Intuitionistic Views on Connexive Constructible Falsity

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Abstract

Intuitionistic logicians generally accept that a negation can be understood as an implication to absurdity. An alternative account of constructive negation is to define it in terms of a primitive notion of falsity. This approach was originally suggested by D. Nelson, who called the operator constructible falsity, as complementing certain constructive aspects of negation. For intuitionistic logicians to be able to understand this new notion, however, it is desirable that constructible falsity has a comprehensive relationship with the traditional intuitionistic negation. This point is especially pressing in H. Wansing's framework of connexive constructible falsity, which exhibits unusual behaviours. From this motivation, this paper enquires what kind of interaction between the two operators can be satisfactory in the framework. We focus on a few natural-looking candidates for such an interaction, and evaluate their relative merits through analyses of their formal properties with both proof-theoretic and semantical means. We in particular note that some interactions allow connexive constructible falsity to provide a different solution to the problem of the failure of the constructible falsity property in intuitionistic logic. An emerging perspective in the end is that intuitionistic logicians may have different preferences depending on whether absurdity is to be understood as the falsehood.

Keywords— Basic systems; Connexive logic; Constructible falsity; Contradictory logics; Intuitionistic logic.

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1 Introduction

The notion of *constructible falsity* (to be denoted by $\sim$) was first introduced by D. Nelson [21] as an operator capturing the constructive procedures to falsify conjunctive and universal statements. Intuitionistic negation (to be denoted by $\neg$), on the other hand, does not fully capture these methods. As a result, while $\vdash \sim(A \wedge B)$ implies $\vdash \sim A$ or $\vdash \sim B$ (*constructible falsity property*) in a Nelsonian system, an analogous property does not hold with respect to $\neg$ in intuitionistic logic.

Constructible falsity can thus be seen as a way to improve the account offered by intuitionistic negation. This does not, however, mean that one accepting such a view has to give up intuitionistic negation as an intuitionistically acceptable operator. Indeed, in Nelson's original system **N3**, intuitionistic negation is definable by taking $\neg A := A \to \sim A$, as noted by A.A. Markov [17]. Alternatively, one may allow the *absurdity* constant $\bot$ inside a system with constructible falsity, and define intuitionistic negation as an implication to absurdity, i.e. $\neg A := A \to \bot$. The accommodation of $\bot$ to the language seems acceptable in light of Nelson's remark that distinguishing the two proof methods for the negation of a universal statement affords one to distinguish the meaning of $\sim\forall xA$ and $\forall xA \to \bot$ [21, p.17]. If one is interested in talking about the meaning of $\bot$ (as part of the latter formula), then it seems unproblematic to have it in one's vocabulary. An axiomatisation of **N3** with both negations as primitive is indeed used by N.N. Vorob'ev [31]. He suggests that such a formalisation is more suitable as a model of mathematical thoughts: his point appears to be that *reductio ad absurdum* used a lot in mathematics corresponds conceptually to $\neg A$ but not to $A \to \sim A$. It is a primitive procedure independent of refutation (corresponding to $\sim$), and the two negations must be treated as primitive, in order to reflect the primitive status of the procedures.

This relationship between the negations change when the paraconsistent variant **N4** (formulated[1] by A. Almukdad and Nelson [1]) of **N3** is considered. Intuitionistic negation is not definable in **N4**, so the choice of whether to include $\bot$ becomes more significant. The version of **N4** with $\bot$ is commonly denoted by $\mathbf{N4}^{\bot}$, and both logics and their extensions are investigated by S.P. Odintsov [23]. The book [15] by N. Kamide and H. Wansing treats the proof theory of both systems and their neighbours.

Consider now a scenario where an *intuitionistic logician* (here we just mean somebody who understands the connectives of intuitionistic logic, without necessarily being an intuitionist or constructivist.) tries to make sense of constructible falsity. In all three (propositional) systems we have mentioned, it is possible to convert each formula to an equivalent formula in which $\sim$ occurs only in front of prime formulas. So in a sense, the understanding of the 'meaning' of $\sim A$ Nelson refers to is reduced to the understanding of $\sim p$ (and $\sim\bot$ if $\bot$ is taken as primitive). How then can an intuitionistic logician grasp the meaning of $\sim p$?

In the case of **N3**, we have $\sim A \to \neg A$ as a theorem, and so an intuitionistic logician may understand $\sim p$ as a strengthening of $\neg p$. This is evidenced by how Markov [17] calls constructible falsity *strong negation*. For **N4** (and $\mathbf{N4}^{\bot}$), on the other hand, there is no

[1]Equivalent systems were already introduced by D. Prawitz [28] and F. von Kutschera [30]: see [32] for more historical details about **N4**.

such or any other constraint that relates $\sim p$ with $\neg p$ or its negand p. It behaves almost like another propositional variable.[2] Therefore it seems an intuitionistic logician would have a harder time understanding the meaning of $\sim p$ in **N4** (and $\mathbf{N4}^{\perp}$) than in **N3**.

An analogous question can be raised for other logics with constructible falsity based on (positive) intuitionistic logic. An especially interesting case is that of the system **C** introduced by Wansing [33]. This system is obtained from **N4** by changing the condition under which an implication is falsified. As a result of this change, **C** satisfies the criteria of *connexive logic* [18, 35]: namely, it validates the theses proposed by Aristotle (AT,AT') and Boethius (BT,BT'):

$$\text{AT: } \sim(\sim A \to A) \quad \text{BT: } (A \to B) \to \sim(A \to \sim B)$$
$$\text{AT': } \sim(A \to \sim A) \quad \text{BT': } (A \to \sim B) \to \sim(A \to B)$$

meanwhile invalidating $(A \to B) \to (B \to A)$ which would hold if $\to$ were a biconditional. A further characteristics of **C** is that it has as theorems a pair of certain formulas A and $\sim A$, i.e. it is a *negation inconsistent*, yet non-trivial system.

C shares with **N4** the characteristics that there is no stipulation for $\sim p$. At the same time, the option of extending it with $\sim A \to \neg A$ is not available: it results in a trivial system because of the negation inconsistency. A different way of extending **C** is proposed by H. Omori and Wansing [26] and later explored Kripke-semantically by G.K. Olkhovikov [24] and algebraically by D. Fazio and Odintsov [9]. The extension **C3** is obtained with the addition of $A \vee \sim A$ as an axiom schema. Since $p \vee \sim p$ holds in **C3**, it is arguably easier for intuitionistic logicians to make sense of $\sim p$ in **C3** than in **C**.[3] On the other hand, they may not be too satisfied with the non-constructivity of the system, such as the failure of the disjunction property.

A question we may ask then is whether there is a satisfactory system of connexive constructible falsity which is more understandable and acceptable for intuitionistic logicians.[4] It is desired that such a system (i) gives a certain stipulation for $\sim p$ as in **C3**, but (ii) remains constructive. In this enquiry, we presuppose the existence of $\perp$ in the language, following the lines of justification mentioned above. Thus more precisely, our concern will be with respect to the expansions[5] $\mathbf{C}^{ab}$ and $\mathbf{C}_3^{ab}$ of **C** and **C3** with the absurdity constant.

In this paper, we shall study formal properties of a few extensions of $\mathbf{C}^{ab}$ which form a natural hierarchy between $\mathbf{C}^{ab}$ and $\mathbf{C}_3^{ab}$ when seen through sequent rules. Our aim is

[2]Here it might be suggested that an intuitionistic logician can understand the meaning of $\sim p$ by an analogy with the behaviour of propositional variables. Encodability of derivations in an **N4**-style system into a two-sorted λ-calculus [34] seems to also support such a view. This can be an answer, but it would not satisfy him if he expected (perhaps mistakenly) to see something 'negative' in the behaviour of $\sim p$ that would justify him to take it as a negation.

[3]It exhibits a property often ascribed to negation, which may well be understood (without endorsement) as a claim of decidability, perhaps by an analogy with classical negation.

[4]As one reviewer pointed out, such a system may be seen to motivate connexive logics from the viewpoint of intuitionistic logic, thus has an affinity with the discussion in [37]. On the other hand, our focus is not directly on the connexive theses themselves.

[5]This type of expansions is already studied in [9], but there is a slight variation, as we shall discuss in the next section.

thereby to find out which notion of connexive constructible falsity is more satisfactory for intuitionistic logicians. We shall concentrate on two candidates for the axiom schemata. The first is the schema of *potential omniscience* $\neg\neg(A \vee {\sim}A)$, which was introduced and investigated by I. Hasuo and R. Kashima [11] in the context of $\mathbf{N4}^{\perp}$. Also, as pointed out by A. Avellone et al. [2], the constructive logic of classical truth by P. Miglioli et al. [19] can be seen as **N3** plus potential omniscience, when the classical truth is identified with the intuitionistic double negation. Within the context of **C**, this schema already appears in the proof of [9, Theorem 49]. The second candidate is the axiom schema $\neg A \to {\sim}A$ whose implication is dual to ${\sim}A \to \neg A$; we shall call the schema *weak negation* on this ground.

The structure of this paper is as follows: Section 2 introduces Hilbert-style systems and sequent calculi for $\mathbf{C}^{ab}$ and its extensions, and shows their equivalence. Section 3 treats Kripke semantics for the systems, and establishes the soundness and completeness of the systems with respect to the sequent calculi following the general argument presented by O. Lahav and A. Avron [16]. Section 4 then applies the results so far to observe some properties (with an emphasis on negation inconsistency) of the extensions with potential omniscience/weak negation, which can be informative for the evaluation of the systems by intuitionistic logicians. In Section 5, we introduce another type of sequent calculus, formulated originally for **N4** in [15], with better proof-theoretic properties. In particular, we show that the calculus for potential omniscience enjoys the subformula property. In Section 6, we make an observation concerning the relation between $\sim$ and $\neg$, which provides a new perspective on the connexive constructible falsity. Lastly, section 7 sums up the insights to evaluate the relative advantages of the systems.

2 Proof Systems

In this section, we shall introduce Hilbert-style axiomatic systems as well as Gentzen-style sequent calculi for the logics that concern us.

2.1 Hilbert-style Systems

The main language $\mathcal{L}$ we shall consider in this paper is defined by the following form.

$$A ::= p \mid {\sim}A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \perp.$$

If we remove $\perp$ from the definition, it defines another propositional language $\mathcal{L}^{+}$. In both languages, $(A \leftrightarrow B) := (A \to B) \wedge (B \to A)$ and in $\mathcal{L}$, $\neg A := A \to \perp$. The set of all formulas in $\mathcal{L}$ ($\mathcal{L}^{+}$) will be denoted by Form (Form^{+}). The set of subformulas of a formula A and of a set Γ of formulas will be denoted by $Sub(A)$ and $Sub(\Gamma)$.

The *complexity* $c(A)$ of a formula A is defined by the following clauses: $c(p) = c(\perp) = 0$, $c({\sim}A) = c(A) + 1$ and $c(A \circ B) = c(A) + c(B) + 2$ for $\circ \in \{\wedge, \vee, \to\}$.

We first introduce Wansing's system **C** [33] in $\mathcal{L}^{+}$ and its expansion $\mathbf{C}^{ab}$ in $\mathcal{L}$ which becomes the basis of our enquiry.

Definition 2.1. The system **C** in $\mathcal{L}^{+}$ is defined by the next axiom schemata and a rule.

$$(A{\to}(B{\to}C)){\to}((A{\to}B){\to}(A{\to}C)) \quad \text{(S)}$$
$$A{\to}(B{\to}A) \quad \text{(K)}$$
$$A{\to}(B{\to}(A{\wedge}B)) \quad \text{(CI)}$$
$$(A_1{\wedge}A_2){\to}A_i \quad \text{(CE)}$$
$$A_i{\to}(A_1{\vee}A_2) \quad \text{(DI)}$$
$$(A{\to}C){\to}((B{\to}C){\to}((A{\vee}B){\to} C)) \quad \text{(DE)}$$
$${\sim}(A{\wedge}B){\leftrightarrow}({\sim}A{\vee}{\sim}B) \quad \text{(NC)}$$
$${\sim}(A{\vee}B){\leftrightarrow}({\sim}A{\wedge}{\sim}B) \quad \text{(ND)}$$
$${\sim}(A{\to}B){\leftrightarrow}(A{\to}{\sim}B) \quad \text{(NI)}$$
$${\sim}{\sim}A{\leftrightarrow}A \quad \text{(NN)}$$
$$\frac{A \qquad A \to B}{B} \quad \text{(MP)}$$

A *derivation* in **C** of A from a set of formulas Γ is a finite sequence $B_1, \ldots, B_n \equiv A$ such that each B_i is either an instance of one of the axiom schemata, an element of Γ, or obtained from the preceding elements by means of (MP). Then for derivability, we write $\Gamma \vdash_h \Delta$ if there is a derivation of a disjunction $A_1 \vee \ldots \vee A_n$ from Γ, where Δ is a non-empty set of formulas and $A_1, \ldots, A_n \in \Delta$.

We shall write $A_1, \ldots, A_m, \Gamma \vdash_h \Delta, B_1, \ldots, B_n$ for $\{A_1, \ldots, A_m\} \cup \Gamma \vdash_h \Delta \cup \{B_1, \ldots, B_n\}$. If Δ is a singleton, we will occasionally omit the parentheses.

The second system is obtained from **C** by expanding the language.

Definition 2.2. The system $\mathbf{C}^{ab}$ in $\mathcal{L}$ is defined from the axiomatisation of **C** by an additional axiom schema:

$$\bot \to A \quad \text{(EFQ)}$$

The relation $\Gamma \vdash_{hab} \Delta$ is defined as before, except that we allow Δ to be empty. $\Gamma \vdash_{hab} \emptyset$ will mean that there is a derivation of $\bot$ from Γ.

We now define a few more systems from $\mathbf{C}^{ab}$. Among these, $\mathbf{C}^{ab}_3$ is an expansion of the system **C3** [25] with $\bot$.

Definition 2.3. The systems $\mathbf{C}^{ab}_{po}$, $\mathbf{C}^{ab}_{wn}$ and $\mathbf{C}^{ab}_3$ are each defined with a respective additional axiom schema.

$$\neg\neg(A \vee {\sim}A) \quad \text{(PO)}$$
$$\neg A \to {\sim}A \quad \text{(WN)}$$
$$A \vee {\sim}A \quad (3)$$

We shall use $\vdash_{hpo}$, $\vdash_{hwn}$ and $\vdash_{h3}$ for the derivability of the systems.

Remark 2.4. One possible option in defining the systems is to have ${\sim}\bot$ (or equivalently, $A \to {\sim}\bot$) as an additional axiom schema, as is done in the case of $\mathbf{N4}^{\bot}$, see e.g. [15, 23, 22]. Intuitively, it states that what is absurd is false. This option is indeed adopted in the systems $\mathbf{C}^{\bot}$, $\mathbf{C3}^{\bot}$ and their extensions in [9]. On the other hand, we are *not* assuming this, chiefly due to ${\sim}\neg A$ being one of its consequences. This means that every intuitionistic negation is false, which seems to be a very strong claim.[6] For another reason, ${\sim}\bot$ is actually provable in the current definition of $\mathbf{C}^{ab}_{wn}$ and $\mathbf{C}^{ab}_3$, so for these systems we do not need the formula

[6]T.M. Ferguson [10] however suggests that it may be possible to motivate the feature using an adequate Brouwer-Heyting-Kolmogorov interpretation.

as an axiom schema (thus $\mathbf{C}_3^{ab}$ is equivalent to $\mathbf{C3}^{\perp}$). This suggests that each system has already in mind, so to speak, whether and what to say about the falsity of $\perp$. This may be worthwhile to be respected (we shall have a few more words on this topic in the conclusion).

Since (MP) is the only rule present in the systems, it is straightforward to observe that the deduction theorem holds for each of the above systems.

Theorem 2.5. For $* \in \{ab, po, wn, 3\}$ we have:

$$\Gamma, A \vdash_{h*} B \text{ if and only if } \Gamma \vdash_{h*} A \to B.$$

Proof. The 'only if' direction is shown by induction on the depth of derivation. The 'if' direction follows by (MP). ☐

It is helpful at this stage to note the (non-strict) relative strength of the systems.

Proposition 2.6. The following statements hold.

(i) If $\Gamma \vdash_{hab} \Delta$ then $\Gamma \vdash_{hpo} \Delta$.

(ii) If $\Gamma \vdash_{hpo} \Delta$ then $\Gamma \vdash_{hwn} \Delta$.

(iii) If $\Gamma \vdash_{hwn} \Delta$ then $\Gamma \vdash_{h3} \Delta$.

Proof. (i) is immediate from the definition; (ii) follows since it follows from (WN) that $\vdash_{hwn} \neg{\sim}A \to \neg\neg A$, from which (PO) follows. For (iii), from (3) it follows that $\vdash_{h3} (A \to {\sim}A) \to {\sim}A$, and also $\vdash_{h3} \neg A \to (A \to {\sim}A)$ by (EFQ); so (WN) follows. ☐

2.2 Sequent Calculi

We shall next introduce (multi-succedent) sequent calculi for $\mathbf{C}^{ab}$, $\mathbf{C}_{po}^{ab}$ and $\mathbf{C}_{wn}^{ab}$. We shall use the framework in which each *sequent* $\Gamma \Rightarrow \Delta$ is such that Γ and Δ are finite *sets*[7] of formulas (cf. e.g. the system $\mathbf{LJ}^{\{\ \}}$ in [3, p.64]). The empty set will be denoted by a blank.

First we define the calculus for $\mathbf{C}^{ab}$.

Definition 2.7. The calculus $\mathbf{GC}^{ab}$ is defined by the following rules.

$$A \Rightarrow A \ (\text{Ax})$$

$$\perp \Rightarrow \ (\text{L}\perp) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \ (\text{Cut})$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \ (\text{LW}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \ (\text{RW})$$

[7] In this setting, it is important to note that different sequents can be derived from the same rule when applied to the same sequent. For instance, consider (L${\sim}{\sim}$) applied to $\{A, B\} \Rightarrow \{C\}$: then we can derive $\{{\sim}{\sim}A, B\} \Rightarrow \{C\}$, but we may also derive $\{{\sim}{\sim}A, A, B\} \Rightarrow \{C\}$, if the antecedent set is conceived as $\{A\} \cup \{A, B\}$.

$$\frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ (\mathrm{L}\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\mathrm{R}\wedge)$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\mathrm{L}\vee) \qquad \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\ (\mathrm{R}\vee)$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma' \Rightarrow \Delta'}{A \to B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathrm{L}\to) \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\mathrm{R}\to)$$

$$\frac{{\sim}A, \Gamma \Rightarrow \Delta \qquad {\sim}B, \Gamma \Rightarrow \Delta}{{\sim}(A \wedge B), \Gamma \Rightarrow \Delta}\ (\mathrm{L}{\sim}\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}A_i}{\Gamma \Rightarrow \Delta, {\sim}(A_1 \wedge A_2)}\ (\mathrm{R}{\sim}\wedge)$$

$$\frac{{\sim}A_i, \Gamma \Rightarrow \Delta}{{\sim}(A_1 \vee A_2), \Gamma \Rightarrow \Delta}\ (\mathrm{L}{\sim}\vee) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim}A \qquad \Gamma \Rightarrow \Delta, {\sim}B}{\Gamma \Rightarrow \Delta, {\sim}(A \vee B)}\ (\mathrm{R}{\sim}\vee)$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad {\sim}B, \Gamma' \Rightarrow \Delta'}{{\sim}(A \to B), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\mathrm{L}{\sim}\to) \qquad \frac{A, \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \to B)}\ (\mathrm{R}{\sim}\to)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{{\sim}{\sim}A, \Gamma \Rightarrow \Delta}\ (\mathrm{L}{\sim}{\sim}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, {\sim}{\sim}A}\ (\mathrm{R}{\sim}{\sim})$$

where $i \in \{1, 2\}$. We write $\vdash_{gab} \Gamma \Rightarrow \Delta$ if there is a derivation in $\mathbf{GC}^{ab}$ of $\Gamma \Rightarrow \Delta$ from the 0-premise rules, i.e. (Ax), (L$\bot$).

As usual, the formulas in Γ, Δ etc. will be called *contexts*, a non-context formula in the premises of a rule will be called *active*, and a non-context formula in the conclusion of a rule will be called *principal*.

For $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}_{wn}$, we have the following calculi.

Definition 2.8. The calculi $\mathbf{GC}^{ab}_{po}$ and $\mathbf{GC}^{ab}_{wn}$ are respectively defined from $\mathbf{GC}^{ab}$ each with an additional rule:

$$\frac{A, \Gamma \Rightarrow \qquad {\sim}A, \Gamma \Rightarrow}{\Gamma \Rightarrow}\ (\mathrm{gPO}) \qquad \frac{A, \Gamma \Rightarrow \qquad {\sim}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{gWN})$$

The relations $\vdash_{gpo}$ and $\vdash_{gwn}$ are defined analogously to $\vdash_{gab}$

A calculus for $\mathbf{C}^{ab}_{3}$ can also be defined, as is done in [26] for **C3**, by allowing the succedent of both premises in (gWN) to be non-empty. Hence (PO), (WN) and (3) give a natural hierarchy of sequent rules, which can motivate our focus on the axioms.

We proceed to establish the correspondence between the Hilbert-style systems and the sequent calculi.

Proposition 2.9. Let $* \in \{ab, po, wn\}$ and Γ, Δ be finite sets of formulas. Then $\Gamma \vdash_{h*} \Delta$ if and only if $\vdash_{g*} \Gamma \Rightarrow \Delta$.

Proof. The 'only if' direction is shown by induction on the depth of derivation in the Hilbert-style systems. Here we look at the case of (PO) in $\mathbf{GC}^{ab}_{po}$ and (WN) in $\mathbf{GC}^{ab}_{wn}$. For (PO):

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A\vee{\sim}A}\,(\mathrm{R}\vee) \qquad \bot \Rightarrow}{A, \neg(A\vee{\sim}A) \Rightarrow}\,(\mathrm{L}\to) \qquad \dfrac{\dfrac{{\sim}A \Rightarrow {\sim}A}{{\sim}A \Rightarrow A\vee{\sim}A}\,(\mathrm{R}\vee) \qquad \bot\Rightarrow}{{\sim}A, \neg(A\vee{\sim}A)\Rightarrow}\,(\mathrm{L}\to)}{\neg(A\vee{\sim}A)\Rightarrow}\,(\mathrm{gPO})$$

$$\dfrac{\neg(A\vee{\sim}A)\Rightarrow}{\Rightarrow \neg\neg(A\vee{\sim}A)}\,(\mathrm{RW}),(\mathrm{R}\to)$$

(A double line indicates multiple applications of rules). For (WN), we have:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \bot \Rightarrow}{A, \neg A \Rightarrow}\,(\mathrm{L}\to) \qquad \dfrac{{\sim}A \Rightarrow {\sim}A}{{\sim}A, \neg A \Rightarrow {\sim}A}\,(\mathrm{LW})}{\neg A \Rightarrow {\sim}A}\,(\mathrm{gWN})}{\Rightarrow \neg A \to {\sim}A}\,(\mathrm{R}\to)$$

For the 'if' direction, we show by induction on the depth of derivation in the sequent calculi. Here we check the cases of (gPO) for $\mathbf{GC}^{ab}_{po}$ and of (gWN) for $\mathbf{GC}^{ab}_{wn}$. For the former, by I.H. we have $A, \Gamma \vdash_{hpo} \bot$ and ${\sim}A, \Gamma \vdash_{hpo} \bot$. By Theorem 2.5, $\Gamma \vdash_{hpo} \neg A$ and $\Gamma \vdash_{hpo} \neg{\sim}A$. Hence $\Gamma \vdash_{hpo} \neg(A \vee {\sim}A)$ and so by (PO) we conclude $\Gamma \vdash_{hpo} \bot$. For the latter, by I.H. $A, \Gamma \vdash_{hwn} \bot$ and ${\sim}A, \Gamma \vdash_{hwn} B_1 \vee \ldots \vee B_n$ for some $B_1, \ldots, B_n \in \Delta$. By Theorem 2.5 we obtain $\Gamma \vdash_{hwn} \neg A$ and $\Gamma \vdash_{hwn} {\sim}A \to (B_1 \vee \ldots \vee B_n)$. Thus by (WN) we conclude $\Gamma \vdash_{hwn} B_1 \vee \ldots \vee B_n$, i.e. $\Gamma \vdash_{hwn} \Delta$. □

3 Semantics

In this subsection, we shall introduce[8] Kripke semantics for $\mathbf{C}^{ab}$, $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}_{wn}$, and then show that the systems are sound and complete with the semantics.

3.1 Kripke Semantics

We first introduce a Kripke semantics for $\mathbf{C}^{ab}$. The presentation here is a combination of bilateral-style sequent calculi used for $\mathbf{C}$ in [33] and non-deterministic sequent calculi due to O. Lahav and A. Avron [16]. We have a few more words about the latter in Section 3.3.

Definition 3.1. A $\mathbf{C}^{ab}$-*frame* $\mathcal{F}$ is a pair $(W, \leq)$, where W is a non-empty set and $\leq$ is a pre-ordering on W. A $\mathbf{C}^{ab}$-*model* $\mathcal{M}$ is a pair $(\mathcal{F}, \mathcal{V})$, where $\mathcal{F}$ is a $\mathbf{C}^{ab}$-frame, $\mathcal{V} = \{\mathcal{V}^+, \mathcal{V}^-\}$ where $\mathcal{V}^* : \mathsf{Form} \to \mathcal{P}(W)$ for $* \in \{+, -\}$. We shall write $w \in \mathcal{V}^*(A)$ also as $\mathcal{M}, w \Vdash^*_{ab} A$ ($\mathcal{M}$ will be omitted when it is contextually clear). $\mathcal{V}$ must satisfy a general condition below:

$$\text{(Upward Closure): } w \Vdash^*_{ab} A \text{ and } w \leq w' \text{ implies } w' \Vdash^*_{ab} A.$$

for both $* \in \{+, -\}$. Moreover, the next conditions for the connectives must also be satisfied.

[8] A semantics for $\mathbf{C}^{ab}_3$ can be similarly given following [26], but it is outside our focus here.

$$
\begin{array}{ll}
w \Vdash^+_{ab} \bot \Leftrightarrow \text{ never.} & \\
w \Vdash^+_{ab} A{\wedge}B \Leftrightarrow w \Vdash^+_{ab} A \text{ and } w \Vdash^+_{ab} B. & w \Vdash^-_{ab} A{\wedge}B \Leftrightarrow w \Vdash^-_{ab} A \text{ or } w \Vdash^-_{ab} B. \\
w \Vdash^+_{ab} A{\vee}B \Leftrightarrow w \Vdash^+_{ab} A \text{ or } w \Vdash^+_{ab} B. & w \Vdash^-_{ab} A{\vee}B \Leftrightarrow w \Vdash^-_{ab} A \text{ and } w \Vdash^-_{ab} B. \\
w \Vdash^+_{ab} A{\to}B \Leftrightarrow \forall w'{\geq}w(w'{\Vdash^+_{ab}}A{\Rightarrow}w'{\Vdash^+_{ab}}B). & w \Vdash^-_{ab} A{\to}B \Leftrightarrow \forall w'{\geq}w(w'{\Vdash^+_{ab}}A{\Rightarrow}w'{\Vdash^-_{ab}}B). \\
w \Vdash^+_{ab} {\sim}A \Leftrightarrow w \Vdash^-_{ab} A. & w \Vdash^-_{ab} {\sim}A \Leftrightarrow w \Vdash^+_{ab} A.
\end{array}
$$

With respect to a $\mathbf{C}^{ab}$-model $\mathcal{M}$ and a sequent $\Gamma \Rightarrow \Delta$, we shall write $\mathcal{M}, w \vDash_{ab} \Gamma \Rightarrow \Delta$ if $\mathcal{M}, w \Vdash^+_{ab} A$ for all $A \in \Gamma$ implies $\mathcal{M}, w \Vdash^+_{ab} B$ for some $B \in \Delta$. If $\mathcal{M}, w \vDash_{ab} \Gamma \Rightarrow \Delta$ for all w in $\mathcal{M}$, then we shall write $\mathcal{M} \vDash_{ab} \Gamma \Rightarrow \Delta$. Finally, we shall write $\vDash_{ab} \Gamma \Rightarrow \Delta$ if $\mathcal{M} \vDash_{ab} \Gamma \Rightarrow \Delta$ for all $\mathcal{M}$.

Remark 3.2. The forcing relations $\Vdash^+_{ab}/\Vdash^-_{ab}$ may be seen to represent e.g. the concepts of *verification/falsification* or *(support of) truth/(support of) falsity* [33]. In our scenario, intuitionistic logicians can be assumed to understand the former relation, by identifying it with the forcing relation of intuitionistic Kripke semantics (except the one for $\sim$, which encodes $\Vdash^-_{ab}$ in $\Vdash^+_{ab}$). The latter relation, on the other hand, needs an explanation, especially when it comes to $\Vdash^-_{ab} p$ for which no special restriction is given.

Next, we define Kripke semantics for $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}_{wn}$.

Definition 3.3. Kripke semantics for $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}_{wn}$ (we shall use the subscripts $_{po}$ and $_{wn}$ for $\Vdash$ and $\vDash$.) are each defined from the one for $\mathbf{C}^{ab}$ by the addition of the following condition of (Potential Omniscience) and (Weak Negation), respectively:

$$
\begin{array}{ll}
\text{(Potential Omniscience):} & \forall w' \geq w(w' \nVdash^+_{po} A) \text{ implies } \exists x \geq w(x \Vdash^-_{po} A). \\
\text{(Weak Negation):} & \forall w' \geq w(w' \nVdash^+_{wn} A) \text{ implies } w \Vdash^-_{wn} A.
\end{array}
$$

In these semantics, some relationships between e.g. verification and falsification of p are given, so the latter concept should be more easily understood by an intuitionistic logician in terms of the former.

Let us note a difference in character between (Potential Omniscience) and (Weak Negation), despite their similar appearances. The former condition may be restricted to propositional variables and $\bot$, similarly to how upward closure is ensured in ordinary Kripke semantics for intuitionistic logic by requiring it to hold only in the atomic case. In contrast, such a restriction does not generalise for the latter condition.

Proposition 3.4. The following statements hold.

(i) If a $\mathbf{C}^{ab}$-model $\mathcal{M}$ satisfies the following conditions:

$$
\begin{array}{c}
\forall w' \geq w(w' \nVdash^+_{ab} p) \text{ implies } \exists x \geq w(x \Vdash^-_{ab} p). \\
\forall w' \geq w(w' \nVdash^+_{ab} \bot) \text{ implies } \exists x \geq w(x \Vdash^-_{ab} \bot).
\end{array}
$$

then $\mathcal{M}$ is a $\mathbf{C}^{ab}_{po}$-model.

(ii) There exists a $\mathbf{C}^{ab}$-model which satisfies the following conditions:

$$\forall w' \geq w(w' \nVdash^+_{ab} p) \text{ implies } w \Vdash^-_{ab} p.$$
$$\forall w' \geq w(w' \nVdash^-_{ab} p) \text{ implies } w \Vdash^+_{ab} p.$$
$$\forall w' \geq w(w' \nVdash^+_{ab} \bot) \text{ implies } w \Vdash^-_{ab} \bot.$$
$$\forall w' \geq w(w' \nVdash^-_{ab} \bot) \text{ implies } w \Vdash^+_{ab} \bot.$$

while not being a $\mathbf{C}^{ab}_{wn}$-model.

Proof. For (i), we show by induction on the complexity of formulas that (Potential Omniscience) is satisfied in $\mathcal{M}$. The cases when $A \equiv p, \bot$ follow from the assumption.

When $A \equiv B \wedge C$, we show the contrapositive. If $\neg\exists x \geq w(x \Vdash^-_{ab} B \wedge C)$, it must be the case that $\forall x \geq w(x \nVdash^-_{ab} B$ and $x \nVdash^-_{ab} C)$ $(*)$ and so $\forall x \geq w(x \nVdash^-_{ab} B)$. Also, as one of the I.H., $\forall w' \geq w(w' \nVdash^+_{ab} B)$ implies $\exists x \geq w(x \Vdash^-_{ab} B)$. Hence we deduce from these that $\exists w' \geq w(w' \Vdash^+_{ab} B)$. Fix one such w'. By $(*)$, $\forall y \geq w'(y \nVdash^-_{ab} C)$. Then as another I.H., it holds that $\forall w'' \geq w'(w'' \nVdash^+_{ab} C)$ implies $\exists y \geq w'(y \Vdash^-_{ab} C)$. So $\exists w'' \geq w'(w'' \Vdash^+_{ab} C)$. By (Upward Closure), we have $w'' \Vdash^+ B$ for such w'' as well. Therefore $\neg\forall w' \geq w(w' \nVdash^+_{ab} B \wedge C)$, as desired.

When $A \equiv B \vee C$, if $\forall w' \geq w(w' \nVdash^+_{ab} B \vee C)$ we have $\forall w' \geq w(w' \nVdash^+_{ab} B)$. Thus by one of the I.H. $\exists x \geq w(x \Vdash^-_{ab} B)$. Take such an x. Then $\forall x' \geq x(x' \nVdash^+_{ab} C)$ and so $\exists y \geq x(y \Vdash^-_{ab} C)$ from the other I.H.. By (Upward Closure), $y \Vdash^-_{ab} B$ as well; so $\exists x \geq w(x \Vdash^-_{ab} B \vee C)$.

When $A \equiv B \to C$, if $\neg\forall w' \geq w(w' \nVdash^+_{ab} C)$ then $w' \Vdash^+_{ab} C$ and so $w' \Vdash^+_{ab} B \to C$ for some $w' \geq w$. Hence $\neg\forall w' \geq w(w' \nVdash^+_{ab} B \to C)$; consequently $\forall w' \geq w(w' \nVdash^+_{ab} B \to C)$ implies $\exists x \geq w(x \Vdash^-_{ab} B \to C)$. Otherwise, $\forall w' \geq w(w' \nVdash^+_{ab} C)$ and by the I.H $\exists x \geq w(x \Vdash^-_{ab} C)$. Hence $\exists x \geq w(x \Vdash^-_{ab} B \to C)$ and therefore $\forall w' \geq w(w' \nVdash^+_{ab} B \to C)$ implies $\exists x \geq w(x \Vdash^-_{ab} B \to C)$ in this case as well.

Finally, when $A \equiv {\sim}B$, by the I.H. $\forall w' \geq w(w' \nVdash^+_{ab} B)$ implies $\exists x \geq w(x \Vdash^-_{ab} B)$, contraposing which we obtain $\forall w' \geq w(w' \nVdash^-_{ab} B)$ implies $\exists x \geq w(x \Vdash^+_{ab} B)$. Therefore $\forall w' \geq w(w' \nVdash^+_{ab} {\sim}B)$ implies $\exists x \geq w(x \Vdash^-_{ab} {\sim}B)$.

For (ii), suppose $\mathcal{M} = ((W, \leq), \mathcal{V})$ is such that $W = \{w, x, y\}$, $\leq$ is the reflexive closure of $\{(w, x), (w, y)\}$, $\mathcal{V}^+(p) = \{x\}$, $\mathcal{V}^-(p) = \{y\}$, $V^-(\bot) = W$ and for compound formulas $\mathcal{V}$ is defined in accordance with the equivalences in Definition 3.1: e.g. set $x \in \mathcal{V}^-(A \to B)$ if for all $y \geq x(y \in \mathcal{V}^+(A)$ implies $y \in \mathcal{V}^-(B))$. Then the equivalences in Definition 3.1 are naturally satisfied, and (Upward Closure) may be checked by induction on the complexity of formula. Therefore $\mathcal{M}$ is a $\mathbf{C}^{ab}$-model. In addition, $\mathcal{M}$ satisfies the conditions of the proposition at each world. For instance, for the first condition, $\forall u' \geq u(u' \nVdash^+_{ab} p)$ implies $u = y$, but $u \Vdash^-_{ab} p$.

Now, it is readily observed that $\forall w' \geq w(w' \nVdash^+_{ab} p \wedge {\sim}p)$, but $w \nVdash^-_{ab} p \wedge {\sim}p$. Therefore (Weak Negation) is not satisfied for all formulas in $\mathcal{M}$. □

3.2 Soundness

In the next two subsections, we shall establish the soundness and completeness of the three sequent calculi $\mathbf{GC}^{ab}$, $\mathbf{GC}^{ab}_{po}$ and $\mathbf{GC}^{ab}_{wn}$ with respect to their Kripke semantics. We shall treat the soundness direction in this subsection.

Theorem 3.5 (soundness)**.** Let $* \in \{ab, po, wn\}$. Then $\vdash_{g*} \Gamma \Rightarrow \Delta$ implies $\vDash_* \Gamma \Rightarrow \Delta$.

Proof. For $\mathbf{GC}^{ab}$, we can establish the statement by induction on the depth of derivation. For instance, if the last step in the derivation is an instance of (R$\sim\to$):

$$\frac{A, \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \to B)}$$

then by the I.H. $\mathcal{M} \vDash_{ab} A, \Gamma \Rightarrow {\sim}B$. Suppose $\mathcal{M}, w \Vdash^+_{ab} C$ for all $C \in \Gamma$ and $\mathcal{M}, w' \Vdash^+_{ab} A$ for $w' \geq w$. Then by (Upward Closure) $\mathcal{M}, w' \Vdash^+_{ab} C$ for all $C \in \{A\} \cup \Gamma$; thus $\mathcal{M}, w' \Vdash^+_{ab} {\sim}B$ and so $\mathcal{M}, w' \Vdash^-_{ab} B$. Therefore $\mathcal{M}, w \Vdash^-_{ab} A \to B$ and consequently $\mathcal{M}, w \Vdash^+_{ab} {\sim}(A \to B)$.

For $\mathbf{GC}^{ab}_{po}$, we in addition need to check the case for (gPO):

$$\frac{A, \Gamma \Rightarrow \qquad {\sim}A, \Gamma \Rightarrow}{\Gamma \Rightarrow}$$

If $\mathcal{M}, w \Vdash^+_{po} B$ for all $B \in \Gamma$, then $\mathcal{M}, w' \Vdash^+_{po} A$ for $w' \geq w$ leads to a contradiction by (Upward Closure) and the I.H.. Hence $\forall w' \geq w(\mathcal{M}, w' \nVdash^+_{po} A)$. But then $\exists w' \geq w(\mathcal{M}, w' \Vdash^-_{po} A)$ by (Potential Omniscience), which again contradicts the I.H.. Therefore $\mathcal{M}, w \vDash_{po} \Gamma \Rightarrow$ for all w, as required.

For $\mathbf{GC}^{ab}_{wn}$, we need to check the case for (gWN).

$$\frac{A, \Gamma \Rightarrow \qquad {\sim}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

If $\mathcal{M}, w \Vdash^+_{wn} B$ for all $B \in \Gamma$, then we infer $\forall w' \geq w(\mathcal{M}, w' \nVdash^+_{wn} A)$ as in the previous case. By (Weak Negation), $\mathcal{M}, w \Vdash^-_{wn} A$; so by the I.H. $\mathcal{M}, w \Vdash^+_{wn} C$ for some $C \in \Delta$. Therefore $\mathcal{M}, w \vDash_{wn} \Gamma \Rightarrow \Delta$. □

We can also connect the Hilbert-style systems and the Kripke semantics.

Corollary 3.6. Let $* \in \{ab, po, wn\}$ and Γ, Δ be finite sets of formulas. Then $\Gamma \vdash_{h*} \Delta$ implies $\vDash_* \Gamma \Rightarrow \Delta$.

Proof. An immediate consequence of Proposition 2.9 and Theorem 3.5. □

3.3 Completeness

The proof of completeness follows the one given by Lahav and Avron [16] for *basic systems*, which are sequent calculi that satisfy a few natural criteria. In [16] the authors present a general framework for formulating a sound and strongly complete Kripke semantics for a calculus in the class, to which $\mathbf{GC}^{ab}$, $\mathbf{GC}^{ab}_{po}$ and $\mathbf{GC}^{ab}_{wn}$ also belong. The argument here is only slightly altered from the outline given in [16], in order to fit the bilateral-style semantical setting.

Let us first introduce some preliminary notions. In what follows, we keep using the abbreviation with $* \in \{ab, po, wn\}$.

Definition 3.7 (maximal set)**.** A *maximal* set (for $\mathbf{GC}^{ab}/\mathbf{GC}^{ab}_{po}/\mathbf{GC}^{ab}_{wn}$) is a pair (Γ, Δ) of sets of formulas, where:

(i) For any finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, $\nvdash_{g*} \Gamma' \Rightarrow \Delta'$.

(ii) If $A \notin \Gamma$ then $\vdash_{g*} A, \Gamma' \Rightarrow \Delta'$ for some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$.

(iii) If $A \notin \Delta$ then $\vdash_{g*} \Gamma' \Rightarrow \Delta', A$ for some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$.

Lemma 3.8. If $\nvdash_{g*} \Gamma' \Rightarrow \Delta'$ for any finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, then there is a maximal set (Γ^m, Δ^m) (for $\mathbf{GC}^{ab}/\mathbf{GC}^{ab}_{po}/\mathbf{GC}^{ab}_{wn}$) such that $\Gamma \subseteq \Gamma^m$ and $\Delta \subseteq \Delta^m$.

Proof. Let $(B_i)_{i\in\mathbb{N}}$ and $(C_i)_{i\in\mathbb{N}}$ be the sets of formulas not occurring in Γ and Δ, respectively. Let $(A_i)_{i\in\mathbb{N}}$ be such that $A_0 := B_0$, $A_1 := C_0$, $A_2 := B_1$, $A_3 := C_1$, ... We define pairs $(\Gamma_i, \Delta_i)_{i\in\mathbb{N}}$ inductively by the following clauses:

$$(\Gamma_0, \Delta_0) := (\Gamma, \Delta).$$

$$(\Gamma_{2i+1}, \Delta_{2i+1}) := \begin{cases} (\Gamma_{2i} \cup \{A_{2i}\}, \Delta_{2i}) & \text{if } \nvdash_{g*} \Gamma' \Rightarrow \Delta' \text{ for any finite} \\ & \Gamma' \subseteq \Gamma_{2i} \cup \{A_{2i}\} \ \&\ \Delta' \subseteq \Delta_{2i}. \\ (\Gamma_{2i}, \Delta_{2i}) & \text{otherwise.} \end{cases}$$

$$(\Gamma_{2i+2}, \Delta_{2i+2}) := \begin{cases} (\Gamma_{2i+1}, \Delta_{2i+1} \cup \{A_{2i+1}\}) & \text{if } \nvdash_{g*} \Gamma' \Rightarrow \Delta' \text{ for any finite} \\ & \Gamma' \subseteq \Gamma_{2i+1} \ \&\ \Delta' \subseteq \Delta_{2i+1} \cup \{A_{2i+1}\}. \\ (\Gamma_{2i+1}, \Delta_{2i+1}) & \text{otherwise.} \end{cases}$$

Let $(\Gamma^m, \Delta^m) := (\bigcup_i \Gamma_i, \bigcup_i \Delta_i)$. We need to check that (i)–(iii) of Definition 3.7 hold for the pair (Γ^m, Δ^m).

For (i), if $\vdash_{g*} \Gamma' \Rightarrow \Delta'$ for some finite $\Gamma' \subseteq \Gamma^m$ and $\Delta' \subseteq \Delta^m$, then there is i such that $\Gamma' \subseteq \Gamma_i$ and $\Delta' \subseteq \Delta_i$. However we can check by induction that this cannot be the case for any i.

For (ii), if $A \notin \Gamma^m$ then $\Gamma \subseteq \Gamma^m$ implies $A \equiv A_{2i}$ for some i. If $\nvdash_{g*} \Gamma' \Rightarrow \Delta'$ for all finite $\Gamma' \subseteq \Gamma_{2i} \cup \{A\}$ and $\Delta' \subseteq \Delta_{2i}$, then $A \in \Gamma_{2i+1} \subseteq \Gamma^m$, a contradiction. So there must be finite $\Gamma' \subseteq \Gamma_{2i} \subseteq \Gamma^m$ and $\Delta' \subseteq \Delta_{2i} \subseteq \Delta^m$ such that $\vdash_{g*} A, \Gamma' \Rightarrow \Delta'$. For (iii), the argument is analogous. □

Next we introduce the notion of a canonical model.

Definition 3.9 (canonical model)**.** The *canonical model* $\mathcal{M}_c = ((W_c, \leq_c), \mathcal{V}_c)$ for $\mathbf{GC}^{ab}$ (or $\mathbf{GC}^{ab}_{po}/\mathbf{GC}^{ab}_{wn}$) is defined by:

- $W_c := \{(\Gamma, \Delta) : (\Gamma, \Delta) \text{ is a maximal set}\}$.
- $(\Gamma, \Delta) \leq_c (\Gamma', \Delta')$ iff $\Gamma \subseteq \Gamma'$.
- $(\Gamma, \Delta) \in \mathcal{V}^+_c(A)$ iff $A \in \Gamma$ and $(\Gamma, \Delta) \in \mathcal{V}^-_c(A)$ iff $\sim A \in \Gamma$.

Towards completeness, we shall show a few lemmas.

Lemma 3.10 (properties of canonical model)**.** Let $\mathcal{M}_c$ be the canonical model for $\mathbf{GC}^{ab}$ (or $\mathbf{GC}^{ab}_{po}/\mathbf{GC}^{ab}_{wn}$). Then:

(i) The following are equivalent.

(a) $\mathcal{M}_c, (\Gamma, \Delta) \vDash_* \Sigma \Rightarrow \Pi$.

(b) $\Sigma \nsubseteq \Gamma$ or $\Pi \nsubseteq \Delta$.

(c) There are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{g*} \Gamma', \Sigma \Rightarrow \Pi, \Delta'$.

(ii) If $\mathcal{M}_c, (\Gamma', \Delta') \vDash_* \Sigma \Rightarrow \Pi$ for all $(\Gamma', \Delta') \geq_c (\Gamma, \Delta)$, then there is a finite $\Gamma'' \subseteq \Gamma$ such that $\vdash_{g*} \Gamma'', \Sigma \Rightarrow \Pi$.

(iii) $\vdash_{g*} \Sigma \Rightarrow \Pi$ iff $\mathcal{M}_c \vDash_* \Sigma \Rightarrow \Pi$.

Proof. For (i), we shall first check that (a) holds if and only if (b) holds. From (a) to (b), suppose $(\Gamma, \Delta) \vDash_* \Sigma \Rightarrow \Pi$, i.e. $(\Gamma, \Delta) \Vdash^+_* A$ for all $A \in \Sigma$ implies $(\Gamma, \Delta) \Vdash^+_* B$ for some $B \in \Pi$. From the definition of $\mathcal{V}_c$, this can be rephrased as that $\Sigma \subseteq \Gamma$ implies $\Gamma \cap \Pi \neq \emptyset$. Hence $\Sigma \subseteq \Gamma$ and $\Pi \subseteq \Delta$ implies $\Gamma \cap \Delta \neq \emptyset$, which contradicts the maximality of (Γ, Δ). Therefore $\Sigma \nsubseteq \Gamma$ or $\Pi \nsubseteq \Delta$. From (b) to (a), if $\Sigma \nsubseteq \Gamma$ then $(\Gamma, \Delta) \nVdash^+_* A$ for some $A \in \Sigma$. So $(\Gamma, \Delta) \vDash_* \Sigma \Rightarrow \Pi$. If on the other hand $\Pi \nsubseteq \Delta$, then there is $A \in \Pi$ such that $A \notin \Delta$. If in addition $A \notin \Gamma$, then by the definition of a maximal set it must be that $\vdash_{g*} A, \Gamma_1 \Rightarrow \Delta_1$ and $\vdash_{g*} \Gamma_2 \Rightarrow \Delta_2, A$ for some finite $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $\Delta_1, \Delta_2 \subseteq \Delta$. Hence by (Cut) $\vdash_{g*} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$; but this contradicts the maximality of (Γ, Δ). So $A \in \Gamma$ and consequently $(\Gamma, \Delta) \vDash_* \Sigma \Rightarrow \Pi$ as well.

Next we check that (b) holds if and only if (c) holds. From (b) to (c), suppose $\Sigma \nsubseteq \Gamma$ or $\Pi \nsubseteq \Delta$. Consider the former case. Then there is $A \in \Sigma$ such that $A \notin \Gamma$. Now because (Γ, Δ) is maximal, it must be that $\vdash_{g*} A, \Gamma' \Rightarrow \Delta'$ for some $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. Hence $\vdash_{g*} \Gamma', \Sigma \Rightarrow \Pi, \Delta'$ with respect to the Γ' and Δ'. The latter case is analogous. From (c) to (b), if $\Sigma \subseteq \Gamma$ and $\Pi \subseteq \Delta$, then $\vdash_{g*} \Gamma', \Sigma \Rightarrow \Pi, \Delta'$ contradicts the maximality of (Γ, Δ). Hence $\Sigma \nsubseteq \Gamma$ or $\Pi \nsubseteq \Delta$.

For (ii), we show the contrapositive. Suppose for all $\Gamma' \subseteq \Gamma$ we have $\nvdash_{g*} \Gamma', \Sigma \Rightarrow \Pi$. Then for any $\Sigma' \subseteq \Gamma \cup \Sigma$ and $\Pi' \subseteq \Pi$ it holds that $\nvdash_{g*} \Sigma' \Rightarrow \Pi'$. Hence apply Lemma 3.8 to obtain a maximal set (Σ'', Π'') such that $\Gamma \cup \Sigma \subseteq \Sigma''$ and $\Pi \subseteq \Pi''$. Now by (i), $(\Sigma'', \Pi'') \nvDash_* \Sigma \Rightarrow \Pi$ and $(\Gamma, \Delta) \leq_c (\Sigma'', \Pi'')$.

For (iii), if $\vdash_{g*} \Sigma \Rightarrow \Pi$ then by (i) $(\Gamma, \Delta) \vDash_* \Sigma \Rightarrow \Pi$ for all $(\Gamma, \Delta) \in W_c$. For the converse direction, we show the contrapositive. If $\nvdash_{g*} \Sigma \Rightarrow \Pi$ then apply Lemma 3.8 to obtain a maximal set (Σ', Π'). Then by (i) we conclude $(\Sigma', \Pi') \nvDash_* \Sigma \Rightarrow \Pi$. □

Lemma 3.11. The canonical model for $\mathbf{GC}^{ab}$ is indeed a $\mathbf{C}^{ab}$-model.

Proof. It is readily checked that $(W_c, \leq_c)$ is a non-empty pre-ordered set. For (Upward Closure), if $(\Gamma, \Delta) \Vdash^*_{ab} A$ and $(\Gamma', \Delta') \geq_c (\Gamma, \Delta)$ then $A \in \Gamma \subseteq \Gamma'$ for $* = +$ and $\sim A \in \Gamma \subseteq \Gamma'$ for $* = -$. So $(\Gamma', \Delta') \Vdash^*_{ab} A$.

We also need to check the conditions on $\bot$ and compound formulas.

$\underline{\bot}$ If we have $\bot \in \Gamma$ for some $(\Gamma, \Delta) \in W_c$, then the fact that $\vdash_{gab} \bot \Rightarrow \Delta'$ for any finite $\Delta' \subseteq \Delta$ contradicts the maximality of (Γ, Δ). Hence $\bot \notin \Gamma$ and consequently $(\Gamma, \Delta) \nVdash^{+}_{ab} \bot$ for all $(\Gamma, \Delta) \in W_c$.

$\underline{\sim}$ For $\Vdash^{+}_{ab}$, it holds that $(\Gamma, \Delta) \Vdash^{+}_{ab} {\sim}A$ iff ${\sim}A \in \Gamma$ iff $(\Gamma, \Delta) \Vdash^{-}_{ab} A$. For $\Vdash^{-}_{ab}$, if $(\Gamma, \Delta) \Vdash^{-}_{ab} {\sim}A$ but $(\Gamma, \Delta) \nVdash^{+}_{ab} A$ then $(\Gamma, \Delta) \vDash_{ab} A \Rightarrow$. By Lemma 3.10 (i) there are $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{gab} \Gamma', A \Rightarrow \Delta'$. Thus by (L$\sim\sim$) $\vdash_{gab} \Gamma', {\sim}{\sim}A \Rightarrow \Delta'$. Hence by Lemma 3.10 (i) again, $(\Gamma, \Delta) \vDash_{ab} {\sim}{\sim}A \Rightarrow$. But we have $(\Gamma, \Delta) \Vdash^{+}_{ab} {\sim}{\sim}A$ as ${\sim}{\sim}A \in \Gamma$, a contradiction. Therefore $(\Gamma, \Delta) \Vdash^{+}_{ab} A$. Conversely, if $(\Gamma, \Delta) \Vdash^{+}_{ab} A$ then $(\Gamma, \Delta) \vDash_{ab} \Rightarrow A$. By Lemma 3.10 (i) there are $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{gab} \Gamma' \Rightarrow A, \Delta'$. Apply (R$\sim\sim$) to obtain $\vdash_{gab} \Gamma' \Rightarrow {\sim}{\sim}A, \Delta'$. By Lemma 3.10 (i), $(\Gamma, \Delta) \vDash_{ab} \Rightarrow {\sim}{\sim}A$. Therefore $(\Gamma, \Delta) \Vdash^{-}_{ab} {\sim}A$.

$\underline{\wedge}$ For $\Vdash^{+}_{ab}$, suppose $(\Gamma, \Delta) \Vdash^{+}_{ab} A \wedge B$ but $(\Gamma, \Delta) \nVdash^{+}_{ab} A$. Then $(\Gamma, \Delta) \vDash_{ab} A \Rightarrow$. By Lemma 3.10 (i) there are $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{gab} \Gamma', A \Rightarrow \Delta'$. Thus by (L$\wedge$), $\vdash_{gab} \Gamma', A \wedge B \Rightarrow \Delta'$. Hence by Lemma 3.10 (i) again, $(\Gamma, \Delta) \vDash_{ab} A \wedge B \Rightarrow$, a contradiction. Thus $(\Gamma, \Delta) \Vdash^{+}_{ab} A$ and similarly $(\Gamma, \Delta) \Vdash^{+}_{ab} B$. Conversely, if $(\Gamma, \Delta) \Vdash^{+}_{ab} A$ and $(\Gamma, \Delta) \Vdash^{+}_{ab} B$, then $(\Gamma, \Delta) \vDash_{ab} \Rightarrow A$ and $(\Gamma, \Delta) \vDash_{ab} \Rightarrow B$. By Lemma 3.10 (i), there are $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{gab} \Gamma' \Rightarrow A, \Delta'$ and $\vdash_{gab} \Gamma' \Rightarrow B, \Delta'$. Thus by (R$\wedge$), $\vdash_{gab} \Gamma' \Rightarrow A \wedge B, \Delta'$. So by Lemma 3.10 (i), $(\Gamma, \Delta) \vDash_{ab} \Rightarrow A \wedge B$. Therefore $(\Gamma, \Delta) \Vdash^{+}_{ab} A \wedge B$.

Next for $\Vdash^{-}_{ab}$, if $(\Gamma, \Delta) \Vdash^{-}_{ab} A \wedge B$ then $(\Gamma, \Delta) \nVdash^{-}_{ab} A$ and $(\Gamma, \Delta) \nVdash^{-}_{ab} B$ imply $(\Gamma, \Delta) \vDash_{ab} {\sim}A \Rightarrow$ and $(\Gamma, \Delta) \vDash_{ab} {\sim}B \Rightarrow$. By Lemma 3.10 (i) there are $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash_{gab} \Gamma', {\sim}A \Rightarrow \Delta'$ and $\vdash_{gab} \Gamma', {\sim}B \Rightarrow \Delta'$. By $(L{\sim}\wedge)$ we infer $\vdash_{gab} \Gamma', {\sim}(A \wedge B) \Rightarrow \Delta'$; so by Lemma 3.10 (i) again, $(\Gamma, \Delta) \vDash_{ab} {\sim}(A \wedge B) \Rightarrow$. Hence ${\sim}(A \wedge B) \notin \Gamma$ and so $(\Gamma, \Delta) \nVdash^{-}_{ab} A \wedge B$, a contradiction. Therefore either $(\Gamma, \Delta) \Vdash^{-}_{ab} A$ or $(\Gamma, \Delta) \Vdash^{-}_{ab} B$. Conversely, if $(\Gamma, \Delta) \Vdash^{-}_{ab} A$ then ${\sim}A \in \Gamma$ and so $(\Gamma, \Delta) \vDash_{ab} \Rightarrow {\sim}A$. Then $(\Gamma, \Delta) \vDash_{ab} \Rightarrow {\sim}(A \wedge B)$ by Lemma 3.10 (i) and (R$\sim\wedge$). Therefore $(\Gamma, \Delta) \Vdash^{-}_{ab} A \wedge B$. The case when $(\Gamma, \Delta) \Vdash^{-}_{ab} B$ is analogous.

$\underline{\vee}$ Similar to the cases for conjunction.

$\underline{\rightarrow}$ For $\Vdash^{+}_{ab}$, suppose $(\Gamma, \Delta) \Vdash^{+}_{ab} A \rightarrow B$. Then since $\vdash_{gab} A, A \rightarrow B \Rightarrow B$, by Lemma 3.10 (i) we infer $(\Gamma', \Delta') \vDash_{ab} A \Rightarrow B$ for any $(\Gamma', \Delta') \geq_c (\Gamma, \Delta)$; that is to say, $(\Gamma', \Delta') \Vdash^{+}_{ab} A$ implies $(\Gamma', \Delta') \Vdash^{+}_{ab} B$ for all $(\Gamma', \Delta') \geq_c (\Gamma, \Delta)$.

Conversely, if for all $(\Sigma, \Pi) \geq_c (\Gamma, \Delta)$ it holds that $(\Sigma, \Pi) \Vdash^{+}_{ab} A$ implies $(\Sigma, \Pi) \Vdash^{+}_{ab} B$, then by Lemma 3.10 (ii) we infer $\vdash_{gab} \Gamma', A \Rightarrow B$ for some $\Gamma' \subseteq \Gamma$. By (R$\rightarrow$) we obtain $\vdash_{gab} \Gamma' \Rightarrow A \rightarrow B$. Hence by Lemma 3.10 (i) we conclude $(\Gamma, \Delta) \vDash_{ab} \Rightarrow A \rightarrow B$, i.e. $(\Gamma, \Delta) \Vdash^{+}_{ab} A \rightarrow B$. The case for $\Vdash^{-}_{ab}$ is argued in a similar manner, using the already established equivalence for negation. □

Lemma 3.12. The canonical model for $\mathbf{GC}^{ab}_{po}$ ($\mathbf{GC}^{ab}_{wn}$) is indeed a $\mathbf{C}^{ab}_{po}$-model ($\mathbf{C}^{ab}_{wn}$-model).

Proof. For $\mathbf{GC}^{ab}_{po}$, we have to check that the canonical model satisfies (Potential Omniscience). Towards a contradiction, suppose that $(\Gamma', \Delta') \nVdash^{+}_{po} A$ for all $(\Gamma', \Delta') \geq_c (\Gamma, \Delta)$ but $(\Gamma', \Delta') \nVdash^{-}_{po} A$ for all $(\Gamma', \Delta') \geq_c (\Gamma, \Delta)$. Then $(\Gamma', \Delta') \vDash_{po} A \Rightarrow$ and $(\Gamma', \Delta') \vDash_{po} {\sim}A \Rightarrow$ for each such (Γ', Δ'); hence by Lemma 3.10 (ii) we conclude $\vdash_{gpo} \Sigma, A \Rightarrow$ and $\vdash_{gpo} \Sigma, {\sim}A \Rightarrow$

for some $\Sigma \subseteq \Gamma$. By (gPO) , $\vdash_{gpo} \Sigma \Rightarrow$; thus $(\Gamma, \Delta) \vDash_{po} \Rightarrow$ by Lemma 3.10 (i), a contradiction. Therefore we can conclude that $(\Gamma', \Delta') \Vdash^{-}_{po} A$ for some $(\Gamma', \Delta') \geq_c (\Gamma, \Delta)$.

For $\mathbf{GC}^{ab}_{wn}$, we have to check (Weak Negation). Suppose that $(\Gamma', \Delta') \nVdash^{+}_{wn} A$ for all $(\Gamma', \Delta') \geq_c (\Gamma, \Delta)$. Then like in the previous case, $\vdash_{gwn} \Sigma, A \Rightarrow$ for some $\Sigma \subseteq \Gamma$. Also $\vdash_{gwn} \Sigma, {\sim}A \Rightarrow {\sim}A$. Thus by (gWN) $\vdash_{gwn} \Sigma \Rightarrow {\sim}A$; therefore $(\Gamma, \Delta) \vDash_{wn} \Rightarrow {\sim}A$ and so $(\Gamma, \Delta) \Vdash^{-}_{wn} A$. □

Now we are ready to show the completeness theorem.

Theorem 3.13 (completeness)**.** Let $* \in \{ab, po, wn\}$. If $\vDash_* \Gamma \Rightarrow \Delta$ then $\vdash_{g*} \Gamma \Rightarrow \Delta$.

Proof. Suppose $\vDash_* \Gamma \Rightarrow \Delta$. Consider the canonical model $\mathcal{M}_c$, which is by Lemma 3.11 and 3.12 is an appropriate model. Then by Lemma 3.10 (iii), we conclude $\vdash_{g*} \Gamma \Rightarrow \Delta$. □

We consequently obtain the completeness with respect to Hilbert-style systems as well.

Corollary 3.14. Let $* \in \{ab, po, wn\}$ and Γ, Δ be finite sets of formulas. Then $\vDash_* \Gamma \Rightarrow \Delta$ implies $\Gamma \vdash_{h*} \Delta$.

4 Properties of $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}_{wn}$

In this section, we shall look at some properties of $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}_{wn}$ that can be shown from the results we have established so far. They provide useful information when we later discuss which of the systems an intuitionistic logician might prefer.

We begin with separating the Hilbert-style systems. This gives a strict hierarchy of the systems, $\mathbf{C}^{ab}_{3}$, $\mathbf{C}^{ab}_{wn}$, $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}$ when ordered from the strongest to the weakest.

Proposition 4.1. The following statements hold.

(i) $\nvdash_{hab} \neg\neg(A \vee {\sim}A)$.

(ii) $\nvdash_{hpo} \neg A \to {\sim}A$.

(iii) $\nvdash_{hwn} A \vee {\sim}A$.

Proof. For (i), take a model $\mathcal{M} = ((W, \leq), \mathcal{V})$ such that $W = \{w, w'\}$; $\leq$ is the reflexive closure of $\{(w, w')\}$; and $\mathcal{V}$ is defined inductively such that $\mathcal{V}^+(p) = \mathcal{V}^-(p) = \emptyset$ for all p, $\mathcal{V}^-(\bot) = W$, and for compound formulas, $\mathcal{V}^+$ and $\mathcal{V}^-$ are defined according to the equivalences in Definition 3.1. As before, $\mathcal{M}$ is easily checked to be a $\mathbf{C}^{ab}$-model. Now, since $\mathcal{M}, w' \nVdash^{+}_{ab} p$ and $\mathcal{M}, w' \nVdash^{+}_{ab} {\sim}p$, it holds that $\mathcal{M}, w' \nVdash^{+}_{ab} p \vee {\sim}p$. Thus $\mathcal{M} \nvDash_{ab} \Rightarrow \neg\neg(p \vee {\sim}p)$. The statement then holds by Corollary 3.6.

For (ii), consider a model $\mathcal{M}'$ defined analogously to $\mathcal{M}$ with the only difference being that $\mathcal{V}^-(p) = \{w'\}$ for all p. In order to check that $\mathcal{M}'$ is a $\mathbf{C}^{ab}_{po}$-model, we need to show that (Potential Omniscience) holds. First observe that:

$$\mathcal{M}', w' \nVdash^{+}_{po} p \Rightarrow \mathcal{M}', w' \Vdash^{-}_{po} p;$$
$$\mathcal{M}', w' \nVdash^{-}_{po} p \Rightarrow \mathcal{M}', w' \Vdash^{+}_{po} p.$$

Now if for some $x \in W$ it holds that $\forall x' \geq x(\mathcal{M}', x' \nVdash^+_{po} p)$, then $\mathcal{M}', w' \nVdash^+_{po} p$; so $\mathcal{M}', w' \Vdash^-_{po} p$. Hence $\exists y \geq x(y \Vdash^-_{po} p)$. From Proposition 3.4 (i), this and the easily checkable case for $\perp$ are sufficient to establish (Potential Omniscience).

The only thing that is left is to observe that $\mathcal{M}'$ works as a counter-model. For this it suffices to note that $\mathcal{M}', w' \nVdash^+_{po} p$ implies $\mathcal{M}', w \Vdash^+_{po} \neg p$, but $\mathcal{M}', w \nVdash^+_{po} {\sim}p$; Therefore $\mathcal{M}', w \nVdash^+_{po} \neg p \to {\sim}p$.

For (iii), consider a model $\mathcal{M}''$ defined analogously to $\mathcal{M}'$ with the only difference being that $\mathcal{V}^+(p) = \{w'\}$ as well, for all p. In order to check that $\mathcal{M}''$ is a $\mathbf{C}^{ab}_{wn}$-model, we claim that

$$\mathcal{M}'', w' \nVdash^+_{wn} A \Rightarrow \mathcal{M}'', w \Vdash^-_{wn} A;$$
$$\mathcal{M}'', w' \nVdash^-_{wn} A \Rightarrow \mathcal{M}'', w \Vdash^+_{wn} A.$$

The cases when $A \equiv p, \perp$ are immediate. For conjunction, if $w' \nVdash^+_{wn} B \wedge C$ then $w' \nVdash^+_{wn} B$ or $w' \nVdash^+_{wn} C$. Hence by the I.H. $w \Vdash^-_{wn} B$ or $w \Vdash^-_{wn} C$ and so $w \Vdash^-_{wn} B \wedge C$. For the second item, if $w' \nVdash^-_{wn} B \wedge C$ then $w' \nVdash^-_{wn} B$ and $w' \nVdash^-_{wn} C$. By the I.H. $w \Vdash^+_{wn} B$ and $w \Vdash^+_{wn} C$; so $w \Vdash^+_{wn} B \wedge C$. The cases for disjunction are analogous.

For implication, if $w' \nVdash^+_{wn} B \to C$ then $w' \nVdash^+_{wn} C$. By the I.H. $w \Vdash^-_{wn} C$ and so $w \Vdash^-_{wn} B \to C$. The case for the second item is similar. Finally if $w' \nVdash^+_{wn} {\sim}A$ then $w' \nVdash^-_{wn} A$. By the I.H. $w \Vdash^+_{wn} A$ and so $w \Vdash^-_{wn} {\sim}A$. The case for the second item is similar as well.

Now if for some x it holds that $\forall x' \geq x(\mathcal{M}'', x' \nVdash^+_{wn} A)$, then $\mathcal{M}'', w' \nVdash^+_{wn} A$ and so by the claim $\mathcal{M}'', w \Vdash^-_{wn} A$, which implies $\mathcal{M}'', x \Vdash^-_{wn} A$, as required. Finally, to see that $\mathcal{M}''$ invalidates $A \vee {\sim}A$, note that $\mathcal{M}'', w \nVdash^+ p \vee {\sim}p$. ☐

Another point that follows from soundness is that ${\sim}\perp$ is not a theorem of $\mathbf{C}^{ab}_{po}$.

Proposition 4.2. $\vdash_{hpo} \neg\neg{\sim}\perp$ but $\nvdash_{hpo} {\sim}\perp$.

Proof. The former follows from ${\sim}\perp \leftrightarrow (\perp \vee {\sim}\perp)$ and $\neg\neg(\perp \vee {\sim}\perp)$. For the latter, construct a model $\mathcal{M} = ((W, \leq), \mathcal{V})$ where $W = \{w, w'\}$, $\leq$ is the reflexive closure of $\{(w, w')\}$, $\mathcal{V}^+(p) = \mathcal{V}^-(p) = W$, $\mathcal{V}^-(\perp) = \{w'\}$ and otherwise $\mathcal{V}$ is defined according to the equivalences in Definition 3.1. Then it is straightforward to see via Proposition 3.4 that $\mathcal{M}$ is a $\mathbf{C}^{ab}_{po}$-model. Note then $\mathcal{M}, w \nVdash^-_{po} \perp$. Hence $\nvdash_{hpo} {\sim}\perp$ by Corollary 3.6. ☐

Remark 4.3. This observation also implies that $\vdash_{hpo} \neg\neg{\sim}\neg A$ but $\nvdash_{hpo} {\sim}\neg A$. Since the latter may be a controversial formula, it can be taken as an advantage that $\mathbf{C}^{ab}_{po}$ does not prove it. The provability of the former, on the other hand, appears less controversial, because it merely states that the falsity of an intuitionistic negation does not (in the sense of $\neg$) lead to absurdity. It might be even preferable from the perspective of an intuitionistic logician, because it offers some information about the status of ${\sim}\perp$ compared with the case for $\mathbf{C}^{ab}$ where it is left unspecified.

Next we shall point to the strength of $\mathbf{C}^{ab}_{po}$ in expressing *provable contradictions*, i.e. formulas A such that both A and ${\sim}A$ are provable in the system. For this purpose, we shall use the logic **CN** introduced by J. Cantwell [4] plus $\bot$, an expansion already considered in [9]. As pointed out by Omori and Wansing [25], **CN** can be seen as an extension of **C3** with Peirce's law; for more information about **CN** and related systems, see also the two-part papers by P. Égré, L. Rossi and J. Sprenger [6, 7] as well as [8].

Definition 4.4. We define $\mathbf{CN}^{\bot}$ by adding the next axiom schema to $\mathbf{C}_3^{ab}$:

$$((A \to B) \to A) \to A \qquad \text{(PL)}$$

We shall use $\vdash_{hcn}$ to denote the derivability.

$\mathbf{CN}^{\bot}$ works as the classical counterpart of $\mathbf{C}_{po}^{ab}$, as confirmed by expanding Glivenko's theorem to include $\sim$.

Proposition 4.5 (Glivenko's theorem)**.** $\Gamma \vdash_{hcn} A$ if and only if $\Gamma \vdash_{hpo} \neg\neg A$.

Proof. The 'if' direction is immediate. For the 'only if' direction, it follows by induction on the depth of derivation in $\mathbf{CN}^{\bot}$. Most of the cases are as in Glivenko's theorem for intuitionistic logic (see e.g. [27]). The only important case is that of (3), but clearly, (PO) suffices in this case. □

Using this, we can embed provable contradictions of $\mathbf{CN}^{\bot}$ into $\mathbf{C}_{po}^{ab}$ in a simple manner.

Corollary 4.6. A formulas A is a provable contradiction in $\mathbf{CN}^{\bot}$ if and only if $\neg(A \wedge {\sim}A) \to A$ is so in $\mathbf{C}_{po}^{ab}$.

Proof. For the 'only if' direction, if A is a provable contradiction in $\mathbf{CN}^{\bot}$ then by Proposition 4.5 we infer $\vdash_{hpo} \neg\neg(A \wedge {\sim}A)$. hence $\vdash_{hpo} \neg(A \wedge \neg A) \to A$ and $\vdash_{hpo} \neg(A \wedge \neg A) \to {\sim}A$, so by (NI) $\vdash_{hpo} {\sim}(\neg(A \wedge \neg A) \to A)$ as well. For the 'if' direction, if $\neg(A \wedge {\sim}A) \to A$ is a provable contradiction in $\mathbf{C}_{po}^{ab}$ then $\vdash_{hpo} \neg(A \wedge {\sim}A) \to (A \wedge {\sim}A)$ and so $\vdash_{hcn} A \wedge {\sim}A$. Thus the statement follows. □

Remark 4.7. We may also note that the same embedding does not work with respect to $\mathbf{C}^{ab}$. It is straightforward from (3) that $(p \leftrightarrow {\sim}p) \to p$ is a provable contradiction in $\mathbf{CN}^{\bot}$, but we can construct a $\mathbf{C}^{ab}$-model $\mathcal{M} = ((\{w\}, (\{(w,w)\})), \mathcal{V})$ such that $\mathcal{V}^{+}(p) = \mathcal{V}^{-}(p) = \emptyset$: we can show in this model that $\mathcal{M}, w \nVdash_{ab}^{+} \neg(((p \leftrightarrow {\sim}p) \to p) \wedge {\sim}((p \leftrightarrow {\sim}p) \to p)) \to ((p \leftrightarrow {\sim}p) \to p)$.

The above corollary says that $\mathbf{C}_{po}^{ab}$ is as rich as $\mathbf{CN}^{\bot}$ in producing provable contradictions. As we shall see later, $\mathbf{C}_{po}^{ab}$ is a constructive system, so this means that every provable contradiction in $\mathbf{CN}^{\bot}$ has a constructive counterpart. At the same time, one may wonder whether this is due to (PO) being a rather strong principle. There can be a worry that the system is not acceptable to someone who is interested in provable contradictions but is inclined to stay in **C**. The following observation, based on the conservativity of Jankov's logic over positive intuitionistic logic [13], addresses such worry to some extent.

Proposition 4.8. Let Γ and Δ be finite sets of formulas in $\mathcal{L}^+$. Then $\Gamma \vdash_{hab} \Delta$ iff $\Gamma \vdash_{hpo} \Delta$ iff $\Gamma \vdash_{hwn} \Delta$.

Proof. By Proposition 2.6, it is sufficient to check that $\Gamma \vdash_{hwn} \Delta$ implies $\Gamma \vdash_{hab} \Delta$. We shall show the contrapositive of this implication. If $\Gamma \nvdash_{hab} \Delta$ then $\nvDash_{ab} \Gamma \Rightarrow \Delta$ by Corollary 3.14. Thus there is a $\mathbf{C}^{ab}$-model $\mathcal{M} = ((W, \leq), \mathcal{V})$ such that $\mathcal{M}, w_0 \nvDash_{ab} \Gamma \Rightarrow \Delta$ for some $w_0 \in W$. Define a new model $\mathcal{M}' = ((W', \leq'), \mathcal{V}')$ as follows.

- $W' := W \cup \{u\}$.
- $\leq' := \leq \cup \{(w, u) : w \in W'\}$.
- $\mathcal{V}'^*$ for $* \in \{+, -\}$ is defined inductively, by:
  - $\mathcal{V}'^*(p) := \mathcal{V}^*(p) \cup \{u\}$.
  - $\mathcal{V}'^-(\bot) := W'$.
  - otherwise, the equivalences in Definition 3.1 are followed.

We claim $\mathcal{M}'$ is a $\mathbf{C}^{ab}_{wn}$-model. It is immediate from the definition that all the equivalences in Definition 3.1 hold. Then it is also straightforward to check that (Upward Closure) is satisfied.

We need also to check that (Weak Negation) holds. Towards this, we shall first show by induction that for any $w \in W'$:

$$\mathcal{M}', u \nVdash^+_{wn} A \Longrightarrow \mathcal{M}', w \Vdash^-_{wn} A \text{ and } \mathcal{M}', u \nVdash^-_{wn} A \Longrightarrow \mathcal{M}', w \Vdash^+_{wn} A.$$

Since $\mathcal{M}', u \Vdash^+_{wn} p$ and $\mathcal{M}', u \Vdash^-_{wn} p$ for all p, the cases for propositional variables hold. For $\bot$, the statements hold because $\mathcal{M}', w \Vdash^-_{wn} \bot$ for all $w \in W'$.

For conjunction, first if $\mathcal{M}', u \nVdash^+_{wn} A \land B$, then $\mathcal{M}', u \nVdash^+_{wn} A$ or $\mathcal{M}', u \nVdash^+_{wn} B$. By the I.H. $\mathcal{M}', w \Vdash^-_{wn} A$ or $\mathcal{M}', w \Vdash^-_{wn} B$; hence $\mathcal{M}', w \Vdash^-_{wn} A \land B$. Next, if $\mathcal{M}', u \nVdash^-_{wn} A \land B$ then $\mathcal{M}', u \nVdash^-_{wn} A$ and $\mathcal{M}', u \nVdash^-_{wn} B$. By the I.H. $\mathcal{M}', w \Vdash^+_{wn} A$ and $\mathcal{M}', w \Vdash^+_{wn} B$; hence $\mathcal{M}', w \Vdash^+_{wn} A \land B$. The cases for disjunction are analogous.

For implication, first if $\mathcal{M}', u \nVdash^+_{wn} A \to B$ then $\mathcal{M}', u \nVdash^+_{wn} B$. Thus by the I.H. $\mathcal{M}', w \Vdash^-_{wn} B$ and consequently $\mathcal{M}', w \Vdash^-_{wn} A \to B$. Similarly, if $\mathcal{M}', u \nVdash^-_{wn} A \to B$ then $\mathcal{M}', u \nVdash^-_{wn} B$, so by the I.H. $\mathcal{M}', w \Vdash^+_{wn} B$ and $\mathcal{M}', w \Vdash^+_{wn} A \to B$.

For negation, first if $\mathcal{M}', u \nVdash^+_{wn} {\sim}A$ then $\mathcal{M}', u \nVdash^-_{wn} A$. By the I.H. $\mathcal{M}', w \Vdash^+_{wn} A$; hence $\mathcal{M}', w \Vdash^-_{wn} {\sim}A$. Similarly, if $\mathcal{M}', u \nVdash^-_{wn} {\sim}A$ then $\mathcal{M}', u \nVdash^+_{wn} A$. By the I.H. $\mathcal{M}', w \Vdash^-_{wn} A$ and so $\mathcal{M}', w \Vdash^-_{wn} {\sim}A$.

Now, if $\forall w' \geq w(\mathcal{M}', w' \nVdash^+_{wn} A)$ then in particular $\mathcal{M}, u \nVdash^+_{wn} A$. By what we have established, we infer $\mathcal{M}, x \Vdash^-_{wn} A$ for any $x \in W'$. Therefore $\mathcal{M}, w \Vdash^-_{wn} A$. It is thus established that (Weak Negation) is satisfied. Consequently $\mathcal{M}'$ is an $\mathbf{C}^{ab}_{wn}$-model.

In order to establish the proposition itself, we shall observe that

$$\mathcal{M}, w \Vdash^*_{ab} A \text{ if and only if } \mathcal{M}', w \Vdash^*_{wn} A$$

for $* \in \{+, -\}$, $w \in W$ and A in $\mathcal{L}^+$. The cases for propositional variables hold by stipulation.

For conjunction, first, $\mathcal{M}, w \Vdash^+_{ab} A \wedge B$ holds if and only if $\mathcal{M}, w \Vdash^+_{ab} A$ and $\mathcal{M}, w \Vdash^+_{ab} B$. By the I.H. this is equivalent to $\mathcal{M}', w \Vdash^+_{wn} A$ and $\mathcal{M}', w \Vdash^+_{wn} B$ and hence to $\mathcal{M}', w \Vdash^+_{wn} A \wedge B$. Similarly, $\mathcal{M}, w \Vdash^-_{ab} A \wedge B$ holds if and only if $\mathcal{M}, w \Vdash^-_{ab} A$ or $\mathcal{M}, w \Vdash^-_{ab} B$ holds. By the I.H. this is equivalent to that $\mathcal{M}', w \Vdash^-_{wn} A$ or $\mathcal{M}', w \Vdash^-_{wn} B$ and hence to $\mathcal{M}', w \Vdash^-_{wn} A \wedge B$. The cases for disjunction are similar.

For implication, first, $\mathcal{M}, w \Vdash^+_{ab} A \to B$ holds if $\forall w' \geq w(\mathcal{M}, w' \Vdash^+_{ab} A \Rightarrow \mathcal{M}, w' \Vdash^+_{ab} B)$. By the I.H. this is equivalent to $\forall w' \geq w(\mathcal{M}', w' \Vdash^+_{wn} A \Rightarrow \mathcal{M}', w' \Vdash^+_{wn} B)$. Furthermore, as is easily checkable, $\mathcal{M}', u \Vdash^*_{wn} C$ for $* \in \{+, -\}$ and C in $\mathcal{L}^+$. Therefore $\mathcal{M}', u \Vdash^+_{wn} A$ implies $\mathcal{M}', u \Vdash^+_{wn} B$ as well. Thus $\mathcal{M}, w \Vdash^+_{ab} A \to B$ is equivalent to $\forall w' \geq' w(\mathcal{M}, w' \Vdash^+_{wn} A \Rightarrow \mathcal{M}, w' \Vdash^+_{wn} B)$, i,e. $\mathcal{M}', w \Vdash^+_{wn} A \to B$. Similarly for the case for $\Vdash^-$.

For negation, $\mathcal{M}, w \Vdash^+_{ab} {\sim}A$ holds if and only if $\mathcal{M}, w \Vdash^-_{ab} A$. By the I.H., this is equivalent to $\mathcal{M}', w \Vdash^-_{wn} A$ and therefore to $\mathcal{M}, w \Vdash^+_{ab} {\sim}A$. The case for $\Vdash^-$ is analogous.

We are now ready to observe that $\mathcal{M}', w_0 \Vdash^+_{wn} A$ for all $A \in \Gamma$ but $\mathcal{M}', w_0 \nVdash^+_{wn} B$ for all $B \in \Delta$. Therefore $\nvDash_{wn} \Gamma \Rightarrow \Delta$ and by Corollary 3.6 we conclude $\Gamma \nvdash_{hwn} \Delta$. □

5 More on Sequent Calculus

In this section, we shall introduce another type of sequent calculi for $\mathbf{C}^{ab}$, $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}_{wn}$ which have certain proof-theoretic advantages. We shall show their cut-eliminability and make a few observations related to constructivity and the subformula property.

5.1 Bilateral-style Sequent Calculi

The calculi we shall consider are based on the subformula calculus **Sn4** for **N4**, introduced by N. Kamide and H. Wansing [14, 15]. As the name suggests, this type of calculi shows a better behaviour with respect to the subformula property than the type of calculi of Definition 2.7. In addition, it has a more bilateral flavour (see e.g. [5, 29, 36]) as well, which might be preferable from certain philosophical perspectives.

In this type of calculus, a sequent (for distinction, we shall call it a *b-sequent*) is of the form $\Gamma|\Delta \Rightarrow^* \Pi$, where Γ, Δ are finite sets of formulas, Π is a set of formulas with at most one element, and $* \in \{+, -\}$. Let us first look at a calculus for $\mathbf{C}^{ab}$.

Definition 5.1. The calculus $\mathbf{SC}^{ab}$ is defined by the following rules:

$$A| \ \Rightarrow^- A \ (\mathrm{Ax}-) \qquad\qquad |A \Rightarrow^+ A \ (\mathrm{Ax}+)$$

$$|\bot \Rightarrow^* \ (\mathrm{L}\bot+)$$

$$\frac{\Gamma|\Delta \Rightarrow^- A \qquad A, \Gamma'|\Delta' \Rightarrow^* \Pi}{\Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^* \Pi}\ (\mathrm{Cut}-) \qquad \frac{\Gamma|\Delta \Rightarrow^+ A \qquad \Gamma'|\Delta', A \Rightarrow^* \Pi}{\Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^* \Pi}\ (\mathrm{Cut}+)$$

$$\frac{\Gamma|\Delta \Rightarrow^* \Pi}{A, \Gamma|\Delta \Rightarrow^* \Pi}\ (\mathrm{LW}-) \qquad \frac{\Gamma|\Delta \Rightarrow^*}{\Gamma|\Delta \Rightarrow^- C}\ (\mathrm{RW}-)$$

$$\frac{\Gamma|\Delta \Rightarrow^* \Pi}{\Gamma|\Delta, A \Rightarrow^* \Pi}\ (\text{LW}+) \qquad \frac{\Gamma|\Delta \Rightarrow^*}{\Gamma|\Delta \Rightarrow^+ C}\ (\text{RW}+)$$

$$\frac{A, \Gamma|\Delta \Rightarrow^* \Pi \qquad B, \Gamma|\Delta \Rightarrow^* \Pi}{A \wedge B, \Gamma|\Delta \Rightarrow^* \Pi}\ (\text{L}\wedge-) \qquad \frac{\Gamma|\Delta \Rightarrow^- A_i}{\Gamma|\Delta \Rightarrow^- A_1 \wedge A_2}\ (\text{R}\wedge-)$$

$$\frac{\Gamma|\Delta, A_i \Rightarrow^* \Pi}{\Gamma|\Delta, A_1 \wedge A_2 \Rightarrow^* \Pi}\ (\text{L}\wedge+) \qquad \frac{\Gamma|\Delta \Rightarrow^+ A \qquad \Gamma|\Delta \Rightarrow^+ B}{\Gamma|\Delta \Rightarrow^+ A \wedge B}\ (\text{R}\wedge+)$$

$$\frac{A_i, \Gamma|\Delta \Rightarrow^* \Pi}{A_1 \vee A_2, \Gamma|\Delta \Rightarrow^* \Pi}\ (\text{L}\vee-) \qquad \frac{\Gamma|\Delta \Rightarrow^- A \qquad \Gamma|\Delta \Rightarrow^- B}{\Gamma|\Delta \Rightarrow^- A \vee B}\ (\text{R}\vee-)$$

$$\frac{\Gamma|\Delta, A \Rightarrow^* \Pi \qquad \Gamma|\Delta, B \Rightarrow^* \Pi}{\Gamma|\Delta, A \vee B \Rightarrow^* \Pi}\ (\text{L}\vee+) \qquad \frac{\Gamma|\Delta \Rightarrow^+ A_i}{\Gamma|\Delta \Rightarrow^+ A_1 \vee A_2}\ (\text{R}\vee+)$$

$$\frac{\Gamma|\Delta \Rightarrow^+ A \qquad B, \Gamma'|\Delta' \Rightarrow^* \Pi}{A \to B, \Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^* \Pi}\ (\text{L}{\to}-) \qquad \frac{\Gamma|\Delta, A \Rightarrow^- B}{\Gamma|\Delta \Rightarrow^- A \to B}\ (\text{R}{\to}-)$$

$$\frac{\Gamma|\Delta \Rightarrow^+ A \qquad \Gamma'|\Delta', B \Rightarrow^* \Pi}{\Gamma, \Gamma'|\Delta, \Delta', A \to B \Rightarrow^* \Pi}\ (\text{L}{\to}+) \qquad \frac{\Gamma|\Delta, A \Rightarrow^+ B}{\Gamma|\Delta \Rightarrow^+ A \to B}\ (\text{R}{\to}+)$$

$$\frac{\Gamma|\Delta, A \Rightarrow^* \Pi}{{\sim}A, \Gamma|\Delta \Rightarrow^* \Pi}\ (\text{L}{\sim}-) \qquad \frac{\Gamma|\Delta \Rightarrow^+ A}{\Gamma|\Delta \Rightarrow^- {\sim}A}\ (\text{R}{\sim}-)$$

$$\frac{A, \Gamma|\Delta \Rightarrow^* \Pi}{\Gamma|\Delta, {\sim}A \Rightarrow^* \Pi}\ (\text{L}{\sim}+) \qquad \frac{\Gamma|\Delta \Rightarrow^- A}{\Gamma|\Delta \Rightarrow^+ {\sim}A}\ (\text{R}{\sim}+)$$

where $i \in \{1, 2\}$. The derivability in $\mathbf{SC}^{ab}$ will be denoted by $\vdash_{sab}$. If the rules (Cut−) and (Cut+) are removed from $\mathbf{SC}^{ab}$, it defines the cut-free system $\mathbf{SC}^{ab}$-(Cut), whose derivability is denoted by $\vdash^{cf}_{sab}$.

Next we define the bilateral-style calculi for $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}_{wn}$.

Definition 5.2. The calculus $\mathbf{SC}^{ab}_{po}$ is defined from $\mathbf{SC}^{ab}$ by the following rules.

$$\frac{p, \Gamma|\Delta \Rightarrow^* \qquad \Gamma|\Delta, p \Rightarrow^*}{\Gamma|\Delta \Rightarrow^*}\ (\text{sPO}) \qquad \frac{\bot, \Gamma|\Delta \Rightarrow^*}{\Gamma|\Delta \Rightarrow^*}\ (\text{L}\bot-)$$

The calculus $\mathbf{SC}^{ab}_{wn}$ is defined from $\mathbf{SC}^{ab}$ by the following rule.

$$\frac{A, \Gamma|\Delta \Rightarrow^* \Pi \qquad \Gamma|\Delta, A \Rightarrow^*}{\Gamma|\Delta \Rightarrow^* \Pi}\ (\text{sWN})$$

The derivability in $\mathbf{SC}^{ab}$ and $\mathbf{SC}^{ab}_{wn}$ will be denoted by $\vdash_{spo}$ and $\vdash_{swn}$. The derivability in the cut-free systems $\mathbf{SC}^{ab}_{po}$-(Cut) and $\mathbf{SC}^{ab}_{wn}$-(Cut) will be denoted by $\vdash^{cf}_{spo}$ and $\vdash^{cf}_{swn}$.

The rule (sPO) is modelled after the rule (Gem-at) for the systems **G3ip** +(Gem-at) in [20] and **G3C3at** in [26]. We should also note already that eliminating cut in $\mathbf{SC}^{ab}_{wn}$ does not give too many benefits, for (gWN) can similarly remove an arbitrary formula.

Before moving onto the proof of cut-elimination, let us observe the correspondence between the bilateral-style calculi and Hilbert-style calculi. For this purpose, we shall use the notations $\sim\Gamma := \{\sim A : A \in \Gamma\}$, $\emptyset^+, \emptyset^- := \bot$, $\{C\}^+ := C$ and $\{C\}^- := \sim C$.

Proposition 5.3. Let $\dagger \in \{ab, po, wn\}$. If $\vdash_{s\dagger} \Gamma|\Delta \Rightarrow^* \Pi$ then $\sim\Gamma, \Delta \vdash_{h\dagger} \Pi^*$.

Proof. By induction on the depth of derivation in $\mathbf{SC}^{ab}_{\dagger}$. For instance, when the last rule applied is an instance of (L$\to$−):

$$\frac{\Gamma|\Delta \Rightarrow^+ A \qquad B, \Gamma'|\Delta' \Rightarrow^- C}{A \to B, \Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^- C}$$

Then from the I.H. $\sim\Gamma, \Delta \vdash_{h\dagger} A$ and $\sim B, \sim\Gamma', \Delta' \vdash_{h\dagger} \sim C$. It is now straightforward to observe from (NC) and Theorem 2.5 that $\sim(A \to B), \sim\Gamma, \sim\Gamma', \Delta, \Delta' \vdash_{h\dagger} \sim C$. □

For the other direction, we need a couple of lemmas for $\mathbf{C}^{ab}_{po}$.

Lemma 5.4. The following statements hold.

(i) If $\vdash_{spo} A \wedge B, \Gamma|\Delta \Rightarrow^* C$ then $\vdash_{spo} A, \Gamma|\Delta \Rightarrow^* C$ and $\vdash_{spo} B, \Gamma|\Delta \Rightarrow^* C$.

(ii) If $\vdash_{spo} A \vee B, \Gamma|\Delta \Rightarrow^* C$ then $\vdash_{spo} A, B, \Gamma|\Delta \Rightarrow^* C$.

(iii) If $\vdash_{spo} A \to B, \Gamma|\Delta \Rightarrow^* C$ then $\vdash_{spo} B, \Gamma|\Delta \Rightarrow^* C$.

(iv) If $\vdash_{spo} \sim A, \Gamma|\Delta \Rightarrow^* C$ then $\vdash_{spo} \Gamma|\Delta, A \Rightarrow^* C$.

(v) If $\vdash_{spo} \Gamma|\Delta, A \wedge B \Rightarrow^* C$ then $\vdash_{spo} \Gamma|\Delta, A, B \Rightarrow^* C$.

(vi) If $\vdash_{spo} \Gamma|\Delta, A \vee B \Rightarrow^* C$ then $\vdash_{spo} \Gamma|\Delta, A \Rightarrow^* C$ and $\vdash_{spo} \Gamma|\Delta, B \Rightarrow^* C$.

(vii) If $\vdash_{spo} \Gamma|\Delta, A \to B \Rightarrow^* C$ then $\vdash_{spo} \Gamma|\Delta, B \Rightarrow^* C$.

(viii) If $\vdash_{spo} \Gamma|\Delta, \sim A \Rightarrow^* C$ then $\vdash_{spo} A, \Gamma|\Delta \Rightarrow^* C$.

Proof. By (Cut−) and (Cut+). □

Lemma 5.5. If $\vdash_{spo} A, \Gamma|\Delta \Rightarrow^*$ and $\vdash_{spo} \Gamma|\Delta, A \Rightarrow^*$ then $\vdash_{spo} \Gamma|\Delta \Rightarrow^*$.

Proof. By induction on the complexity of A. The cases when $A \equiv p$ and $A \equiv \bot$ follow from (sPO) and (L$\bot$−), respectively. If $A \equiv B \wedge C$, then $B \wedge C, \Gamma|\Delta \Rightarrow^*$ and $\Gamma|\Delta, B \wedge C \Rightarrow^*$. By Lemma 5.4 it holds that $B, \Gamma|\Delta \Rightarrow^*$; $C, \Gamma|\Delta \Rightarrow^*$ and $\Gamma|\Delta, B, C \Rightarrow^*$. Hence we obtain the next derivation.

$$\dfrac{\dfrac{\Gamma|\Delta, B, C \Rightarrow^* \qquad \dfrac{B, \Gamma|\Delta \Rightarrow^*}{B, \Gamma|\Delta, C \Rightarrow^*}\,(\mathrm{LW}+)}{\Gamma|\Delta, C \Rightarrow^*}\,(\mathrm{I.H.}) \qquad C, \Gamma|\Delta \Rightarrow^*}{\Gamma|\Delta \Rightarrow^*}\,(\mathrm{I.H.})$$

The other cases are analogous. □

Proposition 5.6. For $\dagger \in \{ab, po, wn\}$, if $\Gamma \vdash_{h\dagger} A$ then $\vdash_{s\dagger} \ |\Gamma \Rightarrow^{+} A$.

Proof. By induction on the depth of derivation in $\mathbf{C}^{ab}_{\dagger}$. As an example, one direction of (NI) is:

$$\dfrac{\dfrac{|A \Rightarrow^{+} A \qquad \dfrac{B| \ \Rightarrow^{-} B}{|{\sim}B \Rightarrow^{-} B}\,(\mathrm{L}{\sim}+)}{\dfrac{|A, A \to {\sim}B \Rightarrow^{-} B}{\dfrac{|A \to {\sim}B \Rightarrow^{-} A \to B}{|A \to {\sim}B \Rightarrow^{+} {\sim}(A \to B)}\,(\mathrm{R}{\sim}+)}\,(\mathrm{R}{\to}-)}\,(\mathrm{L}{\to}+)}{| \ \Rightarrow^{+} (A \to {\sim}B) \to {\sim}(A \to B)}\,(\mathrm{R}{\to}+)$$

The other cases are checked similarly. For (PO), we need to appeal to Lemma 5.5. □

5.2 Cut-elimination

For cut-elimination, the argument will be a standard one, but as in [14, 15], we have to take care of two types of cut rules. We begin with introducing a couple of notions: suppose we have a derivation in $\mathbf{SC}^{ab}_{\dagger}$ ($\dagger \in \{ab, po, wn\}$) in which there is an application of cut (i.e. either (Cut−) or (Cut+)). Then by the *grade* of the cut, we shall mean the complexity of the cutformula (the formula A in (Cut−) and (Cut+).) By the *height* of the cut, we shall mean the number of b-sequents that occur in the subderivation which has the conclusion of cut as the endsequent.

Let us first establish a couple of lemmas.

Lemma 5.7. Let $\dagger \in \{ab, po, wn\}$. Then $\vdash^{cf}_{s\dagger} \Gamma|\Delta \Rightarrow^{+}$ if and only if $\vdash^{cf}_{s\dagger} \Gamma|\Delta \Rightarrow^{-}$.

Proof. By induction on the depth of derivation. □

Lemma 5.8. Let $\dagger \in \{ab, po, wn\}$ and suppose there is a derivation of a b-sequent in $\mathbf{SC}^{ab}_{\dagger}$ in which (Cut−) or (Cut+) is applied only at the last step. Then there is a derivation of the b-sequent $\mathbf{SC}^{ab}_{\dagger}$ in which there is no application of (Cut−) nor (Cut+).

Proof. We shall establish the statement by double induction, with the main induction on the grade of (Cut−)/(Cut+), and the subinduction on the height of (Cut−)/(Cut+). We divide into cases depending on which rules are applied to obtain the premises of the (Cut−)/(Cut+).

First we consider the cases where one of the premises is an instance of one of the 0-premise rules (Ax−), (Ax+) or (L⊥+). Then for the first two cases, the subderivation ending with the other premise is the desired derivation. If the right premise is (L⊥+) and the left premise is (RW+), then the subderivation ending with the premise of the (RW+) is either the desired derivation or is different from it only by the sign on the arrow: in this case apply Lemma 5.7. If the right premise is one of the other rules, e.g. (L→−), the derivation must have the following form.

$$\dfrac{\dfrac{\Gamma|\Delta \Rightarrow^{+} A \qquad B, \Gamma'|\Delta' \Rightarrow^{+} \bot}{A \to B, \Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^{+} \bot}\,(\mathrm{L}{\to}{-}) \qquad |\bot \Rightarrow^{*}}{A \to B, \Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^{*}}\,(\mathrm{Cut}+)$$

Then we can construct the following derivation:

$$\dfrac{\Gamma|\Delta \Rightarrow^{+} A \qquad \dfrac{B, \Gamma'|\Delta \Rightarrow^{+} \bot \qquad |\bot \Rightarrow^{*}}{B, \Gamma'|\Delta' \Rightarrow^{*}}\,(\mathrm{Cut}+)}{A \to B, \Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^{*}}\,(\mathrm{L}{\to}{-})$$

Since the new instance of (Cut+) is of lower height, it is possible to apply the I.H. to the subderivation ending with the instance of (Cut+); so we obtain a cut-free derivation of the endsequent.

Secondly, if one of the premises is obtained by an application of a weakening rule (i.e. (LW−), (LW+), (RW−) or (RW+)), then we can argue similarly to the previous cases, along with possible applications of weakening rules.

Thirdly, assume both of the premises are obtained through non-0-premise and non-weakening rules, but the cutformula is not principal in one of them. Consider, as a first example, the case of (Cut−) where the left premise is obtained through (sWN).

$$\dfrac{\dfrac{A, \Gamma|\Delta \Rightarrow^{-} B \qquad \Gamma|\Delta, A \Rightarrow^{-}}{\Gamma|\Delta \Rightarrow^{-} B}\,(\mathrm{sWN}) \qquad B, \Gamma'|\Delta' \Rightarrow^{*} C}{\Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^{*} C}\,(\mathrm{Cut}-)$$

Then we can construct the following derivation (the dashed line indicates applications of Lemma 5.7 and weakening).

$$\dfrac{\dfrac{A, \Gamma|\Delta \Rightarrow^{-} B \qquad B, \Gamma'|\Delta' \Rightarrow^{*} C}{A, \Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^{*} C}\,(\mathrm{Cut}-) \qquad \dfrac{\Gamma|\Delta, A \Rightarrow^{-}}{\Gamma, \Gamma'|\Delta, \Delta', A \Rightarrow^{*}}\,\dashrightarrow}{\Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^{*} C}\,(\mathrm{sWN})$$

We can then apply the I.H. to the subderivation ending with the instance of (Cut−). As a second example, consider the case of (Cut+) for $\mathbf{SC}^{ab}_{po}$ where the right premise is obtained through (sPO).

$$\dfrac{\Gamma|\Delta \Rightarrow^{+} A \qquad \dfrac{p, \Gamma'|\Delta', A \Rightarrow^{*} \qquad \Gamma'|\Delta', A, p \Rightarrow^{*}}{\Gamma'|\Delta', A \Rightarrow^{*}}\,(\mathrm{sPO})}{\Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^{*}}\,(\mathrm{Cut}+)$$

Then we can construct the next derivation:

$$\dfrac{\dfrac{\Gamma|\Delta \Rightarrow^{+} A \qquad p, \Gamma'|\Delta', A \Rightarrow^{*}}{p, \Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^{*}}\,(\mathrm{Cut}+) \qquad \dfrac{\Gamma|\Delta \Rightarrow^{+} A \qquad \Gamma'|\Delta', A, p \Rightarrow^{*}}{\Gamma, \Gamma'|\Delta, \Delta', p \Rightarrow^{*}}\,(\mathrm{Cut}+)}{\Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^{*}}\,(\mathrm{sPO})$$

Then we can apply the I.H. to the subderivations ending with an application of (Cut+).

Finally, assume that both of the premises are obtained through non-0-premise and non-weakening rules, and the cutformula is principal in both of them. Here we look at the cases for (Cut−) and the cutformula is an implication:

$$\dfrac{\dfrac{\Gamma|\Delta, A \Rightarrow^- B}{\Gamma|\Delta \Rightarrow^- A \to B}(\mathrm{R}{\to}-) \qquad \dfrac{\Gamma'|\Delta' \Rightarrow^+ A \qquad B, \Gamma''|\Delta'' \Rightarrow^* C}{A \to B, \Gamma', \Gamma''|\Delta', \Delta'' \Rightarrow^* C}(\mathrm{L}{\to}-)}{\Gamma, \Gamma', \Gamma''|\Delta, \Delta', \Delta'' \Rightarrow^* C}(\mathrm{Cut}-)$$

Then we can construct the following derivation:

$$\dfrac{\dfrac{\Gamma'|\Delta' \Rightarrow^+ A \qquad \Gamma|\Delta, A \Rightarrow^- B}{\Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^- B}(\mathrm{Cut}+) \qquad B, \Gamma''|\Delta'' \Rightarrow^* C}{\Gamma, \Gamma', \Gamma''|\Delta, \Delta', \Delta'' \Rightarrow^* C}(\mathrm{Cut}-)$$

Now we can first apply the I.H. to the (Cut+) to get a cut-free derivation of $\Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^- B$; then we can apply the I.H. to the (Cut−) because it has a lower grade. Other cases are similarly argued. □

Lemma 5.8 is enough to establish the cut-eliminability of the systems.

Theorem 5.9 (cut-elimination)**.** Let $\dagger \in \{ab, po, wn\}$. Then $\vdash_{s\dagger} \Gamma|\Delta \Rightarrow^* \Pi$ if and only if $\vdash^{cf}_{s\dagger} \Gamma|\Delta \Rightarrow^* \Pi$.

Proof. From Lemma 5.8, it is possible to transform a derivation with (Cut+) and (Cut−) into a cut-free one by removing, step by step, one of the uppermost instances of (Cut+) or (Cut−). □

5.3 Properties of Cut-free Systems

An immediate corollary of Theorem 5.9 is the following *subformula property* of $\mathbf{SC}^{ab}$:

Corollary 5.10 (subformula property)**.** If $\vdash_{sab} \Gamma|\Delta \Rightarrow^* \Pi$ then there is a derivation of the b-sequent in which all formulas are a subformula of $\Gamma \cup \Delta \cup \Pi$.

Proof. By inspection of the rules in $\mathbf{SC}^{ab}$-(Cut). □

On the other hand, the same argument does not show that the systems $\mathbf{SC}^{ab}_{po}$ and $\mathbf{SC}^{ab}_{wn}$ enjoy the subformula property: they have rules which can eliminate a formula, which leads to its not occurring in the endsequent as a subformula. Despite this, we shall see that when it comes to $\mathbf{SC}^{ab}_{po}$, it is always possible to convert any derivation into a derivation in which all formulas occurring also occur in the endsequent as a subformula.

Following the example of *analytic cut* (see e.g. [12, 27]), we shall call an instance of (sPO)/(L⊥−) *analytic*, if the active formula occurs in the conclusion of the rule as a subformula. Our aim here is to eliminate non-analytic instances of the rules that affect the subformula property.

Lemma 5.11. Let N be either a propositional variable or $\bot$. For any derivation of $\Gamma|\Delta \Rightarrow^* \Pi$ in $\mathbf{SC}_{po}^{ab}$-(Cut), suppose all instances of (sPO)/(L$\bot-$) in the derivation are analytic. Then $N \notin Sub((\Gamma \setminus \{N\}) \cup \Delta \cup \Pi)$ implies there is a derivation of $\Gamma \setminus \{N\}|\Delta \Rightarrow^* \Pi$ in which all instances of (sPO)/(L$\bot-$) are analytic.

Proof. We show by induction on the depth of derivation. If the derivation is an instance of (Ax$-$):

$$A| \;\Rightarrow^- A.$$

Then $N \notin Sub((\{A\}\setminus\{N\})\cup\{A\})$ implies $\{A\}\setminus\{N\} = \{A\}$. So the derivation is the desired derivation of $\{A\} \setminus \{N\}| \;\Rightarrow^- A$. For (Ax+) and (L$\bot$+), the statement follows trivially.

Suppose the derivation ends with an instance of (LW$-$):

$$\frac{\Gamma|\Delta \Rightarrow^* \Pi}{A, \Gamma|\Delta \Rightarrow^* \Pi}$$

Then $N \notin Sub((\{A\} \cup \Gamma) \setminus \{N\} \cup \Delta \cup \Pi)$ implies $N \notin Sub(\Gamma \setminus \{N\} \cup \Delta \cup \Pi)$. Hence by the I.H. there is a derivation of $\Gamma \setminus \{N\}|\Delta \Rightarrow^* \Pi$ in which all instances of (sPO)/(L$\bot-$) are analytic. Now if $A \equiv N$ then this is a desired derivation of $(\{A\} \cup \Gamma) \setminus \{N\}|\Delta \Rightarrow^* \Pi$. If on the other hand $A \not\equiv N$, then apply (LW$-$) to obtain a desired derivation.

Suppose the derivation ends with an instance of (L$\wedge-$):

$$\frac{A, \Gamma|\Delta \Rightarrow^* \Pi \qquad B, \Gamma|\Delta \Rightarrow^* \Pi}{A \wedge B, \Gamma|\Delta \Rightarrow^* \Pi}$$

Then $N \notin Sub(((\{A \wedge B\} \cup \Gamma) \setminus \{N\}) \cup \Delta \cup \Pi)$ implies $N \notin Sub(((\{A\} \cup \Gamma) \setminus \{N\}) \cup \Delta \cup \Pi)$, $N \notin Sub(((\{B\} \cup \Gamma) \setminus \{N\}) \cup \Delta \cup \Pi)$ and in particular $N \not\equiv A, B$. Thus by the I.H. there are derivations of $(\{A\} \cup \Gamma) \setminus \{N\}|\Delta \Rightarrow^* \Pi$ and $(\{B\} \cup \Gamma) \setminus \{N\}|\Delta \Rightarrow^* \Pi$ in which all instances of (sPO)/(L$\bot-$) are analytic. Now, because $N \not\equiv A, B$ we can apply (L$\wedge-$) to obtain a desired derivation of $(\{A \wedge B\} \cup \Gamma) \setminus \{N\}|\Delta \Rightarrow^* \Pi$ (note $N \not\equiv A \wedge B$ since it is not a compound formula).

Suppose the derivation ends with an instance of (L$\rightarrow-$):

$$\frac{\Gamma|\Delta \Rightarrow^+ A \qquad B, \Gamma'|\Delta' \Rightarrow^* \Pi}{A \to B, \Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^* \Pi}$$

Then $N \notin Sub(((\{A \to B\}\cup\Gamma\cup\Gamma')\setminus\{N\})\cup\Delta\cup\Delta'\cup\Pi)$ implies $N \notin Sub((\Gamma\setminus\{N\})\cup\Delta\cup\{A\})$ and $N \notin Sub(((\{B\} \cup \Gamma') \setminus \{N\}) \cup \Delta' \cup \Pi)$. Hence by the I.H. we have the derivations of $\Gamma \setminus \{N\}|\Delta \Rightarrow^+ A$ and $(\{B\} \cup \Gamma') \setminus \{N\}|\Delta' \Rightarrow^* \Pi$. Noting $B, A \to B \not\equiv N$, we can apply (L$\rightarrow-$) to obtain $(\{A \to B\} \cup \Gamma \cup \Gamma') \setminus \{N\}|\Delta, \Delta' \Rightarrow^* \Pi$.

Suppose the derivation ends with an instance of (L$\sim-$):

$$\frac{\Gamma|\Delta, A \Rightarrow^* \Pi}{\sim A, \Gamma|\Delta \to^* \Pi}$$

Then $N \notin Sub(((\{\sim A\} \cup \Gamma) \setminus \{N\}) \cup \Delta \cup \Pi)$ implies $N \notin Sub((\Gamma \setminus \{N\}) \cup \Delta \cup \{A\} \cup \Pi)$. So there is a derivation of $\Gamma \setminus \{N\}|\Delta, A \Rightarrow^* \Pi$ in which all instances of (sPO)/(L$\bot$$-$) are analytic. Apply (L$\sim$$-$) to obtain a desired derivation of $(\{\sim A\} \cup \Gamma) \setminus \{N\}|\Delta \Rightarrow^* \Pi$.

Suppose the derivation ends with an instance of (L$\sim$+):

$$\frac{A, \Gamma|\Delta \Rightarrow^* \Pi}{\Gamma|\Delta, \sim A \Rightarrow^* \Pi}$$

Then $N \notin Sub((\Gamma \setminus \{N\}) \cup \Delta \cup \{\sim A\} \cup \Pi)$ implies $N \notin Sub(((\{A\} \cup \Gamma) \setminus \{N\}) \cup \Delta \cup \Pi)$. Hence by the I.H. there is a derivation of $(\{A\} \cup \Gamma) \setminus \{N\}|\Delta \Rightarrow^* \Pi$ wherein all instances of (sPO)/(L$\bot$$-$) are analytic. Noting $A \not\equiv N$, we can apply (L$\sim$+) to obtain a desired derivation of $(\Gamma) \setminus \{N\}|\Delta, \sim A \Rightarrow^* \Pi$.

Suppose the derivation ends with an instance of (sPO):

$$\frac{p, \Gamma|\Delta \Rightarrow^* \qquad \Gamma|\Delta, p \Rightarrow^*}{\Gamma|\Delta \Rightarrow^*}$$

Then by assumption the instance must be analytic, and $N \notin Sub((\Gamma \setminus \{N\}) \cup \Delta)$ implies $N \notin Sub(((\{p\} \cup \Gamma) \setminus \{N\}) \cup \Delta)$. Now if $N \equiv p$, then by the I.H. there is a derivation of $(\{p\} \cup \Gamma) \setminus \{N\}|\Delta \Rightarrow^*$, and this derivation is also a desired derivation of $\Gamma \setminus \{N\}|\Delta \Rightarrow^*$. If $N \not\equiv p$, then $N \notin Sub((\Gamma \setminus \{N\}) \cup \Delta \cup \{p\})$ as well. So we have derivations of $(\{p\} \cup \Gamma) \setminus \{N\}|\Delta \Rightarrow^*$ and $\Gamma \setminus \{N\}|\Delta, p \Rightarrow^*$. As $N \not\equiv p$, we can apply (sPO) to obtain a desired derivation of $\Gamma \setminus \{N\}|\Delta \Rightarrow^*$: note in particular that the application remains analytic because $N \not\equiv p$.

Other cases can be argued analogously. □

Theorem 5.12. If $\vdash^{cf}_{spo} \Gamma|\Delta \Rightarrow^* \Pi$, then there is a derivation of the b-sequent in which all instances of (sPO) and (L$\bot$$-$) are analytic.

Proof. Given a derivation of $\Gamma|\Delta \Rightarrow^* \Pi$, we consider a topmost instance of non-analytic (sPO) or (L$\bot$$-$). By definition, the active formula N in its (left) premise $N, \Gamma'|\Delta' \Rightarrow^\dagger$ does not occur in the conclusion $\Gamma'|\Delta' \Rightarrow^\dagger$ as a subformula. This means we can apply Lemma 5.11 to obtain a subderivation of $\Gamma'|\Delta' \Rightarrow^\dagger$ in which all instances of (sPO)/(L$\bot$+) are analytic. This reduces the number of non-analytic (sPO)/(L$\bot$$-$) in the new overall derivation, and so we can eliminate all instances by repeating the process. □

Hence we can conclude that the subformula property holds for $\mathbf{SC}^{ab}_{po}$ as well.

Corollary 5.13 (subformula property)**.** If $\vdash_{spo} \Gamma|\Delta \Rightarrow^* \Pi$ then there is a derivation of the b-sequent in which all formulas are a subformula of $\Gamma \cup \Delta \cup \Pi$.

Proof. By inspection on the rules in $\mathbf{SC}^{ab}_{po}$-(Cut) restricted with analytic instances of (sPO) and (L$\bot$$-$). □

Next, let us move on to consider the constructivity of the systems, conceived here by means of the disjunction property. We begin with the cases for $\mathbf{SC}^{ab}$ and $\mathbf{SC}^{ab}_{po}$, where, analogously to the case for intuitionistic logic [27], the property holds as a consequence of cut-elimination.

Corollary 5.14 (disjunction property)**.** Let Γ and Δ be finite sets of formulas such that there is no occurrence of $\{\wedge, \sim\}$ in Γ and no occurrence of $\{\vee, \sim\}$ in Δ. Then for $\dagger \in \{ab, po\}$, $\vdash_{s\dagger} \Gamma|\Delta \Rightarrow^{+} A \vee B$ implies $\vdash_{s\dagger} \Gamma|\Delta \Rightarrow^{+} A$ or $\vdash_{s\dagger} \Gamma|\Delta \Rightarrow^{+} B$.

Proof. Suppose that a cut-free derivation of such a b-sequent is given. Then following a path in the derivation upwards, we can construct a finite sequence $s_0, \dots, s_n$ of b-sequents such that s_0 is $\Gamma|\Delta \Rightarrow^{+} A \vee B$, s_{i+1} is the premise of s_i whose succedent is $A \vee B$, and s_n does not have a premise whose succedent is $A \vee B$. Note that the choice of s_{i+1} is uniquely made, as we do not meet an application of (L$\wedge-$) nor (L$\vee+$).

Then s_n cannot be an instance of (Ax+) because Δ does not contain a disjunctive formula. It is not difficult to similarly check other rules to see that the rule applied to obtain s_n must be either (RW+) or (R$\vee$+). Consider the latter case, and assume that the succedent in the premise is A. Then take the premise as s'_n. The we can successively define new b-sequents s'_i whose only difference is that the succedents are A. In particular, each s'_i for $i < n$ is obtained by an application of the same rule. This gives a desired derivation of $\Gamma|\Delta \Rightarrow^{+} A$. It is analogously argued when the rule applied is (RW+). $\square$

The constructible falsity property of the systems (with the same class of antecedent formulas) then follows immediately from the disjunction property. On the other hand, the general disjunction property does not hold with respect to $\mathbf{SC}^{ab}_{wn}$, when conceived with the same class of formulas in the antecedent.

Proposition 5.15. $\vdash_{swn} \; |\neg(p \wedge q) \Rightarrow^{+} \sim p \vee \sim q$ but $\nvdash_{swn} \; |\neg(p \wedge q) \Rightarrow^{+} \sim p$ and $\nvdash_{swn}$ $|\neg(p \wedge q) \Rightarrow^{+} \sim q$

Proof. The first part is verified by the next derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p| \Rightarrow^{-} p}{p|\neg(p\wedge q)\Rightarrow^{-} p}}{p|\neg(p\wedge q)\Rightarrow^{+}\sim p}}{p|\neg(p\wedge q)\Rightarrow^{+}\sim p\vee\sim q} \quad \overline{\overline{q|\neg(p\wedge q)\Rightarrow^{+}\sim p\vee\sim q}}}{p\wedge q|\neg(p\wedge q)\Rightarrow^{+}\sim p\vee\sim q} \quad \dfrac{|p\wedge q\Rightarrow^{+}p\wedge q \qquad |\bot\Rightarrow^{+}}{|p\wedge q, \neg(p\wedge q)\Rightarrow^{+}}}{|\neg(p\wedge q)\Rightarrow^{+}\sim p\vee\sim q}\ (\mathrm{sWN})
$$

As for the second part, by Proposition 5.3 and Corollary 3.6, it suffices to provide counter-models for $\vDash_{wn} \neg(p \wedge q) \Rightarrow \sim p$ and $\vDash_{wn} \neg(p \wedge q) \Rightarrow \sim q$. For the former, let $\mathcal{M} = ((W, \leq), \mathcal{V})$ be a $\mathbf{C}^{ab}_{wn}$-model such that $W = \{w\}$, $\leq = \{(w, w)\}$, $\mathcal{V}^{+}(p) = \mathcal{V}^{-}(q) = W$ and $\mathcal{V}^{-}(p) = \mathcal{V}^{+}(q) = \mathcal{V}^{-}(\bot) = \emptyset$. Otherwise, $\mathcal{V}^{+}$ and $\mathcal{V}^{-}$ are defined according to the equivalences in Definition 3.1. Then we can inductively check $w \Vdash^{+}_{wn} A$ or $w \Vdash^{-}_{wn} A$ for all A; thus (Weak Negation) is satisfied. Now clearly, $w \Vdash^{+}_{wn} \neg(p \wedge q)$ but $w \nVdash^{+} \sim p$. The latter case is analogous. $\square$

It therefore appears that $\mathbf{C}^{ab}_{wn}$ does not enjoy the same level of constructivity[9] as $\mathbf{C}^{ab}$ and $\mathbf{C}^{ab}_{po}$. This suggest that $\mathbf{C}^{ab}_{wn}$ may not be fully acceptable[10] to an intuitionistic logician similarly to the case for **C3**.

6 Mixed Constructible Falsity

In this section, we shall look at a further interaction between $\neg$ and $\sim$ which holds in many systems in the vicinity of $\mathbf{C}^{ab}$. We first introduce the notion of *reduced formula*, commonly used for systems with constructible falsity since Nelson [21], as a preliminary notion.

Definition 6.1. For each formula A in $\mathcal{L}$, we define its *reduced formula* $r(A)$ by the following clauses.

$$
\begin{aligned}
r(p) &= p. & r({\sim}p) &= {\sim}p.\\
r(\bot) &= \bot. & r({\sim}\bot) &= {\sim}\bot.\\
r(A \wedge B) &= r(A) \wedge r(B). & r({\sim}(A \wedge B)) &= r({\sim}A) \vee r({\sim}B).\\
r(A \vee B) &= r(A) \vee r(B). & r({\sim}(A \vee B)) &= r({\sim}A) \wedge r({\sim}B).\\
r(A \to B) &= r(A) \to r(B). & r({\sim}(A \to B)) &= r(A) \to r({\sim}B).\\
& & r({\sim}{\sim}A) &= r(A).
\end{aligned}
$$

We shall set $r(\Gamma) := \{r(A) : A \in \Gamma\}$. Reduced formulas for **C** are already discussed by Wansing [33]. Some standard properties shown therein hold in the current setting as well:

Lemma 6.2. The following statements hold.

(i) $\vdash_{hab} A \leftrightarrow r(A)$.

(ii) $\vdash_{hab} r({\sim}A) \leftrightarrow {\sim}r(A)$.

(iii) $\vdash_{hab} {\sim}A \leftrightarrow {\sim}r(A)$.

Proof. We shall show (i) and (ii) by induction on the complexity of A. Here we shall treat the cases where $A \equiv {\sim}(B \to C)$.

For (i), $r({\sim}(B \to C)) \equiv r(B) \to r({\sim}C)$. By the I.H. $\vdash_{hab} B \leftrightarrow r(B)$ and $\vdash_{hab} {\sim}C \leftrightarrow r({\sim}C)$. The equivalence then follows from (NI).

For (ii), $r({\sim}{\sim}(B \to C)) \equiv r(B) \to r(C)$ and ${\sim}r({\sim}(B \to C)) \equiv {\sim}(r(B) \to r({\sim}C))$. By the I.H. $\vdash_{hab} r({\sim}{\sim}C) \leftrightarrow {\sim}r({\sim}C)$; so from this and (NI) the statement holds.

Now, $\vdash_{hab} {\sim}A \leftrightarrow r({\sim}A)$ from (i) and $\vdash_{hab} r({\sim}A) \leftrightarrow {\sim}r(A)$ from (ii); so (iii) follows. □

Next we introduce a class of formulas in $\mathcal{L}$.

[9] We do not know if $\mathbf{SC}^{ab}_{wn}$ enjoys the disjunction property with the empty antecedent.

[10] It might be argued that **C** is already disfavourable for a similar reason: $\Gamma \vdash A \vee B$ (where Γ is disjunction-free) implies $\Gamma \vdash A$ or $\Gamma \vdash B$ in intuitionistic logic but not in **C**. In this case, however, the two logics have different languages, so it is less clear that we can draw the conclusion that **C** is less constructive than intuitionistic logic.

Definition 6.3. Let $\mathcal{F}$ be a class of formulas in $\mathcal{L}$ given by the next clauses.

$$F ::= \bot \mid (A \wedge F) \mid (F \wedge A) \mid (F \vee F) \mid (A \to F).$$

With respect to this class, we have the following couple of lemmas.

Lemma 6.4. Let $\dagger \in \{ab, po\}$. If $\vdash_{s\dagger} \Gamma|\Delta \Rightarrow^*$, then $\bigwedge r({\sim}\Gamma) \wedge \bigwedge r(\Delta) \in \mathcal{F}$.

Proof. We show by induction on the depth of derivation. By Theorem 5.9, it suffices to consider the cut-free derivations. Also we may check via soundness that the antecedent is non-empty.

The derivation cannot be an instance of (Ax−) or (Ax+). If it is an instance of (L⊥+), then $r(\bot) \equiv \bot \in \mathcal{F}$.

Otherwise, the derivation ends with an instance of a left rule or (sPO). If it ends with an instance of (LW−):

$$\frac{\Gamma|\Delta \Rightarrow^*}{A, \Gamma|\Delta \Rightarrow^*}$$

then by the I.H. $\bigwedge r({\sim}\Gamma) \wedge \bigwedge r(\Delta) \in \mathcal{F}$ Hence $r({\sim}A) \wedge \bigwedge r({\sim}\Gamma) \wedge \bigwedge r(\Delta) \in \mathcal{F}$, as required. The case for (LW+) is analogous.

If the derivation ends with an instance of (L∧−):

$$\frac{A, \Gamma|\Delta \Rightarrow^* \qquad B, \Gamma|\Delta \Rightarrow^*}{A \wedge B, \Gamma|\Delta \Rightarrow^*}$$

then by the I.H. $r({\sim}A) \wedge \bigwedge r({\sim}\Gamma) \wedge \bigwedge r(\Delta) \in \mathcal{F}$ and $r({\sim}B) \wedge \bigwedge r({\sim}\Gamma) \wedge \bigwedge r(\Delta) \in \mathcal{F}$. Now if there is $C \in {\sim}\Gamma \cup \Delta$ such that $r(C) \in \mathcal{F}$, then the statement follows. Otherwise, it must be that $r({\sim}A), r({\sim}B) \in \mathcal{F}$ Hence $r({\sim}(A \wedge B)) \equiv r({\sim}A) \vee {\sim}(B) \in \mathcal{F}$. Hence the statement follows in all cases. The case for (L∨+) is analogous.

If the derivation ends with an instance of (L∧+):

$$\frac{\Gamma|\Delta, A_i \Rightarrow^*}{\Gamma|\Delta, A_1 \wedge A_2 \Rightarrow^*}$$

then by the I.H. $\bigwedge r({\sim}\Gamma) \wedge \bigwedge r(\Delta) \wedge r(A_i) \in \mathcal{F}$. Hence $\bigwedge r({\sim}\Gamma) \wedge \bigwedge r(\Delta) \wedge r(A_1 \wedge A_2) \in \mathcal{F}$. The case for (L∨−) is analogous.

If the derivation ends with an instance of (L→−):

$$\frac{\Gamma|\Delta \Rightarrow^+ A \qquad B, \Gamma'|\Delta' \Rightarrow^*}{A \to B, \Gamma, \Gamma'|\Delta, \Delta' \Rightarrow^*}$$

then by the I.H. $r({\sim}B) \wedge \bigwedge r({\sim}\Gamma') \wedge \bigwedge r(\Delta') \in \mathcal{F}$. If there is $C \in {\sim}\Gamma' \cup \Delta'$ such that $r(C) \in \mathcal{F}$, then the statement follows. Otherwise, $r({\sim}B) \in \mathcal{F}$, so $r({\sim}(A \to B)) \equiv r(A) \to r({\sim}B) \in \mathcal{F}$. Hence the statement follows in both cases. The case for (L→+) is analogous.

If the derivation ends with an instance of (L∼−):

$$\frac{\Gamma|\Delta, A \Rightarrow^*}{{\sim}A, \Gamma|\Delta \Rightarrow^*}$$

then by the I.H. $\bigwedge r({\sim}\Gamma) \wedge \bigwedge r(\Delta) \wedge r(A) \in \mathcal{F}$. Now if $r(A) \in \mathcal{F}$, then $r({\sim}{\sim}A) \in \mathcal{F}$ and so the statement follows. Otherwise, the statement follows from the I.H.. The case for (L$\sim$+) also follows trivially by the I.H..

If the derivation ends with an instance of (sPO):

$$\frac{p, \Gamma|\Delta \Rightarrow^* \qquad \Gamma|\Delta, p \Rightarrow^*}{\Gamma|\Delta \Rightarrow^*}$$

Then by the I.H. $r({\sim}p) \wedge \bigwedge r({\sim}\Gamma) \wedge r(\Delta) \in \mathcal{F}$ and $\bigwedge r({\sim}\Gamma) \wedge r(\Delta) \wedge r(p) \in \mathcal{F}$. Then because ${\sim}p, p \notin \mathcal{F}$, there must be $A \in {\sim}\Gamma \cup \Delta$ such that $r(A) \in \mathcal{F}$. Thus $\bigwedge r({\sim}\Gamma) \wedge r(\Delta) \in \mathcal{F}$. The case for $(L\bot-)$ is analogous. □

Lemma 6.5. If $A \in \mathcal{F}$ then $\vdash_{hwn} {\sim}A$.

Proof. By induction on the construction of formulas in $\mathcal{F}$. If $A \equiv \bot$, then $\vdash_{hwn} {\sim}\bot$ follows from (WN).

If $A \equiv B \wedge F$, then by the I.H. $\vdash_{hwn} {\sim}F$. Hence $\vdash_{hwn} {\sim}(B \wedge F)$ by (DI) and (NC). The case $A \equiv F \wedge B$ is analogous.

If $A \equiv F_1 \vee F_2$, then by the I.H. $\vdash_{hwn} {\sim}F_1$ and $\vdash_{hwn} {\sim}F_2$. Hence $\vdash_{hwn} {\sim}(F_1 \vee F_2)$ by (CI) and (ND).

If $A \equiv B \to F$, then by the I.H. $\vdash_{hwn} {\sim}F$. Hence $\vdash_{hwn} {\sim}(B \to F)$ by (K) and (NI). □

The lemmas allow us to establish the next relationship between $\neg$ and $\sim$. (The first item is in fact obvious from (WN); an alternative proof is given here for the interest of a posterior remark.)

Theorem 6.6. The following statements hold.

(i) If $\vdash_{hwn} \neg A$ then $\vdash_{hwn} {\sim}A$.

(ii) If $\vdash_{hwn} \neg(A \wedge B)$ then $\vdash_{hwn} {\sim}A$ or $\vdash_{hwn} {\sim}B$.

Proof. For (i), by Proposition 4.5, if $\vdash_{hwn} \neg A$ then $\vdash_{hpo} \neg A$. Thus by Proposition 5.3, $\vdash_{spo} \ | \ \Rightarrow^+ \neg A$; by (Cut+), $\vdash_{spo} \ |A \Rightarrow^+$. Hence $r(A) \in \mathcal{F}$ by Lemma 6.4 and so $\vdash_{hwn} {\sim}r(A)$ by Lemma 6.5. Finally, Lemma 6.2 (iii) implies the desired conclusion.

For (ii), like in (i) if $\vdash_{hwn} \neg(A \wedge B)$ then $r(A \wedge B) \equiv r(A) \wedge r(B) \in \mathcal{F}$. This implies that either $r(A) \in \mathcal{F}$ or $r(B) \in \mathcal{F}$. Then we follow the same path to conclude that $\vdash_{hwn} {\sim}A$ or $\vdash_{hwn} {\sim}B$. □

Therefore in $\mathbf{C}^{ab}_{wn}$, we obtain a sort of ‘mixed constructible falsity’ property, where the witness for an intuitionistically negated conjunction is given in terms of constructible falsity. This property may be seen to offer an alternative answer for intuitionistic logicians to the failure of the constructible falsity property for intuitionistic negation. Instead of introducing an alternative notion of negation which replaces intuitionistic negation (as happens in **N4**), the connexive constructible falsity of $\mathbf{C}^{ab}_{wn}$ complements intuitionistic negation by becoming a witness of an intuitionistically negated conjunction. In this specific sense, **C**-style systems with the property might be called *more intuitionistic* than **N4**-style systems.

Remark 6.7. It is easily seen from the proof of the above theorem that the same properties can be shown with respect to $\mathbf{CN}^{\perp}$ and any intermediate system which falls under the scope of Proposition 4.5. Moreover, consider the ([9]-style) variants of $\mathbf{C}^{ab}$ and $\mathbf{C}^{ab}_{po}$ in which $\sim\perp$ is added as an axiom schema, and the corresponding sequent calculi with an additional axiom $\mid\ \Rightarrow^{-} \perp$. It is not difficult to observe that the additional rule does not affect the cut-elimination and Lemma 6.5. Hence the properties of Theorem 6.6 hold with respect to these variants as well.

7 Concluding Remarks

The question that motivated our enquiry is how an intuitionistic logician can make sense of **C**-style connexive constructible falsity, and whether there is a related system in which it is made more understandable by relating it with intuitionistic negation. We in particular looked at two candidates $\mathbf{C}^{ab}_{po}$ and $\mathbf{C}^{ab}_{wn}$.

Having looked at their properties, we may ask which one is to be preferred. Here it seems $\mathbf{C}^{ab}_{po}$ is largely more advantageous, because it has a better behaviour in the semantics (Proposition 3.4), less controversial status on the falsity of intuitionistic negation (Proposition 4.2), a subformula calculus (Corollary 5.13) and better constructivity (Corollary 5.14). Moreover, it shows a good property for investigating provable contradictions constructively (Corollary 4.6), while staying close to **C** (Proposition 4.8). We would therefore suggest that this could be a system that satisfies an intuitionistic logician enough, both in terms of its comprehensibility[11] and its formal behaviours.

In comparison, $\mathbf{C}^{ab}_{wn}$ fares not as well as $\mathbf{C}^{ab}_{po}$ in many of the above-mentioned aspects, and the less satisfactory constructive status may be particularly worrying for an intuitionistic logician. Nonetheless, its satisfaction of 'the mixed constructible falsity' property can offer an independent motivation for the system. Since some of its disadvantages may well be rectified (e.g. by a subformula calculus or the disjunction property with the empty antecedent), further investigations can offer an improved evaluation.

Lastly, however, we would like to point out that there is another system that can potentially meet the expectation of an intuitionistic logician. It is the system $\mathbf{C}^{\perp}$ in [9], i.e. $\mathbf{C}^{ab}$ with an additional axiom schema $\sim\perp$. As we discussed in Remark 6.7, in this system (WN) holds in *the rule form* (i.e. Theorem 6.6 (i)). This relationship between intuitionistic negation and constructible falsity may be enough for an intuitionistic logician to have an adequate understanding of the latter concept. Therefore it seems, from this perspective, the acceptability of $\sim\neg A$ as a theorem and the reading of $\perp$ as falsehood can have a noticeable influence on the preference of intuitionistic logicians.

[11] Admittedly, the double negation can make the schema more difficult to makes sense even though the inner $A \vee \sim A$ is readily understandable. However, it is our (perhaps idealised) supposition that intuitionistic logicians *do* understand all intuitionistic connectives; so the presence of the double negation does not pose an issue for their comprehension.

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