Precise approximations of Rademacher functionals by Stein's method and De Finetti's theorem

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften
an der Fakultät für Mathematik
der Ruhr-Universität Bochum

vorgelegt von

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Bochum, den 22. April 2024

Acknowledgement

To begin with, I want to express my deep gratitude to all the people, who enriched my years of being a doctoral student and made this time unforgettable.

First of all I sincerely thank my supervisor Peter Eichelsbacher for the possibility to do my PhD in his working group. Associated with it are our many interesting and fruitful discussions, his patience and his expertise concerning the understanding and writing of mathematical papers.

I also want to thank my second supervisor Christoph Thäle, who always offered excellent help, in particular when it came to specific questions.

The joint work with my coauthors Yacine Barhoumi–Andréani and Benedikt Rednoss was fullfilling as well as enjoyable.

Moreover I thank Tanja Schiffmann for her continuous support and commitment.

Benedikt and I were in the same working group, too. Apart from his nice personality and helpfulness, the characteristic I appreciate the most is his ability of always finding the words to make me feel better.

Also I want to mention Carolin Kleemann, another appreciated friend and former colleague. Her kind heart and our walks will have a place in my memory.

The next are Nils Hansen and Florian Lam, with whom I shared an office and more important I see as reliable friends.

Furthermore my other colleagues and friends Nils Heerten, Kathrin Meier, Janina Hübner, Tristan Schiller and Philipp Tuchel made me feel a lot of joy, in particular by eating together in the canteen or by playing *Wizard*.

At last I want to thank my mother Karin Butzek for her love, support and understanding through all this time.

Contents

1. Introduction

1.1. **Preface.** In this doctoral thesis the results from two of the author's papers and a third project are presented: [5], a joint work with Yacine Barhoumi–Andréani and Peter Eichelsbacher, [11], a joint work with Peter Eichelsbacher and Benedikt Rednoss, and a third project, which is a joint work with Peter Eichelsbacher.
The foundation of this thesis are Rademacher random variables, which are the building blocks of our objects of interest appearing either with the explicit probability distribution of a finite set of spins in the Curie–Weiss model or as abstract L^2-functionals depending on possibly infinitely many Rademacher random variables. The results focus on normal approximation, but in the context of fluctuations of the total magnetisation in the Curie–Weiss model also other limit distributions appear. We give a new proof for these fluctuations in Fortet–Mourier and Kolmogorov distance, which implies new interpretations concerning the phase transition of the model. For the above mentioned L^2-functionals we derive Cramér-type moderate deviations and non-uniform Berry–Esseen bounds. As applications we discuss partial sums of an i.i.d. Rademacher sequence, infinite weighted 2-runs and subgraph counting in the Erdős–Rényi random graph. Throughout this thesis Stein's method and the Malliavin–Stein method are important tools to obtain our results.

1.2. **Convergence results in probability theory.** As a starting point we introduce the types of convergence results that will appear throughout this thesis. In probability theory and statistics the *central limit theorem* (CLT) is known as one the most important and most useful results. While historically its first versions appeared in the work of A. de Moivre (1733) and P.–S. Laplace (1812), who used the normal distribution to approximate distributions of their interest as the number of heads resulting from many tosses of a fair coin or the binomial distribution, it were mathematicians as A. Lyapunov (1901), G. Pólya (1920), J. W. Lindeberg (1922) and others, who contributed to the rich history of the CLT, see [48] for an extensive overview.

The classical CLT, see e.g. [9], is stated as follows: We consider a sequence $(X_k)_{k\in\mathbb{N}}$ of independent and identically distributed (i.i.d.) random variables with expectation $\mu := \mathbb{E}[X_1] < \infty$ and variance $\sigma^2 := \mathrm{Var}(X_1) < \infty$, for $n \in \mathbb{N}$ the n-th partial sum $S_n := X_1 + \ldots + X_n$ as well as the standardized n-th partial sum $W_n := (S_n - n\mu)/\sqrt{n\sigma^2}$, then

$$W_n \xrightarrow{d} Z \sim \mathcal{N}(0,1), \tag{1.1}$$

which means that W_n converges in distribution to a standard-normal distributed random variable Z as n tends to infinity. The distribution function of a *normal distributed* random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, also known as *Gaussian distribution*, is given by

$$\mathbb{P}(X \le x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{\frac{(t-\mu)^2}{2\sigma^2}}\, dt,$$

we call the case $Z \sim \mathcal{N}(0,1)$ *standard-normal* and define $\Phi(x) := \mathbb{P}(Z \le x)$.
Indeed a CLT is proven in a great number of situations. In particular over the years (1.1) was generalized for not necessarily identically distributed or dependent random variables, and was extended to the multidimensional case and other limit distributions, e.g. densities proportional to $\exp\left(-\frac{\mu x^{2k}}{(2k)!}\right)$.

For now, we come back to (1.1) and ask, how large the approximation error is. A. C. Berry and C.–G. Esseen gave a first answer in 1941 respectively 1942: Under the assumption, that the third absolute moments of $X_1, \ldots, X_n$ are finite, we have the following bound, see [7] and [43],

$$\sup_{x\in\mathbb{R}} |\mathbb{P}(W_n \le x) - \mathbb{P}(Z \le x)| \le \frac{C \cdot \mathbb{E}\,|X_1|^3}{\sqrt{n}}, \tag{1.2}$$

where C is a constant, which was original 7,59 and has been improved over the years. The left-hand side of (1.2) is essentially the difference of the distribution functions of W_n and Z and is called Kolmogorov distance. We will refer to this notion and other distances of random variables more precisely in section 2.1. The right-hand side of (1.2) can be, apart from seeing it as an approximation error, interpreted as the speed of convergence of (1.1). In this case it is of order $O\left(\frac{1}{\sqrt{n}}\right)$, see section 1.5 for Landau notation $O(.)$, and was proved to be optimal. Due to the fundamental work of Berry and Esseen mathematicians are used to call results as (1.2) *Berry–Esseen-type results*. Note, that we have to distinguish *uniform* and *non-uniform* Berry–Esseen bounds. While uniform bounds as (1.2) have a supremum, here over all $x \in \mathbb{R}$, this is not the case for non-uniform bounds. As a consequence the right-hand side of such bounds has an additional prefactor depending on our real variable x, e.g. a result by [8] for independent and not necessarily identically distributed random variables is given as follows:

$$|\mathbb{P}(W_n \le x) - \mathbb{P}(Z \le x)| \le C \sum_{i=1}^{n} \frac{\mathbb{E}\,|X_i|^3}{1 + |x|^3}. \tag{1.3}$$

So far, we talked about the approximation error in the CLT, but to become more precise we talked about the *absolute error*. We can be also interested in the *relative error* and this motivates the notion of moderate deviations. The theory of moderate deviations goes back to H. Cramér in 1938: Under the assumption, that $\mu = 0$ and $\sigma^2 = 1, W_n$ simplifies to $W_n := S_n/\sqrt{n}$ and we can rewrite (1.1) to

$$\frac{\mathbb{P}(W_n > x)}{\mathbb{P}(Z > x)} \longrightarrow 1 \quad \text{for} \quad x = O(1), \tag{1.4}$$

which is a consequence of the convergence in distribution. Cramér was asking what happens, if x depends on $n \in \mathbb{N}$ such that $x \to \infty$ for $n \to \infty$? Can we find an interval such that (1.4) holds for $0 \le x \le I(n), I(n) \to \infty$? The answer was given by himself: Under the assumption, that the moment generating function $\mathbb{E}[e^{t|X_1|}] < \infty$ for all $0 \le t \le t_0$ with $t_0 > 0$,

$$\frac{\mathbb{P}(W_n > x)}{\mathbb{P}(Z > x)} = 1 + O(1)n^{-1/2}(1 + x^3) \quad \text{for} \quad 0 \le x \le n^{1/6}, \tag{1.5}$$

and the result is optimal, see e.g. [24] and [78]. Reminiscent of (1.5) for a sequence $(Y_n)_{n\in\mathbb{N}}$ of random variables, such that $Y_n \xrightarrow{d} Y$, the *moderate deviation of Cramér-type* is given by

$$\frac{\mathbb{P}(Y_n > x)}{\mathbb{P}(Y > x)} = 1 + \text{error term} \to 1$$

with range $0 \le x \le a_n$, where $a_n \to \infty$ for $n \to \infty$.
Apart from moderate deviations there is also the notion of *large* or *moderate deviation principles* (LDP, MDP), we refer to [26] for a precise definition. The MDP corresponding to (1.4) is given by

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log(\mathbb{P}(S_n/b_n > x)) = -\frac{x^2}{2},$$

where $\sqrt{n} << b_n << n$, so a scaling inbetween a central limit theorem and a law of large numbers. At the end of [30] it is shown that moderate deviations of Cramér-type imply MDPs, but we will not go deeper into that.

1.3. **Results for Rademacher random variables.** Now we come to the concrete objects we will investigate and the concrete results we will prove throughout this thesis.

Although probability theory is a large field in mathematics and we can have all kinds of complicated distributions in mind, already some of the easiest can give fascinating results. As we mentioned before, in 1733 de Moivre was interested in the number of heads resulting from many tosses of a fair coin. If we toss a fair coin, there are two outcomes to expect: heads and tails. Another possible interpretation is success and failure, which we can identify with $+1$ and -1. Random variables, that take only values $+1$ and -1 with probabilty $p \in (0,1)$ respectively $q := 1-p$ are known in the literature as Rademacher random variables. Depending on the source, the classical Rademacher distribution, named after H. Rademacher, is defined for $p = q = \frac{1}{2}$, but the notion can be also used for general p. We will work with Rademacher random variables to construct our objects of interest and among them we can distinguish two main types: First, we treat the Curie–Weiss model as an example for an explicit probability distribution depending on finitely many Rademacher random variables. Secondly, we consider general L^2-functionals over possibly infinitely many Rademacher random variables, sometimes just called L^2-Rademacher-functionals. In what follows we give an overview which results other authors proved concerning our topics and how we continue their considerations with our new results.

In statistical mechanics the *Curie–Weiss model* of n *spins* at temperature $T > 0$ is the joint distibution of the random variables $(X_k^{(\beta)})_{1\le k\le n}$ defined by

$$\mathbb{P}\Big(X_1^{(\beta)} = x_1, \dots, X_n^{(\beta)} = x_n\Big) := \frac{e^{\frac{\beta}{2n}s_n^2}}{\mathcal{Z}_{n,\beta}} \bigotimes_{i=1}^{n} d\varrho(x_i), \tag{1.6}$$

where $\beta := T^{-1}$ is the *inverse temperature*, $s_n := \sum_{k=1}^n x_k$ and $\mathcal{Z}_{n,\beta}$ a normalizing constant to ensure (1.6) is a probability distribution. Moreover we denote by ϱ the distribution of a single spin for $\beta = 0$. In our case the spins $(X_k^{(0)})_{1\le k\le n}$ are i.i.d. Rademacher random variables with $p = q = \frac{1}{2}$, in other words $\varrho = \frac{1}{2}(\delta_{+1} + \delta_{-1})$, see also subsection 2.2.1. This setting is also known as the *classical Curie–Weiss model*.
The Curie–Weiss model was originally introduced by P. Curie in 1895 [25] and refined by P.–E. Weiss in 1907 [100] as an exactly solvable model of ferromagnetism: the ferromagnetic alloys have the property of spontaneously changing their magnetic behaviour when heated, once a certain critical temperature treshold is reached.
Nowadays, it is presented as a mean-field approximation of the more refined Ising model, e.g. as the replacement of an interaction with nearest neighbour $\sum_{i\sim j} X_i X_j$ by an interaction with all other spins $\sum_{i,j} X_i X_j$, see e.g. [49, chapter 2]. Such approximations are frequently performed in probability theory in general and in statistical mechanics in particular. Replacing a complex model with a simpler one whose overall behavior may be examined through explicit computations allows to get an intuition of the features that can be inferred from the original model, sometimes with no alteration. In particular, the Curie–Weiss model does exhibit a *phase transition* with three distinct behaviours at high, critical and low temperature.
We refer to [49, ch. 2] for a friendly introduction to its main properties, or the more classical references, e.g. [10], [54], [93] and [99].
Of particular interest is the distribution of the *(unnormalised) total magnetisation*

$$M_n^{(\beta)} := \sum_{k=1}^{n} X_k^{(\beta)}, \tag{1.7}$$

since this random variable contains all the information of the model, as the distribution of every spin is defined by means of $M_n^{(\beta)}$.

The difference of behaviour of the system when the inverse temperature β varies can be summarised in the following theorem that can be found e.g. in the books [10], [38], [49], [54], [93] and [99] or in the papers [33], [39], [40] and [92]. We have the following fluctuations of the total magnetisation:

(1) For $\beta < 1$ and $\boldsymbol{Z}_\beta \sim \mathcal{N}\left(0, \frac{1}{1-\beta}\right)$,

$$\frac{1}{\sqrt{n}} M_n^{(\beta)} \xrightarrow{d} \boldsymbol{Z}_\beta. \tag{1.8}$$

(2) For $\beta = 1$ and $\boldsymbol{F}_0$ with density $\frac{1}{\mathcal{Z}_0} e^{-\frac{x^4}{12}}$,

$$\frac{1}{n^{3/4}} M_n^{(1)} \xrightarrow{d} \boldsymbol{F}_0. \tag{1.9}$$

(3) For $\beta = 1 - \frac{\gamma}{\sqrt{n}}$ with $\gamma \in \mathbb{R}$ fixed, and $\boldsymbol{F}_\gamma$ with density $\frac{1}{\mathcal{Z}_\gamma} e^{-\gamma \frac{x^2}{2} - \frac{x^4}{12}}$,

$$\frac{1}{n^{3/4}} M_n^{(1-\gamma/\sqrt{n})} \xrightarrow{d} \boldsymbol{F}_\gamma. \tag{1.10}$$

(4) For $\beta > 1$ and δ_x the dirac measure of x,

$$\frac{1}{n} M_n^{(\beta)} \xrightarrow{d} \frac{1}{2}\left(\delta_{+m_\beta} + \delta_{-m_\beta}\right), \qquad m_\beta = \tanh(\beta m_\beta). \tag{1.11}$$

We note that in case (3) the left transition $\gamma > 0$ and the right transition $\gamma < 0$ give the same limiting law, even though the graph of the density displays a very different behaviour, with two different modes that announce the case $\beta > 1$ in the second case. This continuous phase transition is characteristic for the classical Curie–Weiss model, see e.g. [49, § 2.5.3].
Several modifications and follow-ups to (1.8) – (1.11) can be made: universality of the limits when the law of the $(X_k)_k$ is changed, see [39] and [40], dynamical spin-flip version, see [63], concentration properties of the spins around the limit in the case $\beta > 1$, see [14] and [15], moderate and large deviations, see [32] and [38], modification of the Hamiltonian leading to the Curie–Weiss–Potts model, see [34] and [41], the inhomogeneous Curie–Weiss model, see [31], etc. These results allow the variety of techniques used in probability theory to express their power and illustrate a form of richness of the field, both in the questions asked and in the responses that follow.
In a domain as venerable as the Curie–Weiss model, older than 100 years, it seems very difficult to innovate, especially with the original model of ± 1-spins. In addition to the classical studies in [39] and [40] that use the Laplace transform, one should add the classical tools of probability theory when concerned with distributional approximation such as Stein's method of exchangeable pairs, see [14], [15], [16], [33] and [90].
In this thesis we aim for giving yet *another proof* of the old and respectable results (1.8) – (1.11) with the additional result of the speed of convergence in Fortet–Mourier and Kolmogorov distance.
For this intention we will work with a very important feature of the Curie–Weiss random variables (1.6), which is the existence of an exchangeability measure. Here, exchangeability means that the joint distribution of the Curie–Weiss spins does not change if they are permuted. While exchangeable pairs have been used thoroughly by means of Stein's method, writing the spins as i.i.d. random variables conditionally to a measure of mixture is a particularly strong peculiarity that was taken advantage of in several works on the Curie–Weiss model, for instance the papers [13], [53], [66] and [76]. The authors of these papers use in an extensive way the existence of a De Finetti measure of exchangeability for the Curie–Weiss spins to tackle natural probabilistic questions as functional CLT, extension to infinite exchangeability, etc., but none of these problems will be treated here, though; we will focus exclusively on a new approach to (1.8) – (1.11), and for this approach the general theory of

B. De Finetti as well as the De Finetti measure of the Curie–Weiss model will be fundamental, see section 2.3. We want to emphasize that this approach is also promising for other mean-field models.

An important role for our second topic plays a paper written by Z.–S. Zhang, which we cite here in its first version [101] and its latest version [102] to compensate changes in the content. In [102] Zhang was able to prove Cramér-type moderate deviations for unbounded exchangeable pairs (W, W'). For such pairs it holds that (W, W') and (W', W) are equal in distribution. If the difference $W - W'$ is bounded, we call (W, W') bounded, otherwise unbounded. Zhang developed his moderate deviations by stopping the proof of the corresponding Berry–Esseen-type inequalities, he had obtained before by Stein's method, at a certain point and continuing differently. Stein's method is basically a powerful tool by itself to derive upper bounds for differences of probability distributions, originally developed for the normal distribution and later extended to other distributions. We refer to section 2.1 for a formal introduction to Stein's method and exchangeable pairs, which are typically combined with it. Zhang rearranged the fragments of the so-called Stein-equation and the bound of its solution, a technique that was already seen in [19], [46], [82] and [89]. This technique will be the core of the proof of our result.
Our ambition is to prove Cramér-type moderate deviations for L^2-functionals over infinitely many independent Rademacher random variables taking values +1 and –1 only. This new general result intersects with [45], where the authors obtain Cramér-type moderate deviations via p-Wasserstein bounds, and we will refer to that. For L^2-Rademacher-functionals a Kolmogorov bound in the context of normal approximation was shown recently by P. Eichelsbacher, B. Rednoss, C. Thäle and G. Zheng in [36, Theorem 3.1] such that the bounding terms can be expressed in terms of operators of the so-called Malliavin–Stein method, see section 2.2. Normal approximation of L^2-functionals over infinitely many Rademacher random variables was derived already in [73], [61], [62] and [29]. Theorem 3.1 in [36] will be our starting point.

Last, we deal with non-uniform Berry–Esseen results. The first bounds of this type came from Esseen himself in 1945, see [44], for independent and identically distributed random variables with finite third moments. They were improved by [70] in 1965 and generalized by [8] in 1966 for independent and not necessarily identically distributed random variables to (1.3). Moreover the constant C was improved over the following decades.
In 2001, L.H.Y. Chen und Q.–M. Shao [21] generalized (1.3) and proved their bound without assuming the existence of third moments. Another feature of their bound is the truncation of the random variables at 1:

$$|\mathbb{P}(W \leq x) - \Phi(x)| \leq C \sum_{i=1}^{n} \left(\frac{\mathbb{E}[X_i^2] \mathbb{1}_{\{|X_i| > 1+|x|\}}}{(1+|x|)^2} + \frac{\mathbb{E}\,|X_i|^3\,\mathbb{1}_{\{|X_i| \leq 1+|x|\}}}{(1+|x|)^3} \right). \tag{1.12}$$

More precise, in (1.3) and (1.12) it is $W = \sum_{i=1}^n X_i$ for an independent and not necessarily identically distributed sequence $(X_k)_{k\in\mathbb{N}}$ with $\mathbb{E}[X_k] = 0, \mathrm{Var}(X_k) < \infty$ and $\mathrm{Var}(W) = 1$, with respectively without existence of absolute third moments, and C an absolute constant. A few years later Chen und Shao [22] established a similar result under local dependence. They obtained both of their results by a combination of Stein's method and a concentration inequality approach.
In the following years, continuations of the work of Chen und Shao can be found in [3] for translated Poisson approximation or [23] for nonlinear statistics. Moreover papers, which aimed for improvement or lower bounds of the absolute constant in the non-uniform prefactor, or comparable results under stricter moment assumptions, are [57], [58], [79] and [91].

The specific starting point in this thesis is [67], where D.L. Liu, Z. Li, H.C. Wang and Z.J. Chen showed non-uniform Berry–Esseen bounds for normal and nonnormal approximations by unbounded exchangeable pairs (W, W'). They referred to a corresponding uniform bound in [90] and proved their main result without concentration inequalities. Recently their work was generalized in [97] for the normal approximation case under the additional assumption of $\mathbb{E}\,|W - W'|^{2r}$ being of certain order.
When we studied the proof of the main result in [67], our observation was the following: The non-uniform bound consists almost of the same terms, which were constructed by the theory of exchangeable pairs, as the uniform bound in [90], but with a prefactor depending on $z \in \mathbb{R}$. The reason for that type of bound is a strict separation between the mentioned terms and the fragments of the Stein-equation of the corresponding exchangeable pair by the Cauchy–Schwarz inequality. Most of the proof depends on the Stein-equation, whose fragments have to be bounded precisely to lead to the desired prefactor, and not on the exchangeable pair itself. This motivates our attempt to adapt the argumentation in [67] to obtain non-uniform Berry–Esseen results for L^2-Rademacher-functionals in the context of Malliavin–Stein method. Moreover we will derive an analogous result for so-called Poisson-functionals, see subsection 2.2.3 for a short introduction.

1.4. **Overview.** The remaining chapters of this thesis are structured as follows.

Chapter 2 thematizes the preliminaries, meaning methods, models and notions, which will appear in the following chapters. In section 2.1 we present important aspects of Stein's method as the concept of the method, its history, properties of the Stein-equation and its solution, and techniques to bound latter ones. The combination of Stein's method and Malliavin calculus is known as Malliavin–Stein method, which is topic of section 2.2; in particular we introduce operators from Malliavin calculus for L^2-Rademacher- and L^2-Poisson-functionals. In section 2.3 we discuss the De Finetti theorem and write down the corresponding measure of the Curie–Weiss model explicitly. Furthermore we motivate the application of surrogate random variables in probability theory and construct our specific surrogate random variable by a combination of the Gaussian CLT in the particular case of Rademacher random variables and a randomisation of the Rademacher parameter p, which is distributed with respect to the De Finetti measure of the Curie–Weiss model.

In Chapter 3 we investigate the Curie–Weiss model by using surrogate random variables, which are distributed with respect to its De Finetti measure of exchangeability, and give a new proof of the phase portrait of the model. Writing the magnetisation as a sum of i.i.d. Rademacher's randomised by the underlying De Finetti random variable, we show that the apparition of a phase transition can be understood as a competition between these two sources of randomness, the Gaussian randomness coming from the CLT approximation and the randomness in the mixture of the Rademacher's. We consider four cases: sub critical ($\beta < 1$), critical ($\beta = 1$), near critical ($\beta = 1 \pm \frac{\gamma}{\sqrt{n}}$) and super critical ($\beta > 1$). The results are proven in Fortet–Mourier distance, see section 3.1, and Kolmogorov distance, see section 3.2, which implies in particular convergence in distribution. Moreover the results include speeds of convergence. In section 3.3, an appendix, we analyse diverse constants, which are relevant for the underlying De Finetti measure.

In Chapter 4, section 4.1, we derive moderate deviations for normal approximation of L^2-functionals over infinitely many Rademacher random variables. They are based on a bound for the Kolmogorov distance between a general L^2-Rademacher-functional and a Gaussian random variable, continued by an intensive study of the behaviour of operators from the

Malliavin–Stein method along with the moment generating function of the mentioned L^2-functional. We treat the i.i.d.-case as a first application and get the optimal range from Cramér in section 4.2. At last, we study infinite weighted 2-runs with general summable coefficient sequences in section 4.3. Despite their comparatively simple structure, the corresponding proof is challenging. Moreover we look at examples for coefficient sequences where the result is optimal.

In Chapter 5, section 5.1, we obtain non-uniform Berry–Esseen bounds for L^2-Rademacher- and L^2-Poisson-functionals, which include a non-uniform second-order Gaussian Poincaré inequality in the Rademacher case. The foundation for these bounds are the corresponding uniform Berry–Esseen bounds respectively their proofs, where we separate the terms consisting of operators from Malliavin–Stein method and the terms consisting of fragments of the Stein-equation from normal approximation. Latter ones have to be bounded precisely to give us the prefactor of the non-uniform bound, whose order depends on the existence of higher moments of the considered functional. As applications we treat infinite weighted 2-runs and subgraph counting in the Erdős–Rényi random graph in section 5.2.

1.5. **Basic notions and notations.** At last we want to collect some basic notions and notations from probability theory, which partially already appeared and will accompany us through this thesis. If not mentioned explicitly, all appearing random variables are defined on an appropriate *probability space* $(\Omega, \mathcal{A}, \mathbb{P})$. The *expectation* $\mathbb{E}(X)$, the *variance* $\mathrm{Var}(X)$ and the *pth moment* $\mathbb{E}(X^p)$ of a random variable X are computed with respect to the underlying *probability measure* $\mathbb{P}$. If $\mathbb{E}(|X|^p) < \infty$, we write $X \in L^p(\Omega)$.
We call two random variables X and Y *equal in distribution*, if $\mathbb{P}(X \leq t) = \mathbb{P}(Y \leq t)\ \forall\, t \in \mathbb{R}$ and then we write $X \overset{d}{=} Y$. A sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ *converges in distribution* to a random variable Y, if $\mathbb{P}(Y_n \leq t) \longrightarrow \mathbb{P}(Y \leq t)$ as $n \to \infty\ \forall\, t \in \mathbb{R}$ and then we write $Y_n \overset{d}{\longrightarrow} Y$. Alternatively we can also say *law* instead of distribution for these notions.
A distribution very similar to the Rademacher distribution is the *Bernoulli distribution*, classically taking values 0 and 1 only. We will sometimes see this more general and call a random variable B Bernoulli distributed, if it takes values $\pm x$ only, apart from ± 1, with probability p respectively $1-p$ and then we write $B \sim \mathrm{Ber}_{\pm x}(p)$.
Denote by $\mathcal{U}([a,b])$ the *continuous uniform distribution* on an interval $[a,b]$. If a random variable X is *absolutely continuous with Lebesgue-density* f, we write $\mathbb{P}(X \in dt) = f(t)dt$.
Throughout this thesis we use the usual *Landau symbols*; the big-O notation $O(.)$ and the small-o notation $o(.)$ with the meaning that the implicit constant does not depend on the parameters in brackets.
Moreover we define the *supremum norm* $\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|$ for any function $f : \mathbb{R} \to \mathbb{R}$. We denote by $\mathcal{C}^1$ the *continuous and continuously differentiable functions*, while by $\mathcal{C}^\infty$ we mean the *infinitely often continuously differentiable functions*.

2. Preliminaries

2.1. **Stein's method.**

2.1.1. *General concept.* In 1972 Charles Stein established a method to provide explicit bounds for the quality of the approximation of a probability distribution through another, with normal approximation as its original application. Over the years this method has become famous as *Stein's method*. We want to explain its main ideas and refer to [4], [85], [20] and [18] as sources and also for further information. The foundation of Stein's method is an important characterisation of the standard normal distribution, also known as *Stein-Lemma*. According to [4, Lemma 2.1] it is

$$Z \sim \mathcal{N}(0,1) \Leftrightarrow \mathbb{E}[f'(Z) - Zf(Z)] = 0 \tag{2.1}$$

for all continuous and piecewise continuous differentiable $f : \mathbb{R} \to \mathbb{R}$ such that the appearing expectations exist. So if a random variable is in some sense close to $\mathcal{N}(0,1)$, it is likely that the expectation in (2.1) is close to 0. The appropriate way to express this closeness is to work with distances of random variables respectively their probability distributions. All these distances are of the following form: For a fixed class of test functions $\mathcal{H}$, which determines an associated metric, and random variables X and Y we are interested in

$$d_{\mathcal{H}}(X,Y) = \sup_{h\in\mathcal{H}} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]| \,.$$

While $\mathcal{H} = \{\mathbb{1}_A : A \text{ measurable}\}$ and $\mathcal{H} = \{h : \mathbb{R} \to \mathbb{R} : |h(x) - h(y)| \le |x-y|\}$ for the *total variation distance* respectively the *Wasserstein distance* are also possible choices in probability theory, throughout this thesis we will focus on the following two distances:

- For $\mathcal{H} = \{h : \mathbb{R} \to \mathbb{R} : \|h\|_\infty \le 1, \|h'\|_\infty \le 1\}$ the *Fortet-Mourier distance* is given by
$$d_{\mathrm{FM}}(X,Y) := \sup_{h\in\mathcal{H}} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]| \,. \tag{2.2}$$
- For $\mathcal{H} = \{\mathbb{1}_{\{.\le t\}} : t \in \mathbb{R}\}$ the *Kolmogorov distance* is given by
$$d_{\mathrm{Kol}}(X,Y) := \sup_{t\in\mathbb{R}} |\mathbb{P}(X \le t) - \mathbb{P}(Y \le t)| \,. \tag{2.3}$$

Thus the *Stein-equation*, written in the case of Kolmogorov distance, is given by

$$f_z'(w) - wf_z(w) = \mathbb{1}_{\{w\le z\}} - \Phi(z), \tag{2.4}$$

respectively for our random variable of interest

$$\sup_{z\in\mathbb{R}} |\mathbb{E}[f_z'(W) - Wf_z(W)]| = \sup_{z\in\mathbb{R}} |\mathbb{P}(W \le z) - \Phi(z)| \,. \tag{2.5}$$

(2.4) is a differential equation and we can solve it with the following trick: We multiply on both sides by $-e^{-w^2/2}$ and rewrite the new equation to

$$\left(e^{-w^2/2} f_z(w)\right)' = -e^{-w^2/2}\left(\mathbb{1}_{\{w\le z\}} - \Phi(z)\right).$$

Then the solution $f = f_z$ is given by

$$\begin{aligned} f_z(w) &= e^{w^2/2} \int_{-\infty}^{w} \left(\mathbb{1}_{\{x\le z\}} - \Phi(z)\right) e^{-x^2/2} dx \\ &= -e^{w^2/2} \int_{w}^{\infty} \left(\mathbb{1}_{\{x\le z\}} - \Phi(z)\right) e^{-x^2/2} dx \\ &= \begin{cases} \frac{\Phi(w)(1-\Phi(z))}{p(w)} & w \le z, \\ \frac{\Phi(z)(1-\Phi(w))}{p(w)} & w > z, \end{cases} \end{aligned} \tag{2.6}$$

where $p(w) = e^{-w^2/2}/\sqrt{2\pi}$ is the density of $\mathcal{N}(0,1)$. If we combine (2.4) and (2.6) we obtain also the first derivative

$$\begin{aligned} f_z'(w) &= wf_z(w) + \mathbb{1}_{\{w\le z\}} - \Phi(z) \\ &= \begin{cases} wf_z(w) + 1 - \Phi(z) & w < z, \\ wf_z(w) - \Phi(z) & w > z; \end{cases} \\ &= \begin{cases} (\sqrt{2\pi}we^{w^2/2}\Phi(w) + 1)(1-\Phi(z)) & w < z, \\ (\sqrt{2\pi}we^{w^2/2}(1-\Phi(w)) - 1)\Phi(z) & w > z. \end{cases} \end{aligned} \tag{2.7}$$

Now we want to collect useful properties related to (2.6) and (2.7), mostly extracted from [20, Lemma 2.3]:

$$0 < f_z(w) \le \min\left\{\frac{\sqrt{2\pi}}{4}, \frac{1}{|z|}\right\} \text{ for all } w \in \mathbb{R}, \tag{2.8}$$

$$wf_z(w) \text{ is an increasing function of } w \in \mathbb{R}, \tag{2.9}$$

$$|wf_z(w)| \le 1 \text{ for all } w \in \mathbb{R}, \tag{2.10}$$

$$|wf_z(w)| \le (1-\Phi(z)) \text{ for all } w < 0 \le z, \tag{2.11}$$

$$|f_z'(w)| \le 1 \text{ for all } w \in \mathbb{R}, \tag{2.12}$$

$$\frac{1-\Phi(w)}{p(w)} \le \min\left\{\frac{1}{w}, \frac{\sqrt{2\pi}}{2}\right\} \text{ for all } w > 0, \tag{2.13}$$

and if a statement is valid for all $w \in \mathbb{R}$, it can be written in particular with $\|.\|_\infty$.
(2.13) is known as *Mill's ratio* for the standard normal distribution. It can be used together with (2.6) to obtain (2.11) as follows:

$$\begin{aligned} |wf_z(w)| &= |w|\,\sqrt{2\pi}e^{w^2/2}\Phi(w)(1-\Phi(z)) \\ &= (1-\Phi(z))(1-\Phi(|w|))\sqrt{2\pi}\,|w|\,e^{|w|^2/2} \\ &\le (1-\Phi(z)), \end{aligned}$$

where we also used the symmetry of Φ. We will need this more precise bound of $|wf_z(w)|$ for the main result in section 4.1, where we distinguish different cases for $w \in \mathbb{R}$.

Motivated by (2.5) our main task is now to show that $\mathbb{E}[f_z'(W) - Wf_z(W)]$ is small. There are several techniques, which have proven to be very useful for that purpose. One thing they have in common is the idea of *coupling*. By that we mean that our random variable of interest W is coupled with another random variable W', resulting from W by a slight change and so in some sense close to W. This concept differs within the approaches and will become more clear, when we illustrate them now.

2.1.2. *Leave one out approach.* Let $X_1, \dots, X_n$ independent random variables with mean zero and variances $\sigma_1^2, ..., \sigma_n^2$ with $\sum_{i=1}^n \sigma_i^2 = 1$. Further let $W = \sum_{i=1}^n X_i$ and $W^{(i)} = \sum_{j\ne i}^n X_j$, the sum where we *leave the i-th summand out*. The key element is now the independence of $W^{(i)}$ and X_i. Due to that

$$\mathbb{E}[Wf(W)] = \sum_{i=1}^n \mathbb{E}\Big(X_i f\Big(W^{(i)} + X_i\Big)\Big)$$

$$= \sum_{i=1}^{n} \mathbb{E}\Big(X_i^2 \int_0^1 f'\big(W^{(i)} + uX_i\big)\, du\Big) + \sum_{i=1}^{n} \mathbb{E}\Big(X_i f\big(W^{(i)}\big)\Big)$$
$$= \sum_{i=1}^{n} \mathbb{E}\Big(X_i^2 \int_0^1 f'\big(W^{(i)} + uX_i\big)\, du\Big)$$

and

$$\begin{aligned}\mathbb{E}[f'(W)] &= \mathbb{E}\left(\sum_{i=1}^{n} \sigma_i^2 f'(W)\right) \\ &= \mathbb{E}\left(\sum_{i=1}^{n} \sigma_i^2 f'(W^{(i)})\right) + \mathbb{E}\left(\sum_{i=1}^{n} \sigma_i^2 \Big(f'(W) - f'(W^{(i)})\Big)\right) \\ &= \mathbb{E}\left(\sum_{i=1}^{n} X_i^2 f'(W^{(i)})\right) + \mathbb{E}\left(\sum_{i=1}^{n} \sigma_i^2 \Big(f'(W) - f'(W^{(i)})\Big)\right).\end{aligned}$$

Combining the last two equations leads to

$$\begin{aligned}\mathbb{E}[f'(W) - Wf(W)] = &\sum_{i=1}^{n} \mathbb{E}\Big(X_i^2 \int_0^1 \Big(f'\big(W^{(i)}\big) - f'\big(W^{(i)} + uX_i\big)\Big)\, du\Big) \\ &+ \sum_{i=1}^{n} \mathbb{E}\Big(\sigma_i^2 \Big(f'(W) - f'(W^{(i)})\Big)\Big).\end{aligned}$$

Both terms include a difference of first derivatives, so we can use

$$\Big|f'(W) - f'(W^{(i)})\Big| \le |X_i| \, \|f''\|_\infty \tag{2.14}$$

for the second term, if f has a bounded second derivative, see subsection 2.1.5 for a related discussion. We get an analogous bound for the first term since u is bounded by 1. Then

$$|\mathbb{E}[f'(W) - Wf(W)]| \le \|f''\|_\infty \sum_{i=1}^{n} \Big(\mathbb{E}\,|X_i|^3 + \sigma_i^2 \mathbb{E}\,|X_i|\Big) \le 2\,\|f''\|_\infty \sum_{i=1}^{n} \mathbb{E}\,|X_i|^3 \tag{2.15}$$

using $\mathbb{E}[X_i^2]\mathbb{E}\,|X_i| \le \Big(\mathbb{E}\,|X_i|^3\Big)^{2/3}\Big(\mathbb{E}\,|X_i|^3\Big)^{1/3}$ by Hölder's inequality. If we consider i.i.d. scaled random variables $X_i = n^{-1/2}\xi_i$ our bound (2.15) is of the form

$$|\mathbb{E}[f'(W) - Wf(W)]| \le \frac{C \cdot \mathbb{E}\,|\xi|^3}{\sqrt{n}}.$$

The authors of [21] apply the leave one out approach together with concentration inequalities to obtain (non-) uniform Berry–Esseen bounds. In the sequel, [22], the approach is generalized for a setting with local dependence, in which not only one summand is left out, but every summand depending on a random variable with fixed index, to get corresponding (non-) uniform Berry–Esseen bounds.

2.1.3. *Exchangeable pairs.* We call a pair (W, W') of random variables an *exchangeable pair* if $(W, W') \overset{d}{=} (W', W)$. If for some $0 < \lambda < 1$ the exchangeable pair satisfies the *linear regression condition*

$$\mathbb{E}[W - W'|W] = \lambda W, \tag{2.16}$$

then we call (W, W') a λ-*Stein Pair* or just *Stein Pair*, see [6] for an introduction to conditional expectations. Typically in settings with dependency (2.16) does not hold exactly, but with a *remainder random variable* R of small order such that

$$\mathbb{E}[W - W'|W] = \lambda W + R.$$

We define $\Delta := W - W'$. In case Δ is bounded we call the exchangeable pair *bounded*, otherwise *unbounded*. There are some basic properties of a Stein Pair (W, W'), which follow almost immediately from the exchangeability:

- $\mathbb{E}[W] = \mathbb{E}[W'] = 0$, if $R = 0$,
- $\mathbb{E}[\Delta^2] = 2\lambda \operatorname{Var}(W)$, if $\operatorname{Var}(W) < \infty$,
- $\mathbb{E}[F(W, W')] = 0$ for all anti-symmetric measurable functions F such that the expectation exists.

Now we want to give an example for an exchangeable pair. Let $X_1, \dots, X_n$ be independent random variables with mean 0 and variance 1. Next we take copies $X_1', \dots, X_n'$ of $X_1, \dots, X_n$, where $X_i \overset{d}{=} X_i'$ and the X_i' are independent as well as independent from all X_i. Last let I be uniform on $\{1, 2, \dots, n\}$ and independent of all X_i and X_i'. Then we define

$$W := \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \quad \text{and} \quad W' := W - \frac{X_I}{\sqrt{n}} + \frac{X_I'}{\sqrt{n}}.$$

So we remove a randomly chosen summand and add an independent copy of it. It follows mainly from

$$\begin{aligned}
\mathbb{E}(W - W'|W) &= \frac{1}{\sqrt{n}} \mathbb{E}(X_I - X_I'|W) \\
&= \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}(X_i - X_i'|W) \\
&= \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}(X_i|W) - \mathbb{E}(X_i'|W) \\
&= \frac{1}{n} \mathbb{E}\left(W|W\right) \\
&= \frac{1}{n} W
\end{aligned}$$

that (W, W') is a Stein-Pair with $\lambda = \frac{1}{n}$, where we used in particular $\mathbb{E}(X_i'|W) = \mathbb{E}(X_i') = 0$ by independence of our random variables. We refer to [20, Lemma 2.7] and its proof to show how the terms of the Stein-equation in the general exchangeable pair setting look like, namely

$$\mathbb{E}[f'(W) - Wf(W)] = \mathbb{E}\Big(f'(W)\Big(1 - \frac{1}{2\lambda}\mathbb{E}[\Delta^2|W]\Big)\Big) + \frac{1}{2\lambda}\mathbb{E}\Big(\Delta \int_{-\Delta}^0 (f'(W) - f'(W + t))\, dt\Big).$$

It is a priori not obvious whether the first term can be bounded well, but if $\operatorname{Var}(W) = 1$ and we recall $\mathbb{E}[\Delta^2] = 2\lambda \operatorname{Var}(W)$ the success of this representation becomes more clear. Note that these are the original terms to rewrite the Stein-equation, but they have been modified over the years in particular for unbounded exchangeable pairs, see e.g. [33], [90], [102] and [67], where the authors obtain (non-)uniform Berry–Esseen bounds and Cramér-type moderate deviations.

2.1.4. *Size-bias- and zero-bias-transformations.* Since these techniques were not used in our research, we just give a short introduction here.

For a random variable $X \geq 0$ with $\mathbb{E}[X] = \mu < \infty$ we say the random variable X^S has a *size-bias distribution* with respect to X if for all f with $\mathbb{E}\,|Xf(X)| < \infty$ we have

$$\mathbb{E}(Xf(X)) = \mu \mathbb{E}(f(X^S)). \tag{2.17}$$

For a real-valued random variable X with $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = \sigma^2 < \infty$ we say the random variable X^Z has a *zero-bias distribution* with respect to X if for all absolutely continuous f with $\mathbb{E}\,|Xf(X)| < \infty$ we have

$$\mathbb{E}(Xf(X)) = \sigma^2 \mathbb{E}(f'(X^Z)). \tag{2.18}$$

We see immediately how our term of interest $\mathbb{E}(Xf(X))$ is rewritten by the defining equations (2.17) and (2.18). In fact, similar to the previous two subsections bounds obtained by the size-bias or zero-bias approach rely on the difference $\left|X - X^S\right|$ respectively $\left|X - X^Z\right|$, see e.g. sections 5.1 and 5.3 in [20] for related results.

2.1.5. *Further techniques.* In this subsection we want to introduce certain techniques that can be seen as an addition to those mentioned so far. We recall, how we wrote the solution of the Stein-equation and related bounds explicitly in the Kolmogorov case in subsection 2.1.1. This can be also done similarly for general real valued measurable test functions h with $\mathbb{E}[h(Z)] < \infty$. Then we have

$$\begin{aligned} f_h(w) &= e^{w^2/2} \int_{-\infty}^{w} (h(x) - \mathbb{E}[h(Z)])\, e^{-x^2/2} dx \\ &= -e^{w^2/2} \int_{w}^{\infty} (h(x) - \mathbb{E}[h(Z)])\, e^{-x^2/2} dx. \end{aligned}$$

and for absolutely continuous $h : \mathbb{R} \to \mathbb{R}$, by [20, Lemma 2.4],

$$\begin{aligned} \|f_h\|_\infty &\le \min\Big(\sqrt{\pi/2}\, \|h(.) - \mathbb{E}[h(Z)]\|_\infty, 2\, \|h'\|_\infty\Big), \\ \|f_h'\|_\infty &\le \min\Big(2\, \|h(.) - \mathbb{E}[h(Z)]\|_\infty, \sqrt{2/\pi}\, \|h'\|_\infty\Big), \\ \|f_h''\|_\infty &\le 2\, \|h'\|_\infty . \end{aligned}$$

The upper bounds of f_h and its derivatives depend on derivatives of the test functions and for higher derivatives more restrictive assumptions concerning boundness and continuouity of $h, h', h'', \ldots$ are needed. Thus in the field of Stein's method there is an ambition to avoid higher derivatives if possible. In particular if we are interested in Kolmogorov distance the main problem is $\|f_z''\|_\infty = \infty$ and (2.14) cannot be applied. To work against this problem a rather simple but effective *trick* is to replace $f'(W)$ by the rest of the Stein equation (2.4). Furthermore, if there is already a difference of first derivatives we can get even more benefit from it: The structure

$$f'(W+t) - f'(W) = (W+t)f(W+t) - Wf(W) + \mathbb{1}_{\{W+t \le z\}} - \mathbb{1}_{\{W \le z\}}$$

motivates to use the monotonicity (2.9) of $wf(w)$ as well as that the indicator $\mathbb{1}_{\{w \le z\}}$ is a decreasing function in w for further computations. This *monotonicity argument* does not require a second derivative and was applied many times in research related to Stein's method, but it was used in [90] for the first time to prove noticeably simplified bounds in the context of (non-)normal approximation for unbounded exchangeable pairs.

2.1.6. *History.* Now that we have collected many of the formulas and techniques related to Stein's method, we finish this section with a brief overview on its history, see [18]. Although C. Stein already worked on his method in the 1960's, his fundamental papers [94] and [95] were published in 1972 respectively 1986. In these he established his general method as well as the leave one out approach and exchangeable pairs. While the concept of size-bias distributions appeared for the first time in 1989 [2] and was extended in 1996 by [52], zero-bias distributions were considered first in 1997 by [51].
Another reason for the success of Stein's method is that its ideas can be applied to many other distributions apart from normal distribution. Among these are the Poisson- [17], binomial- [95] and gamma-distribution [68], distributions with density proportional to $\exp\Big(-\frac{\mu x^{2k}}{(2k)!}\Big)$ [16] and [33], and more. Especially for continuous distributions there is a strategy how to find characterizations reminiscent of (2.1) known as Stein's *density approach.* The idea is to

choose an expression like $f'(x) + \psi(x)f(x)$ with $\psi(x)$ being $\frac{\varphi'(x)}{\varphi(x)}$, if $\varphi(x)$ is the density of the target distribution. In the case of the standard normal distribution we get

$$f'(x) + \frac{\varphi'(x)}{\varphi(x)}f(x) = f'(x) - \frac{xe^{-\frac{x^2}{2}}}{e^{-\frac{x^2}{2}}}f(x) = f'(x) - xf(x),$$

the lefthand side of the classical Stein-equation. We refer to [96] for technical details and further information about the density approach.
Over the years Stein's method has never stopped to expand in various directions with their own applications such as the Malliavin–Stein method. This method will be considered in the next section. Moreover the so-called Stein–Tikhomirov method was developed by several authors. It combines the original theory of Stein with the theory of characteristic functions, see [98] and also [35].

2.2. **Malliavin–Stein method.**

2.2.1. *Malliavin calculus.* Historically Malliavin calculus was applied first to Gaussian- [72], then to Poisson- [77] and then to Rademacher-functionals [73] — in fact these are all since for other distributions the chaos representations of the corresponding functionals do not exist. We want to summarize the setting and the operators that will appear in our results later on. Due to the focus of this thesis on Rademacher random variables we present the corresponding notions more extensive, while the Poisson case will play a minor role. We refer to all sources mentioned so far and additional to [36] and [60] for further details and information as well as [69] for further results related to the topic.

The Rademacher case. We start with $l^2(\mathbb{N})$, the *space of real square-summable sequences*, formally defined as

$$l^2(\mathbb{N}) := \left\{ (a_k)_{k\in\mathbb{N}} \middle| \, \|a\|^2_{l^2(\mathbb{N})} < \infty \right\},$$

where the norm

$$\|a\|^2_{l^2(\mathbb{N})} := \sum_{k\in\mathbb{N}} a_k^2 \tag{2.19}$$

is induced by the scalar product

$$\langle a, b\rangle := \sum_{k\in\mathbb{N}} a_k b_k, \quad a, b \in l^2(\mathbb{N}).$$

Moreover, by $l^2(\mathbb{N})^{\otimes p}$ we mean the *pth tensor product* of $l^2(\mathbb{N})$ for $p \in \mathbb{N}$. Relevant subsets are $l^2(\mathbb{N})^{\circ p}$, the symmetric functions in $l^2(\mathbb{N})^{\otimes p}$, and $l_0^2(\mathbb{N})^{\circ p}$, the symmetric functions in $l^2(\mathbb{N})^{\otimes p}$ which vanish on diagonals.
Let $(p_k)_{k\in\mathbb{N}}$ a sequence with $p_k \in (0,1)$ and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space such that

$$\Omega := \{-1,+1\}^{\mathbb{N}}, \quad \mathcal{F} := \mathcal{P}(\{-1,+1\})^{\otimes\mathbb{N}}, \quad \mathbb{P} := \bigotimes_{k\in\mathbb{N}} (p_k\delta_{+1} + (1-p_k)\delta_{-1}),$$

where $\delta_{\pm 1}$ is the unit-mass dirac-measure concentrated at ± 1 and $\mathcal{P}(M)$ the power set of a set M. Then we define $X = (X_k)_{k\in\mathbb{N}}$, an i.i.d. sequence of *Rademacher random variables*, on $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\mathbb{P}(X_k = 1) = p_k,$$
$$\mathbb{P}(X_k = -1) = q_k = 1 - p_k,$$

and, if needed, the standardized random variable

$$Y_k = \frac{X_k - p_k + q_k}{2\sqrt{q_k p_k}} \quad \forall\, k \in \mathbb{N}.$$

We are interested in square-integrable random variables $F \in L^2(\Omega, \sigma(X), \mathbb{P})$, with $\sigma(X)$ the σ-field generated by X. According to [80, Proposition 6.7] and section 2.1 in [59] we can write this space as the following direct sum:

$$L^2(\Omega, \sigma(X), \mathbb{P}) = \bigoplus_{n\in\mathbb{N}_0} \mathbb{C}_n, \tag{2.20}$$

where $\mathbb{C}_0 = \mathbb{R}$ and $\mathbb{C}_n = \{J_n(f) : f \in l^2(\mathbb{N})^{\otimes n}\}$, the *nth Rademacher chaos*. $\mathbb{C}_n$ consists of square-integrable n-linear polynomias $J_n(f)$ defined by

$$J_n(f) = \sum_{(i_1,\dots,i_n)\in\Delta^n} f(i_1, \dots, i_n) Y_{i_1} Y_{i_2} \dots Y_{i_n},$$

where $f \in l_0^2(\mathbb{N})^{\circ n}$ and $\Delta_n := \{(i_1, ..., i_n) \in \mathbb{N}^n : i_j \neq i_k \text{ for } j \neq k\}$. $J_n(f)$ is called the *nth discrete multiple integral*. The decomposition (2.20) is known as *Wiener-Itô-Walsh decomposition* and as a consequence we can write $F \in L^2(\Omega, \sigma(X), \mathbb{P})$ as

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} J_n(f_n) \tag{2.21}$$

for a unique sequence of functions $(f_n)_{n\in\mathbb{N}}$ with $f_n \in l_0^2(\mathbb{N})^{\circ n}$.
For $F = f(X) = f(X_1, X_2, ...) \in L^1(\Omega, \sigma(X), \mathbb{P})$ we define the *discrete gradient* $D_k F$ of F at kth coordinate:

$$D_k F := \sqrt{p_k q_k}(F_k^+ - F_k^-),$$
$$DF := (D_1 F, D_2 F, ...),$$

where $F_k^+ := f(X_1, ..., X_{k-1}, +1, X_{k+1}, ...)$ and $F_k^- := f(X_1, ..., X_{k-1}, -1, X_{k+1}, ...)$, $k \in \mathbb{N}$. So we fix the k-th Rademacher random variable of our functional at 1 respectively -1 and are interested in the difference. It follows from the definition that $D_k Y_j = \mathbb{1}_{\{k=j\}}$. Note that throughout the literature it is not unusual to define the discrete gradient D a priori for a Rademacher chaos of fixed order, but D can be extended consistently to

$$\mathbb{D}^{1,2} := Dom(D) = \left\{ F \in L^2(\Omega, \sigma(X), \mathbb{P}) \middle| \mathbb{E}[\|DF\|^2_{l^2(\mathbb{N})}] < \infty \right\},$$

where

$$\mathbb{E}[\|DF\|^2_{l^2(\mathbb{N})}] := \mathbb{E}\left[\sum_{k\in\mathbb{N}} (D_k F)^2\right],$$

see section 2.4.1 in [60]. More precise, if F has a chaos representation (2.21) it holds that

$$D_k F = \sum_{n=1}^{\infty} n J_{n-1}(f_n(\cdot, k)), \tag{2.22}$$

where $f_n(\cdot, k) \in l_0^2(\mathbb{N})^{\circ n-1}$ is a function with one fixed component and $n-1$ variables, see [59, Proposition 2.1.17]. Next we define the *divergence operator* δ, also known as *Skorokhod operator*, and its domain $Dom(\delta)$. For $u := (u_k)_{k\in\mathbb{N}} \in (L^2(\Omega))^{\mathbb{N}}$ with

$$u_k := \sum_{n=1}^{\infty} J_{n-1}(f_n(\cdot, k)),$$

where $f_n \in l_0^2(\mathbb{N})^{\circ n-1} \otimes l^2(\mathbb{N})$ for $n \in \mathbb{N}$, we say that $u \in Dom(\delta)$, if

$$\sum_{n=1}^{\infty} n! \left\| \widetilde{f_n} \mathbb{1}_{\Delta_n} \right\|^2_{l^2(\mathbb{N})^{\otimes n}} < \infty.$$

By $\tilde{f}(k_1, ..., k_n) := \frac{1}{n!} \sum_{\sigma\in\mathcal{G}_n} f(k_{\sigma(1)}, ..., k_{\sigma(n)})$ we mean the canonical symmetrization of a function f in n variables such that $\mathcal{G}_n$ is the symmetric group on $\{1, ..., n\}$. Then, for $u \in Dom(\delta)$, the operator δ is given by

$$\delta(u) := \sum_{n=1}^{\infty} J_n\left(\widetilde{f_n} \mathbb{1}_{\Delta_n}\right). \tag{2.23}$$

Another way to characterize δ is by the duality, see [59, Lemma 2.1.22],

$$\mathbb{E}[\langle DF, u\rangle] = \mathbb{E}[F\delta(u)], \quad F \in \mathbb{D}^{1,2}, u \in Dom(\delta), \tag{2.24}$$

such that we can identify δ as the *adjoint operator* of D. Furthermore we can rewrite its domain to

$$Dom(\delta) = \left\{ u \in L^2(\Omega, l^2(\mathbb{N})) \middle| \exists\, C_u > 0\, \forall\, F \in \mathbb{D}^{1,2} : |\mathbb{E}[\langle DF, u\rangle]| \leq C_u \sqrt{\mathbb{E}[F^2]} \right\}.$$

For

$$F \in Dom(L) = \left\{ F = \mathbb{E}[F] + \sum_{n=1}^{\infty} J_n(f_n) \in L^2(\Omega, \sigma(X), \mathbb{P}) \middle| \sum_{n=1}^{\infty} n^2 n! \, \|f_n\|^2_{l^2(\mathbb{N})^{\otimes n}} < \infty \right\}$$

we define by

$$LF := \sum_{n=1}^{\infty} -nJ_n(f_n),$$
$$L^{-1}F := \sum_{n=1}^{\infty} -\frac{1}{n} J_n(f_n),$$

the *Ornstein–Uhlenbeck operator* L and the *pseudo-inverse Ornstein–Uhlenbeck operator* L^{-1}. It is possible to show that $F \in Dom(L)$ is equivalent to $F \in \mathbb{D}^{1,2}$ and $DF \in Dom(\delta)$; in this case, it holds that

$$L = -\delta D. \tag{2.25}$$

The validity of (2.25) follows mainly from

$$DF = (D_k F)_{k\in\mathbb{N}} = \left(\sum_{n=1}^{\infty} nJ_{n-1}(f_n(\cdot, k)) \right)_{k\in\mathbb{N}} = \left(\sum_{n=1}^{\infty} J_{n-1}(nf_n(\cdot, k)) \right)_{k\in\mathbb{N}}$$

and

$$-\delta DF = -\sum_{n=1}^{\infty} J_n\Big(n\widetilde{f_n}\mathbb{1}_{\Delta_n}\Big) = \sum_{n=1}^{\infty} -nJ_n(f_n)$$

by definitions (2.22) and (2.23), and we refer to [59, Lemma 2.1.25] for details. We want to emphasize that (2.24) and (2.25) will be very important for the connection of the Malliavin operators we introduced and Stein's method, which we will illustrate in subsection 2.2.2. Before that we finish the introduction into Malliavin calculus by giving two examples of L^2-Rademacher-functionals we will investigate in the upcoming chapters.

Infinite weighted 2-runs. Due to their simple dependence structure, runs, and more generally weighted or incomplete U-statistics, lend themselves to normal approximations, see [84], where an exchangeable pair coupling is employed for a normal approximation. In [84] the authors studied even degenerate weighted U-statistics, where either weights are considered which ensures a weak dependence or kernel functions are considered which depend on the sample size n in a specific way. See also subsection 1.2 in [74], where subgraph counts in random graphs are considered. Here we consider infinite weighted 2-runs, where random variables are possibly depending on the whole infinite sequence of i.i.d. Rademacher random variables.
Let $X = (X_i)_{i\in\mathbb{Z}}$ be a double-sided sequence of i.i.d. Rademacher random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$ and let for each $n \in \mathbb{N}$, $(a_i^{(n)})_{i\in\mathbb{Z}}$ be a double-sided summable sequence of real numbers. Usually 2-runs are definded with a square-summable sequence but this will be not enough.
The sequence $(F_n)_{n\in\mathbb{N}}$ of standardized infinite weighted 2-runs is then defined as

$$F_n := \frac{G_n - \mathbb{E}[G_n]}{\sqrt{\mathrm{Var}(G_n)}}, \quad G_n := \sum_{i\in\mathbb{Z}} a_i^{(n)} \xi_i \xi_{i+1}, \quad n \in \mathbb{N}, \tag{2.26}$$

where $\xi_i := \frac{X_i+1}{2}$ for $i \in \mathbb{Z}$. More generally one can consider an infinite weighted d-run defined by

$$G_n(d) := \sum_{i\in\mathbb{Z}} a_i^{(n)} \xi_i \cdots \xi_{i+d-1},$$

which is a weighted degenerate U-statistic of degree d. However, since the analysis for any d is of the cost of a quite cumbersome notation, we will focus on the case where $d = 2$ (2-runs). Since the behaviour of the coefficient sequences $(a_i^{(n)})_{i\in\mathbb{Z}}$ will be very important for our studies we define for $p > 0$ the l^p-*norm* by $\|a\|_{l^p(\mathbb{Z})} := (\sum_{i\in\mathbb{Z}} |a_i|^p)^{1/p}$.
For recent results on 2-runs combined with Malliavin–Stein method see [36], [60] and [73].

Subgraph counting in the Erdős–Rényi random graph. We start with the complete graph on n vertices and keep an edge with probability $p \in [0, 1]$, while we remove it with probability $q := 1 - p$, for all edges independently from each other. The outcome is known as the classical Erdős–Rényi random graph $\mathbf{G}$(n,p) and in many applications p depends on n. We fix a graph G_0 with at least one edge and consider the number W of subgraphs $H \subset \mathbf{G}(n, p)$, which are isomorphic to G_0. Note that we are calling two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ *isomorphic* if there is an edge-preserving bijection $f : V_1 \to V_2$ between their sets of vertices, such that two vertices $v, w \in V_1$ are joined by an edge $\{v, w\} \in E_1$ in G_1 if and only if the vertices $f(v), f(w) \in V_2$ are joined by an edge $\{f(v), f(w)\} \in E_2$ in G_2. The corresponding standardized random variable is then defined as

$$F := \frac{W - \mathbb{E}[W]}{\sqrt{\operatorname{Var}(W)}}, \tag{2.27}$$

which is basically the standardized number of copies of G_0 in $G(n, p)$.
For our result we have to define the important quantity

$$\Psi := \min_{\substack{H\subset G_0\\ e_H\geq 1}} \left\{n^{v_H} p^{e_H}\right\},$$

where v_H denotes the number of vertices of a subgraph H of G_0 and e_H the number of edges, respectively. We give a short summary of the history of optimal uniform Berry–Esseen bounds in the context of subgraph counting, since they will be fundamental for our non-uniform Berry–Esseen bound. The first optimal result valid for arbitrary subgraphs and arbitrary p was shown in [81]. After that, the authors of [35] obtained a result of the same quality, but with an easier proof using the Stein–Tikhomirov method. At last another proof using the Malliavin–Stein method was given in [36].

2.2.2. *Stein's method and Malliavin calculus.* The idea of *Malliavin–Stein method* was developed by I. Nourdin and G. Peccati [71], and combines Stein's method with Malliavin calculus. For the first important steps we recall the lefthandside of the Stein-equation (2.4) and the identities (2.25) and (2.24). Then, for a Rademacher-functional $F \in \mathbb{D}^{1,2}$ we can write

$$\begin{aligned}\mathbb{E}\left[F f_z(F)\right] &= \mathbb{E}\left[(LL^{-1}F) f_z(F)\right]\\ &= \mathbb{E}\left[(-\delta DL^{-1}F) f_z(F)\right]\\ &= \mathbb{E}\left[\langle Df_z(F), -DL^{-1}F\rangle\right].\end{aligned}$$

We continue with a closer look at the k-th component of $Df_z(F)$ that gives us

$$\begin{aligned}D_k f_z(F) &= \sqrt{p_k q_k}\left[f_z(F_k^+) - f_z(F_k^-)\right]\\ &= \sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} f_z'(u)du\\ &= \sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} \left[f_z'(u) - f_z'(F)\right]du + f_z'(F) D_k F\\ &=: R_k + f_z'(F) D_k F\end{aligned}$$

and so

$$\mathbb{E}\left[Ff_z(F)\right] = \mathbb{E}\left[\langle R, -DL^{-1}F\rangle\right] + \mathbb{E}\left[\langle f_z'(F)DF, -DL^{-1}F\rangle\right]$$

for $R = (R_1, R_2, ...)$. Next we include $\mathbb{E}\left[f_z'(F)\right]$ into our computation and receive

$$\begin{aligned}\mathbb{E}\left[f_z'(F) - Ff_z(F)\right] &= \mathbb{E}\left[f_z'(F)\Big(1 - \langle DF, -DL^{-1}F\rangle\Big)\right] - \mathbb{E}\left[\langle R, -DL^{-1}F\rangle\right] \\ &\leq \mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right|\right] + \sum_{k\in\mathbb{N}} \mathbb{E}\left[|R_k| \times \left|DL^{-1}F\right|\right]. \end{aligned} \tag{2.28}$$

Now let us assume that $\mathbb{E}[F] = 0$ and $\mathrm{Var}(F) = 1$. Then the first summand of (2.28) is promising since it holds that $1 = \mathrm{Var}(F) = \mathbb{E}[\langle DF, -DL^{-1}F\rangle]$, by choosing $f(x) = x$ in [61, (2.13)]. For the second summand of (2.28) we have to bound $\mathbb{E}\left[|R_k| \times |DL^{-1}F|\right]$; this is done nicely in the proof of [36, Theorem 3.1] by using the monotonicity of the components of the Stein-equation, see subsection 2.1.5. In the mentioned paper the authors obtain ultimately a discrete second-order Gaussian Poincaré inequality, which expresses the bound of the second summand in terms of the divergence operator δ. In any case the second summand will be challenging in chapter 4.

2.2.3. *The Poisson case.* We want to take a short theoretic look on the Poisson case. Let $(X, \mathcal{X})$ be a standard Borel space with a σ-finite measure μ. By η we denote a *Poisson (random) measure*, also known as *Poisson point process*, on X with *control* μ. Note that η is defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $\mathcal{X}_0 = \{B \in \mathcal{X} : \mu(B) < \infty\}$ such that $\eta = \{\eta(B) : B \in \mathcal{X}_0\}$ is a collection of random variables with the following properties:

- $\eta(B)$ is Poisson distributed with parameter $\mu(B)$ for all $B \in \mathcal{X}_0$.
- If $B_1, ..., B_n \in \mathcal{X}_0$ are disjoint sets, the random variables $\eta(B_1), ..., \eta(B_n)$ are independent.

Denote by $\mathbb{P}_\eta$ the distribution of η and, if needed, by $\hat{\eta}$ the *centered Poisson measure*

$$\hat{\eta}(B) = \eta(B) - \mathbb{E}[\eta(B)] = \eta(B) - \mu(B).$$

Last, we use the notations $L^2(\mu^n)$ and $L^2(\mathbb{P}_\eta)$ for the space of square-integrable functions with respect to μ^n respectively the space of square-integrable functionals with respect to $\mathbb{P}_\eta$. We are interested in functionals $F = F(\eta) \in L^2(\mathbb{P}_\eta)$, which posses similar to the Rademacher case a *chaos expansion*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$

where I_n is the *n-fold Wiener-Itô integral*, also known as *Poisson multiple integral*, with respect to $\hat{\eta}$ and $(f_n)_{n\in\mathbb{N}}$ is a unique sequence of symmetric functions in $L^2(\mu^n)$. We refer to section 3 in [65] for formal details. For such functionals we define the *difference operator* D_xF of F at $x \in X$:

$$\begin{aligned} D_xF(\eta) &:= F(\eta + \delta_x) - F(\eta), \\ DF &: x \mapsto D_xF, \end{aligned}$$

also known as the *add-one-cost operator* since it measures the effect on F of adding the point $x \in X$ to η. From here on the rest is very similar to the Rademacher case — to be precise we recall the Poisson case historical as the predecessor. We are interested in functionals

$$F \in \hat{\mathbb{D}}^{1,2} := Dom(D) = \left\{ F \in L^2(\mathbb{P}_\eta) \middle| \sum_{n=1}^{\infty} n \cdot n! \left\|f_n\right\|_n^2 < \infty \right\},$$

with $\|.\|_n$ the norm in $L^2(\mu^n)$. Supposing $F \in \hat{\mathbb{D}}^{1,2}$ such that $\sum_{n=1}^{\infty} n^2 \cdot n! \, \|f_n\|_n^2 < \infty$, we define the *Ornstein–Uhlenbeck operator* L and the *pseudo-inverse Ornstein–Uhlenbeck operator* L^{-1} as

$$LF := \sum_{n=1}^{\infty} -nI_n(f_n),$$
$$L^{-1}F := \sum_{n=1}^{\infty} -\frac{1}{n}I_n(f_n).$$

In the Poisson case there are also characterizations for the *divergence operator* δ and the other Malliavin operators analogous to (2.24) and (2.25), see e.g. [37, Lemma 2.1].

2.3. **The Curie–Weiss model and a surrogate approach.**

2.3.1. *The De Finetti measure of the Curie–Weiss model.* The celebrated De Finetti theorem [27] by Bruno De Finetti, 1969, can be stated as follows:

Every exchangeable infinite sequence of random variables is a mixture of an i.i.d. sequence.

Here, *exchangeability* for an infinite sequence is understood in the sense of the action of the inductive limit of the symmetric group, e.g. every finite sub-sequence of the sequence is invariant by a (finite) permutation, which means that the joint distribution of the random variables does not change if they are permuted. This notion has been extended in several ways, in a finite-dimensional version by [28], using other groups such as the projective limit of symmetric groups by [50] and orthogonal groups by [75] and [88], etc. See e.g. [1] for an account of subtleties on this theorem and further references, or [28] for a presentation of the topic in relation with the computation of total variation distances.

The Curie–Weiss spins are clearly exchangeable since the measure (1.6) is invariant by permutation of the spins. Since n is fixed, the De Finetti theorem a priori does not apply, but nevertheless it exists a measure $\widetilde{\nu}_{n,\beta} : [0,1] \to [0,1]$ such that

$$\mathbb{P}_n^{(\beta)} \equiv \mathbb{P}_{(X_1^{(\beta)},\dots,X_n^{(\beta)})} = \int_{[0,1]} \mathbb{P}_{(X_1(p),\dots,X_n(p))}\, \widetilde{\nu}_{n,\beta}(dp), \tag{2.29}$$

where $(X_k(p))_{1\le k\le n} \sim p\delta_{+1} + (1-p)\delta_{-1}$ i.i.d. So the joint distribution of $(X_k^{(\beta)})_{1\le k\le n}$ can be written as a *mixture* of the joint distribution of $(X_k(p))_{1\le k\le n}$ and the measure $\widetilde{\nu}_{n,\beta}$.
We can write (2.29) in a more probabilistic way using a random variable $\widetilde{V}_{n,\beta} \sim \widetilde{\nu}_{n,\beta}$ independent of $(X_k(p))_{1\le k\le n, p\in[0,1]}$. We then have the randomisation equality

$$(X_1^{(\beta)},\dots,X_n^{(\beta)}) \overset{d}{=} (X_1(\widetilde{V}_{n,\beta}),\dots,X_n(\widetilde{V}_{n,\beta})). \tag{2.30}$$

For another point of view we can also write

$$X_k(p) = 2\,\mathbb{1}_{\{U_k<p\}} - 1, \qquad U_k \sim \mathcal{U}([0,1]), \tag{2.31}$$

which gives a functional representation of this last randomisation (2.30), e.g.

$$X_k^{(\beta)} = 2\,\mathbb{1}_{\left\{\{U_k<\widetilde{V}_{n,\beta}\}\right\}} - 1.$$

The measure $\widetilde{\nu}_{n,\beta}$ respectively the random variable $\widetilde{V}_{n,\beta}$ is well-known for the Curie–Weiss model. It is given by

$$\widetilde{\nu}_{n,\beta}(dp) = \widetilde{f}_{n,\beta}(p)dp, \qquad \widetilde{f}_{n,\beta}(p) := \frac{1}{\mathcal{Z}_{n,\beta}} e^{-\frac{n}{2\beta}\operatorname{Argtanh}(2p-1)^2 - (\frac{n}{2}+1)\ln(1-(2p-1)^2)} \tag{2.32}$$

with an explicit renormalisation constant $\mathcal{Z}_{n,\beta}$ defined by the equality $\int_0^1 \widetilde{f}_{n,\beta}(p)dp = 1$, see [55, Theorem 5.6, (164), (165)]. We recall that $\operatorname{Argtanh}(x) = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|$ for $|x|<1$.
If instead of considering the parameter $p \in [0,1]$ of the Rademacher random variables, we encode the De Finetti measure with the expectation parameter $t := 2p-1 \in [-1,1]$ and get

$$\nu_{n,\beta}(dt) = f_{n,\beta}(t)dt, \qquad f_{n,\beta}(t) := \frac{1}{\mathcal{Z}_{n,\beta}} e^{-\frac{n}{2\beta}\operatorname{Argtanh}(t)^2 - (\frac{n}{2}+1)\ln(1-t^2)} \tag{2.33}$$

and a randomisation by an independent random variable $V_{n,\beta} \sim \nu_{n,\beta}$. The corresponding version of (2.29) is given by

$$\mathbb{P}_{(X_1^{(\beta)},\dots,X_n^{(\beta)})} = \int_{[-1,1]} \mathbb{P}_{\left(X_1\left(\frac{t+1}{2}\right),\dots,X_n\left(\frac{t+1}{2}\right)\right)}\, \nu_{n,\beta}(dt).$$

2.3.2. *Surrogate random variables in probability theory.* Consider the following classical problem in extreme value theory: compute the fluctuations of $\boldsymbol{H}_n := \max_{1\le k\le n} Z_k$ for independent random variables $(Z_k)_{k\ge 1}$ when $n \to +\infty$. One way to proceed is to note that

$$\{\boldsymbol{H}_n \le x\} = \{\forall k \le n,\ Z_k \le x\} = \{\sum_{k=1}^{n} \mathbb{1}_{\{Z_k > x\}} = 0\}. \tag{2.34}$$

The problem amounts thus to analyse the fluctuations of the parametric random variable

$$S_n(x_n) := \sum_{k=1}^{n} \mathbb{1}_{\{Z_k > x_n\}}, \qquad x_n := \mu_n + \sigma_n x,$$

for given numbers μ_n, σ_n that one has to tune in order to get the corresponding limit distribution. Since $S_n(x_n)$ is a sum of independent $\{0,1\}$-Bernoulli random variables, one can proceed to a Poisson approximation $S_n(x_n) \approx \mathrm{Po}(f(x_n))$, using for instance the Chen–Stein method [47], resulting in:

$$\mathbb{P}(\boldsymbol{H}_n \le x_n) = \mathbb{P}(S_n(x_n) = 0) = \mathbb{P}(\mathrm{Po}(f(x_n)) = 0) + o(1) = e^{-f(x_n)} + o(1).$$

By doing so, one has replaced the estimation of a non linear functional of $(Z_k)_k$, a maximum, by a simpler problem, the estimation of a sum. Such a sum is a *surrogate random variable.* Its study is equivalent to the original problem while being arguably simpler.
The equality (2.34) allows to use a strict equality to replace the original problem by the surrogate problem, and the approximation is only performed at the level of $S_n(x_n)$, but one could reverse the steps or add an additional approximation step in between, e.g. replacing the equality (2.34) by an approximation, as long as the original problem is not fundamentally impacted. This is what we perform now.

2.3.3. *Surrogate magnetisation inequalities.* We consider the following setting: For $(Z_k)_k$ i.i.d. satisfying $\mathbb{E}|Z_1|^3 < \infty$, $\sigma_n^2 := \mathrm{Var}(S_n) = n\,\mathrm{Var}(Z_1)$, $\mu_n := \mathbb{E}(S_n) = n\mathbb{E}(Z_1)$ we define $S_n := \sum_{k=1}^n Z_k$ and $W_n := (S_n - \mu_n)/\sigma_n = \sum_{k=1}^n \hat{Z}_k$, where $\hat{Z}_k := (Z_k - \mathbb{E}(Z_k))/(n\,\mathrm{Var}(Z_k))^{1/2}$. We recall the following non-uniform bound obtained with Stein's method and zero-bias transform, see [51, (13)] or [85, Theorem 3.29], valid for all $h \in \mathcal{C}^1$ with $\|h'\|_\infty < \infty$:

$$|\mathbb{E}(h(W_n)) - \mathbb{E}(h(G))| \le C\,\|h'\|_\infty\,\mathbb{E}\left|W_n^Z - W_n\right|, \tag{2.35}$$

where $G \sim \mathcal{N}(0,1), W_n^Z$ defined by (2.18) and C is an absolute constant. According to [85, Proposition 3.32] it holds that $W_n^Z = W_n - \hat{Z}_I + \hat{Z}_I^Z$ for a random index I satisfying $\mathbb{P}(I = i) = \mathrm{Var}(\hat{Z}_i)$ and being independent of all else. As a consequence

$$\mathbb{E}\left|W_n^Z - W_n\right| = \mathbb{E}\left|\hat{Z}_I^Z - \hat{Z}_I\right| \le \mathbb{E}\left|\hat{Z}_I^Z\right| + \mathbb{E}\left|\hat{Z}_I\right| = \mathbb{E}\left|\hat{Z}_1^Z\right| + \mathbb{E}\left|\hat{Z}_1\right|$$

by the triangle inequality and identical distribution. Now we use [85, Proposition 3.32] and (2.18) for $f(x) = \frac{x|x|}{2}$ to get

$$\mathbb{E}\left|\hat{Z}_1^Z\right| = \frac{1}{(n\,\mathrm{Var}(Z_1))^{1/2}}\mathbb{E}\left|(Z_1 - \mathbb{E}(Z_1))^Z\right| = \frac{1}{(n\,\mathrm{Var}(Z_1))^{1/2}}\frac{\mathbb{E}|(Z_1 - \mathbb{E}(Z_1)|^3}{2\,\mathrm{Var}(Z_1)} < \infty$$

since by our moment assumptions all appearing moments are finite. This simplifies (2.35) to

$$|\mathbb{E}(h(W_n)) - \mathbb{E}(h(G))| \le C\,\|h'\|_\infty$$

for another absolute constant C. If we rescale our test function h linearly by $h \leftarrow h(\frac{\cdot - \mu_n}{\sigma_n})$ we can rewrite the foregoing inequality to

$$|\mathbb{E}(h(S_n)) - \mathbb{E}(h(\sigma_n G + \mu_n))| \le C\,\|h'\|_\infty\,, \tag{2.36}$$

We see in particular that rescaling linearly $h \leftarrow h(\frac{\cdot}{\sqrt{n}})$ gives a speed of convergence in $O\big(\frac{1}{\sqrt{n}}\big)$.

Note that other identities using stronger conditions on the functional space defining the norm and stronger moments conditions allow for stronger speed of convergence, see e.g. [51, Corollary 3.1].
In the particular case of Rademacher random variables $(X_k)_k$ of parameter $p := \mathbb{P}(X_1 = 1)$, we have $\mathbb{E}[X_1] = 2p - 1$ and $\mathrm{Var}(X_1) = 4p(1-p)$, thus

$$\left|\mathbb{E}(h(S_n)) - \mathbb{E}\Big(h\Big(\sqrt{n}\, 2\sqrt{p(1-p)}G + (2p-1)n\Big)\Big)\right| \leq C \, \|h'\|_\infty .$$

Now, taking p at random with a distribution ν and writing $S_n(p)$ to mark the dependency, we get

$$\begin{aligned}
\delta_n(h) &:= \left|\int_{[0,1]} \mathbb{E}(h(S_n(p)))\, \nu(dp) - \int_{[0,1]} \mathbb{E}\Big(h\Big(\sqrt{n}\, 2\sqrt{p(1-p)}G + (2p-1)n\Big)\Big)\, \nu(dp)\right| \\
&\leq \int_{[0,1]} \left|\mathbb{E}(h(S_n(p))) - \mathbb{E}\Big(h\Big(\sqrt{n}\, 2\sqrt{p(1-p)}G + (2p-1)n\Big)\Big)\right| \nu(dp) \\
&\leq C \, \|h'\|_\infty \int_{[0,1]} \nu(dp) = C \, \|h'\|_\infty .
\end{aligned}$$

In the case of the Curie–Weiss model, taking p distributed as

$$\boldsymbol{P}_n^{(\beta)} \sim \widetilde{\nu}_{n,\beta}, \tag{2.37}$$

we finally get with (2.29) and (2.30):

$$\left|\mathbb{E}\Big(h\Big(M_n^{(\beta)}\Big)\Big) - \mathbb{E}\Big(h\Big(\sqrt{n}\, G \times 2\sqrt{\boldsymbol{P}_n^{(\beta)}(1-\boldsymbol{P}_n^{(\beta)})} + n \times (2\boldsymbol{P}_n^{(\beta)} - 1)\Big)\Big)\right| \leq C \, \|h'\|_\infty .$$

We can perform a last change of variables to this expression. Define the random variable

$$\boldsymbol{T}_n^{(\beta)} := 2\boldsymbol{P}_n^{(\beta)} - 1 \sim \nu_{n,\beta} \tag{2.38}$$

which corresponds to the parametrisation of the De Finetti measure $\nu_{n,\beta}$ defined in (2.33) as opposed to the one defined in (2.32). Noting that $p(1-p) = \frac{1-t^2}{4}$, we define the *surrogate magnetisation* by

$$\mathcal{M}_n^{(\beta)} := \sqrt{n}\, G \sqrt{1 - (\boldsymbol{T}_n^{(\beta)})^2} + n\, \boldsymbol{T}_n^{(\beta)} \tag{2.39}$$

so that

$$\left|\mathbb{E}\Big(h\Big(M_n^{(\beta)}\Big)\Big) - \mathbb{E}\Big(h\Big(\mathcal{M}_n^{(\beta)}\Big)\Big)\right| \leq C \, \|h'\|_\infty . \tag{2.40}$$

Using (2.40) in conjunction with the triangle inequality yields with $\boldsymbol{Z}_{n,\beta} \sim \mathcal{N}(0, n/(1-\beta))$

$$\begin{aligned}
\left|\mathbb{E}\Big(h\Big(M_n^{(\beta)}\Big)\Big) - \mathbb{E}(h(\boldsymbol{Z}_{n,\beta}))\right| &\leq \left|\mathbb{E}\Big(h\Big(M_n^{(\beta)}\Big)\Big) - \mathbb{E}\Big(h\Big(\mathcal{M}_n^{(\beta)}\Big)\Big)\right| + \left|\mathbb{E}\Big(h\Big(\mathcal{M}_n^{(\beta)}\Big)\Big) - \mathbb{E}(h(\boldsymbol{Z}_{n,\beta}))\right| \\
&\leq C \, \|h'\|_\infty + \left|\mathbb{E}\Big(h\Big(\mathcal{M}_n^{(\beta)}\Big)\Big) - \mathbb{E}(h(\boldsymbol{Z}_{n,\beta}))\right| .
\end{aligned}$$

It is thus enough to control the convergence of the surrogate random variable $\mathcal{M}_n^{(\beta)}$ towards its limit (up to a rescaling) to obtain the speed of convergence of the original random variable. The explanation of (1.8) – (1.11) relies then entirely on the fact that the structure of $\mathcal{M}_n^{(\beta)}$ defined in (2.39) is particularly simple to understand since G is of order 1 and $\boldsymbol{T}_n^{(\beta)} =: \cos(\boldsymbol{\Theta}_n^{(\beta)}) \in [-1, 1]$:

(1) When $\boldsymbol{T}_n^{(\beta)}$ converges to 0, the first term is approximately Gaussian after rescaling by $\sqrt{n}$ and we need to study the behaviour of $\boldsymbol{T}_n^{(\beta)}\sqrt{n}$ which will be shown to be Gaussian too, and this corresponds to $\beta < 1$;
(2) when $\boldsymbol{T}_n^{(\beta)}$ tends to ± 1, we need to rescale by n and the limit will come from the last term in (2.39), and this corresponds to $\beta > 1$;

(3) last, when both terms are of the same order, e.g. $\sqrt{n}\sin(\mathbf{\Theta}_n^{(\beta)}) \approx n\cos(\mathbf{\Theta}_n^{(\beta)})$ or equivalently $\tan(\mathbf{\Theta}_n^{(\beta)}) = O_{\mathbb{P}}(\sqrt{n})$, the analysis has to be refined and a non standard limit can emerge.

Compared with the expository case of subsection 2.3.2, we defined the surrogate by means of an initial (fundamental) inequality and added an additional (non fundamental) inequality to use it, the approximation coming then next.

3. A surrogate by exchangeability approach to the Curie–Weiss model

This chapter is based on [5]. After our preparations in section 2.3 we are ready to give a new proof of (1.8) – (1.11). In order to control exactly the error originated from the replacement of the original random variable by its surrogate, the approximation in law must be quantitative. As a result, we will not only be concerned with the limits in law presented in (1.8) – (1.11), but also with their speed of convergence in particular distances: The Fortet–Mourier distance (2.2) with test functions having a certain degree of smoothness, and the Kolmogorov distance (2.3) whose test functions are indicators of half infinite intervals of the real line.
Here, not only can one save a considerable computational effort by using the classical CLT approximation for sums of i.i.d.'s, but in addition comes an unexpected bonus that arises as a byproduct of the use of such a surrogate: by integrating a fraction of the randomness of the surrogate, the indicator functions in the Kolmogorov distance are replaced by smooth functions. This transfert from randomness to smoothness is a very agreable surprise that reduces the discontinuous norm estimate to a smooth one for a related random variable, allowing thus to bypass the usual pathologies of discontinuous test functions distances, see section 3.2. This is one of the advantages of the surrogate by exchangeability approach: it does not differentiate between the discontinuous and the continuous probability norms.

3.1. **Application to the Curie–Weiss magnetisation in Fortet–Mourier distance.**

3.1.1. *The case* $\beta < 1$. Define $\boldsymbol{Z}_\beta \sim \mathcal{N}\left(0, \frac{1}{1-\beta}\right)$.

Theorem 3.1 (Fluctuations of the *unnormalised* magnetisation for $\beta < 1$)**.** *If* $\beta < 1$*, we have for all* $h \in \mathcal{C}^1$ *with* $\|h\|_\infty, \|h'\|_\infty < \infty$

$$\left|\mathbb{E}\left(h\left(\frac{M_n^{(\beta)}}{\sqrt{n}}\right)\right) - \mathbb{E}(h(\boldsymbol{Z}_\beta))\right| \leq C\frac{\|h'\|_\infty}{\sqrt{n}} + \frac{1}{n}\left(\frac{\beta}{1-\beta}\|h'\|_\infty + C(\beta)\|h\|_\infty\right) \tag{3.1}$$

for explicit constants $C, C(\beta) > 0$.

Proof. Rescaling $M_n^{(\beta)}$ by $\sqrt{n}$ amounts to do $h \leftarrow h(\frac{\cdot}{\sqrt{n}})$, thus $h' \leftarrow \frac{1}{\sqrt{n}}h'(\frac{\cdot}{\sqrt{n}})$. Substituting in (2.40) yields

$$\left|\mathbb{E}\left(h\left(\frac{M_n^{(\beta)}}{\sqrt{n}}\right)\right) - \mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}\right)\right)\right| \leq \frac{C}{\sqrt{n}}\|h'\|_\infty$$

and the triangle inequality implies then

$$\left|\mathbb{E}\left(h\left(\frac{M_n^{(\beta)}}{\sqrt{n}}\right)\right) - \mathbb{E}(h(\boldsymbol{Z}_\beta))\right| \leq \frac{C}{\sqrt{n}}\|h'\|_\infty + \left|\mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}\right)\right) - \mathbb{E}(h(\boldsymbol{Z}_\beta))\right|. \tag{3.2}$$

Note that we do not get better than the usual normal approximation bound for the magnetisation due to the term $\frac{C}{\sqrt{n}}\|h'\|_\infty$. So far, we are focused on the validity of the approximation, e.g. we want to prove that $\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}$ converges in law to a Gaussian, with speed in the Fortet–Mourier norm of order at least $\frac{1}{\sqrt{n}}$. We thus define

$$\widetilde{\delta}_n(h) := \left|\mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}\right)\right) - \mathbb{E}(h(\boldsymbol{Z}_\beta))\right|, \qquad \boldsymbol{Z}_\beta \sim \mathcal{N}\left(0, \frac{1}{1-\beta}\right). \tag{3.3}$$

Define

$$\boldsymbol{X}_{n,\beta} := \sqrt{n}\,\boldsymbol{T}_n^{(\beta)},$$
$$G_\beta \sim \mathcal{N}\left(0, \frac{\beta}{1-\beta}\right).$$

Supposing that

$$\boldsymbol{X}_{n,\beta} \xrightarrow[n\to+\infty]{d} G_\beta \tag{3.4}$$

we get

$$G\sqrt{1-\frac{(\boldsymbol{X}_{n,\beta})^2}{n}}+\boldsymbol{X}_{n,\beta} \sim_{n\to+\infty} G\sqrt{1-\frac{G_\beta^2}{n}}+G_\beta \xrightarrow[n\to+\infty]{} G+G_\beta.$$

Hence by the decomposition with G independent of G_β we receive

$$G+G_\beta \stackrel{d}{=} \boldsymbol{Z}_\beta \tag{3.5}$$

since G and $\boldsymbol{X}_{n,\beta}$ are independent (hence so are G and G_β).
The distribution of $\boldsymbol{X}_{n,\beta} := \sqrt{n}\,\boldsymbol{T}_n^{(\beta)}$ is given by the rescaling of $\nu_{n,\beta}$ in (2.33):

$$\mathbb{P}(\boldsymbol{X}_{n,\beta}\in dt) = f_{n,\beta}\left(\frac{t}{\sqrt{n}}\right)\frac{dt}{\sqrt{n}} = \frac{1}{\sqrt{n}\,\mathcal{Z}_{n,\beta}}e^{-\frac{n}{2\beta}\operatorname{Argtanh}(\frac{t}{\sqrt{n}})^2-(\frac{n}{2}+1)\ln(1-\frac{t^2}{n})}\mathbb{1}_{\{|t|\le\sqrt{n}\}}dt.$$

Using a Taylor expansion in 0, we easily get

$$\frac{n}{2\beta}\operatorname{Argtanh}\left(\frac{t}{\sqrt{n}}\right)^2+\left(\frac{n}{2}+1\right)\ln\left(1-\frac{t^2}{n}\right) = \frac{1}{\beta}\frac{t^2}{2}-\frac{t^2}{2}+O\left(\frac{t^4}{n}\right) = \left(\frac{1-\beta}{\beta}\right)\frac{t^2}{2}+O\left(\frac{t^4}{n}\right).$$

We can moreover show that, see [55],

$$\sqrt{n}\,\mathcal{Z}_{n,\beta} \xrightarrow[n\to+\infty]{} \sqrt{2\pi}\times\sqrt{\frac{\beta}{1-\beta}}$$

which implies (3.4) and would imply

$$\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}} = G\sqrt{1-(\boldsymbol{T}_n^{(\beta)})^2}+\sqrt{n}\,\boldsymbol{T}_n^{(\beta)} \xrightarrow[n\to+\infty]{d} \boldsymbol{Z}_\beta \tag{3.6}$$

with an additional dominated convergence. This is what we prove now. We have

$$\begin{aligned}
\widetilde{\delta}_n(h) &:= \left|\mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}\right)\right)-\mathbb{E}(h(G+G_\beta))\right| \\
&= \left|\mathbb{E}\left(h\left(G\sqrt{1-\frac{(\boldsymbol{X}_{n,\beta})^2}{n}}+\boldsymbol{X}_{n,\beta}\right)\right)-\mathbb{E}(h(G+G_\beta))\right| \\
&= \left|\int_{[-\sqrt{n},\sqrt{n}]}\mathbb{E}\left(h\left(G\sqrt{1-\frac{x^2}{n}}+x\right)\right)f_{n,\beta}\left(\frac{x}{\sqrt{n}}\right)\frac{dx}{\sqrt{n}}-\mathbb{E}(h(G+G_\beta))\right| \\
&=: \left|\int_{[-\sqrt{n},\sqrt{n}]}h_n(x)g_n(x)dx-\int_{\mathbb{R}}h_G(x)g_\beta(x)dx\right|
\end{aligned}$$

with

$$\begin{aligned}
h_n(x) &:= \mathbb{E}\left(h\left(G\sqrt{1-\frac{x^2}{n}}+x\right)\right), \\
h_G(x) &:= \mathbb{E}\Big(h\left(G+x\right)\Big), \\
g_n(x) &:= \frac{1}{\sqrt{n}}f_{n,\beta}\left(\frac{x}{\sqrt{n}}\right), \\
g_\beta(x) &:= \sqrt{\frac{1-\beta}{2\pi\beta}}e^{-\frac{1-\beta}{\beta}\frac{x^2}{2}}.
\end{aligned}$$

Note that we have used a coupling of $\mathcal{M}_n^{(\beta)}$ and $\boldsymbol{Z}_\beta$ by supposing that the random variable G that is used in each random variable is the same. Such a coupling is always possible, and this is an important feature of the proof. We have moreover

$$\left|\int_{[-\sqrt{n},\sqrt{n}]} h_n g_n - \int_{\mathbb{R}} h_G g_\beta\right| \leq \left|\int_{[-\sqrt{n},\sqrt{n}]} (h_n g_n - h_G g_\beta)\right| + \left|\int_{\mathbb{R}\setminus[-\sqrt{n},\sqrt{n}]} h_G g_\beta\right|$$

with

$$\begin{aligned}\left|\int_{\mathbb{R}\setminus[-\sqrt{n},\sqrt{n}]} h_G g_\beta\right| &\leq \|h_G\|_\infty \int_{\mathbb{R}\setminus[-\sqrt{n},\sqrt{n}]} g_\beta \leq \|h\|_\infty \int_{\mathbb{R}\setminus[-\sqrt{n},\sqrt{n}]} g_\beta \\ &= \|h\|_\infty \, \mathbb{P}\Big(|G_\beta| \geq \sqrt{n}\Big) \\ &\leq \|h\|_\infty \, \mathbb{E}\Big(|G_\beta|^{2k}\Big)\, n^{-k}, \qquad \forall\, k \geq 1,\end{aligned}$$

using Markov's inequality.

Note that the true value of the Gaussian tail is $\mathbb{P}(|G_\beta| \geq x) \leq \frac{1}{\sqrt{2\pi x \sigma_\beta^2}} \exp\Big(-\frac{x^2}{2\sigma_\beta^2}\Big)$, hence $\mathbb{P}(|G_\beta| \geq \sqrt{n}) = O(e^{-n/2})$ since $\beta < 1$, and this power bound is small enough for our purposes. The first integral can be estimated writing

$$\begin{aligned}\left|\int_{[-\sqrt{n},\sqrt{n}]} (h_n g_n - h_G g_\beta)\right| &= \left|\int_{[-\sqrt{n},\sqrt{n}]} [\, h_n(g_n - g_\beta) + (h_n - h_G) g_\beta]\right| \\ &\leq \int_{[-\sqrt{n},\sqrt{n}]} |h_n(g_n - g_\beta)| + \left|\int_{[-\sqrt{n},\sqrt{n}]} (h_n - h_G) g_\beta\right| \\ &\leq \|h_n\|_\infty \int_{[-\sqrt{n},\sqrt{n}]} |g_n - g_\beta| + \|h_n - h_G\|_\infty \int_{[-\sqrt{n},\sqrt{n}]} g_\beta.\end{aligned}$$

It is clear that $|h_n(x)| \leq \|h\|_\infty$. We can thus bound $\|h_n - h_G\|_\infty$ by $2\|h\|_\infty$ but since $\int_{[-\sqrt{n},\sqrt{n}]} g_\beta = \mathbb{P}(|G_\beta| \leq \sqrt{n}) = O(1 - e^{-n/2}) = O(1)$ which does not tend to 0, we must work on $\left|\int_{[-\sqrt{n},\sqrt{n}]} (h_n - h_G) g_\beta\right|$ directly. Since we have the same random variables G and G_β, we have a coupling that allows to write

$$\begin{aligned}\int_{[-\sqrt{n},\sqrt{n}]} (h_n - h_G) g_\beta &= \mathbb{E}\left(h\left(G\sqrt{1 - \frac{G_\beta^2}{n}} + G_\beta \right) \mathbb{1}_{\{|G_\beta| \leq \sqrt{n}\}} - h(G + G_\beta) \mathbb{1}_{\{|G_\beta| \leq \sqrt{n}\}} \right) \\ &= \mathbb{E}\left(G\left(\sqrt{1 - \tfrac{G_\beta^2}{n}} - 1\right) \mathbb{1}_{\{|G_\beta| \leq \sqrt{n}\}} h'\left(G_\beta + G + U\, G\left(\sqrt{1 - \tfrac{G_\beta^2}{n}} - 1\right)\right)\right)\end{aligned}$$

with $U \sim \mathcal{U}([0,1])$ independent of (G, G_β). By using $x_+ := \max\{x, 0\} = x\mathbb{1}_{\{x \geq 0\}}$ and $\{|G_\beta| \leq \sqrt{n}\} = \{1 - G_\beta^2/n \geq 0\}$, we then get

$$\begin{aligned}\left|\int_{[-\sqrt{n},\sqrt{n}]} (h_n - h_G) g_\beta\right| &\leq \mathbb{E}\left(\left| G\left(\sqrt{\left(1 - \frac{G_\beta^2}{n}\right)_+} - 1\right)\right|\right) \|h'\|_\infty \\ &= \mathbb{E}(|G|)\ \|h'\|_\infty \times \mathbb{E}\left(\left|\sqrt{\left(1 - \frac{G_\beta^2}{n}\right)_+} - 1\right|\right) \\ &\leq \frac{\mathbb{E}\Big(G_\beta^2\Big)}{n}\ \|h'\|_\infty,\end{aligned}$$

where we have used the independency of G and G_β, $1 - \sqrt{1-x} \leq x$ for $0 \leq x \leq 1$ and $\mathbb{E}(|G|) \leq \sqrt{\mathbb{E}(G^2)} = 1$.

The important quantity to bound is thus

$$\|h_n\|_\infty \int_{[-\sqrt{n},\sqrt{n}]} |g_n - g_\beta| \le \|h\|_\infty \int_{[-\sqrt{n},\sqrt{n}]} |g_n - g_\beta| \,.$$

We thus need to estimate carefully

$$\begin{aligned}
\delta(g_n, g_\beta) &:= \int_{[-\sqrt{n},\sqrt{n}]} |g_n - g_\beta| \\
&= \int_{[-\sqrt{n},\sqrt{n}]} \left| \frac{1}{\sqrt{n}} f_{n,\beta}\left(\frac{x}{\sqrt{n}}\right) - \sqrt{\frac{1-\beta}{2\pi\beta}} e^{-\frac{1-\beta}{\beta}\frac{x^2}{2}} \right| dx \\
&= 2\int_{[0,\sqrt{n}]} \left| \frac{1}{\sqrt{n}} f_{n,\beta}\left(\frac{x}{\sqrt{n}}\right) - \sqrt{\frac{1-\beta}{2\pi\beta}} e^{-\frac{1-\beta}{\beta}\frac{x^2}{2}} \right| dx
\end{aligned}$$

by symmetry of g_n and g_β.
Define

$$\begin{aligned}
\varphi_n(x) &:= \frac{n}{2\beta} \operatorname{Argtanh}\left(\frac{x}{\sqrt{n}}\right)^2 + \left(\frac{n}{2}+1\right) \ln\left(1 - \frac{x^2}{n}\right), \\
C_\beta &:= \frac{1-\beta}{\beta},
\end{aligned}$$

so that

$$\begin{aligned}
f_{n,\beta}\left(\frac{x}{\sqrt{n}}\right) &= \frac{1}{\mathcal{Z}_{n,\beta}} e^{-\varphi_n(x)}, \\
g_\beta(x) &= \sqrt{\frac{C_\beta}{2\pi}} e^{-C_\beta \frac{x^2}{2}}.
\end{aligned}$$

We then have

$$\delta(g_n, g_\beta) \le \left| \frac{1}{\mathcal{Z}_{n,\beta}\sqrt{n}} - \sqrt{\frac{C_\beta}{2\pi}} \right| \int_{[-\sqrt{n},\sqrt{n}]} e^{-\varphi_n(x)} dx + \sqrt{\frac{C_\beta}{2\pi}} \int_{[-\sqrt{n},\sqrt{n}]} \left| e^{-\varphi_n(x)} - e^{-C_\beta \frac{x^2}{2}} \right| dx.$$

The first quantity is

$$\begin{aligned}
\left| \sqrt{\frac{C_\beta}{2\pi}} \times \mathcal{Z}_{n,\beta}\sqrt{n} - 1 \right| \int_{[-\sqrt{n},\sqrt{n}]} g_n &= \left| \sqrt{\frac{C_\beta}{2\pi}} \times \mathcal{Z}_{n,\beta}\sqrt{n} - 1 \right| \mathbb{P}(|X_n| \le \sqrt{n}) \\
&\le \left| \sqrt{\frac{C_\beta}{2\pi}} \times \mathcal{Z}_{n,\beta}\sqrt{n} - 1 \right|
\end{aligned}$$

and an analysis of its speed of convergence to 0 is performed in Lemma 3.15.
The important quantity is the second one. Define

$$\begin{aligned}
\gamma_n &:= \int_{(-\sqrt{n},\sqrt{n})} \left| e^{-\varphi_n(x)} - e^{-C_\beta \frac{x^2}{2}} \right| dx \\
&= \int_{(-\sqrt{n},\sqrt{n})} \left| e^{-(\varphi_n(x) - C_\beta \frac{x^2}{2})} - 1 \right| e^{-C_\beta \frac{x^2}{2}} dx.
\end{aligned}$$

Supposing that the quantity inside the absolute value in the second line was bounded by a constant D_n on $(-\sqrt{n}, \sqrt{n})$, we would have

$$\gamma_n = O\left(D_n \int_{(-\sqrt{n},\sqrt{n})} e^{-C_\beta \frac{x^2}{2}} dx \right) = O\Big(D_n \mathbb{P}\big(|G_\beta| \le \sqrt{n}\big)\Big) = O(D_n)\,.$$

This is not the case as $\operatorname{Argtanh}^2(\frac{x}{\sqrt{n}}) \to +\infty$ when $x \to \sqrt{n}$; nevertheless, the integral is still definite. By symmetry, we will now work on $[0, \sqrt{n})$ and use a factor 2. Let $\varepsilon \in (0, \sqrt{n})$ to be choosen later. We split $[0, \sqrt{n})$ according to

$$[0, \sqrt{n}) := [0, \sqrt{n} - \varepsilon) \cup [\sqrt{n} - \varepsilon, \sqrt{n})$$

and estimate the integral on each of these subintervals.

• **Main interval:** On $[0, \sqrt{n} - \varepsilon)$, the function $\widetilde{\kappa}_n : x \mapsto \varphi_n(x) - C_\beta \frac{x^2}{2}$ is not monotone. This is mainly due to the fact that the second derivatives of the two functions occuring in the difference do not match in 0. Indeed, we have $\varphi_n(0) = \varphi_n'(0) = 0$ but $\varphi_n''(0) = C_\beta - \frac{2}{n}$. Up to comparing with a triangle inequality G_β and $G_{\beta,n} \sim \mathcal{N}(0, (C_\beta - \frac{2}{n})^{-1})$, we define then for $n \geq n_0(\beta) := [2C_\beta^{-1}]$

$$\begin{aligned} C_{\beta,n} &:= C_\beta - \frac{2}{n}, \\ \kappa_n(x) &:= \varphi_n(x) - C_{\beta,n}\frac{x^2}{2}. \end{aligned} \tag{3.7}$$

The replacement of $G_\beta \sim \mathcal{N}(0, C_\beta^{-1})$ by $G_{\beta,n} \sim \mathcal{N}(0, C_{\beta,n}^{-1})$ can be done up to $O\left(\frac{1}{n}\right)$. Indeed,

$$\begin{aligned} \gamma_n &= \int_{(-\sqrt{n},\sqrt{n})} \left| e^{-\varphi_n(x)} - e^{-C_\beta \frac{x^2}{2}} \right| dx \\ &= \int_{(-\sqrt{n},\sqrt{n})} \left| e^{-\varphi_n(x)} - e^{-C_{\beta,n}\frac{x^2}{2}} + e^{-C_{\beta,n}\frac{x^2}{2}} - e^{-C_\beta \frac{x^2}{2}} \right| dx \\ &\leq \int_{(-\sqrt{n},\sqrt{n})} \left| e^{-\varphi_n(x)} - e^{-C_{\beta,n}\frac{x^2}{2}} \right| dx + \int_{(-\sqrt{n},\sqrt{n})} \left| e^{-C_{\beta,n}\frac{x^2}{2}} - e^{-C_\beta \frac{x^2}{2}} \right| dx \\ &=: 2\sqrt{\frac{2\pi}{C_{\beta,n}}}\, \widetilde{\gamma}_n + \int_{(-\sqrt{n},\sqrt{n})} \left| e^{-C_{\beta,n}\frac{x^2}{2}} - e^{-C_\beta \frac{x^2}{2}} \right| dx \end{aligned}$$

with

$$\begin{aligned} \widetilde{\gamma}_n &:= \sqrt{\frac{C_{\beta,n}}{2\pi}} \int_0^{\sqrt{n}} \left| e^{-\varphi_n(x)} - e^{-C_{\beta,n}\frac{x^2}{2}} \right| dx \\ &= \sqrt{\frac{C_{\beta,n}}{2\pi}} \int_0^{\sqrt{n}} \left| 1 - e^{-\kappa_n(x)} \right| e^{-C_{\beta,n}\frac{x^2}{2}} dx \end{aligned}$$

and

$$\begin{aligned} \int_{-\sqrt{n}}^{\sqrt{n}} \left| e^{-C_{\beta,n}\frac{x^2}{2}} - e^{-C_\beta \frac{x^2}{2}} \right| dx &= \int_{-\sqrt{n}}^{\sqrt{n}} \left| e^{\frac{x^2}{n}} - 1 \right| e^{-C_\beta \frac{x^2}{2}} dx \\ &\leq 3 \int_{\mathbb{R}} \frac{x^2}{n} e^{-C_\beta \frac{x^2}{2}} dx \\ &= \frac{3}{n} \frac{\sqrt{2\pi}}{C_\beta^{3/2}}. \end{aligned}$$

Here, we have used $|e^x - 1| \leq e^{|x|} - 1 \leq |x|\, e^{|x|}$ for all $x \in \mathbb{R}$, $e^{\frac{x^2}{n}} \leq 3$ for $x \in (-\sqrt{n}, \sqrt{n})$ and the second moment of a Gaussian.

The function κ_n thus defined in (3.7) is now strictly increasing and positive on $[0, \sqrt{n})$, hence, so is $1 - e^{-\kappa_n}$. Moreover, thanks to the matching of the derivatives and the fact that $\varphi_n'''(0) = 0$, the Taylor formula with integral remainder gives at the fourth order

$$\kappa_n(x) = \frac{x^4}{6} \int_0^1 (1-\alpha)^3 \kappa_n^{(4)}(\alpha x) d\alpha = \frac{x^4}{6} \int_0^1 (1-\alpha)^3 \varphi_n^{(4)}(\alpha x) d\alpha,$$

where $\kappa_n^{(4)}(x) := \left(\frac{d}{dx}\right)^4 \kappa_n(x)$. The only singularity of all the derivatives of κ_n is in $\sqrt{n}$, which is not in $(0, \sqrt{n}-\varepsilon)$. We can thus write for all $x \in [0, \sqrt{n})$

$$0 \le \kappa_n(x) \le \frac{x^4}{6}\int_0^1 (1-\alpha)^3 d\alpha \left\| \kappa_n^{(4)} \mathbb{1}_{[0,\sqrt{n}-\varepsilon)} \right\|_\infty = \frac{x^4}{24}\left\| \kappa_n^{(4)} \mathbb{1}_{[0,\sqrt{n}-\varepsilon)} \right\|_\infty =: M_n(\varepsilon)\frac{x^4}{24}.$$

Since $\kappa_n^{(4)}$ is positive and increasing on $[0, \sqrt{n}-\varepsilon)$, we have

$$M_n(\varepsilon) = \kappa_n^{(4)}(\sqrt{n}-\varepsilon)$$

hence

$$\begin{aligned}
\int_{[0,\sqrt{n}-\varepsilon)} \left|1 - e^{-\kappa_n(x)}\right| e^{-C_{\beta,n}\frac{x^2}{2}} dx &\le \int_{[0,\sqrt{n}-\varepsilon)} \left(1 - e^{-M_n(\varepsilon)\frac{x^4}{24}}\right) e^{-C_{\beta,n}\frac{x^2}{2}} dx \\
&\le \frac{M_n(\varepsilon)}{24}\int_{[0,\sqrt{n}-\varepsilon)} x^4 e^{-C_{\beta,n}\frac{x^2}{2}} dx \\
&\le \frac{M_n(\varepsilon)}{24}\sqrt{\frac{2\pi}{C_{\beta,n}}}\,\mathbb{E}\Big((G_{\beta,n})^4\Big) \\
&= \frac{M_n(\varepsilon)}{8}\sqrt{\frac{2\pi}{C_{\beta,n}}}\,C_{\beta,n}^{-2},
\end{aligned}$$

where we have used the scaling of the Gaussian and its fourth moment equal to 3.
Last, a computation with SageMath [87] gives

$$\kappa_n^{(4)}(x) = \varphi_n^{(4)}(x) = \frac{2}{n\beta}\frac{P(x/\sqrt{n})\,\mathrm{Argtanh}(x/\sqrt{n}) - Q_{n,\beta}(x/\sqrt{n})}{(1-(x/\sqrt{n})^2)^4}$$

with explicit polynomials

$$\begin{aligned}
P(x) &:= 12\,x(x^2+1), \\
Q_{n,\beta}(x) &:= 3\beta(1+2n^{-1})x^4 - 18(1-\beta-2n^{-1})x^2 - (4-3\beta-6\beta n^{-1}) \\
&=: Q_\beta(x) + \frac{1}{n}\widetilde{Q}_\beta(x), \\
Q_\beta(x) &:= 3\beta x^4 - 18(1-\beta)x^2 - (4-3\beta), \\
\widetilde{Q}_\beta(x) &:= 6(\beta x^4 + 6x^2 + 6\beta).
\end{aligned}$$

Define

$$t := \frac{\varepsilon}{\sqrt{n}} \in (0,1).$$

Then,

$$\begin{aligned}
M_n(\varepsilon) = \varphi_n^{(4)}(\sqrt{n}-\varepsilon) &= \frac{2}{n\beta}\frac{P(1-t)\,\mathrm{Argtanh}(1-t) - Q_{n,\beta}(1-t)}{(1-(1-t)^2)^4} \\
&= \frac{2}{n\beta}\frac{P(1-t)\,\mathrm{Argtanh}(1-t) - Q_\beta(1-t)}{t^4(2-t)^4} - \frac{2}{n^2\beta}\frac{\widetilde{Q}_\beta(1-t)}{t^4(2-t)^4}.
\end{aligned} \tag{3.8}$$

- **Remaining interval:** As κ_n is positive on $[\sqrt{n}-\varepsilon, \sqrt{n})$, we have $1 - e^{-\kappa_n(x)} \le 1$ and

$$\begin{aligned}
\int_{[\sqrt{n}-\varepsilon,\sqrt{n})} \left|1 - e^{-\kappa_n(x)}\right| e^{-C_{\beta,n}\frac{x^2}{2}} dx &\le \int_{[\sqrt{n}-\varepsilon,\sqrt{n})} e^{-C_{\beta,n}\frac{x^2}{2}} dx \\
&= \varepsilon\int_0^1 e^{-C_{\beta,n}\frac{(\sqrt{n}-\varepsilon u)^2}{2}} du \\
&\le \varepsilon\, e^{-C_{\beta,n}\frac{\sqrt{n}}{2}} \le 3\varepsilon\, e^{-C_\beta\frac{\sqrt{n}}{2}},
\end{aligned}$$

where we have used the change of variables $x = \sqrt{n} - \varepsilon u$, $(\sqrt{n} - \varepsilon u)^2 = n(1 - tu)^2 \geq \sqrt{n}$ for n big enough and for all $t, u \in [0, 1]$, in addition to $C_{\beta,n} \frac{\sqrt{n}}{2} = C_\beta \frac{\sqrt{n}}{2} - \frac{1}{\sqrt{n}}$ and $e^{1/\sqrt{n}} \leq 3$.

- **General contribution:** We finally get

$$\sqrt{\frac{2\pi}{C_{\beta,n}}}\,\widetilde{\gamma}_n := \int_{(0,\sqrt{n})} \left|1 - e^{-\kappa_n(x)}\right| e^{-C_{\beta,n}\frac{x^2}{2}} dx \leq M_n(\varepsilon) \frac{C_{\beta,n}^{-2}}{8} + 3\sqrt{\frac{C_{\beta,n}}{2\pi}} \varepsilon\, e^{-C_\beta \frac{\sqrt{n}}{2}}.$$

It remains to choose ε in order to get the order of convergence. In view of (3.8), we can take for instance $t = \frac{3}{4}$, yielding

$$\begin{aligned} M_n(\tfrac{3}{4}\sqrt{n}) &= \frac{2}{n\beta} \frac{P\left(\frac{1}{4}\right) \operatorname{Argtanh}(\frac{1}{4}) - Q_\beta(\frac{1}{4})}{(3/4)^4(3/2-1)^4} - \frac{2}{n^2\beta} \frac{\widetilde{Q}_\beta(\frac{1}{4})}{(3/4)^4(3/2-1)^4} \\ &=: \frac{K_1(\beta)}{n} + \frac{K_2(\beta)}{n^2}. \end{aligned}$$

As a result, we get for explicit constants $K_3(\beta), K_4(\beta) > 0$

$$\sqrt{\frac{2\pi}{C_{\beta,n}}}\,\widetilde{\gamma}_n \leq \frac{K_3(\beta)}{n} + \frac{K_4(\beta)}{n^2} + O\left(\sqrt{n}\, e^{-C_\beta \frac{\sqrt{n}}{2}}\right) = O_\beta\left(\frac{1}{n}\right).$$

- **Conclusion:** We have for all $k \geq 1$

$$\widetilde{\delta}_n(h) \leq \frac{\mathbb{E}\left(|G_\beta|^{2k}\right)}{n^k} \|h\|_\infty + \frac{\mathbb{E}\left(G_\beta^2\right)}{n} \|h'\|_\infty + \delta(g_n, g_\beta) \|h\|_\infty$$

and using Lemma 3.15, we have

$$\begin{aligned} \delta(g_n, g_\beta) &\leq \left| \sqrt{\frac{C_\beta}{2\pi}} \times \mathcal{Z}_{n,\beta} \sqrt{n} - 1 \right| + \gamma_n \\ &\leq \frac{C_\beta^{-9/2}}{4\,n} + 2\sqrt{\frac{2\pi}{C_{\beta,n}}}\,\widetilde{\gamma}_n + \frac{3}{n} \frac{\sqrt{2\pi}}{C_\beta^{3/2}} \\ &= O_\beta\left(\frac{1}{n}\right), \end{aligned} \tag{3.9}$$

which gives the result.

□

Remark 3.2. The surrogate approach here defined allows to understand in a better way the apparition of the limiting Gaussian random variable. In the case $\beta < 1$, the Gaussian CLT is present through the random variable G, and it is the adjunction of the random variable G_β coming from the fluctuations of the randomisation that finally gives $\boldsymbol{Z}_\beta = G + G_\beta$. It is thus a subtle mixture of the two structures, sums of i.i.d.'s and randomisation, that gives the final distribution in this case. In the language of statistical mechanics of phase transitions, when a disorder is present in a statistical system and has a marginal effect, one talks about a *marginally relevant disordered system*, see e.g. [12] and [103] in the context of the KPZ equation or random polymers. It is typically the case here with the decomposition $\boldsymbol{Z}_\beta = G + G_\beta$ since in the case $\beta < 1$ the overall behaviour is still Gaussian.

Nevertheless, when looking at the speed in (3.2) and (3.1), we see that it is only the CLT bound for sums of i.i.d.'s that gives its footprint to the first order and not at all the randomisation in this case, while the speed coming from the randomisation only appears at the second order. This will be the opposite in the next case $\beta = 1$. The regime $\beta < 1$ can thus be considered as the regime where the independent CLT dominates at the level of the speed and the randomisation is marginally relevant; the regime $\beta = 1$ will be the one where the randomisation dominates at the level of the fluctuations, visible as non Gaussian behaviour. From this perspective, the transition is interesting: there is a competition between randomisation and sums of independent random variables.

3.1.2. *The case $\beta = 1$.*

Theorem 3.3 (Fluctuations of the *unnormalised* magnetisation for $\beta = 1$)**.** *Let $\boldsymbol{F}$ be a random variable of law given by $\mathbb{P}(\boldsymbol{F} \in dx) := \frac{1}{\mathcal{Z}_{\boldsymbol{F}}} e^{-\frac{x^4}{12}} dx$ with $\mathcal{Z}_{\boldsymbol{F}} := \int_{\mathbb{R}} e^{-\frac{x^4}{12}} dx = 3^{1/4} 2^{-1/2} \Gamma(1/4)$. Then, for all $h \in \mathcal{C}^1$ with $\|h\|_\infty, \|h'\|_\infty < \infty$*

$$\left| \mathbb{E}\left(h\left(\frac{M_n^{(1)}}{n^{3/4}} \right) \right) - \mathbb{E}(h(\boldsymbol{F})) \right| \leq \left(\frac{C}{\sqrt{n}} + O\left(\frac{1}{n^{3/4}} \right) \right) \left(\|h\|_\infty + \|h'\|_\infty \right), \tag{3.10}$$

where $C > 0$ is an explicit constant.

Proof. Randomising (2.40) and rescaling it by a factor $n^{3/4}$, which amounts to do $h \leftarrow h(\frac{\cdot}{n^{3/4}})$ hence $h' \leftarrow \frac{1}{n^{3/4}} h'(\frac{\cdot}{n^{3/4}})$, yields

$$\left| \mathbb{E}\left(h\left(\frac{M_n^{(1)}}{n^{3/4}} \right) \right) - \mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(1)}}{n^{3/4}} \right) \right) \right| \leq \frac{C}{n^{3/4}} \|h'\|_\infty \tag{3.11}$$

with

$$\frac{\mathcal{M}_n^{(1)}}{n^{3/4}} = \frac{G}{n^{1/4}} \sqrt{1 - (\boldsymbol{T}_n^{(1)})^2} + n^{1/4} \boldsymbol{T}_n^{(1)},$$

where $G \sim \mathcal{N}(0,1)$ is independent of $M_n^{(1)}, \boldsymbol{F}$. Setting

$$\boldsymbol{F}_n := n^{1/4} \boldsymbol{T}_n^{(1)} \tag{3.12}$$

with $\boldsymbol{T}_n^{(1)}$ defined in (2.38) gives

$$\frac{\mathcal{M}_n^{(1)}}{n^{3/4}} = \frac{G}{n^{1/4}} \sqrt{1 - \frac{\boldsymbol{F}_n^2}{\sqrt{n}}} + \boldsymbol{F}_n \tag{3.13}$$

and the triangle inequality gives then the analogue of (3.2)

$$\left| \mathbb{E}\left(h\left(\frac{M_n^{(1)}}{n^{3/4}} \right) \right) - \mathbb{E}(h(\boldsymbol{F})) \right| \leq \frac{C}{n^{3/4}} \|h'\|_\infty + \left| \mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(1)}}{n^{3/4}} \right) \right) - \mathbb{E}(h(\boldsymbol{F})) \right|. \tag{3.14}$$

Nevertheless, one sees from the expression of (3.13) that if $\boldsymbol{F}_n \to \boldsymbol{F}$ in distribution,

$$\frac{\mathcal{M}_n^{(1)}}{n^{3/4}} = \frac{G}{n^{1/4}} \sqrt{1 - \frac{\boldsymbol{F}_n^2}{\sqrt{n}}} + \boldsymbol{F}_n \approx \boldsymbol{F} + \frac{G}{n^{1/4}} \left(1 - \frac{\boldsymbol{F}^2}{2\sqrt{n}} \right) \approx \boldsymbol{F} + \frac{G}{n^{1/4}} + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}} \right).$$

As a result, we will always have at best $\mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(1)}}{n^{3/4}} \right) \right) - \mathbb{E}(h(\boldsymbol{F})) = O\left(\frac{\|h'\|_\infty}{n^{1/4}} \right)$ which is incompatible with the results of [16] and [33] that give a speed in $O\left(\frac{1}{\sqrt{n}} \right)$. Such a discrepancy between this result and (3.11) shows that we have used the "wrong" random variable to compare to, when using the triangle inequality. We should instead incorporate another random variable at a distance $\frac{1}{\sqrt{n}}$ to decrease the distance in $\frac{1}{n^{1/4}}$, possibly at the cost of increasing the distance in $\frac{1}{n^{3/4}}$ in (3.11). Such a replacement can be performed by introducing another related surrogate.

For i.i.d. Rademacher random variables $(X_k)_k$, we take $S_n(p) := \sum_{k=1}^n X_k(p)$. A possible representation of $X_k(p)$ is given by

$$X_k(p) := \mathbb{1}_{\{U_k < p\}} - \mathbb{1}_{\{U_k > p\}} = 2\,\mathbb{1}_{\{U_k < p\}} - 1, \qquad (U_k)_{k \geq 1} \sim \text{i.i.d.}\,\mathcal{U}([0,1]) \tag{3.15}$$

and this representation allows in addition to visualise the randomisation of the parameter p in a functional way.

We can thus use a coupling of $S_n(p)$ and $S_n(q)$ using these uniform random variables. This very coupling is said to be *totally dependent* in the sense that these are the same uniform random variables that are used, e.g. $X_k(q)$ is a measurable function of $X_k(p)$ and vice versa. Define for $p, q \in [0,1]$

$$\begin{aligned}\lambda_n &:= 2n(p-q) = \mathbb{E}(S_n(p) - S_n(q))\,,\\ S_n(p,q) &:= S_n(p) - S_n(q) - \lambda_n,\\ \sigma(p)^2 &:= 4p(1-p).\end{aligned}$$

Then,

$$\begin{aligned}|\mathbb{E}(h(S_n(p))) - \mathbb{E}(h(S_n(q)+\lambda_n))| &\leq \|h'\|_\infty\, \mathbb{E}(|S_n(p,q)|)\\ &\leq \|h'\|_\infty \sqrt{\mathbb{E}\Big(|S_n(p,q)|^2\Big)}\\ &=: \|h'\|_\infty \sqrt{n}\sqrt{\sigma(p)^2 + \sigma(q)^2 - 2\rho(p,q)}\end{aligned}$$

with

$$\mathring{X}_k(p) := X_k(p) - \mathbb{E}(X_k(p)) = 2\Big(\mathbb{1}_{\{U_k<p\}} - p\Big)$$

and

$$\begin{aligned}\rho(p,q) &:= \mathbb{E}\Big(\mathring{X}(p)\,\mathring{X}(q)\Big)\\ &= 4\,\mathbb{E}\Big(\Big(\mathbb{1}_{\{U<p\}} - p\Big)\Big(\mathbb{1}_{\{U<q\}} - q\Big)\Big)\\ &= 4\,\mathbb{E}\Big(\mathbb{1}_{\{U<p\wedge q\}} - p\mathbb{1}_{\{U<q\}} - q\mathbb{1}_{\{U<p\}} + pq\Big)\\ &= 4(p\wedge q - pq)\end{aligned}$$

with

$$p \wedge q := \min\{p,q\}.$$

We thus have

$$\begin{aligned}\sigma(p)^2 + \sigma(q)^2 - 2\rho(p,q) &= \mathbb{E}\Big(\Big(\mathring{X}(p) - \mathring{X}(q)\Big)^2\Big)\\ &= 4(p(1-p) + q(1-q) - 2(p\wedge q - pq))\\ &= 4\Big(p + q - 2(p\wedge q) - [p^2 + q^2 - 2pq]\Big)\\ &= 4\Big(|p-q| - |p-q|^2\Big)\\ &= 4\,|p-q|\,(1 - |p-q|).\end{aligned}$$

We can now write with $\boldsymbol{\lambda}_n := 2n(\boldsymbol{P} - \boldsymbol{Q})$ and $(\boldsymbol{P}, \boldsymbol{Q})$ chosen at random independently from $(U_k)_k$

$$\begin{aligned}\boldsymbol{\delta}_n(h) :=& \left|\mathbb{E}\left(h\left(\frac{S_n(\boldsymbol{P})}{n^{3/4}}\right)\right) - \mathbb{E}(h(\boldsymbol{F}))\right|\\ \leq& \left|\mathbb{E}\left(h\left(\frac{S_n(\boldsymbol{P})}{n^{3/4}}\right)\right) - \mathbb{E}\left(h\left(\frac{S_n(\boldsymbol{Q}) + \boldsymbol{\lambda}_n}{n^{3/4}}\right)\right)\right|\\ &+ \left|\mathbb{E}\left(h\left(\frac{S_n(\boldsymbol{Q}) + \boldsymbol{\lambda}_n}{n^{3/4}}\right)\right) - \mathbb{E}\left(h\left(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1) + \boldsymbol{\lambda}_n}{n^{3/4}}\right)\right)\right|\\ &+ \left|\mathbb{E}\left(h\left(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1) + \boldsymbol{\lambda}_n}{n^{3/4}}\right)\right) - \mathbb{E}(h(\boldsymbol{F}))\right|\end{aligned}$$

$$=: \boldsymbol{\delta}_n^{(1)}(h) + \boldsymbol{\delta}_n^{(2)}(h) + \boldsymbol{\delta}_n^{(3)}(h).$$

Choosing $2\boldsymbol{P}-1 = \boldsymbol{T}_n^{(1)} = n^{-1/4}\boldsymbol{F}_n'$ with $\boldsymbol{F}_n' \overset{d}{=} \boldsymbol{F}_n$ will then give a bound on the magnetisation $\frac{M_n^{(1)}}{n^{3/4}}$. We moreover choose $2\boldsymbol{Q}-1 = n^{-1/4}\boldsymbol{F}_n$ to obtain

$$\begin{aligned}\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1) + \boldsymbol{\lambda}_n}{n^{3/4}} &= \frac{G}{n^{1/4}}\sqrt{1 - \frac{\boldsymbol{F}_n^2}{\sqrt{n}}} + \boldsymbol{F}_n + \frac{\boldsymbol{\lambda}_n}{n^{3/4}} \\ &= \frac{G}{n^{1/4}}\sqrt{1 - \frac{\boldsymbol{F}_n^2}{\sqrt{n}}} + \boldsymbol{F}_n + (\boldsymbol{F}_n' - \boldsymbol{F}_n)\end{aligned}$$

since

$$\boldsymbol{\lambda}_n = 2n(\boldsymbol{P}-\boldsymbol{Q}) = n \times n^{-1/4}(\boldsymbol{F}_n' - \boldsymbol{F}_n).$$

We would then want to couple $(\boldsymbol{F}_n, \boldsymbol{F}_n')$ by setting $\boldsymbol{F}_n' - \boldsymbol{F}_n = -\frac{G}{n^{1/4}}$ with the same G that defines the surrogate magnetisation in $\boldsymbol{\delta}_n^{(3)}(h)$, nevertheless, we also need to remember that $p, q \in [0,1]$, hence that $(p-q) \in [-1,1]$. We thus set

$$\boldsymbol{F}_n' - \boldsymbol{F}_n := -\frac{G}{n^{1/4}}\mathbb{1}_{\{|G|\le\sqrt{n}\}}. \tag{3.16}$$

To be precise, $\boldsymbol{P}$ is the usual randomisation and we choose $\boldsymbol{Q}$ appropriate to get (3.16). With the choice (3.16), we have

$$2(\boldsymbol{P}-\boldsymbol{Q}) = \frac{1}{n^{1/4}}(\boldsymbol{F}_n' - \boldsymbol{F}_n) = -\frac{G}{\sqrt{n}}\mathbb{1}_{\{|G|\le\sqrt{n}\}}$$

hence

$$\boldsymbol{\delta}_n^{(1)}(h) \le \frac{\|h'\|_\infty}{n^{3/4}} \times 2\sqrt{n}\sqrt{\mathbb{E}(|\boldsymbol{P}-\boldsymbol{Q}|\,(1-|\boldsymbol{P}-\boldsymbol{Q}|))} = \frac{\|h'\|_\infty}{\sqrt{2n}}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

Setting $g := h(\cdot + \boldsymbol{\lambda}_n/n^{3/4})$ and using the inequality (2.40) rescaled by a factor $n^{3/4}$ still gives

$$\boldsymbol{\delta}_n^{(2)}(h) \le \frac{C}{n^{3/4}}\mathbb{E}(\|g'\|_\infty) \le \frac{C}{n^{3/4}}\|h'\|_\infty,$$

but now, we have

$$\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1) + \boldsymbol{\lambda}_n}{n^{3/4}} = \frac{G}{n^{1/4}}\left(\sqrt{1 - \frac{\boldsymbol{F}_n^2}{\sqrt{n}}} - 1\right) + \boldsymbol{F}_n + \frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}.$$

Set

$$\begin{aligned}\mathcal{F}_n &:= \frac{G}{n^{1/4}}\left(\sqrt{1 - \frac{\boldsymbol{F}_n^2}{\sqrt{n}}} - 1\right), \\ \boldsymbol{\delta}_n^{(4)}(h) &:= |\mathbb{E}(h(\mathcal{F}_n + \boldsymbol{F}_n)) - \mathbb{E}(h(\boldsymbol{F}_n))|, \\ \boldsymbol{\delta}_n^{(5)}(h) &:= |\mathbb{E}(h(\boldsymbol{F}_n)) - \mathbb{E}(h(\boldsymbol{F}))|,\end{aligned}$$

so that

$$\begin{aligned}\boldsymbol{\delta}_n^{(3)}(h) &= \left|\mathbb{E}\left(h\left(\mathcal{F}_n + \boldsymbol{F}_n + \frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}\right)\right) - \mathbb{E}(h(\boldsymbol{F}))\right| \\ &\le \|h'\|_\infty\,\mathbb{E}\left(\frac{|G|}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}\right) + \boldsymbol{\delta}_n^{(4)}(h) + \boldsymbol{\delta}_n^{(5)}(h)\end{aligned}$$

$$\begin{aligned}
&\leq \|h'\|_\infty \frac{\sqrt{\mathbb{E}\left(|G|^2\right)}}{n^{1/4}} \sqrt{\mathbb{P}\left(|G| > \sqrt{n}\right)} + \boldsymbol{\delta}_n^{(4)}(h) + \boldsymbol{\delta}_n^{(5)}(h) \\
&\leq \|h'\|_\infty \frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/4}} + \boldsymbol{\delta}_n^{(4)}(h) + \boldsymbol{\delta}_n^{(5)}(h)
\end{aligned}$$

for all $k \geq 1$ using Taylor expansion, the triangle-, Cauchy–Schwarz- and Markov's inequality. Moreover,

$$\begin{aligned}
\boldsymbol{\delta}_n^{(4)}(h) &= |\mathbb{E}(h(\mathcal{F}_n + \boldsymbol{F}_n)) - \mathbb{E}(h(\boldsymbol{F}_n))| \\
&\leq \|h'\|_\infty \mathbb{E}(|\mathcal{F}_n|) \\
&\leq \|h'\|_\infty \frac{\mathbb{E}(|G|)\,\mathbb{E}\left(\boldsymbol{F}_n^2\right)}{n^{3/4}} \leq \|h'\|_\infty \frac{\mathbb{E}\left(\boldsymbol{F}_n^2\right)}{n^{3/4}}
\end{aligned}$$

using $\left|1 - \sqrt{1-u}\right| \leq u$ for $u \in (0,1)$ and $\mathbb{E}(|G|) \leq \sqrt{\mathbb{E}(G^2)} = 1$. We will now show that $\boldsymbol{F}_n^2 \to \boldsymbol{F}^2$ in distribution and in L^1, hence that $\mathbb{E}\left(\boldsymbol{F}_n^2\right) = \mathbb{E}\left(\boldsymbol{F}^2\right) + o(1)$, which we will achieve to bound $\boldsymbol{\delta}_n^{(4)}(h)$.

The distribution of $\boldsymbol{F}_n := n^{1/4}\,\boldsymbol{T}_n^{(1)}$ is given by the rescaling of $\nu_{n,\beta}$ in (2.33):

$$\mathbb{P}(\boldsymbol{F}_n \in dt) = f_{n,1}\left(\frac{t}{n^{1/4}}\right)\frac{dt}{n^{1/4}} = \frac{1}{n^{1/4}\,\mathcal{Z}_{n,1}} e^{-\frac{n}{2}\operatorname{Argtanh}\left(\frac{t}{n^{1/4}}\right)^2 - \left(\frac{n}{2}+1\right)\ln\left(1-\frac{t^2}{\sqrt{n}}\right)} \mathbb{1}_{\left\{|t| \leq n^{1/4}\right\}} dt.$$

Lemma 3.16 gives

$$n^{1/4}\,\mathcal{Z}_{n,1} = \mathcal{Z}_{\boldsymbol{F}} + O\left(\frac{1}{\sqrt{n}}\right)$$

and a Taylor expansion in 0 yields

$$\frac{n}{2}\operatorname{Argtanh}\left(\frac{t}{n^{1/4}}\right)^2 + \left(\frac{n}{2}+1\right)\ln\left(1 - \frac{t^2}{\sqrt{n}}\right) = -\frac{t^2}{\sqrt{n}} + \frac{t^4}{12}\left(1 - \frac{6}{n}\right) + O\left(\frac{t^6}{n^{3/2}}\right). \tag{3.17}$$

This implies the convergence in law $\boldsymbol{F}_n \to \boldsymbol{F}$ by looking at the densities, and the result by square integrability of $\boldsymbol{F}$.

We now study $\boldsymbol{\delta}_n^{(5)}(h)$. In the same vein as for $\beta < 1$, we have for all $\varepsilon \in (0,1)$ and setting $\bar{\varepsilon} := 1 - \varepsilon$

$$\begin{aligned}
\boldsymbol{\delta}_n^{(5)}(h) &:= |\mathbb{E}(h(\boldsymbol{F}_n)) - \mathbb{E}(h(\boldsymbol{F}))| \\
&\leq \int_{(-\bar{\varepsilon}n^{1/4}, \bar{\varepsilon}n^{1/4})} |h|\,|f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}| + \left|\mathbb{E}\left(h(\boldsymbol{F})\mathbb{1}_{\left\{|\boldsymbol{F}| > \bar{\varepsilon}n^{1/4}\right\}}\right)\right| \\
&\leq \|h\|_\infty \left(\|f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}\|_{L^1([-\bar{\varepsilon}n^{1/4}, \bar{\varepsilon}n^{1/4}])} + \mathbb{P}|\boldsymbol{F}| > \bar{\varepsilon}n^{1/4}\right) \\
&\leq \|h\|_\infty \left(\|f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}\|_{L^1([-\bar{\varepsilon}n^{1/4}, \bar{\varepsilon}n^{1/4}])} + \frac{\mathbb{E}\left(\boldsymbol{F}^{4k}\right)}{(1-\varepsilon)^{4k}\,n^k}\right),
\end{aligned}$$

using the triangle inequality and Markov's inequality for all $k \geq 1$.

We now show that

$$\|f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}\|_{L^1([-\bar{\varepsilon}n^{1/4}, \bar{\varepsilon}n^{1/4}])} \leq \boldsymbol{\delta}_n^{(6)} + \frac{\mathbb{E}\left(\boldsymbol{F}^2\right)}{\sqrt{n}} + O\left(\frac{1}{n}\right), \tag{3.18}$$

where $\boldsymbol{\delta}_n^{(6)}$ will be defined in (3.19) and bounded in (3.21).

In view of (3.17), we introduce the random variable $\widehat{\boldsymbol{F}}_n$ defined by the density

$$f_{\widehat{\boldsymbol{F}}_n}(x) := \frac{1}{\widehat{\mathcal{Z}}_n} e^{-\widehat{\Phi}_n(x)}, \qquad \widehat{\Phi}_n(x) := \frac{x^4}{12}\left(1 - \frac{6}{n}\right) - \frac{x^2}{\sqrt{n}}, \quad \widehat{\mathcal{Z}}_n := \int_{\mathbb{R}} e^{-\widehat{\Phi}_n}.$$

Define also

$$\Phi_n : x \mapsto \frac{n}{2}\operatorname{Argtanh}\Big(\frac{x}{n^{1/4}}\Big)^2 + \Big(\frac{n}{2}+1\Big)\ln\Big(1-\frac{x^2}{\sqrt{n}}\Big),$$
$$\Phi := \widehat{\Phi}_\infty = \Phi_\infty : x \mapsto \frac{x^4}{12}.$$

Then, we can replace $\boldsymbol{F}$ by $\widehat{\boldsymbol{F}_n}$ up to $O(\frac{1}{\sqrt{n}})$ by writing

$$\|f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}\|_{L^1([-\bar{\varepsilon}n^{1/4},\,\bar{\varepsilon}n^{1/4}])} \le \Big\|f_{\boldsymbol{F}_n} - f_{\widehat{\boldsymbol{F}}_n}\Big\|_{L^1([-\bar{\varepsilon}n^{1/4},\,\bar{\varepsilon}n^{1/4}])} + \Big\|f_{\widehat{\boldsymbol{F}}_n} - f_{\boldsymbol{F}}\Big\|_{L^1(\mathbb{R})}$$

and

$$\begin{aligned}
\Big\|f_{\widehat{\boldsymbol{F}}_n} - f_{\boldsymbol{F}}\Big\|_{L^1(\mathbb{R})} &:= \int_{\mathbb{R}}\bigg|\frac{1}{\widehat{\mathcal{Z}}_n}e^{-\widehat{\Phi}_n} - \frac{1}{\mathcal{Z}_{\boldsymbol{F}}}e^{-\Phi}\bigg| = \frac{1}{\mathcal{Z}_{\boldsymbol{F}}}\int_{\mathbb{R}}\bigg|\frac{\mathcal{Z}_{\boldsymbol{F}}}{\widehat{\mathcal{Z}}_n}e^{-\widehat{\Phi}_n+\Phi} - 1\bigg|e^{-\Phi} \\
&= \frac{1}{\mathcal{Z}_{\boldsymbol{F}}}\int_{\mathbb{R}}\bigg|\Big(\frac{\mathcal{Z}_{\boldsymbol{F}}}{\widehat{\mathcal{Z}}_n}-1\Big)e^{-\widehat{\Phi}_n+\Phi} + e^{-\widehat{\Phi}_n+\Phi} - 1\bigg|e^{-\Phi} \\
&\le \bigg|\frac{\mathcal{Z}_{\boldsymbol{F}}}{\widehat{\mathcal{Z}}_n} - 1\bigg|\frac{1}{\mathcal{Z}_{\boldsymbol{F}}}\int_{\mathbb{R}} e^{-\widehat{\Phi}_n+\Phi}e^{-\Phi} + \frac{1}{\mathcal{Z}_{\boldsymbol{F}}}\int_{\mathbb{R}}\Big|e^{-\widehat{\Phi}_n+\Phi} - 1\Big|e^{-\Phi} \\
&= \frac{1}{\mathcal{Z}_{\boldsymbol{F}}}\Big|\mathcal{Z}_{\boldsymbol{F}} - \widehat{\mathcal{Z}}_n\Big| + \mathbb{E}\Big(\Big|e^{\frac{\boldsymbol{F}^4}{2n}+\frac{\boldsymbol{F}^2}{\sqrt{n}}} - 1\Big|\Big) \\
&= \frac{1}{\mathcal{Z}_{\boldsymbol{F}}}\Big|\int_{\mathbb{R}}\Big(e^{-\widehat{\Phi}_n} - e^{-\Phi}\Big)\Big| + \mathbb{E}\Big(\Big|e^{\frac{\boldsymbol{F}^4}{2n}+\frac{\boldsymbol{F}^2}{\sqrt{n}}} - 1\Big|\Big) \\
&\le 2\,\mathbb{E}\Big(\Big|e^{\frac{\boldsymbol{F}^4}{2n}+\frac{\boldsymbol{F}^2}{\sqrt{n}}} - 1\Big|\Big)
\end{aligned}$$

with

$$\mathbb{E}\Big(\Big|e^{\frac{\boldsymbol{F}^4}{2n}+\frac{\boldsymbol{F}^2}{\sqrt{n}}} - 1\Big|\Big) = \mathbb{E}\bigg(\bigg|1 + \frac{\boldsymbol{F}^4}{2n} + \frac{\boldsymbol{F}^2}{\sqrt{n}} - 1 + O\Big(\frac{\boldsymbol{F}^4}{n}\Big)\bigg|\bigg) = \frac{\mathbb{E}\big(\boldsymbol{F}^2\big)}{\sqrt{n}} + O\Big(\frac{1}{n}\Big)$$

by $e^x = 1 + x + O(x^2)$. We thus have

$$\boldsymbol{\delta}_n^{(5)}(h) \le \|h\|_\infty\bigg(\Big\|f_{\boldsymbol{F}_n} - f_{\widehat{\boldsymbol{F}}_n}\Big\|_{L^1([-\bar{\varepsilon}n^{1/4},\bar{\varepsilon}n^{1/4}])} + \frac{\mathbb{E}\big(\boldsymbol{F}^2\big)}{\sqrt{n}} + O\Big(\frac{1}{n}\Big) + \frac{\mathbb{E}\big(\boldsymbol{F}^{4k}\big)}{(1-\varepsilon)^{4k}\,n^k}\bigg)$$

and we now estimate the remaining norm. We have

$$\begin{aligned}
\boldsymbol{\delta}_n^{(6)} &:= \Big\|f_{\boldsymbol{F}_n} - f_{\widehat{\boldsymbol{F}}_n}\Big\|_{L^1([-\bar{\varepsilon}n^{1/4},\bar{\varepsilon}n^{1/4}])} \qquad (3.19)\\
&= \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}}\Big|f_{\boldsymbol{F}_n} - f_{\widehat{\boldsymbol{F}}_n}\Big| = \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}}\bigg|1 - \frac{f_{\boldsymbol{F}_n}}{f_{\widehat{\boldsymbol{F}}_n}}\bigg| f_{\widehat{\boldsymbol{F}}_n} \\
&= \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}}\bigg|1 - \frac{\widehat{\mathcal{Z}}_n}{n^{1/4}\mathcal{Z}_{n,1}}e^{-(\Phi_n-\widehat{\Phi}_n)}\bigg| f_{\widehat{\boldsymbol{F}}_n} \\
&= \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}}\bigg|1 - e^{-(\Phi_n-\widehat{\Phi}_n)} + \Big(1 - \frac{\widehat{\mathcal{Z}}_n}{n^{1/4}\mathcal{Z}_{n,1}}\Big)e^{-(\Phi_n-\widehat{\Phi}_n)}\bigg| f_{\widehat{\boldsymbol{F}}_n} \\
&\le \bigg|1 - \frac{\widehat{\mathcal{Z}}_n}{n^{1/4}\mathcal{Z}_{n,1}}\bigg|\int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}} e^{-(\Phi_n-\widehat{\Phi}_n)}f_{\widehat{\boldsymbol{F}}_n} + \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}}\Big|1 - e^{-(\Phi_n-\widehat{\Phi}_n)}\Big| f_{\widehat{\boldsymbol{F}}_n} \\
&\le \bigg|1 - \frac{\widehat{\mathcal{Z}}_n}{n^{1/4}\mathcal{Z}_{n,1}}\bigg|\frac{n^{1/4}\mathcal{Z}_{n,1}}{\widehat{\mathcal{Z}}_n} + \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}}\Big|1 - e^{-(\Phi_n-\widehat{\Phi}_n)}\Big| f_{\widehat{\boldsymbol{F}}_n}.
\end{aligned}$$

We have in addition

$$\frac{\widehat{\mathcal{Z}}_n}{\mathcal{Z}_{\boldsymbol{F}}} := \frac{1}{\mathcal{Z}_{\boldsymbol{F}}}\int_{\mathbb{R}} e^{-\widehat{\Phi}_n} = \frac{1}{\mathcal{Z}_{\boldsymbol{F}}}\int_{\mathbb{R}} e^{-\frac{x^4}{12}(1-\frac{6}{n})+\frac{x^2}{\sqrt{n}}}dx = \frac{1}{\mathcal{Z}_{\boldsymbol{F}}\left(1-\frac{6}{n}\right)^{1/4}}\int_{\mathbb{R}} e^{-\frac{x^4}{12}+\frac{x^2}{\sqrt{n-6}}}dx$$
$$= \frac{1}{\left(1-\frac{6}{n}\right)^{1/4}}\mathbb{E}\left(e^{\frac{\boldsymbol{F}^2}{\sqrt{n-6}}}\right) = 1 + \frac{\mathbb{E}\left(\boldsymbol{F}^2\right)}{\sqrt{n}} + O\left(\frac{1}{n}\right),$$

using the change of variables $x(1-6/n)^{-1/4} \mapsto x$ for the third and a taylor expansion of $\exp(.)$ for the fifth equation. Further we used $1/(1-x)^\alpha = 1 + \alpha x - \alpha(\alpha-1)x/2 + ...$ for small x. Lemma 3.16 gives

$$\frac{\mathcal{Z}_{\boldsymbol{F}}}{n^{1/4}\mathcal{Z}_{n,1}} = 1 + O\left(\frac{1}{\sqrt{n}}\right).$$

This implies

$$1 - \frac{\widehat{\mathcal{Z}}_n}{n^{1/4}\mathcal{Z}_{n,1}} = 1 - \frac{\widehat{\mathcal{Z}}_n}{\mathcal{Z}_{\boldsymbol{F}}}\frac{\mathcal{Z}_{\boldsymbol{F}}}{n^{1/4}\mathcal{Z}_{n,1}} = O\left(\frac{1}{\sqrt{n}}\right).$$

Hence

$$\begin{aligned}\boldsymbol{\delta}_n^{(6)} &\le \left|\frac{n^{1/4}\mathcal{Z}_{n,1}}{\widehat{\mathcal{Z}}_n} - 1\right| + \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}} \left|1 - e^{-(\Phi_n - \widehat{\Phi}_n)}\right| f_{\widehat{\boldsymbol{F}}_n} \\ &\le \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}} \left|1 - e^{-(\Phi_n - \widehat{\Phi}_n)}\right| f_{\widehat{\boldsymbol{F}}_n} + O\left(\frac{1}{\sqrt{n}}\right) =: \boldsymbol{\delta}_n^{(7)} + O\left(\frac{1}{\sqrt{n}}\right),\end{aligned} \tag{3.20}$$

having in mind that if $\frac{a_n}{b_n} = 1 + O\left(\frac{1}{\sqrt{n}}\right)$, it is $\frac{b_n}{a_n} = \frac{1}{1+O\left(\frac{1}{\sqrt{n}}\right)} = 1 + O\left(\frac{1}{\sqrt{n}}\right)$. It remains to investigate

$$\boldsymbol{\delta}_n^{(7)} := \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}} \left|1 - e^{-(\Phi_n - \widehat{\Phi}_n)}\right| f_{\widehat{\boldsymbol{F}}_n} = \mathbb{E}\left(\left|1 - e^{-(\Phi_n - \widehat{\Phi}_n)(\widehat{\boldsymbol{F}}_n)}\right| \mathbb{1}_{\{|\widehat{\boldsymbol{F}}_n| \le \bar{\varepsilon}n^{1/4}\}}\right).$$

Define

$$\boldsymbol{\kappa}_n(x) := \Phi_n(x) - \widehat{\Phi}_n(x) = \frac{n}{2}\operatorname{Argtanh}\left(\frac{x}{n^{1/4}}\right)^2 + \left(\frac{n}{2}+1\right)\ln\left(1 - \frac{x^2}{\sqrt{n}}\right) + \frac{x^2}{\sqrt{n}} + \frac{x^4}{2n} - \frac{x^4}{12}.$$

The Taylor expansion (3.17) shows that $\boldsymbol{\kappa}_n^{(k)}(0) = 0$ for $k = 0, 1, ..., 5$, hence

$$\boldsymbol{\kappa}_n(x) = \frac{x^6}{5!}\int_0^1 (1-\alpha)^5 \boldsymbol{\kappa}_n^{(6)}(\alpha x)d\alpha.$$

An analysis of $\boldsymbol{\kappa}_n^{(6)}$ with SageMath [87] in the same vein as for the case $\beta < 1$ shows that $\boldsymbol{\kappa}_n^{(6)} \ge 0$ on $(-\bar{\varepsilon}n^{1/4}\bar{\varepsilon}n^{1/4})$, is an odd function and is strictly increasing on $(0, \bar{\varepsilon}n^{1/4})$. As a result, we can write

$$\begin{aligned}\boldsymbol{\delta}_n^{(7)} &:= \int_{-\bar{\varepsilon}n^{1/4}}^{\bar{\varepsilon}n^{1/4}} \left|1 - e^{-\boldsymbol{\kappa}_n}\right| f_{\widehat{\boldsymbol{F}}_n} \\ &= 2\int_0^{\bar{\varepsilon}n^{1/4}} \left(1 - e^{-\boldsymbol{\kappa}_n}\right) f_{\widehat{\boldsymbol{F}}_n} \\ &\le \frac{2}{6!}\boldsymbol{\kappa}_n^{(6)}(\bar{\varepsilon}n^{1/4})\,\mathbb{E}\left(\widehat{\boldsymbol{F}}_n^6\right) \\ &= \frac{2}{6!}\boldsymbol{\kappa}_n^{(6)}(\bar{\varepsilon}n^{1/4})\,\mathbb{E}\left(\boldsymbol{F}^6\right)\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).\end{aligned}$$

Moreover, we can write

$$\boldsymbol{\kappa}_n(x) = n\,W_1\Big(\frac{x}{n^{1/4}}\Big) + W_2\Big(\frac{x}{n^{1/4}}\Big) - \frac{x^4}{12}$$

with W_1, W_2 explicit and infinitely differentiable on $(0, \overline{\varepsilon}n^{1/4})$. As a result,

$$\boldsymbol{\kappa}_n^{(6)}(x) = \frac{1}{\sqrt{n}}\,W_1^{(6)}\Big(\frac{x}{n^{1/4}}\Big) + \frac{1}{n^{3/2}}W_2^{(6)}\Big(\frac{x}{n^{1/4}}\Big)$$

and

$$\boldsymbol{\kappa}_n^{(6)}(\overline{\varepsilon}n^{1/4}) = \frac{1}{\sqrt{n}}\,W_1^{(6)}(\overline{\varepsilon}) + \frac{1}{n^{3/2}}W_2^{(6)}(\overline{\varepsilon})\,.$$

Choosing $\varepsilon = \frac{3}{4}$ for instance gives then

$$\boldsymbol{\kappa}_n^{(6)}(\overline{\varepsilon}n^{1/4}) = \frac{1}{\sqrt{n}}\,W_1^{(6)}\Big(\frac{1}{4}\Big) + O\Big(\frac{1}{n^{3/2}}\Big)$$

and

$$\boldsymbol{\delta}_n^{(7)} \leq \frac{1}{\sqrt{n}} \times \frac{2}{6!}\,W_1^{(6)}\Big(\frac{1}{4}\Big)\,\mathbb{E}\Big(\boldsymbol{F}^6\Big) + O\Big(\frac{1}{n^{3/2}}\Big) =: \frac{K_6}{\sqrt{n}} + O\Big(\frac{1}{n^{3/2}}\Big)\,.$$

Using (3.19) and (3.20), there exists $K_7 > 0$ such that

$$\boldsymbol{\delta}_n^{(6)} \leq \frac{K_7}{\sqrt{n}} + O\Big(\frac{1}{n^{3/2}}\Big)\,. \tag{3.21}$$

In the end, we have

$$\begin{aligned}
\boldsymbol{\delta}_n(h) &\leq \boldsymbol{\delta}_n^{(1)}(h) + \boldsymbol{\delta}_n^{(2)}(h) + \boldsymbol{\delta}_n^{(3)}(h) \\
&\leq \boldsymbol{\delta}_n^{(1)}(h) + \boldsymbol{\delta}_n^{(2)}(h) + \|h'\|_\infty \frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/4}} + \boldsymbol{\delta}_n^{(4)}(h) + \boldsymbol{\delta}_n^{(5)}(h) \\
&\leq \boldsymbol{\delta}_n^{(1)}(h) + \boldsymbol{\delta}_n^{(2)}(h) + \|h'\|_\infty \frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/4}} + \boldsymbol{\delta}_n^{(4)}(h) \\
&\qquad + \|h\|_\infty\left(\boldsymbol{\delta}_n^{(6)} + \frac{\mathbb{E}\Big(\boldsymbol{F}^2\Big)}{\sqrt{n}} + O\Big(\frac{1}{n}\Big) + \frac{\mathbb{E}\Big(\boldsymbol{F}^{4k}\Big)}{(1-\varepsilon)^{4k}\,n^k}\right) \\
&= \boldsymbol{\delta}_n^{(1)}(h) + \boldsymbol{\delta}_n^{(2)}(h) + \|h'\|_\infty \frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/4}} + \boldsymbol{\delta}_n^{(4)}(h) \\
&\qquad + \|h\|_\infty\left(\frac{K_7}{\sqrt{n}} + O\Big(\frac{1}{n^{3/2}}\Big) + \frac{\mathbb{E}\Big(\boldsymbol{F}^2\Big)}{\sqrt{n}} + O\Big(\frac{1}{n}\Big) + \frac{\mathbb{E}\Big(\boldsymbol{F}^{4k}\Big)}{(1-\varepsilon)^{4k}\,n^k}\right) \\
&\leq \frac{\|h'\|_\infty}{\sqrt{2n}}\left(1 + O\Big(\frac{1}{\sqrt{n}}\Big)\right) + \frac{C}{n^{3/4}}\,\|h'\|_\infty + \|h'\|_\infty \frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/4}} + \|h'\|_\infty \frac{\mathbb{E}\Big(\boldsymbol{F}^2\Big) + o(1)}{n^{3/4}} \\
&\qquad + \|h\|_\infty\left(\frac{K_7}{\sqrt{n}} + O\Big(\frac{1}{n^{3/2}}\Big) + \frac{\mathbb{E}\Big(\boldsymbol{F}^2\Big)}{\sqrt{n}} + O\Big(\frac{1}{n}\Big) + \frac{\mathbb{E}\Big(\boldsymbol{F}^{4k}\Big)}{(1-\varepsilon)^{4k}\,n^k}\right),
\end{aligned}$$

hence the result.

□

Remark 3.4. Note that without using the totally dependent coupling for $(S_n(p), S_n(q))$, we could have taken independent random variables and used the bound $\mathbb{E}(|S_n(p) - S_n(q)|) \leq \sqrt{\mathbb{E}((S_n(p) - S_n(q))^2)} = \sqrt{n}\sqrt{\sigma(p)^2 + \sigma(q)^2}$, but this bound does not provide any useful gain. This particular choice of coupling is thus a critical ingredient of the proof.

Remark 3.5. As announced in Remark 3.2, this is the randomisation that dominates the distance in the case $\beta = 1$. This is already visible at the level of the fluctuations, since they are not Gaussian.

3.1.3. *The case* $\beta_n = 1 \pm \frac{\gamma}{\sqrt{n}}$, $\gamma > 0$.

Theorem 3.6 (Fluctuations of the *unnormalised* magnetisation for $\beta_n = 1 - \frac{\gamma}{\sqrt{n}}$, $\gamma \in \mathbb{R}^*$)**.** *Let $\boldsymbol{F}_\gamma$ be a random variable of law given by*

$$\mathbb{P}(\boldsymbol{F}_\gamma \in dx) := \frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}} e^{-\frac{x^4}{12} - \gamma\frac{x^2}{2}} dx, \qquad \mathcal{Z}_{\boldsymbol{F}_\gamma} := \int_{\mathbb{R}} e^{-\frac{x^4}{12} - \gamma\frac{x^2}{2}} dx.$$

Then, for all $h \in \mathcal{C}^1$ with $\|h\|_\infty, \|h'\|_\infty < \infty$

$$\left|\mathbb{E}\left(h\left(\frac{M_n^{(\beta_n)}}{n^{3/4}}\right)\right) - \mathbb{E}(h(\boldsymbol{F}_\gamma))\right| \le \left(\frac{C}{\sqrt{n}} + O\left(\frac{1}{n^{3/4}}\right)\right)\left(\|h\|_\infty + \|h'\|_\infty\right), \tag{3.22}$$

where $C > 0$ is an explicit constant.

Proof. The proof is an adaption of the case $\beta = 1$. Recalling the coupling (3.15) and the notations that follow, we replace $\boldsymbol{F}_n$ by $\boldsymbol{F}_{n,\gamma}$ with law given by

$$\begin{aligned}\mathbb{P}(\boldsymbol{F}_{n,\gamma} \in dt) &= f_{n,\beta_n}\left(\frac{t}{n^{1/4}}\right)\frac{dt}{n^{1/4}} \\ &= \frac{1}{n^{1/4}\,\mathcal{Z}_{n,\beta_n}} e^{-\frac{n}{2\beta_n}\operatorname{Argtanh}\left(\frac{t}{n^{1/4}}\right)^2 - \left(\frac{n}{2}+1\right)\ln\left(1-\frac{t^2}{\sqrt{n}}\right)} \mathbb{1}_{\{|t| \le n^{1/4}\}} dt.\end{aligned} \tag{3.23}$$

By analogy with (3.16), set for $G \sim \mathcal{N}(0,1)$ independent of $M_n^{(\beta_n)}, \boldsymbol{F}_\gamma$

$$\boldsymbol{P} := \frac{1}{2} + \frac{1}{2n^{1/4}}\boldsymbol{F}_{n,\gamma}, \qquad \boldsymbol{Q} := \frac{1}{2} + \frac{1}{2n^{1/4}}\boldsymbol{F}'_{n,\gamma}, \qquad \boldsymbol{F}'_{n,\gamma} - \boldsymbol{F}_{n,\gamma} = -\frac{G}{n^{1/4}}\mathbb{1}_{\{|G| \le \sqrt{n}\}}.$$

Set also $\boldsymbol{\lambda}_n := 2n(\boldsymbol{P} - \boldsymbol{Q}) = n^{3/4}(\boldsymbol{F}'_{n,\gamma} - \boldsymbol{F}_{n,\gamma})$. Analogously to the case $\beta = 1$, we form

$$\begin{aligned}\boldsymbol{\delta}_n^\gamma(h) &:= \left|\mathbb{E}\left(h\left(\frac{S_n(\boldsymbol{P})}{n^{3/4}}\right)\right) - \mathbb{E}(h(\boldsymbol{F}_\gamma))\right| \\ &\le \left|\mathbb{E}\left(h\left(\frac{S_n(\boldsymbol{P})}{n^{3/4}}\right)\right) - \mathbb{E}\left(h\left(\frac{S_n(\boldsymbol{Q}) + \boldsymbol{\lambda}_n}{n^{3/4}}\right)\right)\right| \\ &\qquad + \left|\mathbb{E}\left(h\left(\frac{S_n(\boldsymbol{Q}) + \boldsymbol{\lambda}_n}{n^{3/4}}\right)\right) - \mathbb{E}\left(h\left(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1) + \boldsymbol{\lambda}_n}{n^{3/4}}\right)\right)\right| \\ &\qquad\qquad + \left|\mathbb{E}\left(h\left(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1) + \boldsymbol{\lambda}_n}{n^{3/4}}\right)\right) - \mathbb{E}(h(\boldsymbol{F}_\gamma))\right| \\ &=: \boldsymbol{\delta}_n^{(\gamma,1)}(h) + \boldsymbol{\delta}_n^{(\gamma,2)}(h) + \boldsymbol{\delta}_n^{(\gamma,3)}(h).\end{aligned}$$

Recalling our arguments for $\boldsymbol{\delta}_n^{(1)}(h), \boldsymbol{\delta}_n^{(2)}(h)$ and $\boldsymbol{\delta}_n^{(3)}(h)$ from the previous case we get

$$\begin{aligned}\boldsymbol{\delta}_n^{(\gamma,1)}(h) &\le \frac{\|h'\|_\infty}{n^{3/4}} \times 2\sqrt{n}\sqrt{\mathbb{E}(|\boldsymbol{P} - \boldsymbol{Q}|\,(1 - |\boldsymbol{P} - \boldsymbol{Q}|))} = \frac{\|h'\|_\infty}{\sqrt{2n}}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right), \\ \boldsymbol{\delta}_n^{(\gamma,2)}(h) &\le \frac{C}{n^{3/4}}\|h'\|_\infty, \\ \boldsymbol{\delta}_n^{(\gamma,3)}(h) &\le \|h'\|_\infty \frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/4}} + \boldsymbol{\delta}_n^{(\gamma,4)}(h) + \boldsymbol{\delta}_n^{(\gamma,5)}(h),\end{aligned}$$

where

$$\mathcal{F}_{n,\gamma} := \frac{G}{n^{1/4}}\left(\sqrt{1-\frac{\boldsymbol{F}_{n,\gamma}^2}{\sqrt{n}}}-1\right),$$
$$\boldsymbol{\delta}_n^{(\gamma,4)}(h) := \left|\mathbb{E}(h(\mathcal{F}_{n,\gamma}+\boldsymbol{F}_{n,\gamma}))-\mathbb{E}(h(\boldsymbol{F}_{n,\gamma}))\right|,$$
$$\boldsymbol{\delta}_n^{(\gamma,5)}(h) := \left|\mathbb{E}(h(\boldsymbol{F}_{n,\gamma}))-\mathbb{E}(h(\boldsymbol{F}_{\gamma}))\right|.$$

We have moreover

$$\boldsymbol{\delta}_n^{(\gamma,4)}(h) \le \|h'\|_\infty \frac{\mathbb{E}\left(\boldsymbol{F}_{n,\gamma}^2\right)}{n^{3/4}}.$$

We now show that $\boldsymbol{F}_{n,\gamma}^2 \to \boldsymbol{F}_\gamma^2$ in law and in L^1, implying that $\mathbb{E}\left(\boldsymbol{F}_{n,\gamma}^2\right) = \mathbb{E}\left(\boldsymbol{F}_\gamma^2\right)+o(1)$. Lemma 3.18 gives

$$n^{1/4}\,\mathcal{Z}_{n,\beta_n} = \mathcal{Z}_{\boldsymbol{F}_\gamma} + O\left(\frac{1}{\sqrt{n}}\right)$$

and a Taylor expansion in 0 yields

$$\begin{aligned}\Phi_{n,\gamma}(t) &:= \frac{n}{2\beta_n}\operatorname{Argtanh}\left(\frac{t}{n^{1/4}}\right)^2+\left(\frac{n}{2}+1\right)\ln\left(1-\frac{t^2}{\sqrt{n}}\right)\\ &= \left(\frac{1-\beta_n}{\beta_n}\sqrt{n}-\frac{2}{\sqrt{n}}\right)\frac{t^2}{2}+\frac{t^4}{12}\left(\frac{4-3\beta_n}{\beta_n}-\frac{6}{n}\right)+O\left(\frac{t^6}{n^{3/2}}\right)\\ &= \left(\frac{\gamma}{\beta_n}-\frac{2}{\sqrt{n}}\right)\frac{t^2}{2}+\frac{t^4}{12}\left(\frac{4-3\beta_n}{\beta_n}-\frac{6}{n}\right)+O\left(\frac{t^6}{n^{3/2}}\right).\end{aligned} \tag{3.24}$$

Since $\beta_n \to 1$, this implies the convergence in law $\boldsymbol{F}_{n,\gamma} \to \boldsymbol{F}_\gamma$ by looking at the densities, and the result by square integrability of $\boldsymbol{F}_\gamma$.
We now study $\boldsymbol{\delta}_n^{(\gamma,5)}(h)$. In the same vein as for $\beta = 1$, we have for all $\varepsilon \in (0,1)$ and setting $\overline{\varepsilon} := 1-\varepsilon$,

$$\begin{aligned}\boldsymbol{\delta}_n^{(\gamma,5)}(h) &:= \left|\mathbb{E}(h(\boldsymbol{F}_{n,\gamma}))-\mathbb{E}(h(\boldsymbol{F}_{\gamma}))\right|\\ &\le \int_{(-\overline{\varepsilon}n^{1/4},\overline{\varepsilon}n^{1/4})}|h|\left|f_{\boldsymbol{F}_{n,\gamma}}-f_{\boldsymbol{F}_\gamma}\right|+\left|\mathbb{E}\left(h(\boldsymbol{F}_\gamma)\mathbb{1}_{\{|\boldsymbol{F}_\gamma|>\overline{\varepsilon}n^{1/4}\}}\right)\right|\\ &\le \|h\|_\infty\left(\left\|f_{\boldsymbol{F}_{n,\gamma}}-f_{\boldsymbol{F}_\gamma}\right\|_{L^1([-\overline{\varepsilon}n^{1/4},\overline{\varepsilon}n^{1/4}])}+\mathbb{P}|\boldsymbol{F}_\gamma|>\overline{\varepsilon}n^{1/4}\right)\\ &\le \|h\|_\infty\left(\left\|f_{\boldsymbol{F}_{n,\gamma}}-f_{\boldsymbol{F}_\gamma}\right\|_{L^1([-\overline{\varepsilon}n^{1/4},\overline{\varepsilon}n^{1/4}])}+\frac{\mathbb{E}\left(\boldsymbol{F}_\gamma^{4k}\right)}{(1-\varepsilon)^{4k}\,n^k}\right)\end{aligned}$$

using the triangle inequality and Markov's inequality for all $k \ge 1$. In view of (3.24), we introduce the random variable $\widehat{\boldsymbol{F}}_{n,\gamma}$ defined by the density

$$\begin{aligned}&f_{\widehat{\boldsymbol{F}}_{n,\gamma}}(x) := \frac{1}{\widehat{\mathcal{Z}}_{n,\gamma}}e^{-\widehat{\Phi}_{n,\gamma}(x)}, \qquad \widehat{\mathcal{Z}}_{n,\gamma} := \int_{\mathbb{R}} e^{-\widehat{\Phi}_{n,\gamma}},\\ &\widehat{\Phi}_{n,\gamma}(x) := \alpha_n^{(1)}\frac{x^4}{12}+\alpha_n^{(2)}\frac{x^2}{2}, \quad \alpha_n^{(1)} := \frac{4-3\beta_n}{\beta_n}-\frac{6}{n}, \qquad \alpha_n^{(2)} := \frac{\gamma}{\beta_n}-\frac{2}{\sqrt{n}},\\ &\Phi_\gamma := \widehat{\Phi}_{\infty,\gamma} = \Phi_{\infty,\gamma} : x \mapsto \frac{x^4}{12}+\gamma\frac{x^2}{2}.\end{aligned}$$

Then, we can replace $\boldsymbol{F}_\gamma$ by $\widehat{\boldsymbol{F}}_{n,\gamma}$ up to $O(\frac{1}{\sqrt{n}})$ by writing

$$\left\|f_{\boldsymbol{F}_{n,\gamma}}-f_{\boldsymbol{F}_\gamma}\right\|_{L^1([-\overline{\varepsilon}n^{1/4},\,\overline{\varepsilon}n^{1/4}])} \le \left\|f_{\boldsymbol{F}_{n,\gamma}}-f_{\widehat{\boldsymbol{F}}_{n,\gamma}}\right\|_{L^1([-\overline{\varepsilon}n^{1/4},\,\overline{\varepsilon}n^{1/4}])}+\left\|f_{\widehat{\boldsymbol{F}}_{n,\gamma}}-f_{\boldsymbol{F}_\gamma}\right\|_{L^1(\mathbb{R})}$$

and

$$\begin{aligned}\left\|f_{\widehat{\boldsymbol{F}}_{n,\gamma}}-f_{\boldsymbol{F}_\gamma}\right\|_{L^1(\mathbb{R})}&:=\int_{\mathbb{R}}\left|\frac{1}{\widehat{\mathcal{Z}}_{n,\gamma}}e^{-\widehat{\Phi}_{n,\gamma}}-\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}e^{-\Phi_\gamma}\right|=\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}\int_{\mathbb{R}}\left|\frac{\mathcal{Z}_{\boldsymbol{F}_\gamma}}{\widehat{\mathcal{Z}}_{n,\gamma}}e^{-\widehat{\Phi}_{n,\gamma}+\Phi_\gamma}-1\right|e^{-\Phi_\gamma}\\&=\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}\int_{\mathbb{R}}\left|\left(\frac{\mathcal{Z}_{\boldsymbol{F}_\gamma}}{\widehat{\mathcal{Z}}_{n,\gamma}}-1\right)e^{-\widehat{\Phi}_{n,\gamma}+\Phi_\gamma}+e^{-\widehat{\Phi}_{n,\gamma}+\Phi_\gamma}-1\right|e^{-\Phi_\gamma}\\&\leq\left|\frac{\mathcal{Z}_{\boldsymbol{F}_\gamma}}{\widehat{\mathcal{Z}}_{n,\gamma}}-1\right|\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}\int_{\mathbb{R}}e^{-\widehat{\Phi}_{n,\gamma}+\Phi_\gamma}e^{-\Phi_\gamma}+\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}\int_{\mathbb{R}}\left|e^{-\widehat{\Phi}_{n,\gamma}+\Phi_\gamma}-1\right|e^{-\Phi_\gamma}\\&=\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}\left|\mathcal{Z}_{\boldsymbol{F}_\gamma}-\widehat{\mathcal{Z}}_{n,\gamma}\right|+\mathbb{E}\left(\left|e^{\left(1-\alpha_n^{(1)}\right)\frac{\boldsymbol{F}_\gamma^4}{12}+\left(\gamma-\alpha_n^{(2)}\right)\frac{\boldsymbol{F}_\gamma^2}{2}}-1\right|\right)\\&=\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}\left|\int_{\mathbb{R}}\left(e^{-\widehat{\Phi}_{n,\gamma}}-e^{-\Phi_\gamma}\right)\right|+\mathbb{E}\left(\left|e^{\left(1-\alpha_n^{(1)}\right)\frac{\boldsymbol{F}_\gamma^4}{12}+\left(\gamma-\alpha_n^{(2)}\right)\frac{\boldsymbol{F}_\gamma^2}{2}}-1\right|\right)\\&\leq 2\,\mathbb{E}\left(\left|e^{\left(1-\alpha_n^{(1)}\right)\frac{\boldsymbol{F}_\gamma^4}{12}+\left(\gamma-\alpha_n^{(2)}\right)\frac{\boldsymbol{F}_\gamma^2}{2}}-1\right|\right).\end{aligned}$$

Using

$$\begin{aligned}1-\alpha_n^{(1)}&=\frac{6}{n}+4\left(1-\frac{1}{\beta_n}\right)=\frac{4\gamma}{\sqrt{n}}+O\left(\frac{1}{n}\right),\\\gamma-\alpha_n^{(2)}&=\gamma\left(1-\frac{1}{\beta_n}\right)-\frac{2}{\sqrt{n}}=\frac{\gamma^2-2}{\sqrt{n}}+O\left(\frac{1}{n}\right)\end{aligned}$$

we get

$$\begin{aligned}\mathbb{E}\left(\left|e^{\left(1-\alpha_n^{(1)}\right)\frac{\boldsymbol{F}_\gamma^4}{12}+\left(\alpha_n^{(2)}-\gamma\right)\frac{\boldsymbol{F}_\gamma^2}{2}}-1\right|\right)&=\mathbb{E}\left(\left|1+\frac{4\gamma}{\sqrt{n}}\frac{\boldsymbol{F}_\gamma^4}{12}+\frac{(\gamma^2-2)}{\sqrt{n}}\frac{\boldsymbol{F}_\gamma^2}{2}-1+O\left(\frac{\boldsymbol{F}_\gamma^4}{n}\right)\right|\right)\\&=\frac{2\gamma\mathbb{E}\left(\boldsymbol{F}_\gamma^2\right)+3(\gamma^2-2)\mathbb{E}\left(\boldsymbol{F}_\gamma^4\right)}{6\sqrt{n}}+O\left(\frac{1}{n}\right).\end{aligned}$$

We now estimate the remaining norm:

$$\begin{aligned}\boldsymbol{\delta}_n^{(\gamma,6)}&:=\left\|f_{\boldsymbol{F}_{n,\gamma}}-f_{\widehat{\boldsymbol{F}}_{n,\gamma}}\right\|_{L^1([-\bar\varepsilon n^{1/4},\,\bar\varepsilon n^{1/4}])}\qquad(3.25)\\&=\int_{-\bar\varepsilon n^{1/4}}^{\bar\varepsilon n^{1/4}}\left|f_{\boldsymbol{F}_{n,\gamma}}-f_{\widehat{\boldsymbol{F}}_{n,\gamma}}\right|=\int_{-\bar\varepsilon n^{1/4}}^{\bar\varepsilon n^{1/4}}\left|1-\frac{f_{\boldsymbol{F}_{n,\gamma}}}{f_{\widehat{\boldsymbol{F}}_{n,\gamma}}}\right|f_{\widehat{\boldsymbol{F}}_{n,\gamma}}\\&=\int_{-\bar\varepsilon n^{1/4}}^{\bar\varepsilon n^{1/4}}\left|1-\frac{\widehat{\mathcal{Z}}_{n,\gamma}}{n^{1/4}\mathcal{Z}_{n,\beta_n}}e^{-(\Phi_{n,\gamma}-\widehat{\Phi}_{n,\gamma})}\right|f_{\widehat{\boldsymbol{F}}_{n,\gamma}}\\&=\int_{-\bar\varepsilon n^{1/4}}^{\bar\varepsilon n^{1/4}}\left|1-e^{-(\Phi_{n,\gamma}-\widehat{\Phi}_{n,\gamma})}+\left(1-\frac{\widehat{\mathcal{Z}}_{n,\gamma}}{n^{1/4}\mathcal{Z}_{n,\beta_n}}\right)e^{-(\Phi_{n,\gamma}-\widehat{\Phi}_{n,\gamma})}\right|f_{\widehat{\boldsymbol{F}}_{n,\gamma}}\\&\leq\left|1-\frac{\widehat{\mathcal{Z}}_{n,\gamma}}{n^{1/4}\mathcal{Z}_{n,\beta_n}}\right|\int_{-\bar\varepsilon n^{1/4}}^{\bar\varepsilon n^{1/4}}e^{-(\Phi_{n,\gamma}-\widehat{\Phi}_{n,\gamma})}f_{\widehat{\boldsymbol{F}}_{n,\gamma}}+\int_{-\bar\varepsilon n^{1/4}}^{\bar\varepsilon n^{1/4}}\left|1-e^{-(\Phi_{n,\gamma}-\widehat{\Phi}_{n,\gamma})}\right|f_{\widehat{\boldsymbol{F}}_{n,\gamma}}\\&\leq\left|1-\frac{\widehat{\mathcal{Z}}_{n,\gamma}}{n^{1/4}\mathcal{Z}_{n,\beta_n}}\right|\frac{n^{1/4}\mathcal{Z}_{n,\beta_n}}{\widehat{\mathcal{Z}}_{n,\gamma}}+\int_{-\bar\varepsilon n^{1/4}}^{\bar\varepsilon n^{1/4}}\left|1-e^{-(\Phi_{n,\gamma}-\widehat{\Phi}_{n,\gamma})}\right|f_{\widehat{\boldsymbol{F}}_{n,\gamma}}.\end{aligned}$$

We have in addition

$$\frac{\widehat{\mathcal{Z}}_{n,\gamma}}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}:=\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}\int_{\mathbb{R}}e^{-\widehat{\Phi}_{n,\gamma}}=\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}}\int_{\mathbb{R}}e^{-\alpha_n^{(1)}\frac{x^4}{12}-\alpha_n^{(2)}\frac{x^2}{2}}dx=:\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}\left(\alpha_n^{(1)}\right)^{1/4}}\int_{\mathbb{R}}e^{-\frac{x^4}{12}-\gamma\frac{x^2}{2}-\gamma_n\frac{x^2}{2}}dx$$

$$= \frac{1}{\left(\alpha_n^{(1)}\right)^{1/4}} \mathbb{E}\left(e^{-\gamma_n \boldsymbol{F}_\gamma^2/2}\right) = 1 + \frac{2\gamma - (\gamma^2+2)\mathbb{E}\left(\boldsymbol{F}_\gamma^2\right)}{2\sqrt{n}} + O\left(\frac{1}{n}\right),$$

since

$$\gamma_n := \frac{\alpha_n^{(2)}}{\sqrt{\alpha_n^{(1)}}} - \gamma = \gamma\left(\left(\frac{1}{1+\frac{\gamma}{\sqrt{n}}} + \frac{2}{\gamma\sqrt{n}}\right)\left(\frac{1}{1-\frac{4\gamma}{\sqrt{n}} + O\left(\frac{1}{n}\right)}\right)^{1/2} - 1\right)$$
$$= \frac{\gamma^2+2}{\sqrt{n}} + O\left(\frac{1}{n}\right)$$

and

$$\frac{1}{\left(\alpha_n^{(1)}\right)^{1/4}} = \frac{1}{\left(1-\frac{4\gamma}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right)^{1/4}} = 1 + \frac{\gamma}{\sqrt{n}} + O\left(\frac{1}{n}\right).$$

Lemma 3.18 gives

$$\frac{\mathcal{Z}_{\boldsymbol{F}_\gamma}}{n^{1/4}\mathcal{Z}_{n,\beta_n}} = 1 + O\left(\frac{1}{\sqrt{n}}\right).$$

This implies

$$1 - \frac{\widehat{\mathcal{Z}}_{n,\gamma}}{n^{1/4}\mathcal{Z}_{n,\beta_n}} = 1 - \frac{\widehat{\mathcal{Z}}_{n,\gamma}}{\mathcal{Z}_{\boldsymbol{F}_\gamma}} \frac{\mathcal{Z}_{\boldsymbol{F}_\gamma}}{n^{1/4}\mathcal{Z}_{n,\beta_n}} = O\left(\frac{1}{\sqrt{n}}\right),$$

hence

$$\begin{aligned}\boldsymbol{\delta}_n^{(\gamma,6)} &\leq \left|\frac{n^{1/4}\mathcal{Z}_{n,\beta_n}}{\widehat{\mathcal{Z}}_{n,\gamma}} - 1\right| + \int_{-\overline{\varepsilon}n^{1/4}}^{\overline{\varepsilon}n^{1/4}} \left|1 - e^{-(\Phi_{n,\gamma} - \widehat{\Phi}_{n,\gamma})}\right| f_{\widehat{\boldsymbol{F}}_{n,\gamma}} \\ &\leq \int_{-\overline{\varepsilon}n^{1/4}}^{\overline{\varepsilon}n^{1/4}} \left|1 - e^{-(\Phi_{n,\gamma} - \widehat{\Phi}_{n,\gamma})}\right| f_{\widehat{\boldsymbol{F}}_{n,\gamma}} + O\left(\frac{1}{\sqrt{n}}\right) =: \boldsymbol{\delta}_n^{(\gamma,7)} + O\left(\frac{1}{\sqrt{n}}\right).\end{aligned} \tag{3.26}$$

Define

$$\begin{aligned}\boldsymbol{\kappa}_{n,\gamma}(x) &:= \Phi_{n,\gamma}(x) - \widehat{\Phi}_{n,\gamma}(x) \\ &= \frac{n}{2\beta_n}\operatorname{Argtanh}\left(\frac{x}{n^{1/4}}\right)^2 + \left(\frac{n}{2}+1\right)\ln\left(1 - \frac{x^2}{\sqrt{n}}\right) - \alpha_n^{(1)}\frac{x^4}{12} - \alpha_n^{(2)}\frac{x^2}{2}.\end{aligned}$$

The Taylor expansion (3.24) shows that $\boldsymbol{\kappa}_{n,\gamma}^{(k)}(0) = 0$ for $k = 0, 1, ..., 5$, hence

$$\boldsymbol{\kappa}_{n,\gamma}(x) = \frac{x^6}{5!}\int_0^1 (1-\alpha)^5 \boldsymbol{\kappa}_{n,\gamma}^{(6)}(\alpha x) d\alpha.$$

An analysis of $\boldsymbol{\kappa}_{n,\gamma}^{(6)}$ with SageMath [87] in the same vein as for the other cases shows that $\boldsymbol{\kappa}_{n,\gamma}^{(6)} \geq 0$ on $(-\overline{\varepsilon}n^{1/4}\,\overline{\varepsilon}n^{1/4})$, is an odd function and is strictly increasing on $(0, \overline{\varepsilon}n^{1/4})$. As a result, we can write

$$\begin{aligned}\boldsymbol{\delta}_n^{(\gamma,7)} &:= \int_{-\overline{\varepsilon}n^{1/4}}^{\overline{\varepsilon}n^{1/4}} \left|1 - e^{-\boldsymbol{\kappa}_{n,\gamma}}\right| f_{\widehat{\boldsymbol{F}}_{n,\gamma}} \\ &= 2\int_0^{\overline{\varepsilon}n^{1/4}} \left(1 - e^{-\boldsymbol{\kappa}_{n,\gamma}}\right) f_{\widehat{\boldsymbol{F}}_{n,\gamma}} \\ &\leq \frac{2}{6!}\boldsymbol{\kappa}_{n,\gamma}^{(6)}(\overline{\varepsilon}n^{1/4})\,\mathbb{E}\left(\widehat{\boldsymbol{F}}_{n,\gamma}^6\right) \\ &= \frac{2}{6!}\boldsymbol{\kappa}_{n,\gamma}^{(6)}(\overline{\varepsilon}n^{1/4})\,\mathbb{E}\left(\boldsymbol{F}_\gamma^6\right)\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).\end{aligned}$$

Moreover, we can write

$$\boldsymbol{\kappa}_{n,\gamma}(x) = \frac{n}{\beta_n} W_1\Big(\frac{x}{n^{1/4}}\Big) + n\, W_2\Big(\frac{x}{n^{1/4}}\Big) + W_3\Big(\frac{x}{n^{1/4}}\Big) - \frac{4-3\beta_n}{\beta_n}\frac{x^4}{12} - \frac{\gamma}{\beta_n}\frac{x^2}{2}$$

with W_1, W_2 and W_3 explicit and infinitely differentiable on $(0, \overline{\varepsilon} n^{1/4})$. As a result,

$$\boldsymbol{\kappa}_{n,\gamma}^{(6)}(x) = \frac{1}{\beta_n\sqrt{n}} W_1^{(6)}\Big(\frac{x}{n^{1/4}}\Big) + \frac{1}{\sqrt{n}} W_2^{(6)}\Big(\frac{x}{n^{1/4}}\Big) + \frac{1}{n^{3/2}} W_3^{(6)}\Big(\frac{x}{n^{1/4}}\Big)$$

and

$$\boldsymbol{\kappa}_{n,\gamma}^{(6)}(\overline{\varepsilon} n^{1/4}) = \frac{1}{\beta_n\sqrt{n}} W_1^{(6)}(\overline{\varepsilon}) + \frac{1}{\sqrt{n}} W_2^{(6)}(\overline{\varepsilon}) + \frac{1}{n^{3/2}} W_3^{(6)}(\overline{\varepsilon})\,.$$

Choosing $\varepsilon = \frac{3}{4}$ for instance gives then

$$\begin{aligned}\boldsymbol{\kappa}_{n,\gamma}^{(6)}(\overline{\varepsilon} n^{1/4}) &= \frac{1}{\beta_n\sqrt{n}} W_1^{(6)}\Big(\frac{1}{4}\Big) + \frac{1}{\sqrt{n}} W_2^{(6)}\Big(\frac{1}{4}\Big) + O\Big(\frac{1}{n^{3/2}}\Big)\\ &= \frac{1}{\sqrt{n}}\Big(W_1^{(6)}\Big(\frac{1}{4}\Big) + W_2^{(6)}\Big(\frac{1}{4}\Big)\Big) + O\Big(\frac{1}{n}\Big)\end{aligned}$$

and

$$\boldsymbol{\delta}_n^{(\gamma,7)} \le \frac{1}{\sqrt{n}} \times \frac{2}{6!}\Big(W_1^{(6)}\Big(\frac{1}{4}\Big) + W_2^{(6)}\Big(\frac{1}{4}\Big)\Big)\,\mathbb{E}\big(\boldsymbol{F}_\gamma^6\big) + O\Big(\frac{1}{n}\Big) =: \frac{K_6^\gamma}{\sqrt{n}} + O\Big(\frac{1}{n}\Big)\,.$$

Using (3.25) and (3.26), there exists $K_7^\gamma > 0$ such that

$$\boldsymbol{\delta}_n^{(\gamma,6)} \le \frac{K_7^\gamma}{\sqrt{n}} + O\Big(\frac{1}{n}\Big)\,. \tag{3.27}$$

Collecting all the previous estimates finally gives the desired result.

□

Remark 3.7. It seems interesting to note the discrepancy between the case $\gamma \ge 0$ where the derivative of the function $x \mapsto -\frac{x^4}{12} - \gamma\frac{x^2}{2}$ only vanishes in 0 and the case $\gamma < 0$ where the derivative has two additional zeroes in $\pm\sqrt{-3\gamma}$. As a result, the density $f_{\boldsymbol{F}_\gamma}$ has two humps in this case, which is close to the last case that we will analyse now.

3.1.4. *The case $\beta > 1$.* We consider the transcendent equation

$$\tanh(x) = \frac{x}{\beta}, \qquad \beta > 1. \tag{3.28}$$

An easy study shows that there exist two solutions to this equation denoted by $\pm x_\beta$ with $x_\beta > 1$. We define

$$\boldsymbol{X}_\beta \sim \mathrm{Ber}_{\pm x_\beta}\big(\tfrac{1}{2}\big), \qquad \boldsymbol{B}_\beta \sim \mathrm{Ber}_{\pm m_\beta}\big(\tfrac{1}{2}\big), \qquad m_\beta := \frac{x_\beta}{\beta} = \tanh(x_\beta).$$

Theorem 3.8 (Fluctuations of the *unnormalised* magnetisation for $\beta > 1$)**.** *If $\beta > 1$, we have for all $h \in \mathcal{C}^1$ with $\|h\|_\infty, \|h'\|_\infty < \infty$*

$$\left|\mathbb{E}\Big(h\Big(\frac{M_n^{(\beta)}}{n}\Big)\Big) - \mathbb{E}(h(\boldsymbol{B}_\beta))\right| \le \Big(\frac{C}{\sqrt{n}} + O_\beta\Big(\frac{1}{n}\Big)\Big)\,\|h'\|_\infty \tag{3.29}$$

for an explicit constant $C > 0$.

Proof. Rescaling $M_n^{(\beta)}$ by n and substituting in (2.40) yields

$$\left|\mathbb{E}\left(h\left(\frac{M_n^{(\beta)}}{n}\right)\right) - \mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(\beta)}}{n}\right)\right)\right| \leq \frac{C'}{n}\,\|h'\|_\infty$$

and the triangle inequality implies the following adaptation of (3.2):

$$\left|\mathbb{E}\left(h\left(\frac{M_n^{(\beta)}}{n}\right)\right) - \mathbb{E}(h(\boldsymbol{B}_\beta))\right| \leq \frac{C'}{n}\,\|h'\|_\infty + \left|\mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(\beta)}}{n}\right)\right) - \mathbb{E}(h(\boldsymbol{B}_\beta))\right|. \tag{3.30}$$

Moreover, we have

$$\frac{\mathcal{M}_n^{(\beta)}}{n} = \frac{G}{\sqrt{n}}\sqrt{1-(\boldsymbol{T}_n^{(\beta)})^2} + \boldsymbol{T}_n^{(\beta)}$$

and, since $\boldsymbol{T}_n^{(\beta)} \in [-1,1]$ a.s., we get $\frac{|G|}{\sqrt{n}}\sqrt{1-(\boldsymbol{T}_n^{(\beta)})^2} \leq \frac{|G|}{\sqrt{n}} \to 0$ in law, hence

$$\left|\mathbb{E}\left(h\left(\frac{\mathcal{M}_n^{(\beta)}}{n}\right)\right) - \mathbb{E}(h(\boldsymbol{B}_\beta))\right| \leq \frac{\|h'\|_\infty}{\sqrt{n}} + \left|\mathbb{E}\Big(h\Big(\boldsymbol{T}_n^{(\beta)}\Big) - \mathbb{E}(h(\boldsymbol{B}_\beta))\Big)\right|.$$

It thus remains to show that $\boldsymbol{T}_n^{(\beta)} \xrightarrow{\mathcal{L}} \boldsymbol{B}_\beta$ and to control its norm. For this, remark that

$$\begin{aligned}\left|\mathbb{E}\Big(h\Big(\boldsymbol{T}_n^{(\beta)}\Big) - \mathbb{E}(h(\boldsymbol{B}_\beta))\Big)\right| &= \left|\mathbb{E}\Big(h\Big(\tanh(\boldsymbol{R}_n^{(\beta)})\Big) - \mathbb{E}(h(\tanh(\boldsymbol{X}_\beta)))\Big)\right| \\ &=: \left|\mathbb{E}\Big(\widetilde{h}\Big(\boldsymbol{R}_n^{(\beta)}\Big) - \mathbb{E}\Big(\widetilde{h}(\boldsymbol{X}_\beta)\Big)\Big)\right|, \qquad \widetilde{h} := h\circ\tanh,\end{aligned}$$

where $\boldsymbol{R}_n^{(\beta)}$ has a law given by

$$\mu_{n,\beta}(dy) := \mathbb{P}\Big(\boldsymbol{R}_n^{(\beta)} \in dy\Big) = e^{-n\varphi_\beta(y)}\frac{dy}{\mathcal{Z}_{n,\beta}}, \qquad \varphi_\beta(y) := \frac{y^2}{2\beta} - \log\cosh(y). \tag{3.31}$$

We now adapt the Laplace method, in the easier case of a global minimum, to show that $\mu_{n,\beta} \to \frac{1}{2}(\delta_{m_\beta} + \delta_{-m_\beta})$ weakly. Since $\int_{\mathbb{R}_+} d\mu_{n,\beta} = \int_{\mathbb{R}_-} d\mu_{n,\beta} = \frac{1}{2}$, we have

$$\begin{aligned}\left|\mathbb{E}\Big(\widetilde{h}\Big(\boldsymbol{R}_n^{(\beta)}\Big) - \mathbb{E}\Big(\widetilde{h}(\boldsymbol{X}_\beta)\Big)\Big)\right| &= \int_{\mathbb{R}_+}[\widetilde{h}(y) - \widetilde{h}(x_\beta)]\mu_{n,\beta}(dy) + \int_{\mathbb{R}_-}[\widetilde{h}(y) - \widetilde{h}(-x_\beta)]\mu_{n,\beta}(dy) \\ &=: \delta_n(\widetilde{h}) + \delta_n(\widetilde{h}(-\cdot)).\end{aligned}$$

It is thus enough to treat the case of $\delta_n(\widetilde{h})$. For this, note that (3.28) is equivalent to $\varphi_\beta'(x_\beta) = 0$, hence that for all $x \geq 0$

$$\varphi_\beta(x) = \varphi_\beta(x_\beta) + (x-x_\beta)^2\int_0^1 \varphi_\beta''(\alpha x + \overline{\alpha}x_\beta)\alpha d\alpha, \qquad \overline{\alpha} := 1-\alpha.$$

As a result

$$\begin{aligned}\delta_n(\widetilde{h}) &= e^{-n\varphi_\beta(x_\beta)}\int_{\mathbb{R}_+}[\widetilde{h}(x) - \widetilde{h}(x_\beta)]e^{-n(\varphi_\beta(x)-\varphi_\beta(x_\beta))}\frac{dx}{\mathcal{Z}_{n,\beta}} \\ &= e^{-n\varphi_\beta(x_\beta)}\int_{\mathbb{R}_+}[\widetilde{h}(x) - \widetilde{h}(x_\beta)]e^{-n(x-x_\beta)^2\int_0^1\varphi_\beta''(\alpha x+\overline{\alpha}x_\beta)\alpha d\alpha}\frac{dx}{\mathcal{Z}_{n,\beta}} \\ &= e^{-n\varphi_\beta(x_\beta)}\int_{-x_\beta\sqrt{n}}^{+\infty}\left[\widetilde{h}\left(x_\beta + \frac{w}{\sqrt{n}}\right) - \widetilde{h}(x_\beta)\right]e^{-w^2\int_0^1\varphi_\beta''(\alpha w/\sqrt{n}+x_\beta)\alpha d\alpha}\frac{dw}{\mathcal{Z}_{n,\beta}\sqrt{n}} \\ &\leq e^{-n\varphi_\beta(x_\beta)}\frac{\|\widetilde{h}'\|_\infty}{\sqrt{n}}\int_{-x_\beta\sqrt{n}}^{+\infty}|w|\,e^{-w^2\int_0^1\varphi_\beta''(\alpha w/\sqrt{n}+x_\beta)\alpha d\alpha}\frac{dw}{\mathcal{Z}_{n,\beta}\sqrt{n}}.\end{aligned}$$

As

$$\varphi_\beta'' : x \mapsto -\frac{\beta-1}{\beta} + \tanh(x)^2$$

is bounded and continuous, dominated convergence and continuity imply

$$\int_0^1 \varphi''_\beta(\alpha w/\sqrt{n} + x_\beta)\alpha d\alpha \xrightarrow[n\to+\infty]{} \int_0^1 \varphi''_\beta(x_\beta)\alpha d\alpha = \frac{\varphi''_\beta(x_\beta)}{2}$$

and $\varphi''_\beta(x_\beta) > 0$ by an easy study. It is also easy to see that x_β is the global minimum of φ_β on $\mathbb{R}_+$, hence that $\varphi(x) - \varphi(x_\beta) > 0$ on $\mathbb{R}_+\backslash\{x_\beta\}$. As a result, dominated convergence applies on this set to give

$$\int_{-x_\beta\sqrt{n}}^{+\infty} |w|\, e^{-w^2\int_0^1 \varphi''_\beta(\alpha w/\sqrt{n}+x_\beta)\alpha d\alpha} dw \xrightarrow[n\to+\infty]{} \int_{\mathbb{R}} |w|\, e^{-\frac{w^2}{2}\varphi''(x_\beta)} dw = \sqrt{\frac{2\pi}{\varphi''_\beta(x_\beta)}}\mathbb{E}(|G|)$$

with $G \sim \mathcal{N}(0,1)$.
In the end, we obtain

$$\begin{aligned}\delta_n(\widetilde{h}) &\le \frac{\|\widetilde{h}'\|_\infty}{\sqrt{n}}\left(\sqrt{\frac{2\pi}{\varphi''_\beta(x_\beta)}}\mathbb{E}(|G|) + o(1)\right) \times \frac{e^{-n\varphi_\beta(x_\beta)}}{\sqrt{n}\mathcal{Z}_{n,\beta}} \\ &= \frac{\|\widetilde{h}'\|_\infty}{2\sqrt{n}}(\mathbb{E}(|G|) + o(1))\end{aligned}$$

using Lemma 3.20.
Last, $\|\widetilde{h}'\|_\infty = \left\|h'\circ\tanh \times \tanh'\right\|_\infty \le \|h'\circ\tanh\|_\infty \left\|\tanh'\right\|_\infty = \|h'\|_\infty$ since $\left\|\tanh'\right\|_\infty = 1$. Using $C := C' + 1$ concludes the proof.

□

Remark 3.9. Our results of this section can be rewritten in the Fortet–Mourier distance as follows.

(1) If $\beta < 1$, according to Theorem 3.1 it holds that

$$d_{\mathrm{FM}}\left(\frac{M_n^{(\beta)}}{\sqrt{n}}, \boldsymbol{Z}_\beta\right) \le \frac{C}{\sqrt{n}} + \frac{D(\beta)}{n}$$

for constants $C, D(\beta) > 0$.

(2) If $\beta = 1$, according to Theorem 3.3 it holds that

$$d_{\mathrm{FM}}\left(\frac{M_n^{(1)}}{n^{3/4}}, \boldsymbol{F}_0\right) \le \frac{C}{\sqrt{n}} + \frac{E}{n^{3/4}}$$

for constants $C, E > 0$.

(3) If $\beta = 1 - \frac{\gamma}{\sqrt{n}}$, according to Theorem 3.6 it holds that

$$d_{\mathrm{FM}}\left(\frac{M_n^{(1+\frac{\gamma}{n})}}{n^{3/4}}, \boldsymbol{F}_\gamma\right) \le \frac{C}{\sqrt{n}} + \frac{E(\gamma)}{n^{3/4}}$$

for constants $C, E(\gamma) > 0$.

(4) If $\beta > 1$, according to Theorem 3.8 it holds that

$$d_{\mathrm{FM}}\left(\frac{M_n^{(\beta)}}{n}, \boldsymbol{B}_\beta\right) \le \frac{C}{\sqrt{n}} + \frac{F(\beta)}{n}$$

for constants $C, F(\beta) > 0$.

Note that the original versions of the theorems show the exact dependency on $\|h\|_\infty$ and $\|h'\|_\infty$, which is more precise.

3.2. **Application to the Curie–Weiss magnetisation in Kolmogorov distance.** Recall that for two random variables X, Y, the Kolmogorov distance is defined by

$$d_{\mathrm{Kol}}(X,Y) := \sup_{t\in\mathbb{R}} |\mathbb{P}(X \le t) - \mathbb{P}(Y \le t)| \,. \tag{3.32}$$

It is thus a functional norm in the same vein as the previous one, using test functions $h = \mathbb{1}_{(-\infty,x]}$. The main difference is nevertheless the lack of differentiability of these test functions that prevents the use of (2.36). An extension of Theorem 3.1 to the Kolmogorov distance case requires thus to find an analogue of this inequality for indicator functions. This is furnished by the classical Berry–Esseen bound for sums of i.i.d. random variables, see e.g. [85, Theorem 3.39].

$$d_{\mathrm{Kol}}\Big(S_n,\, \sigma\sqrt{n}\, G + n\mu\Big) = O\bigg(\frac{1}{\sqrt{n}}\bigg) \,. \tag{3.33}$$

Here, $(Z_k)_k$ is a sequence of i.i.d. random variables satisfying $\mathbb{E}\Big(|Z|^3\Big) < \infty$, $S_n := \sum_{k=1}^n Z_k$, $\mathrm{Var}(S_n) =: n\sigma^2$, $\mathbb{E}(S_n) = n\mu$ and $G \sim \mathcal{N}(0,1)$.
Of course, a randomisation of this inequality will give the same result as in subsection 2.3.3, since one has just changed test functions.

3.2.1. *The case $\beta < 1$.*

Theorem 3.10 (Kolmogorov distance to the Gaussian for the *unnormalised* magnetisation for $\beta < 1$)**.** *With $\boldsymbol{Z}_\beta \sim \mathcal{N}\Big(0, \frac{1}{1-\beta}\Big)$, we have*

$$d_{\mathrm{Kol}}\bigg(\frac{M_n^{(\beta)}}{\sqrt{n}}, \boldsymbol{Z}_\beta\bigg) = O\bigg(\frac{1}{\sqrt{n}}\bigg) \,. \tag{3.34}$$

Proof. We use (3.33) in the particular case of Rademacher random variables $(X_k)_k$ of parameter $p := \mathbb{P}(X_1 = 1)$, with $\mathbb{E}(X_1) = 2p-1$ and $\mathrm{Var}(X_1) = 4p(1-p)$, and then randomise p. Taking p distributed as in (2.37), e.g. $\boldsymbol{P}_n^{(\beta)} \sim \widetilde{\nu}_{n,\beta}$, or equivalently taking $t := 2p-1$ distributed as in (2.38), e.g. $\boldsymbol{T}_n^{(\beta)} \sim \nu_{n,\beta}$ yields

$$\begin{aligned} & d_{\mathrm{Kol}}\bigg(M_n^{(\beta)}, G\sqrt{n}\sqrt{1-(\boldsymbol{T}_n^{(\beta)})^2} + n\,\boldsymbol{T}_n^{(\beta)}\bigg) = O\bigg(\frac{1}{\sqrt{n}}\bigg) \\ \iff\; & d_{\mathrm{Kol}}\bigg(\frac{M_n^{(\beta)}}{\sqrt{n}}, \frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}\bigg) = O\bigg(\frac{1}{\sqrt{n}}\bigg) \end{aligned}$$

by invariance of the norm and using the definition of the surrogate $\mathcal{M}_n^{(\beta)}$ given in (2.39). The triangle inequality then yields

$$\begin{aligned} d_{\mathrm{Kol}}\bigg(\frac{M_n^{(\beta)}}{\sqrt{n}}, \boldsymbol{Z}_\beta\bigg) &\le d_{\mathrm{Kol}}\bigg(\frac{M_n^{(\beta)}}{\sqrt{n}}, \frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}\bigg) + d_{\mathrm{Kol}}\bigg(\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}, \boldsymbol{Z}_\beta\bigg) \\ &= d_{\mathrm{Kol}}\bigg(\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}, \boldsymbol{Z}_\beta\bigg) + O\bigg(\frac{1}{\sqrt{n}}\bigg) \end{aligned}$$

and we are led to analyse

$$d_{\mathrm{Kol}}\bigg(\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}, \boldsymbol{Z}_\beta\bigg) =: \sup_{x\in\mathbb{R}} \delta_n(x)$$

with

$$\delta_n(x) := \bigg|\mathbb{P}\bigg(\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}} \le x\bigg) - \mathbb{P}(\boldsymbol{Z}_\beta \le x)\bigg| \,.$$

Recall that $\boldsymbol{Z}_\beta \overset{d}{=} G + G_\beta$ with $G \sim \mathcal{N}(0,1)$ independent of $G_\beta \sim \mathcal{N}(0, \beta/(1-\beta))$. We thus have

$$\delta_n(x) = \left| \mathbb{P}\left(G\sqrt{1 - \frac{(\boldsymbol{X}_n^{(\beta)})^2}{n}} + \boldsymbol{X}_n^{(\beta)} \leq x \right) - \mathbb{P}(G + G_\beta \leq x) \right|$$

with $\boldsymbol{X}_{n,\beta} := \sqrt{n}\, \boldsymbol{T}_n^{(\beta)} \to G_\beta$ (in law).
Integrating on G and using $\Phi(x) := \mathbb{P}(G \leq x)$, $\boldsymbol{X}_{n,\beta} \overset{d}{=} -\boldsymbol{X}_{n,\beta}$ and $G_\beta \overset{d}{=} -G_\beta$ yields

$$\delta_n(x) = \left| \mathbb{E}\left(\Phi\left(\frac{x + \boldsymbol{X}_n^{(\beta)}}{\sqrt{1 - \frac{(\boldsymbol{X}_n^{(\beta)})^2}{n}}} \right) \right) - \mathbb{E}(\Phi(x + G_\beta)) \right|.$$

Set

$$\begin{aligned} Y_x &:= x + G_\beta, \\ \Psi_{x,n}(u) &:= \frac{x+u}{\sqrt{1 - \frac{u^2}{n}}}, \\ \Xi_n(u) &:= \frac{1}{\sqrt{1 - \frac{u^2}{n}}}, \\ Y_{x,n} &:= \Psi_{x,n}\left(\boldsymbol{X}_n^{(\beta)}\right), \\ \mathcal{Y}_{x,n} &:= \Psi_{x,n}(G_\beta) = Y_x\, \Xi_n(G_\beta). \end{aligned}$$

Recall that the support of the law of $\boldsymbol{X}_n^{(\beta)}$ is $(-\sqrt{n}, \sqrt{n})$. We have

$$\begin{aligned} \delta_n(x) &= |\mathbb{E}(\Phi(Y_x)) - \mathbb{E}(\Phi(Y_{x,n}))| \\ &\leq \left| \mathbb{E}\Big(\Phi(Y_x)\, \mathbb{1}_{\{|G_\beta| \leq \sqrt{n}\}} \Big) - \mathbb{E}\Big(\Phi(\mathcal{Y}_{x,n})\, \mathbb{1}_{\{|G_\beta| \leq \sqrt{n}\}} \Big) \right| \\ &\qquad + \left| \mathbb{E}\Big(\Phi(\mathcal{Y}_{x,n})\, \mathbb{1}_{\{|G_\beta| \leq \sqrt{n}\}} \Big) - \mathbb{E}(\Phi(Y_{x,n})) \right| + \left| \mathbb{E}\Big(\Phi(Y_x)\, \mathbb{1}_{\{|G_\beta| > \sqrt{n}\}} \Big) \right| \\ &=: \delta_n^{(1)}(x) + \delta_n^{(2)}(x) + \delta_n^{(3)}(x). \end{aligned}$$

Since $\Phi(x) := \mathbb{P}(G \leq x) \leq 1$, Markov's inequality gives for all $k \geq 1$

$$\delta_n^{(3)}(x) \leq \mathbb{P}\Big(|G_\beta| > \sqrt{n} \Big) \leq \frac{\mathbb{E}\Big(|G_\beta|^{2k} \Big)}{n^k}.$$

Moreover, using the notation g_β (resp. g_n) for the Lebesgue density of G_β (resp. $\boldsymbol{X}_n^{(\beta)}$) as in the proof of Theorem 3.1, we get

$$\begin{aligned} \delta_n^{(2)}(x) &= \left| \mathbb{E}\Big(\Phi \circ \Psi_{x,n}(G_\beta) \mathbb{1}_{\{|G_\beta| \leq \sqrt{n}\}} \Big) - \mathbb{E}\left(\Phi \circ \Psi_{x,n}(\boldsymbol{X}_n^{(\beta)}) \mathbb{1}_{\left\{\left|\boldsymbol{X}_n^{(\beta)}\right| \leq \sqrt{n}\right\}} \right) \right| \\ &= \left| \int_{(-\sqrt{n}, \sqrt{n})} \Phi \circ \Psi_{x,n} \cdot (g_\beta - g_n) \right| \\ &\leq \|\Phi \circ \Psi_{x,n}\|_\infty \left\| (g_\beta - g_n) \mathbb{1}_{(-\sqrt{n}, \sqrt{n})} \right\|_{L^1(\mathbb{R})} \\ &=: \|\Phi \circ \Psi_{x,n}\|_\infty\, \delta_n(g_n, g_\beta). \end{aligned}$$

Since $0 \leq \Phi \leq 1$, we have $\sup_{x \in \mathbb{R}} \|\Phi \circ \Psi_{x,n}\|_\infty \leq 1$, and (3.9) yields then for all $x \in \mathbb{R}$

$$\delta_n^{(2)}(x) = O_\beta\left(\frac{1}{n}\right).$$

We now estimate $\delta_n^{(1)}(x)$. Setting $\varphi := \Phi(\cdot + x) - \Phi \circ \Psi_{x,n}$, we get

$$\begin{aligned}
\delta_n^{(1)}(x) &= \Big|\mathbb{E}\Big(\Phi(Y_x)\mathbb{1}_{\{|G_\beta|\le\sqrt{n}\}}\Big) - \mathbb{E}\Big(\Phi(\mathcal{Y}_x)\mathbb{1}_{\{|G_\beta|\le\sqrt{n}\}}\Big)\Big| \\
&= \Big|\mathbb{E}\Big(\big(\Phi(x+G_\beta) - \Phi\circ\Psi_{x,n}(G_\beta)\big)\,\mathbb{1}_{\{|G_\beta|\le\sqrt{n}\}}\Big)\Big| \\
&=: \Big|\mathbb{E}\Big(\varphi(G_\beta)\Big(\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}} + \mathbb{1}_{\{(1-\varepsilon)\sqrt{n}\le|G_\beta|\le\sqrt{n}\}}\Big)\Big)\Big| \\
&\le \Big|\mathbb{E}\Big(\varphi(G_\beta)\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big)\Big| + \|\varphi\|_\infty\,\mathbb{P}\Big((1-\varepsilon)\sqrt{n}\le|G_\beta|\Big) \\
&\le \Big|\mathbb{E}\Big(\varphi(G_\beta)\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big)\Big| + 2\,\frac{\mathbb{E}\Big(|G_\beta|^{2k}\Big)}{n^k(1-\varepsilon)^{2k}}
\end{aligned}$$

for all $\varepsilon \in (0,1)$, using Markov's inequality and $\|\varphi\|_\infty \le 2$.
Recall that $\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \ge 0$ for all $x \in \mathbb{R}$ and that $\mathcal{Y}_{n,x} = Y_x\,\Xi_n(G_\beta)$. We have then with $U \sim \mathcal{U}([0,1])$ independent of G_β

$$\begin{aligned}
\delta_n^{(4)}(x) &:= \Big|\mathbb{E}\Big(\varphi(G_\beta)\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big)\Big| = \Big|\mathbb{E}\Big(\big(\Phi(Y_x) - \Phi(\mathcal{Y}_{n,x})\big)\,\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big)\Big| \\
&\le \mathbb{E}\Big(|\mathcal{Y}_{x,n} - Y_x|\,\Phi'(Y_x + U(\mathcal{Y}_{x,n} - Y_x))\,\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big) \\
&= \mathbb{E}\Big(|1-\Xi_n(G_\beta)| \times |Y_x|\,\Phi'(Y_x + Y_xU(\Xi_n(G_\beta)-1))\,\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big) \\
&= \frac{1}{\sqrt{2\pi}}\,\mathbb{E}\Big(\Big(\Xi_n(G_\beta)-1\Big)\times|Y_x|\,e^{-\frac{Y_x^2}{2}(1+U(\Xi_n(G_\beta)-1))^2}\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big) \\
&\le \frac{1}{\sqrt{2\pi}}\,\mathbb{E}\Big(\Big(\Xi_n(G_\beta)-1\Big)\times|Y_x|\,e^{-\frac{Y_x^2}{2}}\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big)
\end{aligned}$$

since on $\{|G_\beta| \le (1-\varepsilon)\sqrt{n}\}$, it is $\Xi_n(G) - 1 = (1 - G^2/n)^{-1/2} - 1 \ge 0$. In particular,

$$\begin{aligned}
\delta_n^{(4)}(x) &\le \frac{1}{\sqrt{2\pi}}\,\mathbb{E}\Big(\Big(\Xi_n(G_\beta)-1\Big)\times\sup_{x\in\mathbb{R}}\{|Y_x|\,e^{-\frac{Y_x^2}{2}}\}\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big) \\
&= \frac{1}{\sqrt{2\pi}}\,\mathbb{E}\Big(\Big(\Xi_n(G_\beta)-1\Big)\,\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big)\times\sup_{y\in\mathbb{R}}\{|y|\,e^{-\frac{y^2}{2}}\} \\
&= \frac{1}{\sqrt{2\pi e}}\mathbb{E}\Big(\Big(\Xi_n(G_\beta)-1\Big)\,\mathbb{1}_{\{|G_\beta|\le(1-\varepsilon)\sqrt{n}\}}\Big)
\end{aligned}$$

since $Y_x = x + G_\beta$ and the supremum of the function $y \mapsto |y|\,e^{-y^2/2}$ is easily seen to be reached uniquely in $y = 1$.
Last, the function $x \mapsto \Xi_n(\sqrt{n}x) - 1$ is clearly integrable on $(-1+\varepsilon, 1-\varepsilon)$, and as a result, using $0 \le \frac{1}{\sqrt{1-t}} - 1 \le t\,\sup_{|u|\le 1-\varepsilon}\left|\frac{d}{du}\frac{1}{\sqrt{1-u}}\right| = \frac{t}{2\varepsilon^{3/2}}$ and $\mathbb{E}\Big(G_\beta^2\Big) = \frac{\beta}{1-\beta}$, we finally get for all $x \in \mathbb{R}$

$$\delta_n^{(4)}(x) \le \frac{\beta}{4\varepsilon^{3/2}(1-\beta)\sqrt{2\pi e}} \times \frac{1}{n}.$$

In the end, we obtain

$$\begin{aligned}
\delta_n(x) &\le \delta_n^{(1)}(x) + \delta_n^{(2)}(x) + \delta_n^{(3)}(x) \\
&\le \delta_n^{(4)}(x) + 2\,\frac{\mathbb{E}\Big(|G_\beta|^{2k}\Big)}{n^k(1-\varepsilon)^{2k}} + \delta_n^{(2)}(x) + \delta_n^{(3)}(x)
\end{aligned}$$

$$\leq \frac{\beta}{4\varepsilon^{3/2}(1-\beta)\sqrt{2\pi e}} \times \frac{1}{n} + 2\,\frac{\mathbb{E}\Big(|G_\beta|^{2k}\Big)}{n^k(1-\varepsilon)^{2k}} + O_\beta\Big(\frac{1}{n}\Big) + \frac{\mathbb{E}\Big(|G_\beta|^{2k}\Big)}{n^k}$$
$$= O_\beta\Big(\frac{1}{n}\Big)$$

having choosen $\varepsilon = \frac{3}{4}$ for instance. Taking the supremum over $x \in \mathbb{R}$ ends the proof. □

Remark 3.11. Analogously with the case of a smooth norm analysed in Remark 3.2, we see that $d_{\mathrm{Kol}}\Big(\frac{\mathcal{M}_n^{(\beta)}}{\sqrt{n}}, \boldsymbol{Z}_\beta\Big) = O_\beta\big(\frac{1}{n}\big)$ which is faster than the speed coming from the sum of i.i.d.s, e.g. from the CLT. Here again, we can check in Kolmogorov distance the phenomenon of "CLT domination" over the randomisation.

3.2.2. *The case* $\beta = 1$.

Theorem 3.12 (Kolmogorov distance to $\boldsymbol{F}$ for the *unnormalised* magnetisation for $\beta = 1$)**.** *Let* $\boldsymbol{F}$ *be a random variable of law given in Theorem 3.3. Then,*

$$d_{\mathrm{Kol}}\Big(\frac{M_n^{(1)}}{n^{3/4}}, \boldsymbol{F}\Big) = O\Big(\frac{1}{\sqrt{n}}\Big) \tag{3.35}$$

Proof. We use (3.33) in the particular case of Rademacher random variables $(X_k)_k$ of parameter $p := \mathbb{P}(X_1 = 1)$, with $\mathbb{E}(X_1) = 2p-1$ and $\mathrm{Var}(X_1) = 4p(1-p)$ and then randomise p. Taking p distributed as in (2.37), e.g. $\boldsymbol{P}_n^{(1)} \sim \widetilde{\nu}_{n,1}$, or equivalently taking $t := 2p-1$ distributed as in (2.38), e.g. $\boldsymbol{T}_n^{(1)} \sim \nu_{n,1}$ yields

$$d_{\mathrm{Kol}}\Big(M_n^{(1)}, G\sqrt{n}\sqrt{1-(\boldsymbol{T}_n^{(1)})^2} + n\,\boldsymbol{T}_n^{(1)}\Big) = O\Big(\frac{1}{\sqrt{n}}\Big)$$
$$\iff d_{\mathrm{Kol}}\Big(\frac{M_n^{(1)}}{n^{3/4}}, \frac{\mathcal{M}_n^{(1)}}{n^{3/4}}\Big) = O\Big(\frac{1}{\sqrt{n}}\Big) \tag{3.36}$$

by invariance of the norm and using the definition of the surrogate $\mathcal{M}_n^{(1)}$ given in (2.39). With the coupling (3.15) and the notations that follow, we get

$$\begin{aligned}
\boldsymbol{\delta}_n(x) &:= \Big|\mathbb{P}\Big(\frac{S_n(\boldsymbol{P})}{n^{3/4}} \leq x\Big) - \mathbb{P}(\boldsymbol{F} \leq x)\Big| \\
&\leq \Big|\mathbb{P}\Big(\frac{S_n(\boldsymbol{P})}{n^{3/4}} \leq x\Big) - \mathbb{P}\Big(\frac{S_n(\boldsymbol{Q}) + \boldsymbol{\lambda}_n}{n^{3/4}} \leq x\Big)\Big| \\
&\quad + \Big|\mathbb{P}\Big(\frac{S_n(\boldsymbol{Q}) + \boldsymbol{\lambda}_n}{n^{3/4}} \leq x\Big) - \mathbb{P}\Big(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1) + \boldsymbol{\lambda}_n}{n^{3/4}} \leq x\Big)\Big| \\
&\quad + \Big|\mathbb{P}\Big(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1) + \boldsymbol{\lambda}_n}{n^{3/4}} \leq x\Big) - \mathbb{P}(\boldsymbol{F} \leq x)\Big| \\
&=: \boldsymbol{\delta}_n^{(1)}(x) + \boldsymbol{\delta}_n^{(2)}(x) + \boldsymbol{\delta}_n^{(3)}(x).
\end{aligned}$$

Bound on $\boldsymbol{\delta}_n^{(2)}(x)$: The Berry–Esseen bound (3.36) gives

$$\begin{aligned}
\boldsymbol{\delta}_n^{(2)}(x) &\leq \sup_{x\in\mathbb{R}} \boldsymbol{\delta}_n^{(2)}(x) = \sup_{y\in\mathbb{R}} \Big|\mathbb{P}\Big(\frac{S_n(\boldsymbol{Q})}{n^{3/4}} \leq y\Big) - \mathbb{P}\Big(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1)}{n^{3/4}} \leq y\Big)\Big| \\
&= O\Big(\frac{1}{\sqrt{n}}\Big).
\end{aligned}$$

Bound on $\boldsymbol{\delta}_n^{(3)}(x)$: One has

$$\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1)+\boldsymbol{\lambda}_n}{n^{3/4}} = \frac{G}{n^{1/4}}\left(\sqrt{1-\frac{\boldsymbol{F}_n^2}{\sqrt{n}}}-1\right)+\boldsymbol{F}_n+\frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}.$$

Similarly to the proof of Theorem 3.3, set

$$\begin{aligned}
\mathcal{F}_n &:= \frac{G}{n^{1/4}}\left(\sqrt{1-\frac{\boldsymbol{F}_n^2}{\sqrt{n}}}-1\right),\\
\boldsymbol{\delta}_n^{(4)}(x) &:= \left|\mathbb{P}\left(\mathcal{F}_n+\boldsymbol{F}_n+\frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}\le x\right)-\mathbb{P}(\mathcal{F}_n+\boldsymbol{F}_n\le x)\right|,\\
\boldsymbol{\delta}_n^{(5)}(x) &:= |\mathbb{P}(\mathcal{F}_n+\boldsymbol{F}_n\le x)-\mathbb{P}(\boldsymbol{F}_n\le x)|\,,\\
\boldsymbol{\delta}_n^{(6)}(x) &:= |\mathbb{P}(\boldsymbol{F}_n\le x)-\mathbb{P}(\boldsymbol{F}\le x)|\,,
\end{aligned}$$

so that

$$\begin{aligned}
\boldsymbol{\delta}_n^{(3)}(x) &= \left|\mathbb{P}\left(\mathcal{F}_n+\boldsymbol{F}_n+\frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}\le x\right)-\mathbb{P}(\boldsymbol{F}\le x)\right|\\
&\le \boldsymbol{\delta}_n^{(4)}(x)+\boldsymbol{\delta}_n^{(5)}(x)+\boldsymbol{\delta}_n^{(6)}(x).
\end{aligned}$$

Integrating on $\boldsymbol{F}_n$ and using $F_{\boldsymbol{F}_n}(x) := \mathbb{P}(\boldsymbol{F}_n\le x)=\int_{-\infty}^x f_{\boldsymbol{F}_n}$ and $\mathcal{F}_n \overset{d}{=} -\mathcal{F}_n$ yields

$$\begin{aligned}
\boldsymbol{\delta}_n^{(5)}(x) &= |\mathbb{E}(F_{\boldsymbol{F}_n}(x+\mathcal{F}_n))-\mathbb{E}(F_{\boldsymbol{F}_n}(x))|\\
&\le \|f_{\boldsymbol{F}_n}\|_\infty\,\mathbb{E}(|\mathcal{F}_n|)\\
&\le \|f_{\boldsymbol{F}_n}\|_\infty\,\frac{\mathbb{E}(|G|)\,\mathbb{E}\left(\boldsymbol{F}_n^2\right)}{n^{3/4}}\\
&\le \left(\frac{1}{\mathcal{Z}_{\boldsymbol{F}}}+O\left(\frac{1}{\sqrt{n}}\right)\right)\frac{\mathbb{E}\left(\boldsymbol{F}^2\right)+o(1)}{n^{3/4}},
\end{aligned}$$

using Lemma 3.17.
Similarly, using $G\overset{d}{=}-G$, we get

$$\boldsymbol{\delta}_n^{(4)}(x) = \left|\mathbb{E}\left(F_{\boldsymbol{F}_n}\left(x+\mathcal{F}_n+\frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}\right)\right)-\mathbb{E}(F_{\boldsymbol{F}_n}(x))\right|$$

and

$$\begin{aligned}
\boldsymbol{\delta}_n^{(4)}(x) &= \left|\mathbb{E}\left(F_{\boldsymbol{F}_n}\left(x+\mathcal{F}_n+\frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}\right)\right)-\mathbb{E}(F_{\boldsymbol{F}_n}(x))\right|\\
&\le \|f_{\boldsymbol{F}_n}\|_\infty\,\mathbb{E}\left(\left|\mathcal{F}_n+\frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}\right|\right)\\
&\le \|f_{\boldsymbol{F}_n}\|_\infty\left(\mathbb{E}(|\mathcal{F}_n|)+\frac{\sqrt{\mathbb{E}\left(|G|^2\right)}}{n^{1/4}}\sqrt{\mathbb{P}\left(|G|>\sqrt{n}\right)}\right)\\
&\le \|f_{\boldsymbol{F}_n}\|_\infty\left(\frac{\mathbb{E}\left(\boldsymbol{F}_n^2\right)}{n^{3/4}}+\frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/4}}\right)\\
&\le \left(\frac{1}{\mathcal{Z}_{\boldsymbol{F}}}+O\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{\mathbb{E}\left(\boldsymbol{F}^2\right)+o(1)}{n^{3/4}}+\frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/4}}\right)
\end{aligned}$$

by Lemma 3.17 and the Cauchy–Schwarz- and Markov's inequality.

We now study $\boldsymbol{\delta}_n^{(6)}(x)$. In the same vein as for $\beta < 1$, we have for all $\varepsilon \in (0,1)$ and setting $\overline{\varepsilon} := 1 - \varepsilon$

$$\begin{aligned}
\boldsymbol{\delta}_n^{(6)}(x) &:= |\mathbb{P}(\boldsymbol{F}_n \leq x) - \mathbb{P}(\boldsymbol{F} \leq x)| = \left| \int_{-\infty}^{x} (f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}) \right| \\
&\leq \int_{\mathbb{R}} |f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}| \\
&\leq \int_{(-\overline{\varepsilon} n^{1/4}, \overline{\varepsilon} n^{1/4})} |f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}| + \int_{\mathbb{R} \setminus [-\overline{\varepsilon} n^{1/4}, \overline{\varepsilon} n^{1/4}]} |f_{\boldsymbol{F}_n}| + \int_{\mathbb{R} \setminus [-\overline{\varepsilon} n^{1/4}, \overline{\varepsilon} n^{1/4}]} |f_{\boldsymbol{F}}| \\
&\leq \|f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}\|_{L^1([-\overline{\varepsilon} n^{1/4}, \overline{\varepsilon} n^{1/4}])} + \mathbb{P}\Big(|\boldsymbol{F}_n| > \overline{\varepsilon} n^{1/4}\Big) + \mathbb{P}\Big(|\boldsymbol{F}| > \overline{\varepsilon} n^{1/4}\Big) \\
&\leq \|f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}\|_{L^1([-\overline{\varepsilon} n^{1/4}, \overline{\varepsilon} n^{1/4}])} + \frac{\mathbb{E}\Big(\boldsymbol{F}_n^{4k}\Big)}{(1-\varepsilon)^{4k} n^k} + \frac{\mathbb{E}\Big(\boldsymbol{F}^{4k}\Big)}{(1-\varepsilon)^{4k} n^k} \\
&\leq \|f_{\boldsymbol{F}_n} - f_{\boldsymbol{F}}\|_{L^1([-\overline{\varepsilon} n^{1/4}, \overline{\varepsilon} n^{1/4}])} + \frac{2\mathbb{E}\Big(\boldsymbol{F}^{4k}\Big) + o(1)}{(1-\varepsilon)^{4k} n^k}
\end{aligned}$$

using the triangle inequality and Markov's inequality for all $k \geq 1$. We then conclude with (3.18) and (3.21).

<u>**Bound on $\boldsymbol{\delta}_n^{(1)}(x)$:**</u> Recall that $\boldsymbol{\lambda}_n = 2n(\boldsymbol{P} - \boldsymbol{Q})$. Then,

$$\begin{aligned}
\sup_{x \in \mathbb{R}} \boldsymbol{\delta}_n^{(1)}(x) &= \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{S_n(\boldsymbol{P})}{n^{3/4}} \leq x \right) - \mathbb{P}\left(\frac{S_n(\boldsymbol{Q}) + \boldsymbol{\lambda}_n}{n^{3/4}} \leq x \right) \right| \\
&= \sup_{x \in \mathbb{R}} \Big| \mathbb{P}\Big(S_n(\boldsymbol{P}) - n(2\boldsymbol{P} - 1) \leq n^{3/4} x - n(2\boldsymbol{P} - 1) \Big) \\
&\qquad\qquad - \mathbb{P}\Big(S_n(\boldsymbol{Q}) - n(2\boldsymbol{Q} - 1) \leq n^{3/4} x - n(2\boldsymbol{P} - 1) \Big) \Big| \\
&\leq \mathbb{E}\Bigg(\sup_{x \in \mathbb{R}} \Big| \mathbb{P}\Big(S_n(\boldsymbol{P}) - n(2\boldsymbol{P} - 1) \leq n^{3/4} x - n(2\boldsymbol{P} - 1) \Big| \boldsymbol{P} \Big) \\
&\qquad\qquad - \mathbb{P}\Big(S_n(\boldsymbol{Q}) - n(2\boldsymbol{Q} - 1) \leq n^{3/4} x - n(2\boldsymbol{P} - 1) \Big| \boldsymbol{P}, \boldsymbol{Q} \Big) \Big| \Bigg) \\
&= \mathbb{E}\Bigg(\sup_{y \in \mathbb{R}} \Big| \mathbb{P}\Big(S_n(\boldsymbol{P}) - n(2\boldsymbol{P} - 1) \leq y \Big| \boldsymbol{P} \Big) - \mathbb{P}\Big(S_n(\boldsymbol{Q}) - n(2\boldsymbol{Q} - 1) \leq y \Big| \boldsymbol{Q} \Big) \Big| \Bigg) \\
&\leq \mathbb{E}\left(|\boldsymbol{P} - \boldsymbol{Q}| \left| \frac{1 - (\boldsymbol{P} + \boldsymbol{Q})}{\boldsymbol{P}(1 - \boldsymbol{P})} \right| \right) + O\left(\frac{1}{\sqrt{n}} \right)
\end{aligned}$$

by Lemma 3.21.
We recall the following definitions from the proof of Theorem 3.3:

- $\boldsymbol{Q} = \dfrac{1}{2} + \dfrac{1}{2n^{1/4}} \boldsymbol{F}_n$,
- $\boldsymbol{P} = \dfrac{1}{2} + \dfrac{1}{2n^{1/4}} \boldsymbol{F}_n'$,
- $2(\boldsymbol{P} - \boldsymbol{Q}) = \dfrac{1}{n^{1/4}} (\boldsymbol{F}_n' - \boldsymbol{F}_n) = -\dfrac{G}{\sqrt{n}} \mathbb{1}_{\{|G| \leq \sqrt{n}\}}$.

As a result

$$\sup_{x\in\mathbb{R}} \boldsymbol{\delta}_n^{(1)}(x) \le \frac{1}{4n^{3/4}}\mathbb{E}\left(|G|\left|\frac{\boldsymbol{F}_n+\boldsymbol{F}_n'}{\left(\frac{1}{2}+\frac{1}{2n^{1/4}}\boldsymbol{F}_n'\right)\left(\frac{1}{2}-\frac{1}{2n^{1/4}}\boldsymbol{F}_n'\right)}\right|\right)+O\left(\frac{1}{\sqrt{n}}\right)$$
$$=\frac{1}{n^{3/4}}\mathbb{E}\left(|G|\left|\frac{\boldsymbol{F}_n+\boldsymbol{F}_n'}{1-\frac{1}{\sqrt{n}}\boldsymbol{F}_n'^2}\right|\right)+O\left(\frac{1}{\sqrt{n}}\right)$$
$$\le \frac{1}{n^{3/4}}\left(\mathbb{E}\left(\left(\frac{\boldsymbol{F}+\boldsymbol{F}'}{1-\frac{1}{\sqrt{n}}\boldsymbol{F}^2}\right)^2\right)+o(1)\right)^{1/2}+O\left(\frac{1}{\sqrt{n}}\right)$$
$$\le \frac{\sqrt{2\,\mathbb{E}\left(\boldsymbol{F}^2\right)}}{n^{3/4}}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)+O\left(\frac{1}{\sqrt{n}}\right)=O\left(\frac{1}{\sqrt{n}}\right).$$

In the end, we obtain

$$\delta_n(x)\le \delta_n^{(1)}(x)+\delta_n^{(2)}(x)+\delta_n^{(3)}(x)$$
$$\le O\left(\frac{1}{\sqrt{n}}\right)+O\left(\frac{1}{\sqrt{n}}\right)+\delta_n^{(4)}(x)+\delta_n^{(5)}(x)+\delta_n^{(6)}(x)$$
$$\le O\left(\frac{1}{\sqrt{n}}\right)+\left(\frac{1}{\mathcal{Z}_{\boldsymbol{F}}}+O\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{2\mathbb{E}\left(\boldsymbol{F}^2\right)+o(1)}{n^{3/4}}+\frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/2}}\right)+\frac{\mathbb{E}\left(\boldsymbol{F}^{4k}\right)}{(1-\varepsilon)^{4k}\,n^k}$$
$$=O\left(\frac{1}{\sqrt{n}}\right)$$

which concludes the proof. □

3.2.3. *The case* $\beta_n = 1\pm\frac{\gamma}{\sqrt{n}}$, $\gamma>0$.

Theorem 3.13 (Kolmogorov distance to $\boldsymbol{F}_\gamma$ for the *unnormalised* magnetisation for $\beta_n = 1-\frac{\gamma}{\sqrt{n}}$, $\gamma\in\mathbb{R}^*$)**.** *Let* $\boldsymbol{F}_\gamma$ *be a random variable of law given in Theorem 3.6. Then,*

$$d_{\mathrm{Kol}}\left(\frac{M_n^{(\beta_n)}}{n^{3/4}},\boldsymbol{F}_\gamma\right)=O\left(\frac{1}{\sqrt{n}}\right). \tag{3.37}$$

Proof. Starting with the approach (3.33) with $\beta=\beta_n$ yields

$$d_{\mathrm{Kol}}\left(\frac{M_n^{(\beta_n)}}{n^{3/4}},\frac{\mathcal{M}_n^{(\beta_n)}}{n^{3/4}}\right)=O\left(\frac{1}{\sqrt{n}}\right), \tag{3.38}$$

where the surrogate $\mathcal{M}_n^{(\beta_n)}$ is given in (2.39). With the coupling (3.15) and the notations in the proof of theorem 3.6, we get

$$\boldsymbol{\delta}_n^{\gamma}(x) := \left|\mathbb{P}\left(\frac{S_n(\boldsymbol{P})}{n^{3/4}}\le x\right)-\mathbb{P}(\boldsymbol{F}_\gamma\le x)\right|$$
$$\le \left|\mathbb{P}\left(\frac{S_n(\boldsymbol{P})}{n^{3/4}}\le x\right)-\mathbb{P}\left(\frac{S_n(\boldsymbol{Q})+\boldsymbol{\lambda}_n}{n^{3/4}}\le x\right)\right|$$
$$+\left|\mathbb{P}\left(\frac{S_n(\boldsymbol{Q})+\boldsymbol{\lambda}_n}{n^{3/4}}\le x\right)-\mathbb{P}\left(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G+n(2\boldsymbol{Q}-1)+\boldsymbol{\lambda}_n}{n^{3/4}}\le x\right)\right|$$
$$+\left|\mathbb{P}\left(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G+n(2\boldsymbol{Q}-1)+\boldsymbol{\lambda}_n}{n^{3/4}}\le x\right)-\mathbb{P}(\boldsymbol{F}_\gamma\le x)\right|$$

$$=: \boldsymbol{\delta}_n^{(\gamma,1)}(x) + \boldsymbol{\delta}_n^{(\gamma,2)}(x) + \boldsymbol{\delta}_n^{(\gamma,3)}(x).$$

Bound on $\boldsymbol{\delta}_n^{(\gamma,2)}(x)$: The Berry–Esseen bound (3.38) gives

$$\boldsymbol{\delta}_n^{(\gamma,2)}(x) \le \sup_{x\in\mathbb{R}} \boldsymbol{\delta}_n^{(\gamma,2)}(x) = \sup_{y\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S_n(\boldsymbol{Q})}{n^{3/4}} \le y \right) - \mathbb{P}\left(\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1)}{n^{3/4}} \le y \right) \right|$$
$$= O\left(\frac{1}{\sqrt{n}}\right).$$

Bound on $\boldsymbol{\delta}_n^{(\gamma,3)}(x)$: We have

$$\frac{2\,\boldsymbol{Q}(1-\boldsymbol{Q})\sqrt{n}\,G + n(2\boldsymbol{Q}-1) + \boldsymbol{\lambda}_n}{n^{3/4}} = \frac{G}{n^{1/4}}\left(\sqrt{1 - \frac{\boldsymbol{F}_{n,\gamma}^2}{\sqrt{n}}} - 1\right) + \boldsymbol{F}_{n,\gamma} + \frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}}.$$

Similarly to the proof of Theorem 3.6, set

$$\mathcal{F}_{n,\gamma} := \frac{G}{n^{1/4}}\left(\sqrt{1 - \frac{\boldsymbol{F}_{n,\gamma}^2}{\sqrt{n}}} - 1\right),$$
$$\boldsymbol{\delta}_n^{(\gamma,4)}(x) := \left| \mathbb{P}\left(\mathcal{F}_{n,\gamma} + \boldsymbol{F}_{n,\gamma} + \frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}} \le x \right) - \mathbb{P}(\mathcal{F}_{n,\gamma} + \boldsymbol{F}_{n,\gamma} \le x) \right|,$$
$$\boldsymbol{\delta}_n^{(\gamma,5)}(x) := \left| \mathbb{P}(\mathcal{F}_{n,\gamma} + \boldsymbol{F}_{n,\gamma} \le x) - \mathbb{P}(\boldsymbol{F}_{n,\gamma} \le x) \right|,$$
$$\boldsymbol{\delta}_n^{(\gamma,6)}(x) := \left| \mathbb{P}(\boldsymbol{F}_{n,\gamma} \le x) - \mathbb{P}(\boldsymbol{F}_{\gamma} \le x) \right|,$$

so that

$$\boldsymbol{\delta}_n^{(\gamma,3)}(x) = \left| \mathbb{P}\left(\mathcal{F}_{n,\gamma} + \boldsymbol{F}_{n,\gamma} + \frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}} \le x \right) - \mathbb{P}(\boldsymbol{F}_\gamma \le x) \right|$$
$$\le \boldsymbol{\delta}_n^{(\gamma,4)}(x) + \boldsymbol{\delta}_n^{(\gamma,5)}(x) + \boldsymbol{\delta}_n^{(\gamma,6)}(x).$$

Integrating on $\boldsymbol{F}_{n,\gamma}$ and using $F_{\boldsymbol{F}_{n,\gamma}}(x) := \mathbb{P}(\boldsymbol{F}_{n,\gamma} \le x) = \int_{-\infty}^x f_{\boldsymbol{F}_{n,\gamma}}$ and $\mathcal{F}_{n,\gamma} \overset{d}{=} -\mathcal{F}_{n,\gamma}$ yields

$$\boldsymbol{\delta}_n^{(\gamma,5)}(x) = \left| \mathbb{E}\Big(F_{\boldsymbol{F}_{n,\gamma}}(x + \mathcal{F}_{n,\gamma})\Big) - \mathbb{E}\Big(F_{\boldsymbol{F}_{n,\gamma}}(x)\Big) \right|$$
$$\le \left\| f_{\boldsymbol{F}_{n,\gamma}} \right\|_\infty \mathbb{E}(|\mathcal{F}_{n,\gamma}|)$$
$$\le \left\| f_{\boldsymbol{F}_{n,\gamma}} \right\|_\infty \frac{\mathbb{E}(|G|)\,\mathbb{E}\Big(\boldsymbol{F}_{n,\gamma}^2\Big)}{n^{3/4}}$$
$$\le \left(\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}} + O\left(\frac{1}{\sqrt{n}}\right) \right) \frac{\mathbb{E}\Big(\boldsymbol{F}_\gamma^2\Big) + o(1)}{n^{3/4}}.$$

Here, we have used Lemma 3.19 for the last inequality. Similarly, using $G \overset{d}{=} -G$, we get

$$\boldsymbol{\delta}_n^{(\gamma,4)}(x) = \left| \mathbb{E}\left(F_{\boldsymbol{F}_{n,\gamma}}\left(x + \mathcal{F}_{n,\gamma} + \frac{G}{n^{1/4}}\mathbb{1}_{\{|G|>\sqrt{n}\}} \right) \right) - \mathbb{E}\Big(F_{\boldsymbol{F}_{n,\gamma}}(x)\Big) \right|$$
$$\le \left(\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}} + O\left(\frac{1}{\sqrt{n}}\right) \right) \left(\frac{\mathbb{E}\Big(\boldsymbol{F}_\gamma^2\Big) + o(1)}{n^{3/4}} + \frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/4}} \right)$$

by the same arguments we used in the previous proof.
We now study $\boldsymbol{\delta}_n^{(\gamma,6)}(x)$. In the same vein as for $\beta = 1$, we have for all $\varepsilon \in (0,1)$ and setting $\overline{\varepsilon} := 1 - \varepsilon$

$$\boldsymbol{\delta}_n^{(\gamma,6)}(x) := \left| \mathbb{P}(\boldsymbol{F}_{n,\gamma} \le x) - \mathbb{P}(\boldsymbol{F}_\gamma \le x) \right|$$

$$
\begin{aligned}
&= \left| \int_{-\infty}^{x} (f_{\boldsymbol{F}_{n,\gamma}} - f_{\boldsymbol{F}_\gamma}) \right| \\
&\leq \int_{\mathbb{R}} \left| f_{\boldsymbol{F}_{n,\gamma}} - f_{\boldsymbol{F}_\gamma} \right| \\
&\leq \left\| f_{\boldsymbol{F}_{n,\gamma}} - f_{\boldsymbol{F}_\gamma} \right\|_{L^1([-\bar{\varepsilon} n^{1/4}, \bar{\varepsilon} n^{1/4}])} + \frac{2\mathbb{E}\left(\boldsymbol{F}_\gamma^{4k}\right) + o(1)}{(1-\varepsilon)^{4k}\, n^k}
\end{aligned}
$$

using the triangle inequality and Markov's inequality for all $k \geq 1$. We then conclude with (3.25) and (3.27).

Bound on $\boldsymbol{\delta}_n^{(\gamma,1)}(x)$: Recall that $\boldsymbol{\lambda}_n = 2n(\boldsymbol{P} - \boldsymbol{Q}) = n^{3/4}(\boldsymbol{F}'_{n,\gamma} - \boldsymbol{F}_{n,\gamma})$. Then,

$$
\begin{aligned}
\sup_{x\in\mathbb{R}} \boldsymbol{\delta}_n^{(\gamma,1)}(x) &= \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S_n(\boldsymbol{P})}{n^{3/4}} \leq x \right) - \mathbb{P}\left(\frac{S_n(\boldsymbol{Q}) + \boldsymbol{\lambda}_n}{n^{3/4}} \leq x \right) \right| \\
&\leq \mathbb{E}\left(|\boldsymbol{P} - \boldsymbol{Q}| \left| \frac{1 - (\boldsymbol{P} + \boldsymbol{Q})}{\boldsymbol{P}(1-\boldsymbol{P})} \right| \right) + O\left(\frac{1}{\sqrt{n}} \right)
\end{aligned}
$$

by Lemma 3.21.

We recall the following definitions from the proof of Theorem 3.6:

- $\boldsymbol{Q} = \dfrac{1}{2} + \dfrac{1}{2n^{1/4}} \boldsymbol{F}_{n,\gamma}$,
- $\boldsymbol{P} = \dfrac{1}{2} + \dfrac{1}{2n^{1/4}} \boldsymbol{F}'_{n,\gamma}$,
- $2(\boldsymbol{P} - \boldsymbol{Q}) = \dfrac{1}{n^{1/4}} \left(\boldsymbol{F}'_{n,\gamma} - \boldsymbol{F}_{n,\gamma} \right) = -\dfrac{G}{\sqrt{n}} \mathbb{1}_{\{|G| \leq \sqrt{n}\}}$.

As a result

$$
\begin{aligned}
\sup_{x\in\mathbb{R}} \boldsymbol{\delta}_n^{(\gamma,1)}(x) &\leq \frac{1}{4n^{3/4}} \mathbb{E}\left(|G| \left| \frac{\boldsymbol{F}_{n,\gamma} + \boldsymbol{F}'_{n,\gamma}}{\left(\frac{1}{2} + \frac{1}{2n^{1/4}} \boldsymbol{F}'_{n,\gamma} \right)\left(\frac{1}{2} - \frac{1}{2n^{1/4}} \boldsymbol{F}'_{n,\gamma} \right)} \right| \right) + O\left(\frac{1}{\sqrt{n}} \right) \\
&\leq \frac{1}{n^{3/4}} \left(\mathbb{E}\left(\left(\frac{\boldsymbol{F}_\gamma + \boldsymbol{F}'_\gamma}{1 - \frac{1}{\sqrt{n}} \boldsymbol{F}_\gamma^2} \right)^2 \right) + o(1) \right)^{1/2} + O\left(\frac{1}{\sqrt{n}} \right) \\
&\leq \frac{\sqrt{2\,\mathbb{E}\left(\boldsymbol{F}_\gamma^2\right)}}{n^{3/4}} \left(1 + O\left(\frac{1}{\sqrt{n}} \right) \right) + O\left(\frac{1}{\sqrt{n}} \right) = O\left(\frac{1}{\sqrt{n}} \right).
\end{aligned}
$$

In the end, we obtain

$$
\begin{aligned}
\delta_n^\gamma(x) &\leq \delta_n^{(\gamma,1)}(x) + \delta_n^{(\gamma,2)}(x) + \delta_n^{(\gamma,3)}(x) \\
&\leq O\left(\frac{1}{\sqrt{n}} \right) + O\left(\frac{1}{\sqrt{n}} \right) + \delta_n^{(\gamma,4)}(x) + \delta_n^{(\gamma,5)}(x) + \delta_n^{(\gamma,6)}(x) \\
&\leq O\left(\frac{1}{\sqrt{n}} \right) + \left(\frac{1}{\mathcal{Z}_{\boldsymbol{F}_\gamma}} + O\left(\frac{1}{\sqrt{n}} \right) \right) \left(\frac{2\mathbb{E}\left(\boldsymbol{F}_\gamma^2\right) + o(1)}{n^{3/4}} + \frac{\sqrt{\mathbb{E}(G^{2k})}}{n^{k+1/2}} \right) + \frac{\mathbb{E}\left(\boldsymbol{F}_\gamma^{4k}\right)}{(1-\varepsilon)^{4k}\, n^k} \\
&= O\left(\frac{1}{\sqrt{n}} \right)
\end{aligned}
$$

which concludes the proof.

∎

3.2.4. *The case $\beta > 1$.* Recall that $\pm x_\beta$ are the solution to the transcendent equation (3.28) and that $m_\beta = \tanh(x_\beta)$, with $\boldsymbol{X}_\beta \sim \mathrm{Ber}_{\pm x_\beta}\left(\frac{1}{2}\right)$ and $\boldsymbol{B}_\beta \sim \mathrm{Ber}_{\pm m_\beta}\left(\frac{1}{2}\right)$.

Theorem 3.14 (Fluctuations of the *unnormalised* magnetisation for $\beta > 1$)**.** *If $\beta > 1$, we have*

$$d_{\mathrm{Kol}}\left(\frac{M_n^{(\beta)}}{n}, \boldsymbol{B}_\beta\right) = O\left(\frac{1}{\sqrt{n}}\right) \tag{3.39}$$

for an explicit constant $C > 0$.

Proof. Using (3.33) and the invariance of d_{Kol} gives

$$\begin{aligned} & d_{\mathrm{Kol}}\left(M_n^{(\beta)}, G\sqrt{n}\sqrt{1-(\boldsymbol{T}_n^{(\beta)})^2} + n\,\boldsymbol{T}_n^{(\beta)}\right) = O\left(\frac{1}{\sqrt{n}}\right) \\ \iff & d_{\mathrm{Kol}}\left(\frac{M_n^{(\beta)}}{n}, \frac{\mathcal{M}_n^{(\beta)}}{n}\right) = O\left(\frac{1}{\sqrt{n}}\right) \end{aligned} \tag{3.40}$$

and the triangle inequality then implies

$$\begin{aligned} d_{\mathrm{Kol}}\left(\frac{M_n^{(\beta)}}{n}, \boldsymbol{B}_\beta\right) &\le d_{\mathrm{Kol}}\left(\frac{M_n^{(\beta)}}{n}, \frac{\mathcal{M}_n^{(\beta)}}{n}\right) + d_{\mathrm{Kol}}\left(\frac{\mathcal{M}_n^{(\beta)}}{n}, \boldsymbol{T}_n^{(\beta)}\right) + d_{\mathrm{Kol}}\left(\boldsymbol{T}_n^{(\beta)}, \boldsymbol{B}_\beta\right) \\ &= d_{\mathrm{Kol}}\left(\frac{\mathcal{M}_n^{(\beta)}}{n}, \boldsymbol{T}_n^{(\beta)}\right) + d_{\mathrm{Kol}}\left(\boldsymbol{T}_n^{(\beta)}, \boldsymbol{B}_\beta\right) + O\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

We have

$$\begin{aligned} d_{\mathrm{Kol}}\left(\frac{\mathcal{M}_n^{(\beta)}}{n}, \boldsymbol{T}_n^{(\beta)}\right) &= d_{\mathrm{Kol}}\left(\boldsymbol{T}_n^{(\beta)} + \frac{G}{\sqrt{n}}\sqrt{1-(\boldsymbol{T}_n^{(\beta)})^2},\, \boldsymbol{T}_n^{(\beta)}\right) \\ &= \sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\boldsymbol{T}_n^{(\beta)} + \frac{G}{\sqrt{n}}\sqrt{1-(\boldsymbol{T}_n^{(\beta)})^2} \le x\right) - \mathbb{P}\left(\boldsymbol{T}_n^{(\beta)} \le x\right)\right| \\ &= \sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{G}{\sqrt{n}} \le \mathcal{Y}_{n,x,\beta}\right) - \mathbb{P}(0 \le \mathcal{Y}_{n,x,\beta})\right| = \sup_{x\in\mathbb{R}}\mathbb{P}\left(0 \le \mathcal{Y}_{n,x,\beta} \le \frac{G}{\sqrt{n}}\right) \end{aligned}$$

with

$$\begin{aligned} \mathcal{Y}_{n,x,\beta} &:= \frac{x - \boldsymbol{T}_n^{(\beta)}}{\sqrt{1-(\boldsymbol{T}_n^{(\beta)})^2}} = x\cosh\left(\boldsymbol{R}_n^{(\beta)}\right) - \sinh\left(\boldsymbol{R}_n^{(\beta)}\right) \\ &= \begin{cases} \sqrt{1-x^2}\sinh\left(\mathrm{Argtanh}(x) - \boldsymbol{R}_n^{(\beta)}\right) & \text{if } |x| < 1 \\ s_x e^{-s_x \boldsymbol{R}_n^{(\beta)}} & \text{if } |x| = 1 \\ \sqrt{x^2-1}\cosh\left(\mathrm{Argtanh}(x^{-1}) - \boldsymbol{R}_n^{(\beta)}\right) & \text{if } |x| > 1, \end{cases} \end{aligned}$$

s_x the sign of x. Thus, for $\varepsilon > 0$ small enough so that $\min_{|x-1|<\varepsilon}(x - m_\beta)_+ > 0$, it is

$$d_{\mathrm{Kol}}\left(\frac{\mathcal{M}_n^{(\beta)}}{n}, \boldsymbol{T}_n^{(\beta)}\right) \le \sup_{|x-1|\le\varepsilon}\mathbb{P}\left(0 \le \mathcal{Y}_{n,x,\beta} \le \frac{G}{\sqrt{n}}\right) + \sup_{|x-1|>\varepsilon}\mathbb{P}\left(0 \le \mathcal{Y}_{n,x,\beta} \le \frac{G}{\sqrt{n}}\right).$$

Then

$$\begin{aligned} \sup_{|x-1|>\varepsilon}\mathbb{P}\left(0 \le \mathcal{Y}_{n,x,\beta} \le \frac{G}{\sqrt{n}}\right) &\le \mathbb{P}\left(\frac{\sqrt{\varepsilon}}{2}e^{\mathrm{Argtanh}(\varepsilon) - \boldsymbol{R}_n^{(\beta)}} \le \frac{G}{\sqrt{n}}\right) + \mathbb{P}\left(\sqrt{2\varepsilon} \le \frac{G}{\sqrt{n}}\right) \\ &= O\left(e^{-nC\varepsilon}\right), \qquad C > 0, \end{aligned}$$

and

$$\sup_{|x-1|<\varepsilon}\mathbb{P}\left(0 \le \mathcal{Y}_{n,x,\beta} \le \frac{G}{\sqrt{n}}\right) \le \sup_{|x-1|<\varepsilon}\mathbb{P}\left((\mathcal{Y}_{n,x,\beta})_+ \le \frac{G}{\sqrt{n}}\right)$$

$$
\begin{aligned}
&\le \sup_{|x-1|<\varepsilon} \mathbb{E}\Big(e^{G}\Big)\,\mathbb{E}\Big(e^{-\sqrt{n}(x\cosh(\boldsymbol{R}_n^{(\beta)})-\sinh(\boldsymbol{R}_n^{(\beta)}))_+}\Big)\\
&\le \sqrt{e}\sup_{|x-1|<\varepsilon} e^{-\sqrt{n}[(x\cosh(x_\beta)-\sinh(x_\beta))_+ + o(1)]}\\
&= \sqrt{e}\,\exp\Big(-\sqrt{n}\cosh(x_\beta)\Big(\min_{|x-1|<\varepsilon}(x-\tanh(x_\beta))_+ + o(1)\Big)\Big)\\
&= O\Big(e^{-C_\varepsilon\sqrt{n}}\Big), \qquad C_\varepsilon>0.
\end{aligned}
$$

It then remains to analyse

$$
\begin{aligned}
d_{\mathrm{Kol}}\Big(\boldsymbol{T}_n^{(\beta)},\boldsymbol{B}_\beta\Big) &:= \sup_{x\in\mathbb{R}}\Big|\mathbb{P}\Big(\boldsymbol{T}_n^{(\beta)}\le x\Big)-\mathbb{P}(\boldsymbol{B}_\beta\le x)\Big|\\
&= \sup_{x\in\mathbb{R}}\Big|\mathbb{P}\Big(\tanh(\boldsymbol{R}_n^{(\beta)})\le x\Big)-\mathbb{P}(\tanh(\boldsymbol{X}_\beta)\le x)\Big|\\
&= \sup_{y\in\mathbb{R}}\Big|\mathbb{P}\Big(\boldsymbol{R}_n^{(\beta)}\le y\Big)-\mathbb{P}(\boldsymbol{X}_\beta\le y)\Big|\\
&= \sup_{y\in\mathbb{R}}\left|\int_{-\infty}^{y} e^{-n\varphi_\beta(x)}\frac{dx}{\mathcal{Z}_{n,\beta}}-\frac{1}{2}\Big(\mathbb{1}_{\{y\ge -x_\beta\}}+\mathbb{1}_{\{y\ge x_\beta\}}\Big)\right|.
\end{aligned}
$$

Since $\max_{A\cup B} f=\max\{\max_A f,\max_B f\}$, it is enough to consider the following quantities:

$$
\begin{aligned}
I_+(y) &:= \int_{y}^{+\infty} e^{-n\varphi_\beta(x)}\frac{dx}{\mathcal{Z}_{n,\beta}}, \qquad y>x_\beta,\\
I_0(y) &:= \int_{-\infty}^{y} e^{-n\varphi_\beta(x)}\frac{dx}{\mathcal{Z}_{n,\beta}}-\frac{1}{2}=\int_{0}^{y} e^{-n\varphi_\beta(x)}\frac{dx}{\mathcal{Z}_{n,\beta}}, \qquad -x_\beta<y\le x_\beta,\\
I_-(y) &:= \int_{-\infty}^{y} e^{-n\varphi_\beta(x)}\frac{dx}{\mathcal{Z}_{n,\beta}}=I_+(-y), \qquad y<-x_\beta.
\end{aligned}
$$

By symmetry, it is enough to consider the case $0<y<x_\beta$ in the case of $I_0(y)$. We thus have for $y\in(0,x_\beta)$

$$
\begin{aligned}
I_0(y) &= \int_0^y e^{-n\varphi_\beta(x)}\frac{dx}{\mathcal{Z}_{n,\beta}} \le y\frac{e^{-n\varphi_\beta(y)}}{\mathcal{Z}_{n,\beta}} = O\Big(\sqrt{n}\,e^{-n(\varphi_\beta(x_\beta-\varepsilon)-\varphi_\beta(x_\beta))}\Big) \text{ with Lemma 3.20}\\
&= O\Big(\sqrt{n}\,e^{-n\varepsilon^2\varphi_\beta''(x_\beta)/2}\Big)
\end{aligned}
$$

having set $y:=x_\beta-\varepsilon$ with $\varepsilon>0$.
In the case of $I_+(y)$, $y=x_\beta+\varepsilon$, Markov's inequality gives

$$
I_+(x_\beta+\varepsilon)=\mathbb{P}\Big(\boldsymbol{R}_n^{(\beta)}\ge x_\beta+\varepsilon\Big)\le\frac{1}{\varepsilon}\mathbb{E}\Big(\Big|\boldsymbol{R}_n^{(\beta)}-x_\beta\Big|\Big)=O\Big(\frac{1}{\sqrt{n}}\Big)
$$

using the computations at the end of the proof of Theorem 3.8. This concludes the proof. ∎

3.3. **Appendix: Analysis of diverse constants.**

3.3.1. *Renormalisation constants.*

Lemma 3.15 (Asymptotic analysis of $\mathcal{Z}_{n,\beta}$ for $\beta < 1$)**.** *We have*

$$\left|\sqrt{\frac{C_\beta}{2\pi}} \times \mathcal{Z}_{n,\beta}\sqrt{n} - 1\right| \leq \frac{1}{n}\frac{C_\beta^{-9/2}}{4}.$$

Proof. Using (2.33), we have

$$\mathcal{Z}_{n,\beta} := \int_{(-1,1)} e^{-\frac{n}{2}F_\beta(t)}\frac{dt}{1-t^2}, \qquad F_\beta(t) := \frac{1}{\beta}\operatorname{Argtanh}(t)^2 + \ln(1-t^2).$$

Setting $x = \operatorname{Argtanh}(t)$ and $y := \sqrt{n}x$ gives

$$\mathcal{Z}_{n,\beta} := \int_{\mathbb{R}} e^{-\frac{n}{2}F_\beta(\tanh(x))}dx = \int_{\mathbb{R}} e^{-\frac{n}{2}F_\beta(\tanh(y/\sqrt{n}))}\frac{dy}{\sqrt{n}}$$

with

$$\begin{aligned}\frac{n}{2}F_\beta(\tanh(y/\sqrt{n})) &= \frac{y^2}{2\beta} + \frac{n}{2}\log\Big(1-\tanh(y/\sqrt{n})^2\Big)\\ &= \frac{y^2}{2}\left(\frac{1}{\beta}-1\right) + \frac{y^2}{2} + \frac{n}{2}\log\Big(1-\tanh(y/\sqrt{n})^2\Big)\\ &=: \frac{y^2}{2}C_\beta + \psi_n(y),\end{aligned}$$

where

$$\begin{aligned}C_\beta &:= \frac{1}{\beta} - 1,\\ \psi_n(y) &:= \frac{y^2}{2} + \frac{n}{2}\log\left(1-\tanh(y/\sqrt{n})^2\right) = \frac{y^2}{2} - n\log\cosh\Big(\tfrac{y}{\sqrt{n}}\Big) =: n\psi\left(\frac{y}{\sqrt{n}}\right), \qquad (3.41)\\ \psi(y) &:= \frac{y^2}{2} - \log\cosh(y)\end{aligned}$$

since $1-\tanh^2 = \cosh^{-2}$. The equality

$$\int_{\mathbb{R}} e^{-\frac{y^2}{2}C_\beta}dy = \sqrt{\frac{2\pi}{C_\beta}}$$

implies that

$$\begin{aligned}\sqrt{\frac{2\pi}{C_\beta}} - \mathcal{Z}_{n,\beta}\sqrt{n} &= \int_{\mathbb{R}} e^{-\frac{y^2}{2}C_\beta}dy - \int_{\mathbb{R}} e^{-\frac{n}{2}F_\beta(\tanh(y/\sqrt{n}))}dy\\ &= \int_{\mathbb{R}}\Big(1-e^{-\psi_n(y)}\Big)e^{-\frac{y^2}{2}C_\beta}dy.\end{aligned}$$

The study of ψ_n with SageMath [87] shows that ψ_n is non negative on $\mathbb{R}$ with only cancelation in 0. This can also be seen with the inequality $\cosh(t) \leq \exp(\frac{t^2}{2})$ that follows from the termwise comparison of the Taylor series of each function. As a result, the previous quantity is positive on $\mathbb{R}^*$. Moreover, in the same vein as for κ_n defined in (3.7), we have $\psi_n(0) = \psi_n'(0) = \psi_n''(0) = \psi_n'''(0) = 0$, and the Taylor formula with integral remainder gives at the fourth order

$$\psi_n(y) = \frac{y^4}{6}\int_0^1 (1-\alpha)^3\psi_n^{(4)}(\alpha y)d\alpha.$$

A computation with SageMath [87] gives

- $\psi_n'(y) = y - \sqrt{n}\tanh\left(\frac{y}{\sqrt{n}}\right)$
- $\psi_n''(y) = \tanh\left(\frac{y}{\sqrt{n}}\right)^2$
- $\psi_n'''(y) = \frac{2}{\sqrt{n}}\tanh\left(\frac{y}{\sqrt{n}}\right)\left(1 - \tanh\left(\frac{y}{\sqrt{n}}\right)^2\right)$
- $\psi_n^{(4)}(y) = \frac{2}{n}\left(1 - \tanh\left(\frac{y}{\sqrt{n}}\right)^2\right)\left(1 - 3\tanh\left(\frac{y}{\sqrt{n}}\right)^2\right).$

Moreover, the function $y \mapsto n\log(1 - \tanh(y/\sqrt{n})^2) = -2n\log\cosh(y/\sqrt{n})$ has bounded derivatives. We thus have

$$0 \le 1 - e^{-\psi_n(y)} \le \psi_n(y) \le \frac{y^4}{24}\left\|\psi_n^{(4)}\right\|_\infty .$$

Since $\left|\psi_n^{(4)}(y)\right| := \frac{2}{n}\left|1 - \tanh(\frac{y}{\sqrt{n}})^2\right|\left|1 - 3\tanh(\frac{y}{\sqrt{n}})^2\right| \le \frac{2}{n}$, we get

$$\begin{aligned} 0 \le \sqrt{\frac{2\pi}{C_\beta}} - \mathcal{Z}_{n,\beta}\sqrt{n} &= \int_{\mathbb{R}}\left(1 - e^{-\psi_n(y)}\right)e^{-\frac{y^2}{2}C_\beta}dy \\ &\le \frac{2}{24\,n}\int_{\mathbb{R}} y^4 e^{-\frac{y^2}{2}C_\beta}dy = \frac{1}{4\,n}\sqrt{2\pi}C_\beta^{-5} \end{aligned}$$

hence the result using the fourth moment of a Gaussian (equal to 3). ☐

Lemma 3.16 (Asymptotic analysis of $\mathcal{Z}_{n,1}$)**.** *We have*

$$\left|n^{1/4}\frac{\mathcal{Z}_{n,1}}{\mathcal{Z}_{\boldsymbol{F}}} - 1\right| = O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. Using (2.33), we have

$$\mathcal{Z}_{n,1} := \int_{(-1,1)} e^{-\frac{n}{2}F_1(t)}\frac{dt}{1-t^2}, \qquad F_1(t) := \operatorname{Argtanh}(t)^2 + \ln(1-t^2).$$

Setting $x = \operatorname{Argtanh}(t)$ and $y := n^{1/4}x$ gives

$$\mathcal{Z}_{n,1} := \int_{\mathbb{R}} e^{-\frac{n}{2}F_1(\tanh(x))}dx = \int_{\mathbb{R}} e^{-\frac{n}{2}F_1(\tanh(y/n^{1/4}))}\frac{dy}{n^{1/4}} =: \int_{\mathbb{R}} e^{-n\,\psi(y/n^{1/4})}\frac{dy}{n^{1/4}}$$

with ψ defined in (3.41).

Since $\mathcal{Z}_{\boldsymbol{F}} = \int_{\mathbb{R}} e^{-\frac{y^4}{12}}dy$, it is

$$\begin{aligned} n^{1/4}\mathcal{Z}_{n,1} - \mathcal{Z}_{\boldsymbol{F}} &= \int_{\mathbb{R}} e^{-n\,\psi(y/n^{1/4})}dy - \int_{\mathbb{R}} e^{-\frac{y^4}{12}}dy \\ &= \int_{\mathbb{R}}\left(1 - e^{-\widetilde{\psi}_n(y)}\right)e^{-n\,\psi(y/n^{1/4})}dy, \end{aligned}$$

where

$$\begin{aligned} \widetilde{\psi}_n(y) &:= \frac{y^4}{12} - n\psi\left(\frac{y}{n^{1/4}}\right) = n\left(\frac{1}{12}\left(\frac{y}{n^{1/4}}\right)^4 - \psi\left(\frac{y}{n^{1/4}}\right)\right) =: n\widetilde{\psi}\left(\frac{y}{n^{1/4}}\right), \\ \widetilde{\psi}(y) &:= \frac{y^4}{12} - \psi(y) = \frac{y^4}{12} - \frac{y^2}{2} + \log\cosh(y). \end{aligned} \tag{3.42}$$

The study of $\widetilde{\psi}_n$ with SageMath [87] shows that it is non negative on $\mathbb{R}$ with only cancelation in 0, see also the Taylor formula at the fourth order below. As a result, the previous quantity is positive on $\mathbb{R}^*$. Moreover, in the same vein as for κ_n defined in (3.7), one has $\widetilde{\psi}_n^{(k)}(0) = 0$ for all $k = 0, 1, ..., 5$, and the Taylor formula with integral remainder gives at the sixth order

$$\widetilde{\psi}_n(y) = \frac{y^6}{120}\int_0^1 (1-\alpha)^5 \widetilde{\psi}_n^{(6)}(\alpha y)d\alpha = \frac{y^6}{120\sqrt{n}}\int_0^1 (1-\alpha)^5 \widetilde{\psi}^{(6)}(\alpha y n^{-1/4})d\alpha.$$

A computation with SageMath [87] gives

- $\widetilde{\psi}'(y) = \tanh(y) + \dfrac{y^3}{3} - y$
- $\widetilde{\psi}''(y) = -\tanh(y)^2 + y^2$
- $\widetilde{\psi}'''(y) = 2\tanh(y)\big(1-\tanh(y)^2\big) + 2y$
- $\widetilde{\psi}^{(4)}(y) = 2\,\dfrac{\sinh(y)^4 + 4\sinh(y)^2}{\cosh(y)^4} \geq 0$
- $\widetilde{\psi}^{(5)}(y) = -8\,\dfrac{(\cosh(y)^4-3)\sinh(y)}{\cosh(y)^5}$
- $\widetilde{\psi}^{(6)}(y) = 4\,\dfrac{4\cosh(y)^4 - 30\cosh(y)^2 + 30}{\cosh(y)^6}.$

Note that the Taylor formula at the fourth order gives $\psi_n(y) = \frac{y^4}{6}\int_0^1 (1-\alpha)^3\widetilde{\psi}^{(4)}(\alpha y/n^{1/4})d\alpha$ and since $\widetilde{\psi}^{(4)} \geq 0$, it is easily seen that $\widetilde{\psi}_n$ is non negative. We thus have

$$0 \leq 1 - e^{-\widetilde{\psi}_n(y)} \leq \widetilde{\psi}_n(y) \leq \frac{y^6}{720}\left\|\widetilde{\psi}_n^{(6)}\right\|_\infty = \frac{y^6}{720\sqrt{n}}\left\|\widetilde{\psi}^{(6)}\right\|_\infty.$$

Since

$$\begin{aligned}\widetilde{\psi}^{(6)}(y) &= 4\,\frac{4\cosh(y)^4 - 30\cosh(y)^2 + 30}{\cosh(y)^6}\\ &= 4\Big(4(1-\tanh(y)^2) - 30(1-\tanh(y)^2)^2 + 30(1-\tanh(y)^2)^3\Big) \leq 136\end{aligned}$$

we get

$$\begin{aligned}0 \leq n^{1/4}\mathcal{Z}_{n,1} - \mathcal{Z}_{\boldsymbol{F}} &= \int_{\mathbb{R}}\Big(1 - e^{-\widetilde{\psi}_n(y)}\Big)\, e^{-n\psi(y/n^{1/4})}dy\\ &\leq \frac{136}{720\sqrt{n}}\int_{\mathbb{R}} y^6 e^{-n\psi(y/n^{1/4})}dy = \frac{136}{720\sqrt{n}}\, n^{1/4}\mathcal{Z}_{n,1}\mathbb{E}\Big(\boldsymbol{F}_n^6\Big).\end{aligned}$$

Hence

$$0 \leq 1 - \frac{\mathcal{Z}_{\boldsymbol{F}}}{n^{1/4}\mathcal{Z}_{n,1}} \leq \frac{136}{720\sqrt{n}}\Big(\mathbb{E}\Big(\boldsymbol{F}^6\Big) + o(1)\Big)$$

which concludes the proof. □

Lemma 3.17 (Asymptotic analysis of $\|f_{\boldsymbol{F}_n}\|_\infty$)**.** *With $\boldsymbol{F}_n$ defined in (3.12), we have*

$$\max_{x\in\mathbb{R}} f_{\boldsymbol{F}_n}(x) = \frac{1}{\mathcal{Z}_{\boldsymbol{F}}} + O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. Using (3.12), (2.38) and (2.33), we get

$$f_{\boldsymbol{F}_n}(x) = \frac{1}{n^{1/4}} f_{n,1}\left(\frac{x}{n^{1/4}}\right)$$

$$:= \frac{1}{n^{1/4}\mathcal{Z}_{n,1}} e^{-\frac{n}{2}\operatorname{Argtanh}\left(\frac{x}{n^{1/4}}\right)^2 - (\frac{n}{2}+1)\ln\left(1-\frac{x^2}{\sqrt{n}}\right)} \mathbb{1}_{\{|x|<n^{1/4}\}}$$
$$=: \frac{e^{\xi_n(x)}}{n^{1/4}\mathcal{Z}_{n,1}} \mathbb{1}_{\{|x|<n^{1/4}\}}$$

and

$$\xi_n'(x) = -n^{3/4}\frac{\operatorname{Argtanh}\left(\frac{x}{n^{1/4}}\right) - \left(1+\frac{2}{n}\right)\frac{x}{n^{1/4}}}{1-(\frac{x}{n^{1/4}})^2}.$$

We have $\xi_n'(0) = 0$, and $\xi_n''(0) = \frac{2}{\sqrt{n}}$, hence, 0 is a minimum of ξ_n and $f_{\boldsymbol{F}_n}$. To find the maxima, set

$$y := \operatorname{Argtanh}\left(\frac{x}{n^{1/4}}\right).$$

We need to analyse the solutions of the equation

$$\frac{\tanh(y)}{y} = \frac{1}{1+\frac{2}{n}}. \tag{3.43}$$

This equation is well known in the study of the Curie–Weiss model as it gives the limiting magnetisation when $\beta > 1$ (see e.g. [56, prop. 8]). An easy study shows that (3.43) has a unique solution y_n on $\mathbb{R}_+$ and by symmetry a unique solution $-y_n$ on $\mathbb{R}_-$, both being global maxima.
Define

$$G(w) := 1 - \frac{\tanh(\sqrt{w})}{\sqrt{w}} = \frac{w}{3} + O(w^2), \qquad \text{when } w \to 0,$$
$$\varepsilon_n = 1 - \frac{1}{1+\frac{2}{n}} = \frac{2}{n+2} \underset{n\to+\infty}{\sim} \frac{2}{n}.$$

Then, the solution y_n of (3.43) is such that $y_n = \sqrt{w_n}$, where

$$G(w_n) = \varepsilon_n.$$

Since G is bijective on $\mathbb{R}_+$, we can define its inverse G^{-1} for the composition $\circ$ of functions, and, both functions being $\mathcal{C}^\infty$,

$$w_n = G^{-1}(\varepsilon_n) = G^{-1}(0) + (G^{-1})'(0)\varepsilon_n + O(\varepsilon_n^2) = 3\varepsilon_n + O(\varepsilon_n^2) = \frac{6}{n} + O\left(\frac{1}{n^2}\right).$$

Hence

$$y_n = \sqrt{w_n} = \sqrt{\frac{6}{n}}\left(1+O\left(\frac{1}{n}\right)\right) = \sqrt{\frac{6}{n}} + O\left(\frac{1}{n^{3/2}}\right),$$
$$\frac{x_n}{n^{1/4}} := \tanh(y_n) = \tanh\left(\sqrt{\frac{6}{n}} + O\left(\frac{1}{n^{3/2}}\right)\right) = \sqrt{\frac{6}{n}} + O\left(\frac{1}{n}\right).$$

In the end, using $H(x) := x + \log(1-\tanh(\sqrt{x})^2) = \frac{x^2}{6} + O(x^3)$, we get

$$\|f_{\boldsymbol{F}_n}\|_\infty = f_{\boldsymbol{F}_n}(x_n) = \frac{1}{n^{1/4}\mathcal{Z}_{n,1}} e^{-\frac{n}{2}\operatorname{Argtanh}\left(\frac{x_n}{n^{1/4}}\right)^2 - (\frac{n}{2}+1)\ln\left(1-\frac{x_n^2}{\sqrt{n}}\right)}$$
$$= \frac{1}{n^{1/4}\mathcal{Z}_{n,1}} e^{-\frac{n}{2}w_n - (\frac{n}{2}+1)\ln\left(1-\tanh(\sqrt{w_n})^2\right)} = \frac{1}{n^{1/4}\mathcal{Z}_{n,1}} e^{-\frac{n}{2}H(w_n) - \ln\left(1-\tanh(\sqrt{w_n})^2\right)}$$
$$= \frac{1}{n^{1/4}\mathcal{Z}_{n,1}} e^{-\frac{n}{12}w_n^2 + O(nw_n^3) + \frac{6}{n} + O(n^{-3/2})}$$

$$= \frac{1}{n^{1/4}\mathcal{Z}_{n,1}} e^{\frac{9}{n}+O(n^{-3/2})} = \frac{1}{n^{1/4}\mathcal{Z}_{n,1}}\Big(1+O\Big(\frac{1}{n}\Big)\Big)$$

and Lemma 3.16 gives $n^{1/4}\mathcal{Z}_{n,1} = \mathcal{Z}_{\boldsymbol{F}} + O\big(\frac{1}{\sqrt{n}}\big)$, concluding the proof. □

Lemma 3.18 (Asymptotic analysis of $\mathcal{Z}_{n,\beta_n}$)**.** *Set $\mathcal{Z}_{\boldsymbol{F}_\gamma} := \int_{\mathbb{R}} e^{-\frac{y^4}{12}-\gamma\frac{y^2}{2}}dy$. Then,*

$$\left| n^{1/4}\frac{\mathcal{Z}_{n,\beta_n}}{\mathcal{Z}_{\boldsymbol{F}_\gamma}} - 1\right| = O\Big(\frac{1}{\sqrt{n}}\Big).$$

Proof. Using (2.33), we have

$$\begin{aligned}
\mathcal{Z}_{n,\beta_n} &:= \int_{(-1,1)} e^{-\left(\frac{n}{2\beta_n}\operatorname{Argtanh}(t)^2+\frac{n}{2}\ln(1-t^2)\right)}\frac{dt}{1-t^2}\\
&= \int_{\mathbb{R}} e^{-\left(\frac{\sqrt{n}}{2\beta_n}y^2 - n\ln\left(\cosh\left(y/n^{1/4}\right)\right)\right)}\frac{dy}{n^{1/4}}\\
&= \int_{\mathbb{R}} e^{-\left(\sqrt{n}(\frac{1}{\beta_n}-1)\frac{y^2}{2} - n\left(\ln\left(\cosh\left(y/n^{1/4}\right)\right)-\left(y/n^{1/4}\right)^2/2\right)\right)}\frac{dy}{n^{1/4}}
\end{aligned}$$

having set $y := n^{1/4}\operatorname{Argtanh}(t)$ and used $1-\tanh(x)^2 = \cosh(x)^{-2}$.
Recall that $\psi(y) := \frac{y^2}{2} - \ln(\cosh(y))$ is defined in (3.41) and set

$$\gamma_n := \sqrt{n}\Big(\frac{1}{\beta_n}-1\Big) = \frac{\gamma}{\beta_n}$$

so that

$$\begin{aligned}
\mathcal{Z}_{n,\beta_n} &= \int_{\mathbb{R}} e^{-\gamma_n\frac{y^2}{2}-n\,\psi(y/n^{1/4})}\frac{dy}{n^{1/4}},\\
n^{1/4}\mathcal{Z}_{n,\beta_n} - \mathcal{Z}_{\boldsymbol{F}_\gamma} &= \int_{\mathbb{R}}\Big(e^{-\gamma_n\frac{y^2}{2}-n\,\psi(y/n^{1/4})} - e^{-\gamma\frac{y^2}{2}-\frac{y^4}{12}}\Big)dy.
\end{aligned}$$

We have moreover with $\gamma_n \geq \gamma$

$$\begin{aligned}
0 \leq \mathcal{Z}_{\boldsymbol{F}_\gamma} - \mathcal{Z}_{\boldsymbol{F}_{\gamma_n}} &= \int_{\mathbb{R}}\Big(e^{-\gamma\frac{y^2}{2}-\frac{y^4}{12}} - e^{-\gamma_n\frac{y^2}{2}-\frac{y^4}{12}}\Big)dy = \int_{\mathbb{R}}\Big(e^{-\gamma\frac{y^2}{2}} - e^{-\gamma_n\frac{y^2}{2}}\Big)e^{-\frac{y^4}{12}}dy\\
&\leq (\gamma_n-\gamma)\int_{\mathbb{R}}\frac{y^2}{2}e^{-\frac{y^4}{12}}dy \leq \frac{\gamma^2}{2\sqrt{n}}\mathcal{Z}_{\boldsymbol{F}}\mathbb{E}\big(\boldsymbol{F}^2\big)
\end{aligned}$$

and the same inequality holds if $\gamma \geq \gamma_n$ but with $0 \leq \mathcal{Z}_{\boldsymbol{F}_{\gamma_n}} - \mathcal{Z}_{\boldsymbol{F}_\gamma}$.
Last, the analysis of $n^{1/4}\mathcal{Z}_{n,\beta_n} - \mathcal{Z}_{\boldsymbol{F}_{\gamma_n}}$ is similar to the previous one with $\beta < 1$, using exactly the same function ψ but with a different rescaling. We form

$$0 \leq n^{1/4}\mathcal{Z}_{n,\beta_n} - \mathcal{Z}_{\boldsymbol{F}_{\gamma_n}} = \int_{\mathbb{R}}\Big(e^{-\gamma_n\frac{y^2}{2}-n\,\psi(y/n^{1/4})} - e^{-\gamma_n\frac{y^2}{2}-\frac{y^4}{12}}\Big)dy =: \int_{\mathbb{R}}\Big(1-e^{-\widetilde{\psi}_n(y)}\Big)\widetilde{f}_{\boldsymbol{F}_{\gamma_n}}(y)dy$$

with $\widetilde{f}_{\boldsymbol{F}_{\gamma_n}} := \mathcal{Z}_{\boldsymbol{F}_{\gamma_n}} f_{\boldsymbol{F}_{\gamma_n}}$ and $\widetilde{\psi}_n := n\widetilde{\psi}(\cdot/n^{1/4})$ is defined in (3.42).
Since $\widetilde{\psi}_n \geq 0$ and $0 \leq 1-e^{-\widetilde{\psi}_n(y)} \leq \widetilde{\psi}_n(y) \leq \frac{y^6}{\sqrt{n}}\left\|\widetilde{\psi}^{(6)}\right\|_\infty \leq 136\,\frac{y^6}{\sqrt{n}}$, we get

$$0 \leq n^{1/4}\mathcal{Z}_{n,\beta_n} - \mathcal{Z}_{\boldsymbol{F}_{\gamma_n}} \leq \frac{136}{720}\frac{1}{\sqrt{n}}\int_{\mathbb{R}} y^6\widetilde{f}_{\boldsymbol{F}_{\gamma_n}}(y)dy = \frac{136}{720}\frac{\mathcal{Z}_{\boldsymbol{F}_{\gamma_n}}\mathbb{E}\big(\boldsymbol{F}_{\gamma_n}^6\big)}{\sqrt{n}} = \frac{136}{720}\frac{\mathcal{Z}_{\boldsymbol{F}_\gamma}\mathbb{E}\big(\boldsymbol{F}_\gamma^6\big)+o(1)}{\sqrt{n}}.$$

In the end,

$$\begin{aligned}
\left|n^{1/4}\mathcal{Z}_{n,\beta_n} - \mathcal{Z}_{\boldsymbol{F}_\gamma}\right| &\leq \left|n^{1/4}\mathcal{Z}_{n,\beta_n} - \mathcal{Z}_{\boldsymbol{F}_{\gamma_n}}\right| + \left|\mathcal{Z}_{\boldsymbol{F}_{\gamma_n}} - \mathcal{Z}_{\boldsymbol{F}_\gamma}\right|\\
&\leq \frac{1}{\sqrt{n}}\Big(\frac{136}{720}\mathcal{Z}_{\boldsymbol{F}_\gamma}\mathbb{E}\big(\boldsymbol{F}_\gamma^4\big) + \frac{\gamma^2}{2}\mathcal{Z}_{\boldsymbol{F}}\mathbb{E}\big(\boldsymbol{F}^2\big) + o(1)\Big)
\end{aligned}$$

which concludes the proof. $\square$

Lemma 3.19 (Asymptotic analysis of $\left\|f_{F_{n,\gamma}}\right\|_\infty$). *With $f_{F_{n,\gamma}}$, the density of $\boldsymbol{F}_{n,\gamma}$, defined in (3.23) and $\beta_n := 1 - \frac{\gamma}{\sqrt{n}}$, we have*

$$\max_{x\in\mathbb{R}} f_{F_{n,\gamma}}(x) = \frac{1}{\mathcal{Z}_{F_\gamma}} + O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. Recalling (3.23) we get

$$\begin{aligned} f_{F_{n,\gamma}}(x) &= \frac{1}{n^{1/4}} f_{n,\beta_n}\left(\frac{x}{n^{1/4}}\right) \\ &:= \frac{1}{n^{1/4}\mathcal{Z}_{n,\beta_n}} e^{-\frac{n}{2}\operatorname{Argtanh}\left(\frac{x}{n^{1/4}}\right)^2 - (\frac{n}{2}+1)\ln\left(1-\frac{x^2}{\sqrt{n}}\right)} \mathbb{1}_{\left\{|x|<n^{1/4}\right\}} \\ &=: \frac{e^{\xi_n(x)}}{n^{1/4}\mathcal{Z}_{n,\beta_n}} \mathbb{1}_{\left\{|x|<n^{1/4}\right\}} \end{aligned}$$

and

$$\xi_n'(x) = -n^{3/4}\frac{\beta_n^{-1}\operatorname{Argtanh}\left(\frac{x}{n^{1/4}}\right) - \left(1+\frac{2}{n}\right)\frac{x}{n^{1/4}}}{1-(\frac{x}{n^{1/4}})^2}.$$

We have $\xi_n'(0) = 0$, and $\xi_n''(0) = \frac{2}{\sqrt{n}}$, hence, 0 is a minimum of ξ_n and $f_{F_{n,\gamma}}$. To find the maxima, set

$$y := \operatorname{Argtanh}\left(\frac{x}{n^{1/4}}\right).$$

We need to analyse the solutions of the equation

$$\frac{\tanh(y)}{y} = \frac{\beta_n}{1+\frac{2}{n}}. \tag{3.44}$$

This equation is well known in the study of the Curie–Weiss model as it gives the limiting magnetisation when $\beta > 1$ (see e.g. [56, prop. 8]). An easy study shows that (3.44) has a unique solution y_n on $\mathbb{R}_+$ and by symmetry a unique solution $-y_n$ on $\mathbb{R}_-$, both being global maxima.

Define

$$\begin{aligned} G(w) &:= 1 - \frac{\tanh(\sqrt{w})}{\sqrt{w}} = \frac{w}{3} + O(w^2), \qquad \text{when } w \to 0, \\ \varepsilon_n &= 1 - \frac{\beta_n}{1+\frac{2}{n}} = 1 - \frac{1-\frac{\gamma}{\sqrt{n}}}{1+\frac{2}{n}} \underset{n\to+\infty}{\sim} \frac{\gamma}{\sqrt{n}}\mathbb{1}_{\{\gamma\neq 0\}} + \frac{2}{n}\mathbb{1}_{\{\gamma=0\}}. \end{aligned}$$

Then, the solution y_n of (3.44) is such that $y_n = \sqrt{w_n}$, where

$$G(w_n) = \varepsilon_n.$$

Since G is bijective on $\mathbb{R}_+$, we can define its inverse G^{-1} for the composition $\circ$ of functions, and, both functions being $\mathcal{C}^\infty$,

$$w_n = G^{-1}(\varepsilon_n) = G^{-1}(0) + (G^{-1})'(0)\varepsilon_n + O(\varepsilon_n^2) = 3\varepsilon_n + O(\varepsilon_n^2).$$

Hence

$$\begin{aligned} y_n &= \sqrt{w_n} = \sqrt{3\varepsilon_n}\left(1 + O\left(\varepsilon_n^2\right)\right) = \sqrt{3\varepsilon_n} + O\left(\varepsilon_n^{3/2}\right), \\ \frac{x_n}{n^{1/4}} &:= \tanh(y_n) = \tanh\left(\sqrt{3\varepsilon_n} + O\left(\varepsilon_n^{3/2}\right)\right) = \sqrt{3\varepsilon_n} + O(\varepsilon_n). \end{aligned}$$

In the end, using $H(x) := x + \log(1-\tanh(\sqrt{x})^2) = \frac{x^2}{6} + O(x^3)$, we get

$$\begin{aligned}
\left\|f_{\boldsymbol{F}_{n,\gamma}}\right\|_\infty = f_{\boldsymbol{F}_{n,\gamma}}(x_n) &= \frac{1}{n^{1/4}\mathcal{Z}_{n,\beta_n}} e^{-\frac{n}{2\beta_n}\operatorname{Argtanh}\left(\frac{x_n}{n^{1/4}}\right)^2 - (\frac{n}{2}+1)\ln\left(1-\frac{x_n^2}{\sqrt{n}}\right)} \\
&= \frac{1}{n^{1/4}\mathcal{Z}_{n,\beta_n}} e^{-\frac{n}{2\beta_n}w_n - (\frac{n}{2}+1)\ln\left(1-\tanh(\sqrt{w_n})^2\right)} \\
&= \frac{1}{n^{1/4}\mathcal{Z}_{n,\beta_n}} e^{-\frac{n}{2\beta_n}H(w_n) - \ln\left(1-\tanh(\sqrt{w_n})^2\right)} \\
&= \frac{1}{n^{1/4}\mathcal{Z}_{n,\beta_n}} e^{-\frac{n}{12\beta_n}w_n^2 + O(nw_n^3) + \frac{6}{n} + O(n^{-3/2})} \\
&= \frac{1}{n^{1/4}\mathcal{Z}_{n,\beta_n}} e^{\frac{\gamma^2}{4}\mathbb{1}_{\{\gamma\neq 0\}} + \frac{9}{n}\mathbb{1}_{\{\gamma=0\}} + o(n^{-1})} = \frac{1}{n^{1/4}\mathcal{Z}_{n,\beta_n}}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)
\end{aligned}$$

and Lemma 3.18 gives $n^{1/4}\mathcal{Z}_{n,\beta_n} = \mathcal{Z}_{\boldsymbol{F}_\gamma} + O\left(\frac{1}{\sqrt{n}}\right)$, concluding the proof. □

Lemma 3.20 (Asymptotic analysis of $\mathcal{Z}_{n,\beta}$ for $\beta > 1$)**.** *Set* $\mathcal{Z}_{n,\beta} := \int_{\mathbb{R}} e^{-n\varphi_\beta(y)}dy$ *with* $\varphi_\beta(y) := \frac{y^2}{2\beta} - \log\cosh(y)$*. Then,*

$$\sqrt{n}\, e^{n\varphi_\beta(x_\beta)}\mathcal{Z}_{n,\beta} = \left(2\sqrt{\frac{2\pi}{\varphi_\beta''(x_\beta)}} + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

Proof. By symmetry of φ_β, we have

$$\begin{aligned}
\sqrt{n}\, e^{n\varphi_\beta(x_\beta)}\mathcal{Z}_{n,\beta} &= \sqrt{n}\int_{\mathbb{R}} e^{-n(\varphi_\beta(x)-\varphi_\beta(x_\beta))}dx = 2\sqrt{n}\int_{\mathbb{R}_+} e^{-n(\varphi_\beta(x)-\varphi_\beta(x_\beta))}dx \\
&= 2\sqrt{n}\int_{\mathbb{R}_+} e^{-n(x-x_\beta)^2\int_0^1 \varphi_\beta''(\alpha x+\overline{\alpha}x_\beta)\alpha d\alpha}dx \\
&= 2\int_{-x_\beta\sqrt{n}}^{+\infty} e^{-\frac{w^2}{2}\int_0^1 \varphi_\beta''\left(\alpha w/\sqrt{n}+x_\beta\right)\alpha d\alpha}dw \\
&\xrightarrow[n\to+\infty]{} 2\int_{\mathbb{R}} e^{-\frac{w^2}{2}\varphi_\beta''(x_\beta)}dw = 2\sqrt{\frac{2\pi}{\varphi_\beta''(x_\beta)}}
\end{aligned}$$

using dominated convergence as in the proof of Theorem 3.8. □

3.3.2. *Kolmogorov distance between two centered sums of Rademacher's.*

Lemma 3.21 (Kolmogorov Distance between two centered sums of Rademacher's)**.** *For all* $t \in (0,1)$*, define* $\mathring{S}_n(t) := S_n(t) - n(2t-1)$ *with* $S_n(t) = \sum_{k=1}^n X_k(t)$ *and* $X_k(t) \sim$ i.i.d. *Rademacher random variables. Then, for all* $p, q \in (0,1)$*, we have*

$$d_{\mathrm{Kol}}\left(\mathring{S}_n(p), \mathring{S}_n(q)\right) \le |p-q|\left|\frac{1-(p+q)}{p(1-p)}\right| + O\left(\frac{1}{\sqrt{n}}\right).$$

Proof. Recall that $\mathbb{E}\left(\mathring{S}_n(t)\right) = 0$ and set $\sigma_n^2(q) := \mathrm{Var}(\mathring{S}_n(q)) = 4q(1-q)n$. We define $G_p \sim \mathcal{N}(0, \sigma_n^2(p))$ and $G_q \sim \mathcal{N}(0, \sigma_n^2(q))$. The triangle inequality yields

$$\begin{aligned}
d_{\mathrm{Kol}}\left(\mathring{S}_n(p), \mathring{S}_n(q)\right) &\le d_{\mathrm{Kol}}\left(\mathring{S}_n(p), G_p\right) + d_{\mathrm{Kol}}\left(\mathring{S}_n(q), G_q\right) + d_{\mathrm{Kol}}(G_p, G_q) \\
&\le d_{\mathrm{Kol}}(G_p, G_q) + O\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}$$

by the Berry–Esseen theorem (3.33). It remains to prove that

$$d_{\mathrm{Kol}}(G_p, G_q) \le |p-q| \left| \frac{1-(p+q)}{p(1-p)} \right|.$$

Stein's method for G_p gives, [85, (2.2)],

$$\begin{aligned} d_{\mathrm{Kol}}(G_p, G_q) &:= \sup_{w \in \mathbb{R}} |\mathbb{P}(G_p \le w) - \mathbb{P}(G_q \le w)| \\ &= \sup_{w \in \mathbb{R}} \left| \mathbb{E}\left(f'_{w,p}(G_q) - \frac{G_q}{\sigma_n^2(p)} f_{w,p}(G_q) \right) \right|, \end{aligned}$$

where

$$f_{w,p}(x) := \frac{1}{f_{G_p}(x)} \mathbb{E}\Big(\mathbb{1}_{\{G_p \le x\}} \Big(\mathbb{1}_{\{G_p \le w\}} - \mathbb{P}(G_p \le w) \Big) \Big), \qquad f_{G_p}(x) := \frac{1}{\sigma_n(p)\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_n(p)^2}}$$

is the solution of the Stein equation for G_p, see [95, (19)] and [85, (2.1)].
Moreover, the Gaussian integration by parts gives for $f_{w,p} \in \mathcal{C}^1$

$$\mathbb{E}(G_q f_{w,p}(G_q)) = \sigma_n^2(q) \mathbb{E}\Big(f'_{w,p}(G_q) \Big)$$

implying

$$\begin{aligned} d_{\mathrm{Kol}}(G_p, G_q) &= \sup_{w \in \mathbb{R}} \left| \mathbb{E}\left(f'_{w,p}(G_q) - \frac{\sigma_n^2(q)}{\sigma_n^2(p)} f'_{w,p}(G_q) \right) \right| \\ &= \left| 1 - \frac{\sigma_n^2(q)}{\sigma_n^2(p)} \right| \sup_{w \in \mathbb{R}} \left| \mathbb{E}\Big(f'_{w,p}(G_q) \Big) \right| \\ &\le \left| 1 - \frac{\sigma_n^2(q)}{\sigma_n^2(p)} \right| \sup_{w \in \mathbb{R}} \left\| f'_{w,p} \right\|_\infty \end{aligned}$$

and, see e.g. [95, (22)],

$$\sup_{w \in \mathbb{R}} \left\| f'_{w,p} \right\|_\infty \le 1.$$

Finally

$$\left| 1 - \frac{\sigma_n^2(q)}{\sigma_n^2(p)} \right| = \left| 1 - \frac{4q(1-q)n}{4p(1-p)n} \right| = |p-q| \left| \frac{1-(p+q)}{p(1-p)} \right|$$

which gives the result. □

4. Cramér-type moderate deviations for L^2-Rademacher-functionals

In this chapter, which is based on [11], we derive moderate deviations for L^2-Rademacher-functionals. For the proof of our main result Theorem 4.1 we will need two auxiliary lemmas. The first one gives us a more precise bound of the moment generating function constructed in the setting of the theorem. In the proof of Theorem 4.1 we split the relevant terms into sub-terms by the use of different indicators and the second lemma helps us to bound one of these. This theoretical part is done in the first section. In the following sections we treat the i.i.d.-case and infinite weighted 2-runs as applications.

4.1. Main Result.

Now we present the main result of this chapter.

Theorem 4.1 (Moderate deviations for L^2-Rademacher-functionals)**.** *Let $F \in \mathbb{D}^{1,2}$ with $\mathbb{E}[F] = 0, \operatorname{Var}(F) = 1$, and*

$$F f_z(F) + \mathbb{1}_{\{F>z\}} \in \mathbb{D}^{1,2} \quad \forall\, z \in \mathbb{R},$$

$$\frac{1}{\sqrt{pq}} DF \left| DL^{-1}F \right| \in Dom(\delta).$$

Assume that there exists a constant $A > 0$ and increasing functions $\gamma_1(t), \gamma_2(t)$ such that $e^{tF} \in \mathbb{D}^{1,2}$ and

$$\text{(A1)} \qquad \mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{tF}\right] \le \gamma_1(t)\mathbb{E}\left[e^{tF}\right],$$

$$\text{(A2)} \qquad \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right|\right)\right| e^{tF}\right] \le \gamma_2(t)\mathbb{E}\left[e^{tF}\right],$$

for all $0 \le t \le A$. For $d_0 \ge 0$, let

$$A_0(d_0) := \max\left\{0 \le t \le A : \frac{t^2}{2}(\gamma_1(t) + \gamma_2(t)) \le d_0\right\}.$$

Then, for any $d_0 \ge 0$,

$$\left|\frac{\mathbb{P}(F > z)}{1 - \Phi(z)} - 1\right| \le 25 e^{d_0}(1 + z^2)(\gamma_1(z) + \gamma_2(z))$$

provided that $0 \le z \le A_0(d_0)$.

In consequence, the following result is achieved.

Theorem 4.2. *Under the assumptions from Theorem 4.1, there is*

$$\left|\frac{\mathbb{P}(F > z)}{1 - \Phi(z)} - 1\right| \le 25 e^{\frac{z^2}{2}(\gamma_1(z)+\gamma_2(z))}(1 + z^2)(\gamma_1(z) + \gamma_2(z))$$

for all $0 \le z \le A$.

As we mentioned before we continue with two auxiliary lemmas.

Lemma 4.3 (Bound for the moment generating function)**.** *Under the assumptions of Theorem 4.1, for $0 \le t \le A$, we have*

$$\mathbb{E}\left[e^{tF}\right] \le \exp\left\{\frac{t^2}{2}(1 + \gamma_1(t) + \gamma_2(t))\right\}. \tag{4.1}$$

Then, for $0 \le t \le A_0(d_0)$,

$$\mathbb{E}\left[e^{tF}\right] \le e^{d_0} e^{t^2/2}. \tag{4.2}$$

Proof. Let $h(t) := \mathbb{E}\left[e^{tF}\right]$. We recall that $\mathbb{E}\left[e^{tF}\right] < \infty$ is implied by $e^{tF} \in \mathbb{D}^{1,2}$ for $0 \le t \le A$, and so, by the continuity of the exponential funtion, we have $h'(t) = \mathbb{E}\left[Fe^{tF}\right]$. It follows with (2.24) and (2.25) that

$$\begin{aligned}\mathbb{E}\left[Fe^{tF}\right] &= \mathbb{E}\left[(LL^{-1}F)e^{tF}\right]\\ &= \mathbb{E}\left[(-\delta DL^{-1}F)e^{tF}\right]\\ &= \mathbb{E}\left[\langle De^{tF}, -DL^{-1}F\rangle\right]. \end{aligned} \tag{4.3}$$

Now we consider the k-th component of De^{tF}, which gives us

$$\begin{aligned}D_k e^{tF} &= \sqrt{p_k q_k}\left[e^{tF_k^+} - e^{tF_k^-}\right]\\ &= t\sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} e^{tu} du\\ &= t\sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} \left[e^{tu} - e^{tF}\right] du + te^{tF} D_k F\\ &=: tR_k + te^{tF} D_k F.\end{aligned}$$

If we define $R := (R_1, R_2, \ldots)$, we can go on from (4.3) by writing

$$\begin{aligned}\mathbb{E}\left[Fe^{tF}\right] &= \mathbb{E}\left[\langle tR, -DL^{-1}F\rangle\right] + \mathbb{E}\left[\langle te^{tF} DF, -DL^{-1}F\rangle\right]\\ &\le t\mathbb{E}\left[e^{tF}\right] + t\mathbb{E}\left|\langle R, -DL^{-1}F\rangle\right| + t\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{tF}\right]. \end{aligned} \tag{4.4}$$

Without loss of generality $F_k^- \le F \le F_k^+$; for the other case we just have to change the sign. Then we can bound R_k as follows.

$$\begin{aligned}R_k &= \sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} \left[e^{tu} - e^{tF}\right] du\\ &\le \sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} \left[e^{tF_k^+} - e^{tF_k^-}\right] du\\ &= \sqrt{p_k q_k}\left[e^{tF_k^+} - e^{tF_k^-}\right]\int_{F_k^-}^{F_k^+} du\\ &= D_k e^{tF} \cdot \frac{1}{\sqrt{p_k q_k}} D_k F\end{aligned}$$

and by combining both cases

$$|R_k| \le \frac{1}{\sqrt{p_k q_k}} D_k e^{tF} \cdot D_k F. \tag{4.5}$$

By condition (A2) and (4.5) we get

$$\begin{aligned}t\mathbb{E}\left|\langle R, -DL^{-1}F\rangle\right| &\le t\mathbb{E}\left[\langle |R|, \left|DL^{-1}F\right|\rangle\right]\\ &\le t\mathbb{E}\left[\langle De^{tF}, \frac{1}{\sqrt{pq}} DF\left|DL^{-1}F\right|\rangle\right]\\ &\le t\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}} DF\left|DL^{-1}F\right|\right)\right| e^{tF}\right]\\ &\le t\gamma_2(t)\mathbb{E}\left[e^{tF}\right]. \end{aligned} \tag{4.6}$$

By condition (A1), for $0 \le t \le A$,

$$t\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{tF}\right] \le t\gamma_1(t)\mathbb{E}\left[e^{tF}\right]. \tag{4.7}$$

Combining (4.4), (4.6) and (4.7), we have for $0 \leq t \leq A$,

$$\begin{aligned} h'(t) &= \mathbb{E}\left[Fe^{tF}\right] \\ &\leq th(t) + \{t(\gamma_1(t) + \gamma_2(t))\}h(t) \\ &= \{1 + \gamma_1(t) + \gamma_2(t)\}\, t\, h(t). \end{aligned}$$

Having in mind that $h(0) = 1$, and γ_1 and γ_2 are increasing, we complete the proof of (4.1) by solving the foregoing differential inequality:

$$\begin{aligned} \log(h(t)) &= \int_0^t \frac{h'(s)}{h(s)} ds \\ &\leq \int_0^t (1 + \gamma_1(s) + \gamma_2(s)) s ds \\ &\leq \int_0^t (1 + \gamma_1(t) + \gamma_2(t)) s ds \\ &= \frac{t^2}{2}(1 + \gamma_1(t) + \gamma_2(t)), \end{aligned}$$

now we apply exp(.) on both sides. At last, (4.2) follows immediately from (4.1) by definition of $A_0(d_0)$. □

Lemma 4.4. *Under the assumptions of Theorem 4.1, we have for* $0 \leq z \leq A_0(d_0)$,

$$\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| Fe^{F^2/2}\mathbb{1}_{\{0\leq F\leq z\}}\right] \leq 6e^{d_0}(1 + z^2)\gamma_1(z) \tag{4.8}$$

and

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}} DF\left|DL^{-1}F\right|\right)\right| Fe^{F^2/2}\mathbb{1}_{\{0\leq F\leq z\}}\right] \leq 6e^{d_0}(1 + z^2)\gamma_2(z). \tag{4.9}$$

Proof. Same as [102] we apply the idea in [19, Lemma 5.2] for this proof. For $a \in \mathbb{R}$, denote $[a] = \max\{n \in \mathbb{N} : n \leq a\}$. Next, we define $H := 1 - \langle DF, -DL^{-1}F\rangle$.

$$\mathbb{E}\left[|H|\, Fe^{F^2/2}\mathbb{1}_{\{0\leq F\leq z\}}\right] = \sum_{j=1}^{[z]} \mathbb{E}\left[|H|\, Fe^{F^2/2}\mathbb{1}_{\{j-1\leq F\leq j\}}\right] + \mathbb{E}\left[|H|\, Fe^{F^2/2}\mathbb{1}_{\{[z]\leq F\leq z\}}\right].$$

For the first term we get

$$\begin{aligned} \sum_{j=1}^{[z]} \mathbb{E}\left[|H|\, Fe^{F^2/2}\mathbb{1}_{\{j-1\leq F\leq j\}}\right] &\leq \sum_{j=1}^{[z]} je^{(j-1)^2/2 - j(j-1)}\mathbb{E}\left[|H|\, e^{jF}\mathbb{1}_{\{j-1\leq F\leq j\}}\right] \\ &\leq 3\sum_{j=1}^{[z]} je^{-j^2/2}\mathbb{E}\left[|H|\, e^{jF}\mathbb{1}_{\{j-1\leq F\leq j\}}\right] \end{aligned}$$

and similarly, for the second

$$\begin{aligned} \mathbb{E}\left[|H|\, Fe^{F^2/2}\mathbb{1}_{\{[z]\leq F\leq z\}}\right] &\leq ze^{[z]^2/2 - [z]z}\mathbb{E}\left[|H|\, e^{zF}\mathbb{1}_{\{[z]\leq F\leq z\}}\right] \\ &\leq 3ze^{-z^2/2}\mathbb{E}\left[|H|\, e^{zF}\mathbb{1}_{\{[z]\leq F\leq z\}}\right]. \end{aligned}$$

For both terms, we used similar manipulations, namely for $j - 1 \leq F \leq j$:

- $e^{(j-1)^2/2 - j(j-1)} = e^{j^2/2 - j + 1/2 - j^2 + j} = e^{-j^2/2}e^{1/2} \leq 3e^{-j^2/2}$.
- $e^{(j-F)^2/2} \leq e^{1/2} \Leftrightarrow e^{F^2/2} \leq e^{-j^2/2 + jF + 1/2} \Leftrightarrow e^{F^2/2} \leq e^{(j-1)^2/2 - j(j-1)}e^{jF}$.

And for $[z] \leq F \leq z$:

- $e^{[z]^2/2 - [z]z} = e^{[z]^2/2 - [z]z + z^2/2 - z^2/2} = e^{(z-[z])^2/2}e^{-z^2/2} \leq 3e^{-z^2/2}$.
- $e^{(z-F)^2/2} \leq e^{(z-[z])^2/2} \Leftrightarrow e^{F^2/2} \leq e^{(z-[z])^2/2 + zF - z^2/2} \Leftrightarrow e^{F^2/2} \leq e^{[z]^2/2 - [z]z}e^{zF}$.

By condition (A1) and (4.2), and recalling that γ_1 is increasing, for any $0 \le x \le z \le A_0(d_0)$

$$e^{-x^2/2}\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{xF}\right] \le \gamma_1(x)\mathbb{E}[e^{xF - x^2/2}] \le e^{d_0}\gamma_1(x) \le e^{d_0}\gamma_1(z).$$

By the foregoing inequalities,

$$\mathbb{E}\left[|H|\, Fe^{F^2/2}\mathbb{1}_{\{0\le F\le z\}}\right] \le 3e^{d_0}\gamma_1(z)\left(\sum_{j=1}^{[z]} j + z\right) \le 6e^{d_0}(1+z^2)\gamma_1(z).$$

The other statement of the lemma is completely analogous. □

Now we are ready to prove our two theorems.

Proof of Theorem 4.1. We note at first that

$$|\mathbb{P}(F > z) - (1 - \Phi(z))| = |1 - \mathbb{P}(F \le z) - (1 - \Phi(z))| = |\Phi(z) - \mathbb{P}(F \le z)|.$$

By Stein's method and the proof of [36, Theorem 3.1] we have for $z \in \mathbb{R}$

$$|\mathbb{P}(F > z) - (1 - \Phi(z))| = |\mathbb{E}[f_z'(F) - Ff_z(F)]| \le J_1 + J_2$$

with

$$J_1 := \mathbb{E}\left|f_z'(F)\big(1 - \langle DF, -DL^{-1}F\rangle\big)\right|,$$

$$J_2 := \mathbb{E}\left[(Ff_z(F) + \mathbb{1}_{\{F>z\}})\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right].$$

For the upcoming estimation we can split J_2 into two terms, namely

$$|J_2| \le \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\left|Ff_z(F) + \mathbb{1}_{\{F>z\}}\right|\right] \le J_{21} + J_{22}$$

with

$$J_{21} := \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\left|Ff_z(F)\right|\right],$$

$$J_{22} := \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F>z\}}\right].$$

Using the same arguments as in the proof of [102, Proposition 4.1], in particular (2.6), (2.10) and (2.11), we have

$$J_{21} \le J_{23} + J_{24} + J_{25}$$

with

$$J_{23} := (1 - \Phi(z)) \cdot \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F<0\}}\right],$$

$$J_{24} := \sqrt{2\pi} \cdot (1 - \Phi(z)) \cdot \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right| Fe^{F^2/2}\mathbb{1}_{\{0\le F\le z\}}\right],$$

$$J_{25} := \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F>z\}}\right] = J_{22}.$$

Thus,

$$|J_2| \le J_{23} + J_{24} + 2 \cdot J_{25}. \tag{4.10}$$

For J_{23}, by condition (A2) with $t = 0$ and noting that γ_2 is increasing,

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F<0\}}\right] \le \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\right] \le \gamma_2(0) \le e^{d_0}\gamma_2(z). \tag{4.11}$$

For J_{24}, by Lemma 4.4, we have

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|Fe^{F^2/2}\mathbb{1}_{\{0\le F\le z\}}\right] \le 6e^{d_0}(1+z^2)\gamma_2(z). \tag{4.12}$$

For J_{25}, by condition (A2) and (4.2), for $0 \le z \le A_0(d_0)$,

$$\begin{aligned}
J_{25} &= \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F>z\}}e^{zF}e^{-zF}\right] \\
&\le \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F>z\}}e^{zF}\right]e^{-z^2} \\
&\le \gamma_2(z)\mathbb{E}\left[e^{zF}\right]e^{-z^2} \\
&\le e^{d_0}\gamma_2(z)e^{-z^2/2}.
\end{aligned}$$

We recall that for $z > 0$

$$e^{-z^2/2} \le \sqrt{2\pi}\cdot(1+z)\cdot(1-\Phi(z)) \le \frac{3\sqrt{2\pi}}{2}\cdot(1+z^2)\cdot(1-\Phi(z)).$$

Then, for $0 \le z \le A_0(d_0)$,

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F>z\}}\right] \le \frac{3e^{d_0}\sqrt{2\pi}}{2}(1+z^2)\gamma_2(z)(1-\Phi(z)). \tag{4.13}$$

Therefore, combining (4.10) – (4.13), for $0 \le z \le A_0(d_0)$, we have

$$|J_2| \le \left(1+6\sqrt{2\pi}+3\sqrt{2\pi}\right)e^{d_0}(1+z^2)\gamma_2(z)(1-\Phi(z)) \le 25e^{d_0}(1+z^2)\gamma_2(z)(1-\Phi(z)).$$

For the remaining term J_1 we have a similar approach after using again Stein's equation:

$$\begin{aligned}
|J_1| &= \mathbb{E}\left|f_z'(F)\left(1-\langle DF,-DL^{-1}F\rangle\right)\right| \\
&\le \mathbb{E}\left[|f_z'(F)|\left|1-\langle DF,-DL^{-1}F\rangle\right|\right] \\
&\le J_{11}+J_{12}+J_{13}
\end{aligned}$$

with

$$\begin{aligned}
J_{11} &:= \mathbb{E}\left[|Ff_z(F)|\left|1-\langle DF,-DL^{-1}F\rangle\right|\right], \\
J_{12} &:= \mathbb{E}\left[(1-\Phi(z))\left|1-\langle DF,-DL^{-1}F\rangle\right|\right], \\
J_{13} &:= \mathbb{E}\left[\mathbb{1}_{\{F>z\}}\left|1-\langle DF,-DL^{-1}F\rangle\right|\right].
\end{aligned}$$

From here on we can identify any of these terms with a corresponding term from the first part of the proof, namely $J_{21} - J_{25}$. Therefore, combining these modified estimations, for $0 \le z \le A_0(d_0)$, we have

$$|J_1| \le \left(1+1+6\sqrt{2\pi}+3\sqrt{2\pi}\right)e^{d_0}(1+z^2)\gamma_1(z)(1-\Phi(z)) \le 25e^{d_0}(1+z^2)\gamma_1(z)(1-\Phi(z)).$$

All in all, we have shown, for $0 \le z \le A_0(d_0)$,

$$|\mathbb{P}(F>z)-(1-\Phi(z))| \le 25e^{d_0}(1+z^2)(\gamma_1(z)+\gamma_2(z))(1-\Phi(z))$$

or equivalently

$$\left|\frac{\mathbb{P}(F > z)}{1-\Phi(z)} - 1\right| \le 25 e^{d_0}(1+z^2)(\gamma_1(z)+\gamma_2(z)). \qquad \square$$

Proof of Theorem 4.2. Let $0 \le z_0 \le A$ be fixed. Choose $d_0 = \frac{z_0^2}{2}(\gamma_1(z_0)+\gamma_2(z_0))$. Per definition there is $0 \le z_0 \le A_0(d_0)$. Hence, we may apply Theorem 4.1, which then implies

$$\begin{aligned}\left|\frac{\mathbb{P}(F > z_0)}{1-\Phi(z_0)} - 1\right| &\le 25 e^{d_0}(1+z_0^2)(\gamma_1(z_0)+\gamma_2(z_0)) \\ &= 25 e^{\frac{z_0^2}{2}(\gamma_1(z_0)+\gamma_2(z_0))}(1+z_0^2)(\gamma_1(z_0)+\gamma_2(z_0))\,. \qquad \square\end{aligned}$$

4.2. **Application: The i.i.d.-case.** As a first application we treat the i.i.d.-case: For our sequence $(X_i)_{i\in\mathbb{N}}$ of Rademacher random variables we consider the standardized nth partial sum

$$F := F_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i := \frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{X_i-(2p-1)}{2\sqrt{pq}}.$$

The classical result can be received:

Corollary 4.5. *Recall the definition of F_n from above. Then*

$$\frac{\mathbb{P}(F_n > z)}{1-\Phi(z)} = 1 + O(1)(1+z^2)\gamma_n(z), \tag{4.14}$$

for $0 \le z \le n^{1/6}$ such that $(1+z^2)\gamma_n(z) \le 1$, where O(1) is bounded by a constant and

$$\gamma_n(z) := e^{O(1)zn^{-1/2}}\frac{(1+z)}{\sqrt{n}}.$$

Remark 4.6. We obtain the optimal range $0 \le z \le n^{1/6}$ from (1.5), but there is no $\log(n)$ in our error term compared to [45, Corollary 2.2].

Proof of Corollary 4.5. Due to finiteness of the sum and therefore boundness it is easy to see that the assumptions

- $F \in \mathbb{D}^{1,2}$,
- $Ff_z(F) + \mathbb{1}_{\{F>z\}} \in \mathbb{D}^{1,2} \quad \forall z \in \mathbb{R}$,
- $\frac{1}{\sqrt{pq}} DF\,|DL^{-1}F| \in Dom(\delta)$,
- $e^{tF} \in \mathbb{D}^{1,2}$

are valid. Now we start to compute the terms appearing in (A1) and (A2). By definition

$$F_k^+ = \frac{1}{\sqrt{n}}\left(\sum_{\substack{i=1\\ i\neq k}}^n \frac{X_i-(2p-1)}{2\sqrt{pq}} + \frac{1-(2p-1)}{2\sqrt{pq}}\right)$$

$$F_k^- = \frac{1}{\sqrt{n}}\left(\sum_{\substack{i=1\\ i\neq k}}^n \frac{X_i-(2p-1)}{2\sqrt{pq}} + \frac{-1-(2p-1)}{2\sqrt{pq}}\right)$$

and we get, for fixed $n \in \mathbb{N}$,

$$D_kF = \sqrt{pq}\left(F_k^+ - F_k^-\right) = \frac{2\sqrt{pq}}{2\sqrt{pq}\sqrt{n}} = \frac{1}{\sqrt{n}}.$$

On the other hand it holds that $-L^{-1}F = F$ and so

$$-D_k L^{-1} F = \frac{1}{\sqrt{n}}.$$

With these expressions we can compute the scalar product

$$\langle DF, -DL^{-1}F\rangle = \sum_{i=1}^{n}\left(\frac{1}{\sqrt{n}}\right)^2 = 1 = \mathrm{Var}(F).$$

As a consequence

$$\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{tF}\right] = 0,$$

so $\gamma_1(t) = 0$ or we are free to choose something appropriate. We now move on and show that a bound as in condition (A2) exists: By the Cauchy–Schwarz inequality

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}} DF\left|DL^{-1}F\right|\right)\right| e^{tF}\right] \le \left(\mathbb{E}\left[\left(\delta\left(\frac{1}{\sqrt{pq}} DF\left|DL^{-1}F\right|\right)\right)^2 e^{tF}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}}.$$

By [80, Corollary 9.9] it is $\delta(u) = \sum_{k=1}^{n} Y_k u_k$, where Y_k is the kth centered and standardized Rademacher random variable as above. The corollary can be applied since

$$u_k := \frac{D_k F\,|D_k L^{-1} F|}{\sqrt{pq}} = \frac{1}{n\sqrt{pq}}$$

does not depend on X_k. Then

$$\begin{aligned}\left(\mathbb{E}\left[(\delta(u))^2 e^{tF}\right]\right)^{\frac{1}{2}} &= \frac{1}{n\sqrt{pq}}\left(\sum_{k,l=1}^{n} \mathbb{E}\left[Y_k Y_l e^{tF}\right]\right)^{\frac{1}{2}} \\ &\le \frac{1}{n\sqrt{pq}}\left(\sum_{k=1}^{n} \mathbb{E}\left[Y_k^2 e^{tF}\right]\right)^{\frac{1}{2}} + \frac{1}{n\sqrt{pq}}\left(\sum_{\substack{k,l=1\\ k\neq l}}^{n} \mathbb{E}\left[Y_k Y_l e^{tF}\right]\right)^{\frac{1}{2}}. \qquad (4.15)\end{aligned}$$

We estimate the first term of (4.15) without further computation:

$$\frac{1}{n\sqrt{pq}}\left(\sum_{k=1}^{n} \mathbb{E}\left[Y_k^2 e^{tF}\right]\right)^{\frac{1}{2}} \le \frac{C}{\sqrt{n}}\left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}}.$$

For the second term of (4.15) we take a closer look at the summands $\mathbb{E}\left[Y_k Y_l e^{tF}\right]$. Our strategy is to split F into F_a, the summands depending on Y_k and Y_l, and F_u, the independent summands, such that

$$F_a = \frac{1}{\sqrt{n}}(Y_k + Y_l)\,, \quad F_u = F - F_a.$$

Then by Taylor expansion

$$\begin{aligned}\mathbb{E}[Y_k Y_l e^{tF}] &= \mathbb{E}[Y_k Y_l e^{tF_a} e^{tF_u}] \\ &= \mathbb{E}[Y_k Y_l e^{tF_u}] + t\mathbb{E}[Y_k Y_l F_a e^{tF_u}] + t^2\mathbb{E}[Y_k Y_l F_a^2 r_2(tF_a) e^{tF_u}/2],\end{aligned}$$

where $|r_2(tF_a)| \le e^{t|F_a|}$.

0-order-term: By independence

$$\mathbb{E}[Y_k Y_l e^{tF_u}] = \mathbb{E}[Y_k]\mathbb{E}[Y_l]\mathbb{E}[e^{tF_u}] = 0.$$

1st-order-term: By definition of F_a and by independence

$$t\mathbb{E}[Y_k Y_l F_a e^{tF_u}] = \frac{t}{\sqrt{n}}\left(\mathbb{E}[Y_k^2 Y_l e^{tF_u}] + \mathbb{E}[Y_k Y_l^2 e^{tF_u}]\right)$$

$$= \frac{t}{\sqrt{n}}\Big(\mathbb{E}[Y_k^2]\mathbb{E}[Y_l]\mathbb{E}[e^{tF_u}] + \mathbb{E}[Y_k]\mathbb{E}[Y_l^2]\mathbb{E}[e^{tF_u}]\Big)$$
$$= 0.$$

2nd-order-term: Finally, having in mind that $|r_2(tF_a)| \leq e^{t|F_a|}$ we can bound the last term of our Taylor expansion by

$$t^2 \left|\mathbb{E}[Y_k Y_l F_a^2 r_2(tF_a) e^{tF_u}/2]\right| \leq \frac{Ct^2}{n}\mathbb{E}\left[e^{tF}\right] e^{ct}$$

with $c = O\Big(\frac{1}{\sqrt{n}}\Big)$ and so

$$\frac{1}{n\sqrt{pq}}\left(\sum_{\substack{k,l=1\\k\neq l}}^{n} \mathbb{E}\left[Y_k Y_l e^{tF}\right]\right)^{\frac{1}{2}} = \frac{1}{n\sqrt{pq}}\left(\sum_{\substack{k,l=1\\k\neq l}}^{n} t^2 \left|\mathbb{E}[Y_k Y_l F_a^2 r_2(tF_a) e^{tF_u}/2]\right|\right)^{\frac{1}{2}}$$
$$\leq \frac{Ce^{ct}}{\sqrt{n}} t\Big(\mathbb{E}\left[e^{tF}\right]\Big)^{\frac{1}{2}}.$$

Summarizing everything we have done so far a bound as in condition (A2) is obtained by

$$\mathbb{E}\left[|\delta(u)|\, e^{tF}\right] \leq \Big(\mathbb{E}\left[(\delta(u))^2\, e^{tF}\right]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\left[e^{tF}\right]\Big)^{\frac{1}{2}}$$
$$\leq \left(\left(\sum_{k\in\mathbb{Z}} \mathbb{E}\left[u_k^2 e^{tF}\right]\right)^{\frac{1}{2}} + \left(\sum_{\substack{k,l\in\mathbb{Z}\\k\neq l}} \mathbb{E}\left[u_k u_l X_k X_l e^{tF}\right]\right)^{\frac{1}{2}}\right)\Big(\mathbb{E}\left[e^{tF}\right]\Big)^{\frac{1}{2}}$$
$$\leq \frac{Ce^{ct}}{\sqrt{n}}(1+t)\mathbb{E}\left[e^{tF}\right]$$
$$=: \gamma_2(t)\mathbb{E}\left[e^{tF}\right].$$

□

4.3. **Application: Infinite weighted 2-runs.** Our moderate deviation is given as follows.

Theorem 4.7. *Recall the definition of F_n given by (2.26). Then*

$$\frac{\mathbb{P}(F_n > z)}{1-\Phi(z)} = 1 + O(1)(1+z^2)\gamma_n(z), \tag{4.16}$$

for $0 \le z \le \min\{C_n^{-1/3}, C_n^{-2}, \mathrm{Var}(G_n)^{1/2}\}$ *such that* $(1+z^2)\gamma_n(z) \le 1$*, where O(1) is bounded by a constant only depending on the coefficient sequence* $(a_i^{(n)})_{i\in\mathbb{Z}}$ *and*

$$\gamma_n(z) := e^{O(1)z(\mathrm{Var}(G_n))^{-1/2}}\Big((1+z^{1/2}+z)C_n\Big), \quad C_n := \frac{\left\|a^{(n)}\right\|^2_{l^4(\mathbb{Z})}}{\mathrm{Var}(G_n)}.$$

Remark 4.8. The constant C_n has an important meaning. It is the order of the corresponding Kolmogorov distance in [36, Theorem 1.1]. Depending on the coefficient sequence, C_n can behave differently: By (4.18) and (4.39) C_n is in general bounded by a constant, but it can be a constant itself, see e.g. $a_i^{(n)} = \frac{1}{i^2}$. So, to make (4.16) tend to 0 and the range increase in n, the condition $C_n \to 0$ for $n \to \infty$ is sufficient. We give now examples, where this is the case and where the resulting rate is optimal.

Example 4.9. We consider $a_i^{(n)} = \mathbb{1}_{\{|i|\le n\}}\,\forall\, i \in \mathbb{Z}$, which is obviously a summable sequence. Then $\left\|a^{(n)}\right\|^2_{l^4(\mathbb{Z})} = O\Big(n^{1/2}\Big)$, $\mathrm{Var}(G_n) = O(n)$ and $C_n = O\Big(n^{-1/2}\Big)$. The moderate deviation we get is

$$\frac{\mathbb{P}(F_n > z)}{1-\Phi(z)} = 1 + O(1)(1+z^2)\gamma_n(z), \tag{4.17}$$

for $0 \le z \le n^{1/6}$ such that $(1+z^2)\gamma_n(z) \le 1$, where

$$\gamma_n(z) := e^{O(1)zn^{-1/2}}\Big((1+z^{1/2}+z)n^{-1/2}\Big).$$

In order to discuss the quality of this result, we use a lower Kolmogorov bound known from [42, Theorem 1(c)], which got later refined by [83, Corollary 3.12]. Since G_n is almost surely an integer between $-n$ and n, said results imply that the Kolmogorov distance for normal approximation of F_n is bounded from below by $c_0 \cdot (\mathrm{Var}(G_n))^{-\frac{1}{2}}$ for some constant $c_0 > 0$. As $\mathrm{Var}(G_n)$ is of order n, we conclude that C_n being of order $n^{-\frac{1}{2}}$ is optimal.

Example 4.10. We generalize the previous example to $a_i^{(n)} = n^{-\beta}\mathbb{1}_{\{|i|\le n^\alpha\}}\,\forall\, i\in\mathbb{Z}, \alpha\in\mathbb{R}, \beta > 0$. Then $\left\|a^{(n)}\right\|^2_{l^4(\mathbb{Z})} = O\Big(n^{(\alpha-4\beta)/2}\Big)$, $\mathrm{Var}(G_n) = O\Big(n^{\alpha-2\beta}\Big)$ and $C_n = O\Big(n^{-\alpha/2}\Big)$. If we choose $\beta = 0$ and $\alpha \ge 1$ the moderate deviation we get is of the same form as (4.17) with range $0 \le z \le n^{\alpha/6}$ respectively $0 \le z \le (\mathrm{Var}(G_n))^{\alpha/6}$. Using the same argumentation as in the previous example, we see that the rate of C_n is again optimal.

Proof of Theorem 4.7. In what follows, we show that all the assumptions of Theorem 4.1 are verified. Note that although here the Rademacher random variables are indexed by $\mathbb{Z}$ instead of $\mathbb{N}$, Theorem 4.1 can be fully carried to this setting. Since the coefficient sequence $(a_i^{(n)})_{i\in\mathbb{Z}}$ is in $l^1(\mathbb{Z})$ we have $\left\|a^{(n)}\right\|_{l^p(\mathbb{Z})} < \infty\,\forall\, p\in\mathbb{N}$. By definition $\mathbb{E}[F] = 0$ and $\mathrm{Var}(F) = 1$, and the rewritten random variable

$$F := F_n = \frac{1}{\sqrt{\mathrm{Var}(G_n)}}\sum_{i\in\mathbb{Z}} a_i^{(n)}\left[\xi_i\xi_{i+1} - \frac{1}{4}\right] = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\sum_{i\in\mathbb{Z}} a_i^{(n)}\left[X_i + X_iX_{i+1} + X_{i+1}\right]$$

is bounded. In particular we will use $\mathbb{E}\,|G_n| \le \left\|a^{(n)}\right\|_{l^1(\mathbb{Z})}$ and

$$\frac{1}{16}\left\|a^{(n)}\right\|^2_{l^2(\mathbb{Z})} \le \mathrm{Var}(G_n) = \frac{3}{16}\sum_{i\in\mathbb{Z}}(a_i^{(n)})^2 + \frac{1}{8}\sum_{i\in\mathbb{Z}} a_i^{(n)}a_{i+1}^{(n)} \le \frac{5}{16}\left\|a^{(n)}\right\|^2_{l^2(\mathbb{Z})}. \tag{4.18}$$

Regarding the assumptions that $\frac{1}{\sqrt{pq}} DF\,|DL^{-1}F| \in Dom(\delta)$ and $Ff_z(F)+\mathbb{1}_{\{F>z\}} \in \mathbb{D}^{1,2}\,\forall\, z \in \mathbb{R}$, we follow the argumentation of [36], see in particular Remark 3.5 in there. F is an element of the sum of the first and second Rademacher chaos, see the beginning of the proof of Theorem 1.1 in [36], and by hypercontractivity we find that $F \in L^4(\Omega)$. Following the calculations in the proof of Lemma 3.7 in [29] with $u_k = \frac{1}{\sqrt{p_k q_k}} D_kF\,|D_kL^{-1}F|$ for $k \in \mathbb{Z}$, it can be shown that assumption (2.14) in Proposition 2.2 in [61] is satisfied. This implies that $u = \frac{1}{\sqrt{pq}} DF\,|DL^{-1}F| \in Dom(\delta)$. Further, it implies that $\mathbb{E}[(Ff_z(F)+\mathbb{1}_{\{F>z\}})\delta(u)] = \mathbb{E}[\langle D(Ff_z(F)+\mathbb{1}_{\{F>z\}}), u\rangle]$, which is why we do not need to verify whether $Ff_z(F)+\mathbb{1}_{\{F>z\}}$ is an element of $\mathbb{D}^{1,2}\,\forall\, z \in \mathbb{R}$.
We start to compute the terms appearing in (A1) and (A2). By definition

$$F_k^+ = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(\sum_{\substack{i\in\mathbb{Z}\\ i\neq k-1,k}} a_i^{(n)}\left[X_i + X_iX_{i+1} + X_{i+1}\right] + a_{k-1}^{(n)}(2X_{k-1}+1) + a_k^{(n)}(2X_{k+1}+1)\right)$$

$$F_k^- = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(\sum_{\substack{i\in\mathbb{Z}\\ i\neq k-1,k}} a_i^{(n)}\left[X_i + X_iX_{i+1} + X_{i+1}\right] - a_{k-1}^{(n)} - a_k^{(n)}\right)$$

and we get, for fixed $n \in \mathbb{N}$,

$$D_kF = \frac{1}{2}\left(F_k^+ - F_k^-\right) = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(a_{k-1}^{(n)}(X_{k-1}+1) + a_k^{(n)}(X_{k+1}+1)\right). \tag{4.19}$$

Further we obtain

$$-L^{-1}F = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\sum_{i\in\mathbb{Z}} a_i^{(n)}\left[X_i + \frac{1}{2}X_iX_{i+1} + X_{i+1}\right]$$

and so

$$-D_kL^{-1}F = \frac{1}{8\sqrt{\mathrm{Var}(G_n)}}\left(a_{k-1}^{(n)}(X_{k-1}+2) + a_k^{(n)}(X_{k+1}+2)\right). \tag{4.20}$$

Now it is easy to see that $F \in \mathbb{D}^{1,2}$. By definition $\mathrm{Var}(F) = 1$, so F is in particular square-integrable. The condition $\mathbb{E}(\sum_{k\in\mathbb{Z}}(D_kF)^2) < \infty$ follows from (4.19) and the fact that $(a_i^{(n)})_{i\in\mathbb{Z}}$ is in $l^1(\mathbb{Z})$ respectively $l^2(\mathbb{Z})$. Concerning the assumption $e^{tF} \in \mathbb{D}^{1,2}\,\forall\, t \in \mathbb{R}$ we notice at first that the boundness of F implies the square-integrability of e^{tF}. We compute further

$$\begin{aligned}
\mathbb{E}\left(\sum_{k\in\mathbb{Z}}(D_ke^{tF})^2\right) &= \mathbb{E}\left(\sum_{k\in\mathbb{Z}}\left(X_k\sqrt{pq}e^{tF}\left(1 - e^{-t\frac{X_k}{\sqrt{pq}}D_kF}\right)\right)^2\right)\\
&= \mathbb{E}\left(\sum_{k\in\mathbb{Z}} X_k^2\,pq\,e^{2tF}\left|1 - e^{-t\frac{X_k}{\sqrt{pq}}D_kF}\right|^2\right)\\
&\leq \mathbb{E}\left(\sum_{k\in\mathbb{Z}} pq\,e^{2tF}\,t^2\frac{X_k^2}{pq}(D_kF)^2 e^{2t\frac{|X_k|}{\sqrt{pq}}|D_kF|}\right)\\
&= \mathbb{E}\left(t^2e^{2tF}\sum_{k\in\mathbb{Z}}(D_kF)^2e^{2t\left|F_k^+ - F_k^-\right|}\right),
\end{aligned}$$

where we used $|e^x - 1| \leq e^{|x|} - 1 \leq |x|\,e^{|x|}\,\forall\, x \in \mathbb{R}$ and equation (12.2) in [80]. The boundness follows then from the boundness of the terms we have studied so far. With (4.19) and (4.20) we can compute the scalar product

$$\langle DF, -DL^{-1}F\rangle = \frac{1}{32\operatorname{Var}(G_n)}\Bigg(\sum_{k\in\mathbb{Z}}(a_{k-1}^{(n)})^2\left[X_{k-1}^2+3X_{k-1}+2\right] + (a_k^{(n)})^2\left[X_{k+1}^2+3X_{k+1}+2\right] + a_{k-1}^{(n)}a_k^{(n)}\left[3X_{k-1}+2X_{k-1}X_{k+1}+3X_{k+1}+4\right]\Bigg).$$

If we choose $f(x)=x$ in [61, (2.13)] we are able to write

$$1=\mathbb{E}[\langle DF, -DL^{-1}F\rangle] = \frac{1}{32\operatorname{Var}(G_n)}\sum_{k\in\mathbb{Z}}3(a_{k-1}^{(n)})^2+4a_{k-1}^{(n)}a_k^{(n)}+3(a_k^{(n)})^2.$$

We use the Cauchy–Schwarz inequality for

$$\mathbb{E}\left[\left|1-\langle DF, -DL^{-1}F\rangle\right|e^{tF}\right]\le\left(\mathbb{E}\left[\left(1-\langle DF, -DL^{-1}F\rangle\right)^2e^{tF}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}}$$

and deal with the double sum resulting from the square of

$$\langle DF, -DL^{-1}F\rangle - 1 = \frac{1}{32\operatorname{Var}(G_n)}\Bigg(\sum_{k\in\mathbb{Z}}(a_{k-1}^{(n)})^2 3X_{k-1}+(a_k^{(n)})^2 3X_{k+1} + a_{k-1}^{(n)}a_k^{(n)}\left[3X_{k-1}+2X_{k-1}X_{k+1}+3X_{k+1}\right]\Bigg).$$

Then we can write $\left(1-\langle DF, -DL^{-1}F\rangle\right)^2 = B_1+\ldots+B_9$ with

$$B_1=\frac{9}{1024(\operatorname{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_{k-1}^{(n)})^2(a_{l-1}^{(n)})^2X_{k-1}X_{l-1},$$

$$B_2=\frac{9}{1024(\operatorname{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_k^{(n)})^2(a_l^{(n)})^2X_kX_l,$$

$$B_3=\frac{9}{1024(\operatorname{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_{k-1}^{(n)})^2(a_l^{(n)})^2X_{k-1}X_l,$$

$$B_4=\frac{9}{1024(\operatorname{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_k^{(n)})^2(a_{l-1}^{(n)})^2X_kX_{l-1},$$

$$B_5=\frac{3}{1024(\operatorname{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_{k-1}^{(n)})^2(a_{l-1}^{(n)})(a_l^{(n)})X_{k-1}\left[3X_{l-1}+2X_{l-1}X_{l+1}+3X_{l+1}\right],$$

$$B_6=\frac{3}{1024(\operatorname{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_k^{(n)})^2(a_{l-1}^{(n)})(a_l^{(n)})X_k\left[3X_{l-1}+2X_{l-1}X_{l+1}+3X_{l+1}\right],$$

$$B_7=\frac{3}{1024(\operatorname{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_{k-1}^{(n)})(a_k^{(n)})(a_{l-1}^{(n)})^2X_{l-1}\left[3X_{k-1}+2X_{k-1}X_{k+1}+3X_{k+1}\right],$$

$$B_8=\frac{3}{1024(\operatorname{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_{k-1}^{(n)})(a_k^{(n)})(a_l^{(n)})^2X_l\left[3X_{k-1}+2X_{k-1}X_{k+1}+3X_{k+1}\right],$$

where $B_1=B_2=B_3=B_4$ and $B_5=B_6=B_7=B_8$ by symmetry and change of variables. The last missing term is given by

$$B_9=\frac{1}{1024(\operatorname{Var}(G_n))^2}\Bigg(\sum_{k,l\in\mathbb{Z}}(a_{k-1}^{(n)})(a_k^{(n)})(a_{l-1}^{(n)})(a_l^{(n)})\cdot\left[3X_{k-1}+2X_{k-1}X_{k+1}+3X_{k+1}\right] \cdot\left[3X_{l-1}+2X_{l-1}X_{l+1}+3X_{l+1}\right]\Bigg).$$

So basically we have to deal with three classes of subterms in total. Since they will be multiplied with e^{tF}, we have to study

$$\mathbb{E}[X_i e^{tF}], \quad \mathbb{E}[X_i X_j e^{tF}], \quad \mathbb{E}[X_i X_j X_k e^{tF}], \quad \mathbb{E}[X_i X_j X_k X_l e^{tF}]$$

for $i \neq j \neq k \neq l$ — if two or more indices are equal, it is just one of the terms from before or immediately $\mathbb{E}[e^{tF}]$. This is done in the following lemma and we will refer to it, in particular the inequalities shown in its proof.

Lemma 4.11. *In the setting of Theorem 4.7 we have for $i \neq j \neq k \neq l \in \mathbb{Z}$*

$$\begin{aligned}\left|\mathbb{E}[X_i e^{tF}]\right| \leq\ & \frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \cdot \left(\left|a_{i-1}^{(n)}\right| + \left|a_i^{(n)}\right|\right) \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ &+ \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1 \in \{i-1,i\} \\ n_1 \in \{i-2,i+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ &+ \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2, n_2 \in \{i-1,i\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.21}$$

$$\begin{aligned}\left|\mathbb{E}[X_i X_j e^{tF}]\right| \leq\ & \frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \cdot \left|a_{\min(i,j)}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \mathbb{1}_{\{|i-j|=1\}} \\ &+ \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1 \in \{i-1,i,j-1\} \\ n_1 \in \{i-2,i+1,j-2,j+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ &+ \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2, n_2 \in \{i-1,i,j-1,j\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.22}$$

$$\begin{aligned}\left|\mathbb{E}[X_i X_j X_k e^{tF}]\right| \leq\ & \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1 \in \{i-1,i,j-1,j,k-1,k\} \\ n_1 \in \{i-2,i+1,j-2,j+1,k-2,k+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ &+ \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2, n_2 \in \{i-1,i,j-1,j,k-1,k\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.23}$$

$$\begin{aligned}\left|\mathbb{E}[X_i X_j X_k X_l e^{tF}]\right| \leq\ & \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1 \in \{i-1,i,j-1,j,k-1,k,l-1,l\} \\ n_1 \in \{i-2,i+1,j-2,j+1,k-2,k+1,l-2,l+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ &+ \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2, n_2 \in \{i-1,i,j-1,j,k-1,k,l-1,l\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.24}$$

Proof. The first key element of our strategy is to split F into F_a, the summands that depend on the X's multiplied with e^{tF}, and F_u, the summands that are independent. We should have in mind that F_a and F_u are not necessarily independent from each other. To get this dependency structure under control we will make use of several Taylor expansions of the exponential. Note that there are remainder functions $r_1, r_2 : \mathbb{R} \to \mathbb{R}$, such that $e^x = 1 + x \cdot r_1(x) = 1 + x + \frac{x^2}{2} \cdot r_2(x)$ with $|r_1(x)|, |r_2(x)| \leq e^{\max\{0,x\}} \leq e^{|x|}$ for all $x \in \mathbb{R}$. So, the second key element is an iterated Taylor expansion on e^{tF} according to the following scheme: For a finite index set I, let there be real numbers $(x_j)_{j\in I}$, $(y_j)_{j\in I}$ and z. Then by iterated

Taylor expansion there is

$$
\begin{aligned}
e^{z+\sum_{j\in I} x_j} &= 1\cdot e^z + \sum_{j\in I} x_j \cdot e^z + \frac{1}{2}\Big(\sum_{j\in I} x_j\Big)^2 r_2\Big(\sum_{j\in I} x_j\Big)\cdot e^z \\
&= 1\cdot e^z + \sum_{j\in I} x_j \cdot 1\cdot e^{z-y_j} + \sum_{j\in I} x_j\cdot y_j r_1(y_j)\cdot e^{z-y_j} + \frac{1}{2}\Big(\sum_{j\in I} x_j\Big)^2 r_2\Big(\sum_{j\in I} x_j\Big)\cdot e^z.
\end{aligned} \tag{4.25}
$$

We remind on the short notation $A_k := a_k^{(n)}\left[X_k + X_kX_{k+1} + X_{k+1}\right]$ for the upcoming computations. In the case of $\mathbb{E}[X_i e^{tF}]$:

$$
F_a = \frac{1}{4\sqrt{\operatorname{Var}(G_n)}}(A_{i-1}+A_i)
$$

and by independence

$$
\begin{aligned}
\mathbb{E}[X_i e^{tF}] = \mathbb{E}[X_i e^{tF_a}e^{tF_u}] &= \mathbb{E}[X_i e^{tF_u}] + t\mathbb{E}[X_iF_a e^{tF_u}] + t^2\mathbb{E}[X_iF_a^2 r_2(tF_a)e^{tF_u}/2] \\
&= t\mathbb{E}[X_iF_a e^{tF_u}] + t^2\mathbb{E}[X_iF_a^2 r_2(tF_a)e^{tF_u}/2],
\end{aligned}
$$

where we chose $z = F_u$ and $\sum_{j\in I} x_j = F_a$ — note that I will increase with every case since the number of multiplied X's increases. For the first order term we split F_u in the same manner as before, $F_u = F_{u_a} + F_{u_u}$, such that $F_{u_a} = (A_{i-2}+A_{i+1})/4\sqrt{\operatorname{Var}(G_n)}$ and use the iteration from (4.25). Then

$$
X_iF_a = \frac{1}{4\sqrt{\operatorname{Var}(G_n)}}\Big(a_{i-1}^{(n)}(X_{i-1}X_i + X_{i-1} + 1) + a_i^{(n)}(1 + X_{i+1} + X_iX_{i+1})\Big)
$$

and

$$
\begin{aligned}
t\mathbb{E}[X_iF_a e^{tF_u}] &= t\mathbb{E}\left[X_iF_a e^{tF_{u_a}}e^{tF_{u_u}}\right] \\
&= t\mathbb{E}\left[X_iF_a e^{tF_{u_u}}\right] + t^2\mathbb{E}\left[X_iF_aF_{u_a} r_1(tF_{u_a})e^{tF_{u_u}}\right],
\end{aligned}
$$

so $y_j = F_{u_a}$. From here on we get e^{tF} back by bounding the difference of the independent part and F, e.g.

$$
|F_{u_u} - F| = |F_{u_a} + F_a| \le c = O\left(\frac{1}{\sqrt{\operatorname{Var}(G_n)}}\right) \to 0 \text{ as } n\to\infty.
$$

Since the exact constant is not important, we always write just c if we use that type of estimation, and in the same way C for prefactors. Thus

$$
\begin{aligned}
t\left|\mathbb{E}[X_iF_a e^{tF_u}]\right| \le& \frac{Ct}{4\sqrt{\operatorname{Var}(G_n)}}\cdot\left(\left|a_{i-1}^{(n)}\right| + \left|a_i^{(n)}\right|\right)\cdot\mathbb{E}\left[e^{tF}\right]e^{ct} \\
&+ \frac{Ct^2}{16\operatorname{Var}(G_n)}\cdot\sum_{\substack{m_1\in\{i-1,i\}\\ n_1\in\{i-2,i+1\}}}\left|a_{m_1}^{(n)}\right|\left|a_{n_1}^{(n)}\right|\cdot\mathbb{E}\left[e^{tF}\right]e^{ct}.
\end{aligned} \tag{4.26}
$$

For the second order term we just bound

$$
t^2\left|\mathbb{E}[X_iF_a^2 r_2(tF_a)e^{tF_u}/2]\right| \le \frac{Ct^2}{32\operatorname{Var}(G_n)}\cdot\sum_{m_2,n_2\in\{i-1,i\}}\left|a_{m_2}^{(n)}\right|\left|a_{n_2}^{(n)}\right|\cdot\mathbb{E}\left[e^{tF}\right]e^{ct}. \tag{4.27}
$$

In the case of $\mathbb{E}[X_iX_je^{tF}]$:

$$F_a = \begin{cases} \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(A_i + A_{j-1} + A_j\right), & i = j+1, \\ \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(A_{i-1} + A_i + A_{j-1} + A_j\right), & |i-j| \geq 2, \\ \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(A_{i-1} + A_i + A_j\right), & j = i+1, \end{cases}$$

and

$$\mathbb{E}[X_iX_je^{tF}] = t\mathbb{E}[X_iX_jF_ae^{tF_u}] + t^2\mathbb{E}[X_iX_jF_a^2r_2(tF_a)e^{tF_u}/2].$$

For the first order term we compute as a preparation

$$\begin{aligned} X_iX_jF_a = a_{i-1}^{(n)}(X_{i-1}X_iX_j + X_{i-1}X_j + X_j) &+ a_i^{(n)}(X_j + X_{i+1}X_j + X_iX_{i+1}X_j) \\ &+ a_{j-1}^{(n)}(X_iX_{j-1}X_j + X_iX_{j-1} + X_i) \\ &+ a_j^{(n)}(X_i + X_iX_{j+1} + X_iX_jX_{j+1}). \end{aligned}$$

In particular we have to consider the special case $|i-j| = 1$ and assume $i = j+1$. If not, we just have to swap i and j. Under our assumption the last equation reduces to

$$\begin{aligned} X_iX_jF_a = a_{i-1}^{(n)}(X_i + 1 + X_j) &+ a_i^{(n)}(X_j + X_{i+1}X_j + X_iX_{i+1}X_j) \\ &+ a_{j-1}^{(n)}(X_iX_{j-1}X_j + X_iX_{j-1} + X_i). \end{aligned}$$

From here on we assume that the indices apperaring in upcoming F_a's and F_{u_a}'s are all different. If not, there is only an effect on the number of coefficients and so the constants, but not on the order of our bound. Having that in mind we split F_u in the same manner as before, $F_u = F_{u_a} + F_{u_u}$, such that $F_{u_a} = (A_{i-2} + A_{i+1} + A_{j-2} + A_{j+1})/4\sqrt{\mathrm{Var}(G_n)}$. Then

$$t\mathbb{E}[X_iX_jF_ae^{tF_u}] = t\mathbb{E}[X_iX_jF_ae^{tF_{u_u}}] + t^2\mathbb{E}[X_iX_jF_aF_{u_a}r_1(tF_{u_a})e^{tF_{u_u}}]$$

and thus for $i = j+1$

$$\begin{aligned} t\left|\mathbb{E}[X_iX_jF_ae^{tF_u}]\right| \leq &\frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \cdot \left|a_{i-1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct} \\ &+ \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1 \in \{i-1,i,j-1\} \\ n_1 \in \{i-2,i+1,j-2,j+1\}}} \left|a_{m_1}^{(n)}\right|\left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \end{aligned} \tag{4.28}$$

In the case $|i-j| \geq 2$ the first term of the last inequality does not appear since all the indices in $X_iX_jF_a$ are different:

$$t\left|\mathbb{E}[X_iX_jF_ae^{tF_u}]\right| \leq \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1 \in \{i-1,i,j-1,j\} \\ n_1 \in \{i-2,i+1,j-2,j+1\}}} \left|a_{m_1}^{(n)}\right|\left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \tag{4.29}$$

For the second order term we just bound

$$t^2\left|\mathbb{E}[X_iX_jF_a^2r_2(tF_a)e^{tF_u}/2]\right| \leq \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2 \in \{i-1,i,j-1,j\}} \left|a_{m_2}^{(n)}\right|\left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \tag{4.30}$$

In the case of $\mathbb{E}[X_iX_jX_ke^{tF}]$:

$$F_a = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}(A_{i-1} + A_i + A_{j-1} + A_j + A_{k-1} + A_k)$$

and

$$\mathbb{E}[X_iX_jX_ke^{tF}] = t\mathbb{E}[X_iX_jX_kF_ae^{tF_u}] + t^2\mathbb{E}[X_iX_jX_kF_a^2r_2(tF_a)e^{tF_u}/2].$$

For the first order term we compute as a preparation

$$\begin{aligned}X_iX_jX_kF_a =& a_{i-1}^{(n)}(X_{i-1}X_iX_jX_k + X_{i-1}X_jX_k + X_jX_k)\\&+ a_i^{(n)}(X_jX_k + X_{i+1}X_jX_k + X_iX_{i+1}X_jX_k)\\&+ a_{j-1}^{(n)}(X_iX_{j-1}X_jX_k + X_iX_{j-1}X_k + X_iX_k)\\&+ a_j^{(n)}(X_iX_k + X_iX_{j+1}X_k + X_iX_jX_{j+1}X_k)\\&+ a_{k-1}^{(n)}(X_iX_jX_{k-1}X_k + X_iX_jX_{k-1} + X_iX_j)\\&+ a_k^{(n)}(X_iX_j + X_iX_jX_{k+1} + X_iX_jX_kX_{k+1}).\end{aligned}$$

And by our assumption $i \neq j \neq k$ in every summand at least one X will remain. We split $F_u = F_{u_a} + F_{u_u}$, such that $F_{u_a} = (A_{i-2} + A_{i+1} + A_{j-2} + A_{j+1} + A_{k-2} + A_{k+1})\,/4\sqrt{\mathrm{Var}(G_n)}$. Then

$$\begin{aligned}t\mathbb{E}[X_iX_jX_kF_ae^{tF_u}] &= t\mathbb{E}[X_iX_jX_kF_ae^{tF_{u_a}}e^{tF_{u_u}}]\\&= t\mathbb{E}[X_iX_jX_kF_ae^{tF_{u_u}}] + t^2\mathbb{E}[X_iX_jX_kF_aF_{u_a}r_1(tF_{u_a})e^{tF_{u_u}}]\\&= t^2\mathbb{E}[X_iX_jX_kF_aF_{u_a}r_1(tF_{u_a})e^{tF_{u_u}}]\end{aligned}$$

by independence and thus

$$t\left|\mathbb{E}[X_iX_jX_kF_ae^{tF_u}]\right| \leq \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i,j-1,j,k-1,k\}\\ n_1\in\{i-2,i+1,j-2,j+1,k-2,k+1\}}} \left|a_{m_1}^{(n)}\right|\left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \quad (4.31)$$

For the second order term we just bound

$$t^2\left|\mathbb{E}[X_iX_jX_kF_a^2r_2(tF_a)e^{tF_u}/2]\right| \leq \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i,j-1,j,k-1,k\}} \left|a_{m_2}^{(n)}\right|\left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \quad (4.32)$$

In the case of $\mathbb{E}[X_iX_jX_kX_le^{tF}]$:

$$F_a = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}(A_{i-1} + A_i + A_{j-1} + A_j + A_{k-1} + A_k + A_{l-1} + A_l)$$

and

$$\mathbb{E}[X_iX_jX_kX_le^{tF}] = t\mathbb{E}[X_iX_jX_kX_lF_ae^{tF_u}] + t^2\mathbb{E}[X_iX_jX_kX_lF_a^2r_2(tF_a)e^{tF_u}/2].$$

For the first order term we compute as a preparation

$$\begin{aligned}X_iX_jX_kX_lF_a =& a_{i-1}^{(n)}(X_{i-1}X_iX_jX_kX_l + X_{i-1}X_jX_kX_l + X_jX_kX_l)\\&+ a_i^{(n)}(X_jX_kX_l + X_{i+1}X_jX_kX_l + X_iX_{i+1}X_jX_kX_l)\\&+ a_{j-1}^{(n)}(X_iX_{j-1}X_jX_kX_l + X_iX_{j-1}X_kX_l + X_iX_kX_l)\\&+ a_j^{(n)}(X_iX_kX_l + X_iX_{j+1}X_kX_l + X_iX_jX_{j+1}X_kX_l)\\&+ a_{k-1}^{(n)}(X_iX_jX_{k-1}X_kX_l + X_iX_jX_{k-1}X_l + X_iX_jX_l)\\&+ a_k^{(n)}(X_iX_jX_l + X_iX_jX_{k+1}X_l + X_iX_jX_kX_{k+1}X_l)\\&+ a_{l-1}^{(n)}(X_iX_jX_kX_{l-1}X_l + X_iX_jX_kX_{l-1} + X_iX_jX_k)\\&+ a_k^{(n)}(X_iX_jX_k + X_iX_jX_kX_{l+1} + X_iX_jX_kX_lX_{l+1}).\end{aligned}$$

And by our assumption $i \neq j \neq k \neq l$ in every summand at least one X will remain. F_{u_a} is given by $F_{u_a} = (A_{i-2} + A_{i+1} + A_{j-2} + A_{j+1} + A_{k-2} + A_{k+1} + A_{l-2} + A_{l+1})/4\sqrt{\mathrm{Var}(G_n)}$ this time. Then

$$t\mathbb{E}[X_iX_jX_kX_lF_ae^{tF_u}] = t^2\mathbb{E}[X_iX_jX_kX_lF_aF_{u_a}r_1(tF_{u_a})e^{tF_{u_u}}]$$

by independence and thus

$$t\left|\mathbb{E}[X_iX_jX_kX_lF_ae^{tF_u}]\right| \leq \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i,j-1,j,k-1,k,l-1,l\}\\ n_1\in\{i-2,i+1,j-2,j+1,k-2,k+1,l-2,l+1\}}} \left|a_{m_1}^{(n)}\right|\left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \quad (4.33)$$

For the second order term we just bound

$$t^2\left|\mathbb{E}[X_iX_jX_kX_lF_a^2r_2(tF_a)e^{tF_u}/2]\right| \leq \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i,j-1,j,k-1,k,l-1,l\}} \left|a_{m_2}^{(n)}\right|\left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \quad (4.34)$$

□

Now we are ready to deal with all three classes of subterms and choose B_2, B_6 and B_9 as representatives:

First class of subterms, $B_1 - B_4$:

$$\mathbb{E}[B_2e^{tF}] = \frac{9}{1024(\mathrm{Var}(G_n))^2}\Bigg(\sum_{k\in\mathbb{Z}}(a_k^{(n)})^4\mathbb{E}[e^{tF}] + \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}}(a_k^{(n)})^2(a_l^{(n)})^2\mathbb{E}[X_kX_le^{tF}] + \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|\geq 2}}(a_k^{(n)})^2(a_l^{(n)})^2\mathbb{E}[X_kX_le^{tF}]\Bigg). \quad (4.35)$$

The first one is the easiest by

$$B_{21} := \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{k\in\mathbb{Z}}(a_k^{(n)})^4\mathbb{E}[e^{tF}] = \frac{9}{1024(\mathrm{Var}(G_n))^2}\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4\mathbb{E}[e^{tF}].$$

By using (4.28), (4.29) and (4.30) it remains to bound

$$\begin{aligned} B_{22} &:= \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}}(a_k^{(n)})^2(a_l^{(n)})^2\mathbb{E}[X_kX_le^{tF}] \\ &\leq \frac{9}{1024(\mathrm{Var}(G_n))^2}(B_{221}+B_{222}+B_{223})\,\mathbb{E}\left[e^{tF}\right]e^{ct}, \end{aligned}$$

$$\begin{aligned} B_{23} &:= \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{\substack{k,l\in\mathbb{Z}\\|k-l|\geq 2}}(a_k^{(n)})^2(a_l^{(n)})^2\mathbb{E}[X_kX_le^{tF}] \\ &\leq \frac{9}{1024(\mathrm{Var}(G_n))^2}(B_{231}+B_{232})\,\mathbb{E}\left[e^{tF}\right]e^{ct}, \end{aligned}$$

such that

$$B_{221} := \frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}}(a_k^{(n)})^2(a_l^{(n)})^2\left|a_{\min(k,l)}^{(n)}\right|,$$

$$B_{222} := \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}} (a_k^{(n)})^2(a_l^{(n)})^2 \cdot \sum_{\substack{m_1\in\{k-1,k,l-1,l\}\\n_1\in\{k-2,k+1,l-2,l+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right|,$$

$$B_{223} := \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}} (a_k^{(n)})^2(a_l^{(n)})^2 \cdot \sum_{m_2,n_2\in\{k-1,k,l-1,l\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right|,$$

$$B_{231} := \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|\geq 2}} (a_k^{(n)})^2(a_l^{(n)})^2 \cdot \sum_{\substack{m_1\in\{k-1,k,l-1,l\}\\n_1\in\{k-2,k+1,l-2,l+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right|,$$

$$B_{232} := \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|\geq 2}} (a_k^{(n)})^2(a_l^{(n)})^2 \cdot \sum_{m_2,n_2\in\{k-1,k,l-1,l\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right|.$$

Then by the inequality of arithmetic and geometric means, from here on AM-GM inequality,

$$\begin{aligned}
B_{221} &= \frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \left(\sum_{k\in\mathbb{Z}} (a_k^{(n)})^2(a_{k-1}^{(n)})^2 \left|a_{k-1}^{(n)}\right| + \sum_{k\in\mathbb{Z}} (a_k^{(n)})^2(a_{k+1}^{(n)})^2 \left|a_k^{(n)}\right| \right)\\
&= \frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \left(\sum_{k\in\mathbb{Z}} \sqrt[5]{\left|a_k^{(n)}\right|^{10} \left|a_{k-1}^{(n)}\right|^{10} \left|a_{k-1}^{(n)}\right|^{5}} + \sum_{k\in\mathbb{Z}} \sqrt[5]{\left|a_k^{(n)}\right|^{10} \left|a_{k+1}^{(n)}\right|^{10} \left|a_k^{(n)}\right|^{5}} \right)\\
&\leq \frac{Ct}{20\sqrt{\mathrm{Var}(G_n)}} \left(\sum_{k\in\mathbb{Z}} 2\left|a_k^{(n)}\right|^5 + 3\left|a_{k-1}^{(n)}\right|^5 + \sum_{k\in\mathbb{Z}} 3\left|a_k^{(n)}\right|^5 + 2\left|a_{k+1}^{(n)}\right|^5 \right)\\
&\leq \frac{Ct}{\sqrt{\mathrm{Var}(G_n)}} \left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^5.
\end{aligned}$$

If we look at $B_{222} - B_{232}$, we can change the order of summation since all summands are non-negative. And we become even bigger if we add the missing indices:

$$B_{223} \leq \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \sum_{k,l\in\mathbb{Z}} \sum_{m_2\in\{k-1,k,l-1,l\}} \sum_{n_2\in\{k-1,k,l-1,l\}} (a_k^{(n)})^2(a_l^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right|.$$

From here on we treat different cases, but every time we can use the AM-GM inequality:
Case 1: $m_2 \in \{k-1,k\}$ and $n_2 \in \{k-1,k\}$:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} (a_k^{(n)})^2(a_l^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| &= \sum_{k\in\mathbb{Z}} (a_k^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \sum_{l\in\mathbb{Z}} (a_l^{(n)})^2\\
&\leq C \left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4 \left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^2.
\end{aligned}$$

Case 2: $m_2 \in \{l-1,l\}$ and $n_2 \in \{l-1,l\}$:

$$\begin{aligned}
\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} (a_k^{(n)})^2(a_l^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| &= \sum_{l\in\mathbb{Z}} (a_l^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \sum_{k\in\mathbb{Z}} (a_k^{(n)})^2\\
&\leq C \left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4 \left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^2.
\end{aligned}$$

Case 3: $m_2 \in \{k-1,k\}$ and $n_2 \in \{l-1,l\}$:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} (a_k^{(n)})^2(a_l^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| &= \sum_{k\in\mathbb{Z}} (a_k^{(n)})^2 \left|a_{m_2}^{(n)}\right| \sum_{l\in\mathbb{Z}} (a_l^{(n)})^2 \left|a_{n_2}^{(n)}\right|\\
&\leq C \left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6.
\end{aligned}$$

Case 4: $m_2 \in \{l-1, l\}$ and $n_2 \in \{k-1, k\}$:

$$\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(a_k^{(n)})^2(a_l^{(n)})^2\left|a_{m_2}^{(n)}\right|\left|a_{n_2}^{(n)}\right| = \sum_{l\in\mathbb{Z}}(a_l^{(n)})^2\left|a_{m_2}^{(n)}\right|\sum_{k\in\mathbb{Z}}(a_k^{(n)})^2\left|a_{n_2}^{(n)}\right| \leq C\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6.$$

According to (4.18) in case 1 and 2 the norm $\left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^2$ vanishes directly with the variance in the prefactor. Summarizing for B_{223}:

$$B_{223} \leq \frac{Ct^2}{32}\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4 + \frac{Ct^2}{32\,\mathrm{Var}(G_n)}\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6.$$

Analogously we get basically bounds of the same order for B_{222}, B_{231} and B_{232}. Combining our bounds for the subterms of (4.35) gives us

$$\mathbb{E}[B_2e^{tF}] \leq Ce^{ct}\left(t^2\frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} + (1+t^2)\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} + t\frac{\left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^5}{(\mathrm{Var}(G_n))^{5/2}}\right)\mathbb{E}[e^{tF}].$$

Second class of subterms, $B_5 - B_8$: We write $\mathbb{E}[B_6e^{tF}] = B_{61} + B_{62} + B_{63}$ such that

$$B_{61} := \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_k^{(n)})^2(a_{l-1}^{(n)})(a_l^{(n)})\mathbb{E}[X_kX_{l-1}e^{tF}],$$
$$B_{62} := \frac{6}{1024(\mathrm{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_k^{(n)})^2(a_{l-1}^{(n)})(a_l^{(n)})\mathbb{E}[X_kX_{l-1}X_{l+1}e^{tF}],$$
$$B_{63} := \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_k^{(n)})^2(a_{l-1}^{(n)})(a_l^{(n)})\mathbb{E}[X_kX_{l+1}e^{tF}].$$

B_{61} and B_{63} are analogous to B_{22} and B_{23} since they have the same structure: Two coefficients with k-index, two coefficients with l-index, one X with k-index and one X with l-index. And with that the arguments are the same. Looking at the remaining B_{62} two indices of the X's are equal if and only if $k = l+1$ or $k = l-1$. In the latter case B_{62} reduces to

$$\frac{C}{(\mathrm{Var}(G_n))^2}\sum_{k\in\mathbb{Z}}(a_k^{(n)})^3(a_{k+1}^{(n)})\mathbb{E}[X_{k+2}e^{tF}],$$

so we can use (4.26) and (4.27) giving us upper bounds of order

$$O\left(t\frac{\left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^5}{(\mathrm{Var}(G_n))^{5/2}}\right) \quad\text{and}\quad O\left(t^2\frac{\left\|a^{(n)}\right\|_{l^6(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right).$$

And the same for $k = l+1$. At last, if neither $k = l-1$ nor $k = l+1$ we can use (4.31) and (4.32) giving us upper bounds of order

$$O\left(t^2\frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right) \quad\text{and}\quad O\left(t^2\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2}\right).$$

Combining our bounds for B_{61}, B_{62} and B_{63} we get

$$\mathbb{E}[B_6e^{tF}] \leq Ce^{ct}\left(t^2\frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} + t^2\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} + t\frac{\left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^5}{(\mathrm{Var}(G_n))^{5/2}} + t^2\frac{\left\|a^{(n)}\right\|_{l^6(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right)\mathbb{E}[e^{tF}].$$

Third class of subterms: It consists only of B_9 and so we have to deal with $\mathbb{E}[B_9 e^{tF}]$. Multiplying all the X's inside we get products of lengths two, three and four. The first two cases are already solved and a product of length four appears only one time, namely

$$\frac{C}{(\mathrm{Var}(G_n))^2} \sum_{k,l\in\mathbb{Z}} (a_{k-1}^{(n)})(a_k^{(n)})(a_{l-1}^{(n)})(a_l^{(n)})\mathbb{E}[X_{k-1}X_{k+1}X_{l-1}X_{l+1}e^{tF}].$$

We have two pairs of two equal indices of the X's if and only if $k = l$ and then we are in the situation of B_{21}. Note that it is impossible that three or more indices are equal. If two indices are equal and two indices are different, e.g. $k - 1 = l + 1$ we are in the $2X$-case and can use (4.28), (4.29) and (4.30). At last, if all four indices are different, most of the work is done by (4.33) and (4.34) leading to upper bounds of order

$$O\left(t^2 \frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right) \quad \text{and} \quad O\left(t^2 \frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2}\right).$$

Combining our bounds from all the different cases we get

$$\mathbb{E}[B_9 e^{tF}] \le Ce^{ct}\left(t^2 \frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} + (1+t^2)\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} + t\frac{\left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^5}{(\mathrm{Var}(G_n))^{5/2}} + t^2\frac{\left\|a^{(n)}\right\|_{l^6(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right)\mathbb{E}[e^{tF}].$$

Summarizing everything we have done so far a bound as in condition (A1) is obtained by

$$\begin{aligned}
\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{tF}\right] &\le \left(\mathbb{E}\left[\left(1 - \langle DF, -DL^{-1}F\rangle\right)^2 e^{tF}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}} \\
&= \left(\mathbb{E}\left[\sum_{i=1}^{9} B_i e^{tF}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}} \\
&\le \sum_{i=1}^{9}\left(\mathbb{E}\left[B_i e^{tF}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}} \\
&\le \widetilde{\gamma_1}(t)\mathbb{E}\left[e^{tF}\right]
\end{aligned}$$

such that

$$\begin{aligned}
\widetilde{\gamma_1}(t) &= Ce^{ct}\left(t\frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^3}{(\mathrm{Var}(G_n))^{3/2}} + (1+t)\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^2}{\mathrm{Var}(G_n)} + t^{1/2}\frac{\left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^{5/2}}{(\mathrm{Var}(G_n))^{5/4}} + t\frac{\left\|a^{(n)}\right\|_{l^6(\mathbb{Z})}^3}{(\mathrm{Var}(G_n))^{3/2}}\right) \\
&=: Ce^{ct}\left(tC_{n,1} + (1+t)C_{n,2} + t^{1/2}C_{n,3} + tC_{n,4}\right).
\end{aligned}$$

We now move on and show that a bound as in condition (A2) exists: Again, by the Cauchy–Schwarz inequality

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right| e^{tF}\right] \le \left(\mathbb{E}\left[\left(\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right)^2 e^{tF}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}}.$$

By [80, Corollary 9.9] it is $\delta(u) = \sum_{k=0}^{\infty} Y_k u_k$, where Y_k is the kth centered and standardized Rademacher random variable. In our case $Y_k = X_k$ and the corollary can be applied since $u_k := D_kF\left|D_kL^{-1}F\right|/\sqrt{p_kq_k}$ does not depend on X_k. Then

$$\left(\mathbb{E}\left[(\delta(u))^2 e^{tF}\right]\right)^{\frac{1}{2}} = \left(\sum_{k,l\in\mathbb{Z}} \mathbb{E}\left[u_k u_l X_k X_l e^{tF}\right]\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k\in\mathbb{Z}} \mathbb{E}\left[u_k^2 e^{tF}\right] \right)^{\frac{1}{2}} + \left(\sum_{\substack{k,l\in\mathbb{Z}\\k\neq l}} \mathbb{E}\left[u_k u_l X_k X_l e^{tF}\right] \right)^{\frac{1}{2}}. \tag{4.36}$$

For the upcoming computations we recall

$$D_k F = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}} \left(a_{k-1}^{(n)}(X_{k-1}+1) + a_k^{(n)}(X_{k+1}+1) \right),$$
$$-D_k L^{-1} F = \frac{1}{8\sqrt{\mathrm{Var}(G_n)}} \left(a_{k-1}^{(n)}(X_{k-1}+2) + a_k^{(n)}(X_{k+1}+2) \right).$$

and so

$$D_k F(-D_k L^{-1}F) = \frac{1}{32\,\mathrm{Var}(G_n)} \Big((a_{k-1}^{(n)})^2 \left[3X_{k-1}+3\right] + (a_k^{(n)})^2 \left[3X_{k+1}+3\right] + a_{k-1}^{(n)} a_k^{(n)} \left[3X_{k-1} + 2X_{k-1}X_{k+1} + 3X_{k+1} + 4\right] \Big).$$

The square of the righthandside is of a familiar form: Every summand consists of a product of length four of coefficients with index $k \pm ...$ multiplied with something bounded, and so as before we get immediately or by the AM-GM-inequality

$$\sum_{k\in\mathbb{Z}} \mathbb{E}\left[u_k^2 e^{tF}\right] \leq C \frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} \mathbb{E}\left[e^{tF}\right].$$

For the remaining term of (4.36) we adapt the strategy that is used in the proof of Lemma 4.11 — see its beginning for a detailed explanation. Set

$$A_k := a_k^{(n)} \left[X_k + X_k X_{k+1} + X_{k+1}\right]$$

so that

$$F_a = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}} (A_{k-2} + A_{k-1} + A_k + A_{k+1} + A_{l-2} + A_{l-1} + A_l + A_{l+1})$$

and by Taylor expansion

$$\mathbb{E}[u_k u_l X_k X_l e^{tF}] = \mathbb{E}[u_k u_l X_k X_l e^{tF_a} e^{tF_u}]$$
$$= \mathbb{E}[u_k u_l X_k X_l e^{tF_u}] + t\mathbb{E}[u_k u_l X_k X_l F_a e^{tF_u}] + t^2 \mathbb{E}[u_k u_l X_k X_l F_a^2 r_2(tF_a) e^{tF_u}/2].$$

0-order-term: By independence

$$\mathbb{E}[u_k u_l X_k X_l e^{tF_u}] = \mathbb{E}[u_k u_l X_k X_l]\mathbb{E}[e^{tF_u}].$$

Since by definition X_k and X_l respectively u_k are independent, we just have to check whether the same goes for X_k and u_l, which is leading to two cases.
Case 1: $l \notin \{k-1, k+1\}$

$$\mathbb{E}[u_k u_l X_k X_l] = \mathbb{E}[X_k]\mathbb{E}[u_k u_l X_l] = 0.$$

Case 2: $l \in \{k-1, k+1\}$

$$\sum_{k\in\mathbb{Z}} \mathbb{E}[u_k u_l X_k X_l]\mathbb{E}[e^{tF_u}] \leq C \frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} \mathbb{E}\left[e^{tF}\right] e^{ct},$$

following our usual argumentation.
1st-order-term: We split F_u in the same manner as before, $F_u = F_{u_a} + F_{u_u}$, such that

$F_{u_a} = (A_{k-3} + A_{k+2} + A_{l-3} + A_{l+2})/4\sqrt{\mathrm{Var}(G_n)}$ and use another Taylor expansion of degree 1. Then

$$\begin{aligned} t\mathbb{E}[u_k u_l X_k X_l F_a e^{tF_u}] &= t\mathbb{E}[u_k u_l X_k X_l F_a e^{tF_{u_a}} e^{tF_{u_u}}] \\ &= t\mathbb{E}[u_k u_l X_k X_l F_a e^{tF_{u_u}}] + t^2\mathbb{E}[u_k u_l X_k X_l F_a F_{u_a} r_1(tF_{u_a}) e^{tF_{u_u}}]. \end{aligned} \tag{4.37}$$

Note that F_{u_u} is — as part of F_u — independent of X_k, X_l, u_k and u_l, but also independent of F_a since we removed F_{u_a}, the depending part of F_u. As a consequence

$$\mathbb{E}[u_k u_l X_k X_l F_a e^{tF_{u_u}}] = \mathbb{E}[u_k u_l X_k X_l F_a]\mathbb{E}[e^{tF_{u_u}}].$$

Our next observation is $\mathbb{E}[u_k u_l X_k X_l F_a] = 0$ for $|k-l| \geq 5$ — in this case all appearing indices are different and the claim follows ultimately from independence. We treat the remaining case $|k-l| \leq 4$ as four subcases $|k-l| = i$ for $i \in \{1,2,3,4\}$, but here we just write down the first one $|k-l| = 1$ as the others are analogous and so there outcome. In the mentioned case, if $l = k+1$, we receive

$$t\,|u_k u_l X_k X_l F_a| \leq \frac{Ct}{(\mathrm{Var}(G_n))^{5/2}} \sum_{\substack{i_1,i_2 \in \{k-1,k\} \\ i_3,i_4 \in \{k,k+1\} \\ i_5 \in \{k-2,k-1,k,k+1,k+2\}}} \left|a_{i_1}^{(n)}\right| \left|a_{i_2}^{(n)}\right| \left|a_{i_3}^{(n)}\right| \left|a_{i_4}^{(n)}\right| \left|a_{i_5}^{(n)}\right|$$

and very similar for $l = k-1$. For both every summand consists of a product of five coefficients with index $k \pm ...$, and so we get

$$\sum_{\substack{k,l \in \mathbb{Z} \\ k \neq l \\ |k-l|=1}} t\left|\mathbb{E}[u_k u_l X_k X_l F_a]\mathbb{E}[e^{tF_{u_u}}]\right| \leq Ct \frac{\left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^5}{(\mathrm{Var}(G_n))^{5/2}} \mathbb{E}\left[e^{tF}\right] e^{ct}.$$

Having in mind that $|r_1(tF_{u_a})| \leq e^{t|F_{u_a}|}$ we can bound the second term of (4.37) by using

$$t^2\,|u_k u_l X_k X_l F_a F_{u_a}| \leq \frac{Ct^2}{(\mathrm{Var}(G_n))^3} \sum_{\substack{i_1,i_2 \in \{k-1,k\} \\ i_3,i_4 \in \{l-1,l\} \\ i_5 \in \{k-2,k-1,k,k+1,l-2,l-1,l,l+1\} \\ i_6 \in \{k-3,k+2,l-3,l+2\}}} \left|a_{i_1}^{(n)}\right| \left|a_{i_2}^{(n)}\right| \left|a_{i_3}^{(n)}\right| \left|a_{i_4}^{(n)}\right| \left|a_{i_5}^{(n)}\right| \left|a_{i_6}^{(n)}\right|$$

and every summand consists of a product of length six of either three coefficients with index $k \pm ...$ and three coefficients with index $l \pm ...$, or four coefficients with index $k \pm ...$ and two coefficients with index $l \pm ...$or the other way around. Combining the cases we get

$$\sum_{\substack{k,l \in \mathbb{Z} \\ k \neq l}} t^2\left|\mathbb{E}[u_k u_l X_k X_l F_a F_{u_a} r_1(tF_{u_a}) e^{tF_{u_u}}]\right| \leq Ct^2 \left(\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} + \frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} \right) \mathbb{E}\left[e^{tF}\right] e^{ct}.$$

2nd-order-term: Finally, having in mind that $|r_2(tF_a)| \leq e^{t|F_a|}$ we can bound the last term of our original Taylor expansion by using

$$t^2\left|u_k u_l X_k X_l F_a^2\right| \leq \frac{Ct^2}{(\mathrm{Var}(G_n))^3} \sum_{\substack{i_1,i_2 \in \{k-1,k\} \\ i_3,i_4 \in \{l-1,l\} \\ i_5,i_6 \in \{k-2,k-1,k,k+1,l-2,l-1,l,l+1\}}} \left|a_{i_1}^{(n)}\right| \left|a_{i_2}^{(n)}\right| \left|a_{i_3}^{(n)}\right| \left|a_{i_4}^{(n)}\right| \left|a_{i_5}^{(n)}\right| \left|a_{i_6}^{(n)}\right|$$

and get the analogous bound

$$\sum_{\substack{k,l \in \mathbb{Z} \\ k \neq l}} t^2\left|\mathbb{E}[u_k u_l X_k X_l F_a^2 r_2(tF_a) e^{tF_u}/2]\right| \leq Ct^2 \left(\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} + \frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} \right) \mathbb{E}\left[e^{tF}\right] e^{ct}.$$

Summarizing everything we have done so far a bound as in condition (A2) is obtained by

$$\begin{aligned}
\mathbb{E}\left[|\delta(u)|\, e^{tF}\right] &\le \Big(\mathbb{E}\left[(\delta(u))^2\, e^{tF}\right]\Big)^{\frac{1}{2}} \Big(\mathbb{E}\left[e^{tF}\right]\Big)^{\frac{1}{2}} \\
&\le \left(\left(\sum_{k\in\mathbb{Z}} \mathbb{E}\left[u_k^2 e^{tF}\right]\right)^{\frac{1}{2}} + \left(\sum_{\substack{k,l\in\mathbb{Z}\\ k\neq l}} \mathbb{E}\left[u_k u_l X_k X_l e^{tF}\right]\right)^{\frac{1}{2}}\right) \Big(\mathbb{E}\left[e^{tF}\right]\Big)^{\frac{1}{2}} \\
&\le \widetilde{\gamma_2}(t)\mathbb{E}\left[e^{tF}\right]
\end{aligned}$$

such that

$$\begin{aligned}
\widetilde{\gamma_2}(t) &= Ce^{ct}\left(t\frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^3}{(\mathrm{Var}(G_n))^{3/2}} + (1+t)\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^2}{\mathrm{Var}(G_n)} + t^{1/2}\frac{\left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^{5/2}}{(\mathrm{Var}(G_n))^{5/4}}\right) \\
&=: Ce^{ct}\Big(tC_{n,1} + (1+t)C_{n,2} + t^{1/2}C_{n,3}\Big)\,.
\end{aligned}$$

In a final step we want to simplify our bounds by comparing the constants $C_{n,i}$ with each other. To do so, we will use, for $m \ge 2$:

$$\begin{aligned}
\left\|a^{(n)}\right\|_{l^m(\mathbb{Z})}^m &= \sum_{k\in\mathbb{Z}}\Big(|a_k|^{m-1}\,|a_k|\Big) \\
&\le \sqrt{\sum_{k\in\mathbb{Z}}\Big(|a_k|^{m-1}\Big)^2 \sum_{k\in\mathbb{Z}}|a_k|^2} \\
&= \sqrt{\sum_{k\in\mathbb{Z}}\Big(|a_k|^{m-1}\Big)^2}\cdot C\cdot(\mathrm{Var}(G_n))^{1/2} \qquad (4.38) \\
&\le \sum_{k\in\mathbb{Z}}|a_k|^{m-1}\cdot C\cdot(\mathrm{Var}(G_n))^{1/2} \\
&= \left\|a^{(n)}\right\|_{l^{m-1}(\mathbb{Z})}^{m-1}\cdot C\cdot(\mathrm{Var}(G_n))^{1/2} \qquad (4.39)
\end{aligned}$$

by the Cauchy–Schwarz inequality. Then (4.38) and (4.39) imply

$$\begin{aligned}
C_{n,1} &= \sum_{k\in\mathbb{Z}}|a_k|^3\,(\mathrm{Var}(G_n))^{-3/2} \le \left(\sum_{k\in\mathbb{Z}}|a_k|^4\right)^{1/2}(\mathrm{Var}(G_n))^{-1} = C_{n,2}, \\
C_{n,3} &= \left(\sum_{k\in\mathbb{Z}}|a_k|^5\right)^{1/2}(\mathrm{Var}(G_n))^{-5/4} \le \left(\sum_{k\in\mathbb{Z}}|a_k|^4\,(\mathrm{Var}(G_n))^{1/2}\right)^{1/2}(\mathrm{Var}(G_n))^{-5/4} = C_{n,2}, \\
C_{n,4} &= \left(\sum_{k\in\mathbb{Z}}|a_k|^6\right)^{1/2}(\mathrm{Var}(G_n))^{-3/2} \le \left(\sum_{k\in\mathbb{Z}}|a_k|^5\,(\mathrm{Var}(G_n))^{1/2}\right)^{1/2}(\mathrm{Var}(G_n))^{-3/2} \le C_{n,2}.
\end{aligned}$$

So, we choose $\gamma_1(t) = \gamma_2(t) := Ce^{ct}\Big((1+t^{1/2}+t)C_n\Big)$ for (A1) and (A2), and $C_n := C_{n,2}$.

□

5. Non-uniform Berry–Esseen bounds for L^2-Rademacher- and L^2-Poisson-functionals

In this chapter we obtain non-uniform Berry–Esseen bounds for L^2-Rademacher and Poisson-functionals. The starting points are the corresponding uniform bounds, which were proven in [36] for the Rademacher respectively in [64] for the Poisson case in the context of normal approximation by Malliavin–Stein method. The core of our proof is then to show non-uniform bounds for the fragments of the Stein-equation; these bounds require the existence of higher moments of the functionals and lead ultimately to the important prefactor of our non-uniform Berry–Esseen bounds. In the second section we study two applications in the Rademacher case, namely infinte weighted 2-runs and subgraph counting in the Erdős–Rényi random graph.

5.1. **Main Results.** We present our first main result:

Theorem 5.1 (Non-uniform Berry–Esseen bound for Rademacher-functionals)**.** *Let $F \in \mathbb{D}^{1,2}$ with $\mathbb{E}[F] = 0, \mathrm{Var}(F) = 1$ and $\mathbb{E}[F^{2k}] < C$ for fixed $k \in \mathbb{N}$. Further*

$$F f_z(F) + \mathbb{1}_{\{F>z\}} \in \mathbb{D}^{1,2} \quad \forall z \in \mathbb{R},$$

$$\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right| \in Dom(\delta).$$

Then, for any $z \in \mathbb{R}$,

$$|\mathbb{P}(F \leq z) - \Phi(z)| \leq \frac{C}{(1+|z|)^k} \left(\left(\mathbb{E}\Big(1 - \langle DF, -DL^{-1}F\rangle\Big)^2 \right)^{1/2} + \left(\mathbb{E}\left(\delta\left(\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right| \right) \right)^2 \right)^{1/2} \right),$$

and C is a constant depending on $k \in \mathbb{N}$.

Proof of Theorem 5.1. By Stein's method and the proof of [36, Theorem 3.1] we have for $z \in \mathbb{R}$

$$|\mathbb{P}(F \leq z) - \Phi(z)| = |\mathbb{E}[f_z'(F) - F f_z(F)]| \leq J_1 + J_2 \tag{5.1}$$

with

$$J_1 := \mathbb{E}\left| f_z'(F)\Big(1 - \langle DF, -DL^{-1}F\rangle\Big) \right|,$$
$$J_2 := \mathbb{E}\left[(F f_z(F) + \mathbb{1}_{\{F>z\}}) \delta\left(\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right| \right) \right].$$

For the upcoming computation we split J_2 into two terms, namely

$$|J_2| \leq \mathbb{E}\left[\left| \delta\left(\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right| \right) \right| \left| F f_z(F) + \mathbb{1}_{\{F>z\}} \right| \right] \leq J_{21} + J_{22}$$

with

$$J_{21} := \mathbb{E}\left[\left| \delta\left(\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right| \right) \right| |F f_z(F)| \right],$$
$$J_{22} := \mathbb{E}\left[\left| \delta\left(\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right| \right) \right| \mathbb{1}_{\{F>z\}} \right].$$

Now we continue by applying the Cauchy–Schwarz inequality on every of our terms of interest, such that

$$J_1 \leq (\mathbb{E}\,|f_z'(F)|^2)^{1/2}\Big(\mathbb{E}\big(1-\langle DF,-DL^{-1}F\rangle\big)^2\Big)^{1/2},$$

$$J_{21} \leq (\mathbb{E}\,|Ff_z(F)|^2)^{1/2}\left(\mathbb{E}\left(\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right)^2\right)^{1/2},$$

$$J_{22} \leq (\mathbb{P}(F>z))^{1/2}\left(\mathbb{E}\left(\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right)^2\right)^{1/2}.$$

To bound the fragments of the Stein-equation we refer to the proof of [67, Theorem 2], where the authors considered Stein-equations fo non-normal approximation, but their arguments are also valid for normal approximation by choosing $g(x)=x$ in their framework. In particular, in their condition (A4) we can choose τ arbitrary, e.g. $\tau=\frac{1}{2}$. We extend our adaption of the authors work by also referring to the proof of [67, Theorem 3]. So, in what follows we will mention the relevant passages of both proofs explicitly. Note, that our appearing constants will very likely depend on $k\in\mathbb{N}$, but we will just write C. The core of our proof is to show that

$$(\mathbb{E}\,|f_z'(F)|^2)^{1/2} \leq \frac{C}{(1+|z|)^k}, \tag{5.2}$$

$$(\mathbb{E}\,|Ff_z(F)|^2)^{1/2} \leq \frac{C}{(1+|z|)^k}, \tag{5.3}$$

$$(\mathbb{P}(F>z))^{1/2} \leq \frac{C}{(1+|z|)^k}. \tag{5.4}$$

case $z>0$:
Proof of (5.2) and (5.4): We follow the proof of inequality (12) in [67], such that we write

$$\mathbb{E}\,|f_z'(F)|^2 = \mathbb{E}[f_z'(F)^2\mathbb{1}_{\{F\leq 0\}}] + \mathbb{E}[f_z'(F)^2\mathbb{1}_{\{0<F\leq z/2\}}] + \mathbb{E}[f_z'(F)^2\mathbb{1}_{\{F>z/2\}}]. \tag{5.5}$$

and bound these terms separately. First, by inequality (14) in [67], we get

$$\mathbb{E}[f_z'(F)^2\mathbb{1}_{\{F\leq 0\}}] \leq \frac{Ce^{-z^2}}{z^2} = \frac{Cz^{2k-2}e^{-z^2}}{z^{2k}} \leq \frac{C}{z^{2k}},$$

using in particular $z^{2l}e^{-z^2}\leq C\;\forall\, l\in\mathbb{N}$. Secondly,

$$\begin{aligned}\mathbb{E}[f_z'(F)^2\mathbb{1}_{\{0<F\leq z/2\}}] &\leq \left[C\left(1+\frac{z}{2}e^{z^2/8}\right)^2 e^{-z^2}\frac{1}{z^2}\right] \\ &\leq \left[C\left(\frac{e^{-z^2}}{z^2}+e^{-3z^2/4}\right)\right] \\ &= \left[C\left(\frac{z^{2k-2}e^{-z^2}}{z^{2k}}+\frac{z^{2k}e^{-3z^2/4}}{z^{2k}}\right)\right] \\ &\leq \frac{C}{z^{2k}}.\end{aligned}$$

This technique of adding the desired exponent of z was used in the middle of the proof of [67, Theorem 3]. Last, we can bound the third term of (5.5) and show also (5.4) by using (2.12) and Markov's inequality for

$$\mathbb{P}(F>z) \leq \mathbb{P}(F^{2k}>z^{2k}) \leq \frac{\mathbb{E}[F^{2k}]}{z^{2k}} < \frac{C}{z^{2k}} \quad \forall\, z>0.$$

Proof of (5.3): We adapt the proof of inequality (17) in [67] for the following bounds:

$$\begin{aligned}\mathbb{E}[(Ff_z(F))^2] &= \mathbb{E}\left[\left(f_z'(F) - \left(\mathbb{1}_{\{F\le z\}} - \Phi(z)\right)\right)^2\right] \\ &\le C\left(\mathbb{E}\left[(f_z'(F))^2\right] + \mathbb{E}\left[\left(\mathbb{1}_{\{F\le z\}} - \Phi(z)\right)^2\right]\right) \\ &\le C\left(\frac{C}{z^{2k}} + \mathbb{E}\left[\left(\mathbb{1}_{\{F\le z\}} - \Phi(z)\right)^2\right]\right), \end{aligned} \tag{5.6}$$

where we used the Stein-equation (2.4) as well as the AM-GM-inequality and (5.2) for the first respectively the second inequality. For the remaining expectation we obtain

$$\mathbb{E}\left[\left(\mathbb{1}_{\{F\le z\}} - \Phi(z)\right)^2\right] = \mathbb{E}\left[(1-\Phi(z))^2\,\mathbb{1}_{\{F\le z\}}\right] + \mathbb{E}\left[\Phi(z)^2\mathbb{1}_{\{F>z\}}\right] \le \frac{C}{z^{2k}} \tag{5.7}$$

by Markov's inequality and inequality (13) in [67].

case $z \le 0$:

To treat this case, first of all we can go into the proof of [36, Theorem 3.1], having in mind that $1 = \mathbb{1}_{\{F>z\}} + \mathbb{1}_{\{F\le z\}}$, and receive a bound very similar to (5.1), where J_1 stays the same but the indicator inside J_2 changes to $\mathbb{1}_{\{F\le z\}}$. A direct consequence of this procedure is a modified (5.4) that we prove with the same arguments:

$$\mathbb{P}(F\le z) \le \mathbb{P}(F^{2k} > z^{2k}) \le \frac{\mathbb{E}[F^{2k}]}{z^{2k}} < \frac{C}{z^{2k}} \quad \forall\, z\le 0.$$

Proof of (5.2) and (5.3): As the authors explain themselves at the end of the proof of [67, Theorem 2] no big modifications of their argumentation are needed. The modified version of (5.5) is given by

$$\mathbb{E}\,|f_z'(F)|^2 = \mathbb{E}[f_z'(F)^2\mathbb{1}_{\{F\le z/2\}}] + \mathbb{E}[f_z'(F)^2\mathbb{1}_{\{z/2\le F\le 0\}}] + \mathbb{E}[f_z'(F)^2\mathbb{1}_{\{F>0\}}],$$

and similar arguments lead to (5.2) in this case.

To prove (5.3) in the case $z\le 0$, we can keep (5.6) as in the previous case, but slightly change (5.7) to

$$\mathbb{E}\left[\left(\mathbb{1}_{\{F\le z\}} - \Phi(z)\right)^2\right] = 2\mathbb{P}(F\le z) + 2\Phi(z)^2 \le \frac{C}{z^{2k}}$$

by the AM-GM- and Markov's inequality as well as inequality (25) in [67].

To obtain (5.2) – (5.4) only a few steps are missing. Having in mind that (2.10) and (2.12) hold, it is possible to consider the minimum of our bounds and 1 for all substantial subterms. Similar to [67] this goes along with

$$\min\left(1, \frac{C}{|z|^{2k}}\right) \le \frac{C}{(1+|z|)^{2k}}, \tag{5.8}$$

which we would like to show explicitly with respect to our more general exponent of z, compared to [67]. If $|z|^{2k} \le C$, the minimum is 1 and (5.8) is equivalent to

$$(1+|z|)^{2k} \le C,$$

which is true, as our assumption implies $|z|^l \le C$ for $1 \le l \le 2k$. If $|z|^{2k} > C$, the minimum is $\frac{C}{|z|^{2k}}$ and (5.8) is equivalent to

$$C\frac{(1+|z|)^{2k}}{|z|^{2k}} \le C.$$

We show this last inequality by the Binomial theorem, which leads to

$$\begin{aligned} C\frac{(1+|z|)^{2k}}{|z|^{2k}} &= C\sum_{l=0}^{2k}\binom{2k}{l}|z|^{l-2k} \\ &\le C\left(\frac{1}{|z|^{2k}}+\cdots+C_{2k,s}\frac{1}{|z|^{s}}+\cdots+1\right) \\ &< C\left(\frac{1}{C}+\cdots+C_{2k,s}\frac{1}{C^{s/2k}}+\cdots+1\right) \\ &= 1+\cdots+C_{2k,s}C^{1-s/2k}+\cdots+C \\ &\le C \end{aligned}$$

where $C_{2k,s}$ is a constant to bound the corresponding binomial coefficient.
Finally, we receive the desired prefactor of the theorem by taking the squareroot at last.

□

Remark 5.2. Since we adapted the arguments of [67] and [97] we were able to prove non-uniform Berry–Esseen bounds without use of concentration inequalities. Although we can write down our general result for any $k \in \mathbb{N}$, the case $k = 3$ is of special interest for us: It visualizes that we achieve a prefactor of order $(1+|z|)^{-3}$ only at the cost of existing sixth moments of our functionals. If we compare our result to the classical i.i.d.-result, we see that we are unfortunately not as good as the authors of [21]. Nevertheless our result makes it possible to obtain non-uniform Berry–Esseen bounds without much effort, if the uniform case was already studied.

A direct consequence of Theorem 5.1 is the following result, mostly obtained by a second-order Gaussian Poincaré inequality given by [36, Theorem 4.1]:

Theorem 5.3 (Non-uniform second-order Gaussian Poincaré inequality)**.** *Let $F \in \mathbb{D}^{1,2}$ with $\mathbb{E}[F] = 0, \mathrm{Var}(F) = 1$ and $\mathbb{E}[F^{2k}] < C$ for fixed $k \in \mathbb{N}$. Further*

$$F f_z(F) + \mathbb{1}_{\{F>z\}} \in \mathbb{D}^{1,2} \quad \forall z \in \mathbb{R},$$

$$\frac{1}{\sqrt{pq}} DF \left| DL^{-1}F \right| \in Dom(\delta).$$

Then, for any $z \in \mathbb{R}$,

$$|\mathbb{P}(F \le z) - \Phi(z)| \le \frac{C}{(1+|z|)^k}\left(\frac{\sqrt{15}}{2}\sqrt{B_1} + \frac{\sqrt{3}}{2}\sqrt{B_2} + 2\sqrt{B_3} + 2\sqrt{6}\sqrt{B_4} + 2\sqrt{3}\sqrt{B_5}\right),$$

where

$$\begin{aligned} B_1 &:= \sum_{j,k,l\in\mathbb{N}} \sqrt{\mathbb{E}[(D_jF)^2(D_kF)^2]}\sqrt{\mathbb{E}[(D_lD_jF)^2(D_lD_kF)^2]}, \\ B_2 &:= \sum_{j,k,l\in\mathbb{N}} \frac{1}{p_lq_l}\mathbb{E}[(D_lD_jF)^2(D_lD_kF)^2], \\ B_3 &:= \sum_{k\in\mathbb{N}} \frac{1}{p_kq_k}\mathbb{E}[(D_kF)^4], \\ B_4 &:= \sum_{k,l\in\mathbb{N}} \frac{1}{p_kq_k}\sqrt{\mathbb{E}[(D_kF)^4]}\sqrt{\mathbb{E}[(D_lD_kF)^4]}, \\ B_5 &:= \sum_{k,l\in\mathbb{N}} \frac{1}{p_kq_kp_lq_l}\mathbb{E}[(D_lD_kF)^4], \end{aligned}$$

and C is a constant depending on $k \in \mathbb{N}$.

Proof of Theorem 5.3. By Theorem 5.1 we already know that

$$|\mathbb{P}(F \le z) - \Phi(z)| \le \frac{C}{(1+|z|)^k}\left(\left(\mathbb{E}\Big(1 - \langle DF, -DL^{-1}F\rangle\Big)^2\right)^{1/2} + \left(\mathbb{E}\left(\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right)^2\right)^{1/2}\right).$$

Moreover in the proof of [36, Theorem 4.1] it is shown that

$$\left(\mathbb{E}\Big(1 - \langle DF, -DL^{-1}F\rangle\Big)^2\right)^{1/2} \le \frac{\sqrt{15}}{2}\sqrt{B_1} + \frac{\sqrt{3}}{2}\sqrt{B_2},$$

$$\left(\mathbb{E}\left(\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right)^2\right)^{1/2} \le 2\sqrt{B_3} + 2\sqrt{6}\sqrt{B_4} + 2\sqrt{3}\sqrt{B_5}.$$

□

It is also possible to formulate an analogous version of Theorem 5.1 for Poisson functionals:

Theorem 5.4 (Non-uniform Berry–Esseen bound for Poisson-functionals)**.** *Let $F \in \hat{\mathbb{D}}^{1,2}$ with $\mathbb{E}[F] = 0, \operatorname{Var}(F) = 1$ and $\mathbb{E}[F^{2k}] < C$ for fixed $k \in \mathbb{N}$. Further*

$$\mathbb{E}\int_X\int_X \left[D_y\left(D_xF\left|D_xL^{-1}F\right|\right)\right]^2 \mu^2(dx,dy) < \infty,$$

$$Ff_z(F) \in \hat{\mathbb{D}}^{1,2} \quad \forall z \in \mathbb{R}.$$

Then, for any $z \in \mathbb{R}$,

$$|\mathbb{P}(F \le z) - \Phi(z)| \le \frac{C}{(1+|z|)^k}\left(\left(\mathbb{E}\Big(1 - \langle DF, -DL^{-1}F\rangle\Big)^2\right)^{1/2} + \left(\mathbb{E}\Big(\delta\Big(DF\left|DL^{-1}F\right|\Big)\Big)^2\right)^{1/2}\right),$$

and C is a constant depending on $k \in \mathbb{N}$.

Proof of Theorem 5.4. By Stein's method and the proof of [64, Theorem 1.12](special case $\mathbb{E}[F] = 0, \sigma^2 = 1$) we have for $z \in \mathbb{R}$

$$|\mathbb{P}(F \le z) - \Phi(z)| = |\mathbb{E}[f_z'(F) - Ff_z(F)]| \le J_1 + J_2$$

with

$$J_1 := \mathbb{E}\left|f_z'(F)\Big(1 - \langle DF, -DL^{-1}F\rangle\Big)\right|,$$

$$J_2 := \mathbb{E}\left[(Ff_z(F) + \mathbb{1}_{\{F>z\}})\delta\Big(DF\left|DL^{-1}F\right|\Big)\right].$$

Since we consider normal approximation, from here on the proof is just an adaption of the proof of Theorem 5.1. □

Remark 5.5. We summarize our results: We have shown non-uniform Berry–Esseen bounds for Rademacher- and Poisson-functionals. The core of the proofs, including [67, Theorem 2], is to start from the known terms of the corresponding uniform Berry–Esseen bound and separate them by the Cauchy–Schwarz inequality into a part, which highly depends on the used technique as exchangeable pairs or Malliavin–Stein, and a part, which consists of a fragment of a Stein-equation. While the first ones can be kept almost as the uniform Berry–Esseen bound, the latter ones have to be bounded precisely, which requires the existence of higher moments for our arguments.

5.2. **Applications.**

5.2.1. *Application: Infinte weighted 2-runs.* We recall some basic notations and properties. Let $X = (X_i)_{i\in\mathbb{Z}}$ be a double-sided sequence of *i.i.d.* Rademacher random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$ and let for each $n \in \mathbb{N}, (a_i^{(n)})_{i\in\mathbb{Z}}$ be a double-sided square-summable sequence of real numbers.
The sequence $(F_n)_{n\in\mathbb{N}}$ of standardized infinte weighted 2-runs is then defined as

$$F_n := \frac{G_n - \mathbb{E}[G_n]}{\sqrt{\mathrm{Var}(G_n)}}, \quad G_n := \sum_{i\in\mathbb{Z}} a_i^{(n)} \xi_i \xi_{i+1}, \quad n \in \mathbb{N},$$

where $\xi_i := \frac{X_i+1}{2}$ for $i \in \mathbb{Z}$. Since it is often nice to work with centered random variables we rewrite F_n as

$$F := F_n = \frac{1}{\sqrt{\mathrm{Var}(G_n)}} \sum_{i\in\mathbb{Z}} a_i^{(n)} \left[\xi_i\xi_{i+1} - \frac{1}{4}\right] = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}} \sum_{i\in\mathbb{Z}} a_i^{(n)} \left[X_i + X_iX_{i+1} + X_{i+1}\right].$$

We recall further $\left\|a^{(n)}\right\|_{l^p(\mathbb{Z})} := \left(\sum_{i\in\mathbb{Z}} |a_i|^p\right)^{1/p} < \infty \, \forall p \geq 2$ as $(a_i^{(n)})_{i\in\mathbb{Z}} \in l^2(\mathbb{Z})$, as well as $\mathbb{E}[X^k] = 1$ for k even and $\mathbb{E}[X^k] = 0$ for k odd, and $\mathrm{Var}(G_n) = O\left(\left\|a^{(n)}\right\|^2_{l^2(\mathbb{Z})}\right)$ by (4.18).
Our corresponding result is given as follows.

Theorem 5.6 (Non-uniform Berry–Esseen bound for 2-runs)**.** *In the setting of infinite weighted 2-runs from above we have*

$$|\mathbb{P}(F \leq z) - \Phi(z)| \leq \frac{C}{(1+|z|)^3} \left(\frac{\left\|a^{(n)}\right\|^2_{l^4(\mathbb{Z})}}{\|a^{(n)}\|^2_{l^2(\mathbb{Z})}} \right),$$

and C is a constant depending on the coefficient sequence $(a_i^{(n)})_{i\in\mathbb{Z}}$.

Proof. We want to apply Theorem 5.1 for $k = 3$, so we have to show that

$$\mathbb{E}[F_n^6] = \frac{1}{4^6(\mathrm{Var}(G_n))^3} \sum_{(i,j,k,l,m,r)\in\mathbb{Z}^6} a_i^{(n)} a_j^{(n)} a_k^{(n)} a_l^{(n)} a_m^{(n)} a_r^{(n)} \mathbb{E}(A_i \cdot \ldots \cdot A_r) \leq C < \infty, \tag{5.9}$$

where $A_i = [X_i + X_iX_{i+1} + X_{i+1}]$. If we compute $A_i \cdot \ldots \cdot A_r$ for a fixed multi-index (i, j, k, l, m, r), we get 3^6 summands in total. Among these summands the following cases are possible (apart from the designation of the indices, e.g. i or $i + 1$):

(i) 6 single X's, 0 pairs, so 6 X's in total,
(ii) 5 single X's, 1 pair, so 7 X's in total,
(iii) 4 single X's, 2 pairs, so 8 X's in total,
(iv) 3 single X's, 3 pairs, so 9 X's in total,
(v) 2 single X's, 4 pairs, so 10 X's in total,
(vi) 1 single X, 5 pairs, so 11 X's in total,
(vii) 0 single X's, 6 pairs, so 12 X's in total

and by a *pair* we denote a term of the form X_iX_{i+1}.
case (i): When is $\mathbb{E}[X_iX_jX_kX_lX_mX_r] \neq 0$? Since the odd moments of X are equal to 0 and the random variables are independent this happens if and only if the numbers of equal indices are even, namely

- 6 equal indices, e.g. $\mathbb{E}[X_i^6] = 1$,
- 4 and 2 equal indices, e.g. $\mathbb{E}[X_i^4] \cdot \mathbb{E}[X_i^2] = 1$,
- 2 and 2 and 2 equal indices, e.g. $\mathbb{E}[X_i^2] \cdot \mathbb{E}[X_j^2] \cdot \mathbb{E}[X_k^2] = 1$.

Note, that the indices of $a^{(n)}$ and X do not have to be exactly the same, but up to a natural number. In any case the product of the $a^{(n)\prime}s$ has length six and with respect to the subcases of equal indices mentioned before, we can bound the corresponding subterms by the AM-GM-inequality, cf. page 82 for examples of this strategy, to get

$$\frac{C\left\|a^{(n)}\right\|_{l^6(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} = O(1)$$

and

$$\frac{C\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4\left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^2}{(\mathrm{Var}(G_n))^3} = O(1)$$

and

$$\frac{C\left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} = O(1),$$

where we also used $\mathrm{Var}(G_n) = O\Big(\left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^2\Big)$ and $\left\|a^{(n)}\right\|_{l^m(\mathbb{Z})}^m \leq \left\|a^{(n)}\right\|_{l^{m-1}(\mathbb{Z})}^{m-1} \cdot C \cdot \mathrm{Var}(G_n))^{1/2}$, see (4.39).
case (ii): Since $i \neq i+1$ it is $\mathbb{E}[X_iX_{i+1}X_jX_kX_lX_mX_r] = 0$ for any index combiantion of this type.
case (iii): Since $i \neq i+1$ and $j \neq j+1$ there can not be more than six equal indices in $\mathbb{E}[X_iX_{i+1}X_jX_{j+1}X_kX_lX_mX_r]$ for any index combination of this type. In other words, equal indices are from (up to six) different index sets. Note further that although the number of $X's$ increases with every case, the number of $a^{(n)\prime}s$ always stays at six. So up to a certain degree we have a freedom of choice how we construct our $\left\|a^{(n)}\right\|_{l^m(\mathbb{Z})}$, but $m \leq 6$. Basically our bound will be a mixture of the norms appearing in case (i).
cases (iv) and (vi): These cases are analogous to (ii).
cases (v) and (vii): These cases are analogous to (iii).
Putting all the cases together we have shown (5.9). Now we refer to the uniform bound in [36, Theorem 1.1] to receive the desired result. □

5.2.2. *Application: Subgraph counting in the Erdős–Rényi random graph.* Again, we recall some basic notations. We start with the complete graph on n vertices and keep an edge with probability $p \in [0,1]$, while we remove it with probability $q := 1-p$, for all edges independently from each other. The outcome is known as the classical Erdős–Rényi random graph $\mathbf{G}$(n,p) and in many applications p depends on n. We fix a graph G_0 with at least one edge and consider the number W of subgraphs $H \subset \mathbf{G}(n,p)$, which are isomorphic to G_0. The corresponding standardized random variable is then defined as

$$F := \frac{W - \mathbb{E}[W]}{\sqrt{\mathrm{Var}(W)}},$$

which is basically the standardized number of copies of G_0 in $G(n,p)$.
For our result we have to define the important quantity

$$\Psi := \min_{\substack{H \subset G_0 \\ e_H \geq 1}} \left\{ n^{v_H} p^{e_H} \right\},$$

where v_H denotes the number of vertices of a subgraph H of G_0 and e_H the number of edges, respectively. Our corresponding result is given as follows.

Theorem 5.7 (Non-uniform Berry–Esseen bound for subgraph counts)**.** *In the setting of subgraph counts in the Erdős–Rényi random graph from above we have*

$$|\mathbb{P}(F \leq z) - \Phi(z)| \leq \frac{C}{(1+|z|)^k} O\Big((q\Psi)^{-\frac{1}{2}}\Big),$$

and C is a constant depending on $k \in \mathbb{N}$.

Proof. In [86, Theorem 2] a central limit theorem for F was shown and the core of the proof was the method of moments. The idea of this method is to show that the moments of F converge to the moments of the standard normal distribution, which is uniquely determined by its moments. So, in particular all moments of F are bounded and we can apply Theorem 5.1. Now we refer to the uniform bound in [36, Theorem 1.2] to receive the desired result. □

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