, the desired result is obtained as follows. In case of expansions of the algebra related to **FDE**, we can simplify even further, as we observed in [32, Theorem 4.8].

**Theorem 37.** *Given any expansion* $\mathcal{F}$ *of the algebra* $\langle\{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\},\wedge,\vee\rangle$ *the following (1) and (2) are equivalent: (1)* $\mathcal{F}$ *is functionally complete; (2) all of the* $\delta_a$*'s as well as* $\mathsf{C}_a$*'s (*$a \in \{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\}$*) are definable, where* $\delta_a(b) = \mathbf{t}$*, if* $a = b$*, otherwise* $\delta_a(b) = \mathbf{f}$*; and* $\mathsf{C}_a(b) = a$*, for all* $a,b \in \mathcal{V}$*.*

Similarly, we obtain the next result for the three-element cases, where $\mathbf{i} \in \{\mathbf{b},\mathbf{n}\}$.

**Theorem 38.** *Given any expansion* $\mathcal{F}$ *of the algebra* $\langle\{\mathbf{t},\mathbf{i},\mathbf{f}\},\wedge,\vee\rangle$ *the following (1) and (2) are equivalent: (1)* $\mathcal{F}$ *is functionally complete; (2) all of the* $\delta_a$*'s as well as* $\mathsf{C}_a$*'s (*$a \in \{\mathbf{t},\mathbf{i},\mathbf{f}\}$*) are definable, where* $\delta_a$ *and* $\mathsf{C}_a$ *are defined as in Theorem 37.*

Building on these results, we obtain the following.

**Theorem 39.** ***FDE***$^{16}$*,* ***K3***$^{4}$ *and* ***LP***$^{4}$ *are definitionally complete.*

*Proof.* For $\mathbf{FDE}^{16}$, in view of the above theorem, it suffices to prove that all of the $\delta_a$s as well as $\mathsf{C}_a$'s ($a \in \{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\}$) are definable in $\langle\{\mathbf{t},\mathbf{b},\mathbf{n},\mathbf{f}\},\sim_{16},\wedge,\vee\rangle$, and this can be done as follows: $\delta_{\mathbf{t}}(x) := \neg(\sim_{16} x \vee \sim_{16}\sim_{16} x)$, $\delta_{\mathbf{b}}(x) := \neg(\neg\sim_{16} x \vee \sim_{16}\sim_{16} x)$, $\delta_{\mathbf{n}}(x) := \neg(\neg\sim_{16}\sim_{16} x \vee \sim_{16} x)$, $\delta_{\mathbf{f}}(x) := \neg\neg(\sim_{16} x \wedge \sim_{16}\sim_{16} x)$, $\mathsf{C}_{\mathbf{t}}(x) := x \vee \sim_{16}\sim_{16} x$, $\mathsf{C}_{\mathbf{b}}(x) := \sim_{16}(x \wedge \sim_{16}\sim_{16} x)$, $\mathsf{C}_{\mathbf{n}}(x) := \sim_{16}(x \vee \sim_{16}\sim_{16} x)$, and $\mathsf{C}_{\mathbf{f}}(x) := x \wedge \sim_{16}\sim_{16} x$, where $\neg x := \sim_{16}(\sim_{16}\sim_{16}((x \wedge \sim_{16} x) \wedge \sim_{16}(x \wedge \sim_{16} x)) \wedge ((x \wedge \sim_{16} x) \vee \sim_{16}(x \wedge \sim_{16} x)))$.

For $\mathbf{K3}^4$, in view of the above theorem, it suffices to prove that all of the $\delta_a$'s as well as $\mathsf{C}_a$'s ($a \in \{\mathbf{t},\mathbf{n},\mathbf{f}\}$) are definable in $\langle\{\mathbf{t},\mathbf{n},\mathbf{f}\},\sim_4,\wedge,\vee\rangle$, and this can be done as follows: $\delta_{\mathbf{t}}(x) := x \wedge \sim_4(x \wedge \sim_4\sim_4 x)$, $\delta_{\mathbf{n}}(x) := \sim_4\sim_4(x \vee \sim_4 x)$, $\delta_{\mathbf{f}}(x) := \sim_4\sim_4(x \vee \sim_4\sim_4 x)$, $\mathsf{C}_{\mathbf{t}}(x) := \sim_4(x \wedge \sim_4 x \wedge \sim_4\sim_4 x)$, $\mathsf{C}_{\mathbf{n}}(x) := \sim_4\sim_4(x \wedge \sim_4 x \wedge \sim_4\sim_4 x)$, and $\mathsf{C}_{\mathbf{f}}(x) := x \wedge \sim_4 x \wedge \sim_4\sim_4 x$.

Similarly, for $\mathbf{LP}^4$, in view of the above theorem, it suffices to prove that all of the $\delta_a$'s as well as $\mathsf{C}_a$'s ($a \in \{\mathbf{t},\mathbf{b},\mathbf{f}\}$) are definable in $\langle\{\mathbf{t},\mathbf{b},\mathbf{f}\},\sim_4,\wedge,\vee\rangle$, and this can be done as follows: $\delta_{\mathbf{t}}(x) := x \wedge \sim_4\sim_4(x \wedge \sim_4 x)$, $\delta_{\mathbf{b}}(x) := \sim_4(x \vee \sim_4\sim_4 x)$, $\delta_{\mathbf{f}}(x) := \sim_4(x \vee \sim_4 x)$, $\mathsf{C}_{\mathbf{t}}(x) := \sim_4\sim_4(x \wedge \sim_4 x \wedge \sim_4\sim_4 x)$, $\mathsf{C}_{\mathbf{b}}(x) := \sim_4(x \wedge \sim_4 x \wedge \sim_4\sim_4 x)$, and $\mathsf{C}_{\mathbf{f}}(x) := x \wedge \sim_4 x \wedge \sim_4\sim_4 x$. This completes the proof. ◁

**Remark 40.** Note that it is not difficult to see that other variants are *not* functionally complete.

Finally, we add a brief remark on the Post completeness.

**Definition 41.** The logic $\mathbf{L}$ is *Post complete* iff for every formula $A$ such that $\nvdash A$, the extension of $\mathbf{L}$ by $A$ becomes trivial, i.e., $\vdash_{\mathbf{L}\cup\{A\}} B$ for any $B$.

**Theorem 42** (Tokarz, [45])**.** *Definitionally complete logics are Post complete.*

In view of Theorems 39 and 42, we obtain the following result.

**Corollary 43.** $\mathbf{FDE}^{16}$*,* $\mathbf{K3}^{4}$ *and* $\mathbf{LP}^{4}$ *are Post complete.*

**Remark 44.** Note that the converse of Theorem 42 does not hold, i.e., there are logics that are Post complete without being definitionally complete, such as the negation-free fragment of classical propositional logic. Therefore, one may ask if other variants of **FDE**, **LP** and **K3** are Post complete. The answer is that in our case, none of the variants other than $\mathbf{FDE}^{16}$, $\mathbf{K3}^{4}$ and $\mathbf{LP}^{4}$ are Post complete, as observed in the following proposition.

**Proposition 45.** *None of the variants other than* $\mathbf{FDE}^{16}$*,* $\mathbf{K3}^{4}$ *and* $\mathbf{LP}^{4}$ *are Post complete.*

*Proof.* The results hold by considering extensions by (ECQ) or (LEM). ◁

5.3. **Variable Sharing Property and Admissibility of Contraposition.** Let us now turn our attention to two more properties that **FDE** is well known for enjoying, namely, the variable sharing property and the admissibility of the rule of contraposition. We will first deal with the variable sharing property, by recalling the definition.

**Definition 46.** A logic **L** satisfies the *variable sharing property* iff for all $A, B \in \mathsf{Form}$, $A \vdash_{\mathbf{L}} B$ implies that $A$ and $B$ share at least one propositional variable.

**Remark 47.** Usually, the variable sharing property is stated with respect to the conditional included in the object language, but since we do not have conditionals in the language, we will consider the version above.[6]

Then, we obtain the following result.

**Theorem 48.** $\mathbf{FDE}^{1}$*,* $\mathbf{FDE}^{2}$*,* $\mathbf{FDE}^{9}$ *and* $\mathbf{FDE}^{10}$ *satisfy the variable sharing property. The other systems, including the variants of* **K3** *and* **LP***, do not satisfy the variable sharing property.*

*Proof.* Suppose $A \vdash^{i}_{\mathbf{FDE}} B$ $(i \in \{1,2,9,10\})$, but that $A$ and $B$ do not share any propositional variables. Then, if we consider a valuation $v$ that assigns the value **b** to all the variables in $A$ and the value **n** to all the variables in $B$, then we obtain $v(A) = \mathbf{b}$ and $v(B) = \mathbf{n}$. Indeed, by a simple inductive proof, we may observe that both values **b** and **n** are closed under the set of operations $\{\sim_i, \wedge, \vee\}$, where $i \in \{1,2,9,10\}$. Then the above valuation is a counter-model for $A \vdash^{i}_{\mathbf{FDE}} B$, an absurdity in view of our assumption.

For the latter half, it easy to check that one of the rules of the unilateral natural deduction system serves as a counterexample of the variable sharing property. ◁

**Remark 49.** Our result shows that there are no sub-classical variants of **FDE** with the variable sharing property (VSP), but there are three other systems if we widen our scope beyond sub-classicality.

[6] As a referee pointed out, there is a system with the variable sharing property in the original form, but not in the above form. Such examples include the logic determined by the matrix $M_0$ presented by Nuel Belnap in [6].

Let us now turn to the status of the rule of contraposition, which is seen as crucial for the understanding of negation by, for example, the advocates of the so-called *Australian plan* for negation.[7] We first clarify the form of contraposition we have in mind.

**Definition 50.** A logic **L** admits the *rule of contraposition* iff for all $A, B \in \mathsf{Form}$,

(Contra) $A \vdash_{\mathbf{L}} B$ implies $\sim B \vdash_{\mathbf{L}} \sim A$.

Then, as is well known, **FDE** admits the rule of contraposition. The easiest way to see this is from the perspective of the star semantics, defined as follows.

**Definition 51.** A *Routley interpretation* is a structure $\langle W, *, v\rangle$, where $W$ is a set of worlds, $*\colon W \longrightarrow W$ is a function with $w^{**} = w$, and $v\colon W \times \mathsf{Prop} \longrightarrow \{0,1\}$. The function $v$ is extended to an assignment $I$ of truth values for all pairs of worlds and formulas by the conditions:

(1) $I(w,p)=v(w,p)$, (3) $I(w,A\wedge B)=1$ iff $I(w,A)=1$ and $I(w,B)=1$,
(2) $I(w,\sim A)=1$ iff $I(w^*,A)\neq 1$, (4) $I(w,A\vee B)=1$ iff $I(w,A)=1$ or $I(w,B)=1$.

**Definition 52.** A formula $A$ is a *star semantic consequence* of $\Gamma$ ($\Gamma \models^*_{\mathbf{FDE}} A$) iff for all Routley interpretations $\langle W, *, v\rangle$ and for all $w \in W$, if $I(w,B) = 1$ for all $B \in \Gamma$ then $I(w,A) = 1$.

Then, the following result is well known, due to Richard Routley and Valerie Routley (cf. [39]).[8]

**Theorem 53** (Routley & Routley)**.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{\mathbf{FDE}} A$ *iff* $\Gamma \models^*_{\mathbf{FDE}} A$.

As a corollary, we obtain that **FDE** satisfies the rule of contraposition. Now, the question is that if there are other systems within the variations we are considering that satisfy the rule of contraposition. The answer is yes, since $\mathbf{FDE}^7$ also admits (Contra), and for the purpose of establishing this result, we introduce a variation of Routley interpretations as follows.

**Definition 54.** Let *one-step Routley interpretation* be a structure $\langle W, *, v\rangle$ as in Routley interpretation, except that $w^{**} = w$ is replaced by $w^* = w^{**}$.

**Remark 55.** One-step here means that it starts to "loop" after one application of the star operator. We can also consider $n$-step Routley interpretations in general, but we will not consider them in this paper.

**Definition 56.** A formula $A$ is a *one step star semantic consequence* of $\Gamma$ ($\Gamma \models^{*1}_{\mathbf{FDE}} A$) iff for all one step Routley interpretations $\langle W, *, v\rangle$ and for all $w \in W$, if $I(w,B) = 1$ for all $B \in \Gamma$ then $I(w,A) = 1$.

Then, we obtain the following result.

**Theorem 57.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash^7_{\mathbf{FDE}} A$ *iff* $\Gamma \models^{*1}_{\mathbf{FDE}} A$.

[7] For one of the most recent discussions on this topic, see [7; 8]. Note also that the Dunn semantics offers the key insight for the so-called *American plan* for negation.

[8] Or, more precisely as Dunn writes in [16, p. 440], the star semantics was "actually mathematically in 1957 anticipated by A. Białynicki-Birula and H. Rasiowa, and shown equivalent by Dunn in 1966."

*Proof.* For the soundness direction, we establish the result by a straightforward verification that each rule is truth-preserving. We only deal with $(\sim_7\sim_7 1)$ and $(\sim_7\sim_7 2)$. To this end, it suffices to observe that $I(w,\sim_7 A)=1$ iff $I(w^*,A)\neq 1$ and that $I(w,\sim_7\sim_7 A)=1$ iff $I(w^*,\sim_7 A)\neq 1$ iff $I(w^{**},A)=1$ iff $I(w^*,A)=1$ (by $w^*=w^{**}$), and thus for all $A\in\mathsf{Form}$ and for all $w\in W$, we obtain $I(w,\sim_7 A\vee\sim_7\sim_7 A)=1$ and $I(w,\sim_7 A\wedge\sim_7\sim_7 A)\neq 1$.

For the completeness direction, we make use of the completeness result with respect to the four-valued semantics (cf. Theorem 27), and establish that if $\Gamma\nvDash^7_{\mathbf{FDE}} A$ then $\Gamma\nvDash^{*1}_{\mathbf{FDE}} A$. So, assume that $\Gamma\nvDash^7_{\mathbf{FDE}} A$. Then, there is a four-valued $\mathbf{FDE}^7$ interpretation $v_0$ such that $v_0(B)\in\mathcal{D}$ for all $B\in\Gamma$, and $v_0(A)\notin\mathcal{D}$. Given $v_0$, we define a one-step Routley interpretation as follows: $W:=\{a,b\}$, $a^*=b$ and $b^*=b$, and

$$v(a,p)=1 \text{ iff } v_0(p)\in\{\mathbf{t},\mathbf{b}\},\qquad v(b,p)=1 \text{ iff } v_0(p)\in\{\mathbf{t},\mathbf{n}\}.$$

Then, once we show that the above condition holds for all $A\in\mathsf{Form}$, we obtain the desired result. That the above condition holds for all $A\in\mathsf{Form}$ can be proved by induction on the construction of $A$. The base case, for atomic formulas, is obvious by the definition. For the induction step, the cases are split based on the connectives. Since the cases for conjunction and disjunction can be done in exactly the same way as we do for **FDE**, we will focus on the case for negation, namely, the case when $A$ is of the form $\sim_7 B$. Then,

(1) $v(a,\sim_7 B)=1$ iff $v(a^*,B)\neq 1$ iff $v(b,B)\neq 1$ iff $v_0(B)\notin\{\mathbf{t},\mathbf{n}\}$ (by IH) iff $v_0(\sim_7 B)=\mathbf{t}$ (by the truth table) iff $v_0(\sim_7 B)\in\{\mathbf{t},\mathbf{b}\}$ (since $v_0(\sim_7 B)$ is never $\mathbf{b}$ by the truth table).

(2) $v(b,\sim_7 B)=1$ iff $v(b^*,B)\neq 1$ iff $v(b,B)\neq 1$ iff $v_0(B)\notin\{\mathbf{t},\mathbf{n}\}$ (by IH) iff $v_0(\sim_7 B)=\mathbf{t}$ (by the truth table) iff $v_0(\sim_7 B)\in\{\mathbf{t},\mathbf{n}\}$ (since $v_0(\sim_7 B)$ is never $\mathbf{n}$ by the truth table).

This completes the proof. ◁

As an immediate corollary, we obtain the following.

**Corollary 58.** *$\mathbf{FDE}^7$ admits (Contra).*

**Remark 59.** Note that (Contra) is not admissible for the other systems. First, for **K3** and **LP**, this is immediate since (ECQ) and (Contra) will establish (LEM), and (LEM) and (Contra) will establish (ECQ). Second, for the variants of **FDE** that are negation inconsistent or negation incomplete, it is not difficult to prove the desired results. Indeed, assume (Contra) and take $B$ to be an instance of the negation inconsistent formula. Then, we obtain $p\vdash B$ holds, and thus by (Contra), we obtain $\sim B\vdash\sim p$. But, since $B$ is an instance of the negation inconsistent formula, we obtain $\vdash\sim p$, but this is absurd. The proof is similar for the negation incomplete case. Finally, for the rest of systems, we show specific counterexamples.

1. For $\mathbf{FDE}^2$: $p\vdash^2_{\mathbf{FDE}}\sim_2\sim_2 p$ but $\sim_2\sim_2\sim_2 p\nvdash^2_{\mathbf{FDE}}\sim_2 p$.
2. For $\mathbf{FDE}^3$: $q\vdash^3_{\mathbf{FDE}}\sim_3 p\vee\sim_3\sim_3 p$ but $\sim_3(\sim_3 p\vee\sim_3\sim_3 p)\nvdash^3_{\mathbf{FDE}}\sim_3 q$.
3. For $\mathbf{FDE}^5$: $q\vdash^5_{\mathbf{FDE}}\sim_5(p\wedge\sim_5 p\wedge\sim_5\sim_5 p)$ but $\sim_5\sim_5(p\wedge\sim_5 p\wedge\sim_5\sim_5 p)\nvdash^5_{\mathbf{FDE}}\sim_5 q$.
4. For $\mathbf{FDE}^6$: $p\wedge\sim_6 p\wedge\sim_6\sim_6 p\vdash^6_{\mathbf{FDE}} q$ but $\sim_6 q\nvdash^5_{\mathbf{FDE}}\sim_6(p\wedge\sim_6 p\wedge\sim_6\sim_6 p)$.

5. For $\mathbf{FDE}^9$: $\sim_9 \sim_9 p \vdash^9_{\mathbf{FDE}} p$ but $\sim_9 p \nvdash^9_{\mathbf{FDE}} \sim_9 \sim_9 \sim_9 p$.
6. For $\mathbf{FDE}^{10}$: $\sim_{10} p \vee \sim_{10} q \vdash^{10}_{\mathbf{FDE}} \sim_{10}(p \wedge q)$ but $\sim_{10} \sim_{10}(p \wedge q) \nvdash^{10}_{\mathbf{FDE}} \sim_{10}(\sim_{10} p \vee \sim_{10} q)$.
7. For $\mathbf{FDE}^{11}$: $q \vdash^{11}_{\mathbf{FDE}} p \vee \sim_{11} p \vee \sim_{11} \sim_{11} p$ but $\sim_{11}(p \vee \sim_{11} p \vee \sim_{11} \sim_{11} p) \nvdash^{11}_{\mathbf{FDE}} \sim_{11} q$.

**Remark 60.** In view of the above result, none of the *contra-classical* variants can be captured by the Australian plan with the local consequence relation. In other words, if one is in deep favor of (Contra), then the contra-classical variants cannot be captured. However, one may still work with the Australian plan, but take pointed models and define the semantic consequence relation in terms of truth preservation at the distinguished point. Whether this way will allow the Australian plan advocates to capture any of the contra-classical variants or not is an interesting question that we will leave to interested readers.

## 6. Reflections: Too Many Varieties?

Given all the variants, one may conclude that there are far too many options, and wonder about the implications of all this. This, of course, is a natural and even a pressing question. For the purpose of addressing the question, at least partially, we will make use of non-deterministic semantics. More specifically, we will consider some family of negations under certain classification, put them together along the framework of non-deterministic semantics, and explore the shared property for those negations. Let us first recall the basic definition of non-deterministic semantics (cf. [3] for an overview).

**Definition 61.** A *non-deterministic matrix* (*Nmatrix* for short) for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and for every $n$-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding $n$-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$. We say that $M$ is (in)finite if so is $\mathcal{V}$. A *legal valuation* in an Nmatrix $M$ is a function $v : \mathsf{Form} \rightarrow \mathcal{V}$ that satisfies the following condition for every $n$-ary connective $*$ of $\mathcal{L}$ and $A_1, \ldots, A_n \in \mathsf{Form}$:

(gHom) $$v(*(A_1, \ldots, A_n)) \in \tilde{*}(v(A_1), \ldots, v(A_n)).$$

The condition (gHom) can be interpreted as a generalized homomorphism condition.

Let us now consider four kinds of non-deterministic matrices. The first one is obtained by putting together the truth table for subclassical negations, with a motivation to explore the common core of subclassical variants of **FDE**.

**Definition 62.** A *four-valued subclassical **FDE**-Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$, and for every $n$-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding $n$-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the braces for sets):

| $A$ | $\tilde{\sim} A$ |
|---|---|
| **t** | **f** |
| **b** | **t**, **b** |
| **n** | **n**, **f** |
| **f** | **t** |

| $A \tilde{\wedge} B$ | **t** | **b** | **n** | **f** |
|---|---|---|---|---|
| **t** | **t** | **b** | **n** | **f** |
| **b** | **b** | **b** | **f** | **f** |
| **n** | **n** | **f** | **n** | **f** |
| **f** | **f** | **f** | **f** | **f** |

| $A \tilde{\vee} B$ | **t** | **b** | **n** | **f** |
|---|---|---|---|---|
| **t** | **t** | **t** | **t** | **t** |
| **b** | **t** | **b** | **t** | **b** |
| **n** | **t** | **t** | **n** | **n** |
| **f** | **t** | **b** | **n** | **f** |

A *four-valued subclassical* ***FDE*** *-valuation* in a four-valued subclassical **FDE**-Nmatrix $M$ is a function $v\colon \mathsf{Form} \to \mathcal{V}$ that satisfies (gHom). Finally, $A$ is a *four-valued subclassical* ***FDE*** *-consequence* of $\Gamma$ ($\Gamma \models_{4s} A$) iff for every four-valued subclassical **FDE**-valuation $v$, if $v(B) \in \mathcal{D}$ for every $B \in \Gamma$ then $v(A) \in \mathcal{D}$.

The corresponding (unilateral) natural deduction system is introduced as follows.

**Definition 63.** The natural deduction rules $\mathcal{R}^{sub}_{\mathbf{FDE}}$ for **sub-FDE** are all the rules $\mathcal{R}_{\mathbf{FDE}}$ for **FDE** but replacing $(\sim\sim 1)$ and $(\sim\sim 2)$ by the following rules.

$$\frac{\sim A \quad \sim\sim A}{A}\ (\sim\sim 1) \qquad \frac{A}{\sim A \vee \sim\sim A}\ (\sim\sim 2)$$

Based on these, given any set $\Sigma \cup \{A\}$ of formulas, $\Sigma \vdash^{sub}_{\mathbf{FDE}} A$ iff for some finite $\Sigma' \subseteq \Sigma$, there is a derivation of $A$ from $\Sigma'$ in the calculus whose rule set is $\mathcal{R}^{sub}_{\mathbf{FDE}}$.

**Remark 64.** One can also devise a bilateral natural deduction for **sub-FDE** replacing $(\sim\sim 1)$ and $(\sim\sim 2)$ by the following rules to the bilateral presentation of **FDE**.

$$\frac{\overline{\overline{A}} \quad \overline{\overline{\sim A}}}{A} \qquad \frac{\overline{A} \quad \begin{array}{c}[\![A]\!]\\ \vdots \\ \overline{B}\end{array} \quad \begin{array}{c}[\![\sim A]\!]\\ \vdots \\ \overline{B}\end{array}}{B}$$

The further details are left for the interested readers.

Then, we may establish soundness and completeness results. The soundness is again tedious but not difficult.

**Theorem 65.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \vdash^{sub}_{\boldsymbol{FDE}} A$ *then* $\Gamma \models_{4s} A$.

*Proof.* It can be shown by a straightforward verification that each rule preserves designated values. Here we only spell out the details for the validity of $(\sim\sim 1)$ and $(\sim\sim 2)$.

Ad $(\sim\sim 1)$: Suppose, for reductio, that there is a four-valued subclassical **FDE**-valuation $v_0$ such that $v_0(\sim A) \in \mathcal{D}$, $v_0(\sim\sim A) \in \mathcal{D}$, but $v_0(A) \notin \mathcal{D}$. Then, the first and the third assumption together with the Nmatrices imply that $v_0(A) = \mathbf{f}$, and thus $v_0(\sim\sim A) = \mathbf{f}$. But, this is absurd in view of the second assumption.

Ad $(\sim\sim 2)$: Suppose, for reductio, that there is a four-valued subclassical **FDE**-valuation $v_0$ such that $v_0(A) \in \mathcal{D}$, but $v_0(\sim A \vee \sim\sim A) \notin \mathcal{D}$. Then, the second assumption together with the Nmatrices imply that $v_0(\sim A) \notin \mathcal{D}$ and $v_0(\sim\sim A) \notin \mathcal{D}$. By $v_0(A) \in \mathcal{D}$ and $v_0(\sim A) \notin \mathcal{D}$, we obtain that $v_0(A) = \mathbf{t}$, and thus $v_0(\sim\sim A) = \mathbf{t}$. But, this is absurd in view of $v_0(\sim\sim A) \notin \mathcal{D}$. ◁

For completeness, we prepare a definition and a lemma.

**Definition 66.** For any $\Sigma \subseteq \mathsf{Form}$, let $v^{sub}_{\Sigma}$ from $\mathsf{Form}$ to $\{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ be defined as follows:

$$v^{sub}_{\Sigma}(A) := \begin{cases} \mathbf{t} \text{ iff } \Sigma \vdash^{sub}_{\mathbf{FDE}} A \text{ and } \Sigma \nvdash^{sub}_{\mathbf{FDE}} \sim A; \\ \mathbf{b} \text{ iff } \Sigma \vdash^{sub}_{\mathbf{FDE}} A \text{ and } \Sigma \vdash^{sub}_{\mathbf{FDE}} \sim A; \\ \mathbf{n} \text{ iff } \Sigma \nvdash^{sub}_{\mathbf{FDE}} A \text{ and } \Sigma \nvdash^{sub}_{\mathbf{FDE}} \sim A; \\ \mathbf{f} \text{ iff } \Sigma \nvdash^{sub}_{\mathbf{FDE}} A \text{ and } \Sigma \vdash^{sub}_{\mathbf{FDE}} \sim A. \end{cases}$$

Note that we are defining the canonical valuation in a different manner compared to Definition 25, reflecting the difference of how deterministic and non-deterministic semantics are introduced.

**Lemma 67.** *If $\Sigma$ is a prime theory, then $v^{sub}_{\Sigma}$ is a well-defined four-valued subclassical* ***FDE****-valuation.*

*Proof.* Note first that the well-definedness of $v^{sub}_{\Sigma}$ is obvious. Then the desired result is proved by induction on the number $n$ of connectives. Base case: For atomic formulas, it is obvious by the definition. Induction step: We split the cases based on the connectives. Here we only deal with $\sim$. If $A = {\sim}B$, then we have the following cases.

| Cases | $v_{\Sigma}(B)$ | condition for $B$ | $v_{\Sigma}(A)$ | condition for $A$ i.e., $\sim B$ |
|---|---|---|---|---|
| (i) | $\mathbf{t}$ | $\Sigma \vdash^{sub}_{\mathbf{FDE}} B$ and $\Sigma \nvdash^{sub}_{\mathbf{FDE}} {\sim}B$ | $\mathbf{f}$ | $\Sigma \nvdash^{sub}_{\mathbf{FDE}} {\sim}B$ and $\Sigma \vdash^{sub}_{\mathbf{FDE}} {\sim}{\sim}B$ |
| (ii) | $\mathbf{b}$ | $\Sigma \vdash^{sub}_{\mathbf{FDE}} B$ and $\Sigma \vdash^{sub}_{\mathbf{FDE}} {\sim}B$ | $\mathbf{t},\mathbf{b}$ | $\Sigma \vdash^{sub}_{\mathbf{FDE}} {\sim}B$ |
| (iii) | $\mathbf{n}$ | $\Sigma \nvdash^{sub}_{\mathbf{FDE}} B$ and $\Sigma \nvdash^{sub}_{\mathbf{FDE}} {\sim}B$ | $\mathbf{n},\mathbf{f}$ | $\Sigma \nvdash^{sub}_{\mathbf{FDE}} {\sim}B$ |
| (iv) | $\mathbf{f}$ | $\Sigma \nvdash^{sub}_{\mathbf{FDE}} B$ and $\Sigma \vdash^{sub}_{\mathbf{FDE}} {\sim}B$ | $\mathbf{t}$ | $\Sigma \vdash^{sub}_{\mathbf{FDE}} {\sim}B$ and $\Sigma \nvdash^{sub}_{\mathbf{FDE}} {\sim}{\sim}B$ |

By induction hypothesis, we have the conditions for $B$, for cases (ii) and (iii), it is easy to see that the conditions for $A$ i.e., $\sim B$ are provable. For (i) and (iv), we can use $({\sim}{\sim}2)$ and $({\sim}{\sim}1)$, respectively. ◁

We are now ready to establish the completeness result.

**Theorem 68.** *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \vDash_{4s} A$ then $\Gamma \vdash^{sub}_{\mathbf{FDE}} A$.*

*Proof.* Assume $\Gamma \nvdash^{sub}_{\mathbf{FDE}} A$. Then, by Lemma 24, there is a $\Sigma \supseteq \Gamma$ such that $\Sigma$ is a prime theory and $A \notin \Sigma$, and by Lemma 67, a four-valued subclassical valuation $v^{sub}_{\Sigma}$ can be defined with $v^{sub}_{\Sigma}(B) \in \mathcal{D}$ for every $B \in \Gamma$ and $v^{sub}_{\Sigma}(A) \notin \mathcal{D}$. Thus it follows that $\Gamma \nvDash^{sub}_{\mathbf{FDE}} A$, as desired. ◁

Let us now turn to the second kind of non-deterministic matrices, which is obtained by combining the negations that produce negation inconsistency.

**Definition 69.** A *four-valued negation inconsistent* ***FDE****-Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$, and for every $n$-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding $n$-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$. Definition 62 gives $\tilde{\wedge}$ and $\tilde{\vee}$; $\tilde{\sim}$ is

| $A$ | $\mathbf{t}$ | $\mathbf{b}$ | $\mathbf{n}$ | $\mathbf{f}$ |
|---|---|---|---|---|
| $\tilde{\sim}A$ | $\mathbf{n},\mathbf{f}$ | $\mathbf{t},\mathbf{b}$ | $\mathbf{f}$ | $\mathbf{b}$ |

A *four-valued negation inconsistent* ***FDE****-valuation* in a four-valued negation inconsistent **FDE**-Nmatrix $M$ is a function $v\colon \mathsf{Form} \to \mathcal{V}$ that satisfies (gHom). Finally, $A$ is a *four-valued negation inconsistent* ***FDE****-consequence* of $\Gamma$ ($\Gamma \vDash_{4b} A$) iff for every four-valued negation inconsistent **FDE**-valuation $v$, if $v(B) \in \mathcal{D}$ for every $B \in \Gamma$ then $v(A) \in \mathcal{D}$.

The corresponding (unilateral) natural deduction system is introduced as follows.

**Definition 70.** The natural deduction rules $\mathcal{R}^{b}_{\mathbf{FDE}}$ for **b-FDE** are all the rules $\mathcal{R}_{\mathbf{FDE}}$ for **FDE** but replacing $({\sim}{\sim}1)$ and $({\sim}{\sim}2)$ by the following rule.

$$\frac{}{A \vee {\sim}{\sim}A}\ ({\sim}{\sim})$$

Based on these, given any set $\Sigma \cup \{A\}$ of formulas, $\Sigma \vdash^b_{\mathbf{FDE}} A$ iff for some finite $\Sigma' \subseteq \Sigma$, there is a derivation of $A$ from $\Sigma'$ in the calculus whose rule set is $\mathcal{R}^b_{\mathbf{FDE}}$.

**Remark 71.** One can also devise a bilateral natural deduction for **b-FDE** by replacing $(\sim\sim 1)$ and $(\sim\sim 2)$ by the following rule to the bilateral presentation of **FDE**.

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ \overline{B}\end{array} \quad \begin{array}{c}[\![\sim A]\!]\\ \vdots\\ \overline{B}\end{array}}{B}$$

The further details are left for the interested readers.

Then, we may establish the following result.

**Theorem 72.** *For all $\Gamma \cup \{A\} \subseteq$ Form, $\Gamma \vdash^b_{\mathbf{FDE}} A$ iff $\Gamma \vDash_{4b} A$.*

*Proof.* For the soundness direction, we establish the result by a straightforward verification that each rule preserves designated values. Here we only spell out the details for the validity of $(\sim\sim)$.

Suppose, for reductio, that there is a four-valued negation inconsistent **FDE**-valuation $v_0$ such that $v_0(A \vee \sim\sim A) \notin \mathcal{D}$. Then, together with the Nmatrices, the assumption implies that $v_0(A) \notin \mathcal{D}$ and $v_0(\sim\sim A) \notin \mathcal{D}$. By $v_0(A) \notin \mathcal{D}$, there are two cases. If $v_0(A) = \mathbf{n}$, then $v_0(\sim\sim A) = \mathbf{b}$, which is absurd in view of $v_0(\sim\sim A) \notin \mathcal{D}$. If $v_0(A) = \mathbf{f}$, then $v_0(\sim\sim A) \in \mathcal{D}$ which is absurd in view of $v_0(\sim\sim A) \notin \mathcal{D}$.

For the completeness direction, we need to define $v^b_\Sigma$ as in Definition 66 with an obvious modification, and establish the analogue of Lemma 67. In particular, we need to check the following.

| Cases | $v_\Sigma(B)$ | condition for $B$ | $v_\Sigma(A)$ | condition for $A$ i.e., $\sim B$ |
|---|---|---|---|---|
| (i) | $\mathbf{t}$ | $\Sigma \vdash^b_{\mathbf{FDE}} B$ and $\Sigma \nvdash^b_{\mathbf{FDE}} \sim B$ | $\mathbf{n}, \mathbf{f}$ | $\Sigma \nvdash^b_{\mathbf{FDE}} \sim B$ |
| (ii) | $\mathbf{b}$ | $\Sigma \vdash^b_{\mathbf{FDE}} B$ and $\Sigma \vdash^b_{\mathbf{FDE}} \sim B$ | $\mathbf{t}, \mathbf{b}$ | $\Sigma \vdash^b_{\mathbf{FDE}} \sim B$ |
| (iii) | $\mathbf{n}$ | $\Sigma \nvdash^b_{\mathbf{FDE}} B$ and $\Sigma \nvdash^b_{\mathbf{FDE}} \sim B$ | $\mathbf{f}$ | $\Sigma \nvdash^b_{\mathbf{FDE}} \sim B$ and $\Sigma \vdash^b_{\mathbf{FDE}} \sim\sim B$ |
| (iv) | $\mathbf{f}$ | $\Sigma \nvdash^b_{\mathbf{FDE}} B$ and $\Sigma \vdash^b_{\mathbf{FDE}} \sim B$ | $\mathbf{b}$ | $\Sigma \vdash^b_{\mathbf{FDE}} \sim B$ and $\Sigma \vdash^b_{\mathbf{FDE}} \sim\sim B$ |

By induction hypothesis, we have the conditions for $B$, for cases (i) and (ii), it is easy to see that the conditions for $A$ i.e., $\sim B$ are provable. For (iii) and (iv), we can use $(\sim\sim)$. ◁

**Remark 73.** Note that although we combined the negations that produce negation inconsistency, and thus named the Nmatrix including the phrase "negation inconsistent," it is not clear to us at the time of writing if the resulting system **b-FDE** is negation inconsistent or not.

The third one now is obtained by combining the negations that produce negation incompleteness.

**Definition 74.** A *four-valued negation incomplete **FDE**-Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$, and for every $n$-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding $n$-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$. Definition 62 gives $\tilde{\wedge}$ and $\tilde{\vee}$; $\tilde{\sim}$ is

| $A$ | $\mathbf{t}$ | $\mathbf{b}$ | $\mathbf{n}$ | $\mathbf{f}$ |
|---|---|---|---|---|
| $\tilde{\sim}A$ | $\mathbf{n}$ | $\mathbf{t}$ | $\mathbf{n},\mathbf{f}$ | $\mathbf{t},\mathbf{b}$ |

A *four-valued negation incomplete* ***FDE****-valuation* in a four-valued negation incomplete **FDE**-Nmatrix $M$ is a function $v\colon \mathsf{Form} \to \mathcal{V}$ that satisfies (gHom). Finally, $A$ is a *four-valued negation incomplete* ***FDE****-consequence* of $\Gamma$ ($\Gamma \models_{4n} A$) iff for every four-valued negation inconsistent **FDE**-valuation $v$, if $v(B) \in \mathcal{D}$ for every $B \in \Gamma$ then $v(A) \in \mathcal{D}$.

The corresponding (unilateral) natural deduction system is as follows.

**Definition 75.** The natural deduction rules $\mathcal{R}^n_{\mathbf{FDE}}$ for **n-FDE** are all the rules $\mathcal{R}_{\mathbf{FDE}}$ for **FDE** but replacing $({\sim}{\sim}1)$ and $({\sim}{\sim}2)$ by the following rule.

$$\frac{A \quad {\sim}{\sim}A}{B}\ ({\sim}{\sim})$$

Based on these, given any set $\Sigma \cup \{A\}$ of formulas, $\Sigma \vdash^n_{\mathbf{FDE}} A$ iff for some finite $\Sigma' \subseteq \Sigma$, there is a derivation of $A$ from $\Sigma'$ in the calculus whose rule set is $\mathcal{R}^n_{\mathbf{FDE}}$.

**Remark 76.** One can also devise a bilateral natural deduction for **n-FDE** by replacing $({\sim}{\sim}1)$ and $({\sim}{\sim}2)$ by the following rules to the bilateral presentation of **FDE**.

$$\frac{\overline{A} \quad \overline{\overline{{\sim}A}}}{B}$$

The further details are left for the interested readers.

Then, we may establish the following result.

**Theorem 77.** *For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash^n_{\mathbf{FDE}} A$ iff $\Gamma \models_{4n} A$.*

*Proof.* For the soundness direction, we establish the result by a straightforward verification that each rule preserves designated values. Here we only spell out the details for the validity of $({\sim}{\sim})$.

Suppose, for reductio, that there is a four-valued negation inconsistent **FDE**-valuation $v_0$ such that $v_0(A) \in \mathcal{D}$ and $v_0({\sim}{\sim}A) \in \mathcal{D}$, but $v_0(B) \notin \mathcal{D}$. Then, the first assumption together with the Nmatrices imply that $v_0({\sim}{\sim}A) \notin \mathcal{D}$. But this is absurd in view of the second assumption.

For the completeness direction, we again need to define $v^n_\Sigma$ as in Definition 66 with an obvious modification, and establish the analogue of Lemma 67. In particular, we need to check the following.

| Cases | $v_\Sigma(B)$ | condition for $B$ | $v_\Sigma(A)$ | condition for $A$ i.e., ${\sim}B$ |
|---|---|---|---|---|
| (i) | $\mathbf{t}$ | $\Sigma \vdash^n_{\mathbf{FDE}} B$ and $\Sigma \nvdash^n_{\mathbf{FDE}} {\sim}B$ | $\mathbf{n}$ | $\Sigma \nvdash^n_{\mathbf{FDE}} {\sim}B$ and $\Sigma \nvdash^n_{\mathbf{FDE}} {\sim}{\sim}B$ |
| (ii) | $\mathbf{b}$ | $\Sigma \vdash^n_{\mathbf{FDE}} B$ and $\Sigma \vdash^n_{\mathbf{FDE}} {\sim}B$ | $\mathbf{t}$ | $\Sigma \vdash^n_{\mathbf{FDE}} {\sim}B$ and $\Sigma \nvdash^n_{\mathbf{FDE}} {\sim}{\sim}B$ |
| (iii) | $\mathbf{n}$ | $\Sigma \nvdash^n_{\mathbf{FDE}} B$ and $\Sigma \nvdash^n_{\mathbf{FDE}} {\sim}B$ | $\mathbf{n},\mathbf{f}$ | $\Sigma \nvdash^n_{\mathbf{FDE}} {\sim}B$ |
| (iv) | $\mathbf{f}$ | $\Sigma \nvdash^n_{\mathbf{FDE}} B$ and $\Sigma \vdash^n_{\mathbf{FDE}} {\sim}B$ | $\mathbf{t},\mathbf{b}$ | $\Sigma \vdash^n_{\mathbf{FDE}} {\sim}B$ |

By induction hypothesis, we have the conditions for $B$, for cases (iii) and (iv), it is easy to see that the conditions for $A$ i.e., ${\sim}B$ are provable. For (i) and (ii), we can use $({\sim}{\sim})$. ◁

**Remark 78.** Similarly to the case with **b-FDE**, although we combined the negations that produce negation incompleteness, and thus named the Nmatrix including the phrase "negation incomplete," it is not clear to us at the time of writing if the resulting system **n-FDE** is negation incomplete or not.

Finally, let us consider the *fully* contra-classical kind, by combining all the contra-classical negations.

**Definition 79.** A *four-valued contra-classical **FDE**-Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$, and for every $n$-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding $n$-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$. Definition 62 gives $\tilde{\wedge}$ and $\tilde{\vee}$; $\tilde{\sim}$ is

| $A$ | $\mathbf{t}$ | $\mathbf{b}$ | $\mathbf{n}$ | $\mathbf{f}$ |
|---|---|---|---|---|
| $\tilde{\sim}A$ | $\mathbf{n},\mathbf{f}$ | $\mathbf{t},\mathbf{b}$ | $\mathbf{n},\mathbf{f}$ | $\mathbf{t},\mathbf{b}$ |

A *four-valued contra-classical **FDE**-valuation* in a four-valued contra-classical **FDE**-Nmatrix $M$ is a function $v\colon \mathsf{Form} \to \mathcal{V}$ that satisfies (gHom). Finally, $A$ is a *four-valued contra-classical **FDE**-consequence* of $\Gamma$ ($\Gamma \models_{4c} A$) iff for every four-valued contra-classical **FDE**-valuation $v$, if $v(B) \in \mathcal{D}$ for every $B \in \Gamma$ then $v(A) \in \mathcal{D}$.

The corresponding (unilateral) natural deduction system is introduced as follows.

**Definition 80.** The natural deduction rules $\mathcal{R}^{con}_{\mathbf{FDE}}$ for **con-FDE** are all the rules $\mathcal{R}_{\mathbf{FDE}}$ for **FDE** but eliminating the rules $({\sim}{\sim}1)$ and $({\sim}{\sim}2)$. Based on this, given any set $\Sigma \cup \{A\}$ of formulas, $\Sigma \vdash^{con}_{\mathbf{FDE}} A$ iff for some finite $\Sigma' \subseteq \Sigma$, there is a derivation of $A$ from $\Sigma'$ in the calculus whose rule set is $\mathcal{R}^{con}_{\mathbf{FDE}}$.

**Remark 81.** One can also devise a bilateral natural deduction for **con-FDE** by eliminating $({\sim}{\sim}1)$ and $({\sim}{\sim}2)$. The further details are left for the interested readers.

Then, we may establish the following result.

**Theorem 82.** *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \vdash^{con}_{\mathbf{FDE}} A$ *iff* $\Gamma \models_{4c} A$*.*

*Proof.* For the soundness direction, we having nothing specific to do for rules solely involving negation since we do not have any after eliminating the double negation introduction/elimination rules.

For the completeness, we again need to define $v^{con}_{\Sigma}$ as in Definition 66 with an obvious modification, and establish the analogue of Lemma 67. In particular, we need to check the following.

| Cases | $v_\Sigma(B)$ | condition for $B$ | $v_\Sigma(A)$ | condition for $A$ i.e., ${\sim}B$ |
|---|---|---|---|---|
| (i) | $\mathbf{t}$ | $\Sigma \vdash^{con}_{\mathbf{FDE}} B$ and $\Sigma \nvdash^{con}_{\mathbf{FDE}} {\sim}B$ | $\mathbf{n},\mathbf{f}$ | $\Sigma \nvdash^{con}_{\mathbf{FDE}} {\sim}B$ |
| (ii) | $\mathbf{b}$ | $\Sigma \vdash^{con}_{\mathbf{FDE}} B$ and $\Sigma \vdash^{con}_{\mathbf{FDE}} {\sim}B$ | $\mathbf{t},\mathbf{b}$ | $\Sigma \vdash^{con}_{\mathbf{FDE}} {\sim}B$ |
| (iii) | $\mathbf{n}$ | $\Sigma \nvdash^{con}_{\mathbf{FDE}} B$ and $\Sigma \nvdash^{con}_{\mathbf{FDE}} {\sim}B$ | $\mathbf{n},\mathbf{f}$ | $\Sigma \nvdash^{con}_{\mathbf{FDE}} {\sim}B$ |
| (iv) | $\mathbf{f}$ | $\Sigma \nvdash^{con}_{\mathbf{FDE}} B$ and $\Sigma \vdash^{con}_{\mathbf{FDE}} {\sim}B$ | $\mathbf{t},\mathbf{b}$ | $\Sigma \vdash^{con}_{\mathbf{FDE}} {\sim}B$ |

By induction hypothesis, we have the conditions for $B$, for all the cases, and it is easy to see that the conditions for $A$ i.e., ${\sim}B$ are provable without any additional rules. $\lhd$

**Remark 83.** Somewhat surprisingly, the contra-classicality vanishes in the resulting system that is obtained by combining, with the help of non-deterministic semantics, all the contra-classical variants of **FDE** with respect to negation. In particular, we end up in a subsystem of **FDE**, which is obtained by removing the falsity condition for negation. This is in sharp contrast with the case in which we combined the sub-classical variants of **FDE**. Of course, if we take the combination of all variants of **FDE**, both sub-classical and contra-classical, then the result will be the same as with the case of focusing on contra-classical variants.

**Remark 84.** Given that the corresponding Dunn semantics will be to simply leave the falsity condition for negation unspecified, this system can be also seen as reflecting the position that there is nothing more to negation than expressing falsity. A similar consideration for *classical negation* in the context of expansions of **FDE**, in which there are again 16 candidates as explored in [9], can be found in [43].

## 7. Concluding Remarks

By building on the framework of Dunn semantics, we explored variants of **FDE**, **K3**, and **LP** by fixing the truth condition for negation, but making changes in the falsity condition. We also offered proof systems in the style of natural deduction, both in the unilateral and in the bilateral manner, and established soundness and completeness results for all systems. This was followed by an investigation into the basic properties of the given variants. Our results, for the variants of **FDE**, are summarized in the following table.

| | $\mathbf{FDE}^1$ | $\mathbf{FDE}^2$ | $\mathbf{FDE}^3$ | $\mathbf{FDE}^4$ | $\mathbf{FDE}^5$ | $\mathbf{FDE}^6$ | $\mathbf{FDE}^7$ | $\mathbf{FDE}^8$ |
|---|---|---|---|---|---|---|---|---|
| Subclassical | ✓ | × | ✓ | × | ✓ | × | ✓ | × |
| Contra-classical | × | ✓ | × | ✓ | × | ✓ | × | ✓ |
| Neg. inconsistent | × | × | × | ✓ | × | × | × | ✓ |
| Neg. incomplete | × | × | × | × | × | × | × | × |
| Func. complete | × | × | × | × | × | × | × | × |
| Post complete | × | × | × | × | × | × | × | × |
| Adm. of (Contra) | ✓ | × | × | × | × | × | ✓ | × |
| VSP | ✓ | ✓ | × | × | × | × | × | × |

| | $\mathbf{FDE}^9$ | $\mathbf{FDE}^{10}$ | $\mathbf{FDE}^{11}$ | $\mathbf{FDE}^{12}$ | $\mathbf{FDE}^{13}$ | $\mathbf{FDE}^{14}$ | $\mathbf{FDE}^{15}$ | $\mathbf{FDE}^{16}$ |
|---|---|---|---|---|---|---|---|---|
| Subclassical | × | × | × | × | × | × | × | × |
| Contra-classical | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Neg. inconsistent | × | × | × | ✓ | × | × | × | ✓ |
| Neg. incomplete | × | × | × | × | ✓ | ✓ | ✓ | ✓ |
| Func. complete | × | × | × | × | × | × | × | ✓ |
| Post complete | × | × | × | × | × | × | × | ✓ |
| Adm. of (Contra) | × | × | × | × | × | × | × | × |
| VSP | ✓ | ✓ | × | × | × | × | × | × |

This may seem to be too many variations. With that possible objection in mind, we also explored four combinations of systems, by putting together (i) sub-classical systems, (ii) negation inconsistent systems, (iii) negation incomplete systems, and (iv) contra-classical systems. The resulting systems are semantically described in terms of non-deterministic semantics, and we also offered unilateral and bilateral proof systems that are sound and complete.

Moreover, our results, for the variants of **K3** and **LP**, are summarized in the following table.

| | $\mathbf{K3}^1$ | $\mathbf{K3}^2$ | $\mathbf{K3}^3$ | $\mathbf{K3}^4$ | $\mathbf{LP}^1$ | $\mathbf{LP}^2$ | $\mathbf{LP}^3$ | $\mathbf{LP}^4$ |
|---|---|---|---|---|---|---|---|---|
| Subclassical | ✓ | ✓ | × | × | ✓ | × | ✓ | × |
| Contra-classical | × | × | ✓ | ✓ | × | ✓ | × | ✓ |
| Negation inconsistent | × | × | × | × | × | ✓ | × | ✓ |
| Negation incomplete | × | × | ✓ | ✓ | × | × | × | × |
| Functionally complete | × | × | × | ✓ | × | × | × | ✓ |
| Post complete | × | × | × | ✓ | × | × | × | ✓ |
| Admissibility of (Contra) | × | × | × | × | × | × | × | × |
| Variable sharing property | × | × | × | × | × | × | × | × |

Unsurprisingly, the variants of **K3** and **LP** do not enjoy the variable sharing property and thus fail to be relevance logics. The tweaking of the falsity condition of negation in **K3** may lead to negation incomplete systems, whereas the tweaking of the falsity condition of negation in **LP** may give one a negation inconsistent logic.

There are a number of different directions to pursue for further investigation. Beside those already mentioned in passing, we will note a few more questions. First, let us briefly note that if one emphasizes the symmetry of truth and falsity, and make that carry over for various properties, then among the contra-classical variations, the one with both negation inconsistency and negation incompleteness might be seen as the most favorable, not only satisfying one of them, and that will single out $\mathbf{FDE}^{16}$ as the plausible variant of **FDE**. Given that $\mathbf{FDE}^{16}$ also enjoys the functional completeness, the system, at least from a purely technical perspective, seems worth investigating further.

Second, a related direction to the previous one, is to explore if we can specify further properties, beside the very basic ones we discussed in this paper, so that each of the variants can be singled out by different desiderata. A full answer to this problem seems to contribute substantially to our systematic understanding of both subclassicality and contra-classicality.

Third, given the origin of **FDE** as the first-degree entailment of relevance logics **R** and **E**, we may ask, especially with those having the variable sharing property, if there are variants of relevance logics that will have our variants as their first degree entailment.

Fourth, our variations mainly focused on *deterministic* ones, and only explored four *non-deterministic* ones. However, for the case with **FDE**, from a purely combinatorial perspective assuming the framework of non-deterministic semantics, there are 81 possibilities, and we have only covered 20 of them (16 deterministic and 4 non-deterministic cases). What can be learnt from the other 41 cases is also a problem that seems to be worth addressing.

Finally, but not the least, we focused on a simple propositional language in this paper, but there are a lot of motivations to expand the language both with further propositional connectives (conditionals, modalities, etc.) as well as quantifiers. What kind of insight we gain in these various expansions is yet another direction that is natural and important.

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