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Remarks on Semantic Information and Logic. From Semantic Tetralateralism to the Pentalattice $\mathbb{65536}_5$

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Abstract: A 16-element lattice $\mathbb{16}_{inf}$ of generalized semantical values pre-ordered by set-inclusion as an information order is presented. The propositional logic $\mathbb{Inf}$ of that lattice is axiomatized and a generalization of $\mathbb{16}_{inf}$ to a 65536-element pentalattice is suggested.

Keywords: Informational interpretation; Kripke semantics; Semantic tetralateralism; Generalized semantical values; Negation inconsistency.

1 Introduction

The paper deals with the notion of semantic information carried (or conveyed) by a declarative sentence, especially information carried by a formula in certain propositional languages in a given model in virtue of the meaning of the logical operations. The focus is thus on *logical information* and not on information in terms of the descriptive content of internally structured atomic sentences in first- or higher-order languages. If the information carried by a formula A in a model is represented by sets of states at which A is semantically evaluated, then 'classically' the evaluation gives rise to a distinction between two sets, the set of states at which A is true, A's truth set in the model, and the set of states at which A is false. The information carried by A is given already with A's truth set (also called 'the UCLA proposition expressed by A'), because falsity is identified with untruth and A's truth set determines its complement as A's falsity set. If we shift our attention

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from truth and falsity to information given with respect to the truth or falsity of atomic formulas and ultimately arbitrary formulas, we are dealing with what Nuel Belnap (1976, 1977) has called ‘told values’: **T** (*told true but not false*), **F** (*told false but not true*), **N** (*told neither true nor false*), **B** (*told both true and false*). The information carried by a formula A in a model is then represented by four sets of states, and the set of states at which a formula A is told false need not coincide with the set of states at which A fails to be told true.

The states of a model can be seen as *information states* as they represent the semantic information that is given with a valuation function. With Belnap’s four-valued functions, a state may support the truth or the falsity of an atomic formula, and if no combination of being told is excluded, there may be states at which a given atomic formula is both told true and told false (states that support both the truth and the falsity of the formula) and states at which the formula is neither being told true nor being told false (states that neither support the truth nor the falsity of the formula). As is well known, the set of states can be given a relational or algebraic structure. In Grzegorczyk’s (1964) and Kripke’s (1965) informational interpretation of intuitionistic logic, the non-empty set of states is pre-ordered or partially ordered by a binary relation of possible expansion of information states. The semantics is made many-valued in the relational semantics for Nelson’s constructive logics with strong negation **N3** and **N4**, see (Odintsov, 2008) and references therein, by introducing two separate satisfiability relations, verification (support of truth) and falsification (support of falsity). Informationally interpreted algebraic structures for substructural subsystems of intuitionistic logic and Nelson’s logics, namely models based on semilattice-ordered monoids, have been studied in (Wansing, 1993a; 1993b). Also in Urquhart’s (1972) semilattice semantics for relevance logic the set of states has an algebraic structure, featuring a binary operation of combination of information states (or pieces), see also (Punčochář, 2016), (Weiss, 2022). The ternary relation used in Routley-Meyer models for relevance logic has been given an informational reading by Mares (2009, 2010) and, more recently, Punčochář and Sedlár have developed an *information based semantics* in the context of inquisitive logic (Punčochář, 2019), (Punčochář & Sedlár, 2021).

Whilst the use of such relational and algebraic information structures turned out to be a rich and flexible approach in the study of substructural and other non-classical logics, I will focus on further semantical categories in addition to truth and falsity, respectively support of truth and support of falsity. With the distinction between sense and reference, Gottlob Frege enriched the inventory of basic semantical categories and values. Next to truth and falsity there are meaningfulness and meaninglessness (nonsensicality). Although according to Frege in a scientific language it ought to be the case that the sense of a sentence (the thought expressed by it) determines the sentence's reference (its truth value *The True* or *The False*), Frege nevertheless acknowledged natural language sentences that have a meaning but no reference. The four basic semantic values (*true*, *false*, *meaningful*, and *nonsensical*) induce a set of sixteen told values, including the values *told both meaningful and false* and *told both meaningful and nonsensical*. In this paper I will present two non-classical logics in languages that contain the unary connectives $[m]$ ("it is meaningful that") and $[n]$ ("it is nonsensical that"). One system, **N4mn**, is an expansion of the four-valued constructive and paraconsistent logic **N4**, and it is presented in (Wansing & Ayhan, 2023) as a case study in logical tetralateralism.[2] The other system, $\mathbb{Inf}$, is a logic interpreted on a 16-element lattice $\mathbb{16}_{inf} = (\mathbb{16}, \subseteq)$ of generalized truth values generated from the set of the four basic semantical values by considering its powerset, $\mathbb{16}$. In **N4mn** (and its connexive version **C4mn** defined in Section 5), the information carried by a formula A in a model is represented by 16 sets of states, in $\mathbb{Inf}$ it is represented by one out of 16 semantical values.

The move from metaphysically understood semantical values to informational told values allows one to take a fresh look at logical consequence and hence on logic. On the standard conception, semantic consequence is understood as truth preservation from the premises to the conclusion of an inference, and, from a 'classical' point of view, as untruth preservation from the conclusion to the premises. From the informational point of view, one may think of logic as the study of information flow, see (Mares, 2008), (Wansing, 2022), (Wansing & Odintsov, 2016).[3] Information flow, however, comes in more than one flavor depending on the basic semantic categories. In a valid inference, the information that the premises are true, false, meaningful, respectively nonsensical provides the information that the conclusion is true, false, meaningful, respectively nonsensical; that is, if the premises are told true, false, meaningful, respectively nonsensical, then so is the conclusion.

In the paper, the 16-valued logic **N4mn** is introduced semantically and shown to be faithfully embeddable into positive intuitionistic propositional

[2] The term 'tetralateral' mixes Greek and Latin. Such a mixture is, however, not unusual and can also be found, for example, in the expressions 'tetra-lateral position sensing detectors' and 'tetravalued modal algebras'.

[3] An informational account of entailment in terms of informational content inclusion has been suggested in (Shramko & Wansing, 2021).

logic. The logic $\mathbb{Inf}$ is new. It is introduced as a formula-formula inference system and is shown to be sound and complete with respect to $\mathbb{16}_{inf}$. Semantic consequence is defined with respect to the subset relation as an information order on $\mathbb{16}$, and set intersection (union) as the lattice meet (join) gives rise to a conjunction (disjunction) connective. The presentation ends with the definition of a 65536-element pentalattice, $\mathbb{65536}_5$, with five lattice orderings: an information preorder, a truth preorder, a falsity preorder, a meaningfulness preorder, and a nonsensicality preorder. This step is motivated by the rationale for proceeding from the smallest non-trivial bilattice $FOUR_2$ to the trilattice $SIXTEEN_3$, see Shramko and Wansing (2005, 2011).

2 Meaning and information

In this section I will address some basic terminological and conceptual issues.

The word 'information' is used in different ways in different contexts. Nevertheless, as Luciano Floridi (2010, p. 20 f.) explains:

> Over the past decades, it has become common to adopt a General Definition of Information (GDI) in terms of data + meaning. GDI has become an operational standard, especially in fields that treat data and information as reified entities, that is, stuff that can be manipulated (consider, for example, the now common expressions 'data mining' and 'information management'). A straightforward way of formulating GDI is as a tripartite definition (Table 1): According to (GDI.l), information is made of data. In (GDI.2), 'well formed' means that the data are rightly put together,

Table 1. The General Definition of Information (GDI)

GDI)	σ is an instance of information, understood as semantic content, if and only if: GDI.l) σ consists of n *data*, for $n \geq 1$; GDI.2) the data are well formed; GDI.3) the well-formed data are *meaningful*.

> according to the rules (*syntax*) that govern the chosen system, code, or language being used. . . . Regarding (GDI.3), this is where semantics finally occurs. 'Meaningful' means that the data must comply with the meanings (*semantics*) of the chosen system, code, or language in question.

Given the looseness of the term ‘information’, the GDI is a solid basis to work with. If the data one is interested in are declarative sentences from a natural language or formulas from a formal language (closed formulas in the case of a first- or higher-order language), the data are well-formed, and *the information carried or conveyed by a declarative sentence or formula*, its semantic content, is its meaning. A meaningful compound sentence (formula) consists of subsentences (subformulas), each of which is well-formed and, moreover, meaningful if we assume compositionality of meaning.

There is more to be said about the concept of semantic information, but in what follows by ‘semantic information’ I will understand the meaning of a declarative sentence and, in particular, the meaning of a formula from a given formal language.

3 Semantic tetralateralism

Preparatory to the introduction of the logic $\mathbb{I}\mathrm{n}\mathbb{f}$ in Section 4, we will expand the language of propositional **N4** by two unary connectives, $[m]$ and $[n]$. A formula $[m]A$ is to be read as “it is meaningful that A”, and $[n]A$ is to be understood as “it is nonsensical that A”. The logic of the expanded language will be referred to as **N4mn**. Its semantics is a tetralateralism insofar as it makes use of four different forcing relations.

The propositional language $\mathcal{L}$ of **N4mn** based on a denumerable set of propositional variables Φ is defined in Backus-Naur form as follows:

$$\begin{array}{rl} \text{variables } \Phi: & p \in \Phi \\ \text{formulas:} & A \in \mathit{Form}_{\mathcal{L}}(\Phi) \\ A ::= & p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid {\sim}A \mid [m]A \mid [n]A. \end{array}$$

The language $\mathcal{L}'$ of positive intuitionistic propositional logic, $\mathbf{IPL}^{+}$, is obtained from $\mathcal{L}$ by dropping the unary connectives, i.e., $\sim$, $[m]$, and $[n]$, and the language $\mathcal{L}''$ of the propositional logic **N4** is obtained from $\mathcal{L}$ by dropping $[m]$ and $[n]$.

Definition 1 *A* Kripke frame *is a structure $\langle M, R\rangle$, where M is a nonempty set (of information states), and R is a reflexive and transitive binary relation (of information state expansion) on M.*

Definition 2 *A* valuation $\models$ *on a Kripke frame $\langle M, R\rangle$ is a mapping from the set Φ of propositional variables to the power set 2^M of M such that for any $p \in \Phi$ and any $x, y \in M$, if $x \in \models(p)$ and xRy, then $y \in \models(p)$. We* *will write* $x \models p$ *for* $x \in \models (p)$*. A valuation* $\models$ *is extended to a mapping from the set of all* $\mathcal{L}'$*-formulas to* 2^M *by:*

$$x \models A \to B \text{ iff } \forall y \in M\ [xRy \text{ and } y \models A \text{ imply } y \models B],$$
$$x \models A \wedge B \text{ iff } x \models A \text{ and } x \models B,$$
$$x \models A \vee B \text{ iff } x \models A \text{ or } x \models B.$$

If $\mathcal{F} = \langle M, R\rangle$ *is a Kripke frame, then* $\langle M, R, \models\rangle$ *is a Kripke model for* $\mathbf{IPL}^+$ *based on* $\mathcal{F}$.

The following *heredity condition* holds for $\models$: for any $\mathcal{L}'$-formula A and any $x, y \in M$, if $x \models A$ and xRy, then $y \models A$.

Definition 3 *An* $\mathcal{L}'$*-formula* A *is* true in a Kripke model $\langle M, R, \models\rangle$ *for* $\mathbf{IPL}^+$ *if* $x \models A$ *for any* $x \in M$*, and is* valid on a Kripke frame $\mathcal{F} = \langle M, R\rangle$ *if it is true for every Kripke model for* $\mathbf{IPL}^+$ *based on* $\mathcal{F}$*. An* $\mathcal{L}'$*-formula* A *is said to be* $\mathbf{IPL}^+$-valid *if* A *is valid on every Kripke frame. Let* $\Gamma \cup \{A\}$ *be a set of* $\mathcal{L}'$*-formulas. Semantic consequence (entailment) is defined in terms of truth preservation at each state:* $\Gamma \models A$ *if for every Kripke model* $\langle M, R, \models\rangle$ *for* $\mathbf{IPL}^+$ *and for all* $x \in M$*,* $x \models A$ *if* $x \models B$ *for all* $B \in \Gamma$*. We define the logic* $\mathbf{IPL}^+$ *model-theoretically as the pair* $\langle \mathcal{L}', \{\langle \Gamma, A\rangle \mid \Gamma \models A\}\rangle$.

We turn to the language $\mathcal{L}$ and define *four* separate valuation functions $\models^+$, $\models^-$, $\models^m$, and $\models^n$. These mappings determine for a given propositional variable p the set of states that support the truth, the falsity, the meaningfulness, and the nonsensicality (meaninglessness) of p, respectively. Support of truth, support of falsity, support of meaningfulness, and support of meaninglessness are seen as properties that are independent of each other. In particular, it is not excluded that an information state supports both the truth and the falsity of a given propositional variable or both its meaningfulness and its nonsensicality.

Definition 4 *The valuation functions* $\models^+$*,* $\models^-$*,* $\models^m$*, and* $\models^n$ *on a Kripke frame* $\langle M, R\rangle$ *are mappings from the set* Φ *to the power set* 2^M *of* M *such that for any* $\star \in \{+, -, m, n\}$*, any* $p \in \Phi$ *and any* $x, y \in M$*, if* $x \in \models^\star (p)$ *and* xRy*, then* $y \in \models^\star (p)$*. We will write* $x \models^\star p$ *for* $x \in \models^\star (p)$*. The functions* $\models^+$*,* $\models^-$*,* $\models^m$*, and* $\models^n$ *are extended to mappings from the set of all* $\mathcal{L}$*-formulas to* 2^M *by:*

$$x \models^+ A \wedge B \text{ iff } x \models^+ A \text{ and } x \models^+ B,$$
$$x \models^+ A \vee B \text{ iff } x \models^+ A \text{ or } x \models^+ B,$$

$x \models^+ A \to B$ *iff* $\forall y \in M$ $[xRy$ *and* $y \models^+ A$ *imply* $y \models^+ B]$,
$x \models^+ {\sim}A$ *iff* $x \models^- A$,
$x \models^+ [m]A$ *iff* $x \models^m A$,
$x \models^+ [n]A$ *iff* $x \models^n A$,

$x \models^- A \wedge B$ *iff* $x \models^- A$ *or* $x \models^- B$,
$x \models^- A \vee B$ *iff* $x \models^- A$ *and* $x \models^- B$,
$x \models^- A \to B$ *iff* $x \models^+ A$ *and* $x \models^- B$,
$x \models^- {\sim}A$ *iff* $x \models^+ A$,
$x \models^- [m]A$ *iff* $x \models^n A$,
$x \models^- [n]A$ *iff* $x \models^m A$,

$x \models^m A \circ B$ *iff* $x \models^m A$ *and* $x \models^m B$, *for* $\circ \in \{\wedge, \vee, \to\}$,
$x \models^m \circ A$ *iff* $x \models^m A$, *for* $\circ \in \{\sim, [m], [n]\}$,

$x \models^n A \circ B$ *iff* $x \models^n A$ *or* $x \models^n B$, *for* $\circ \in \{\wedge, \vee, \to\}$,
$x \models^n \circ A$ *iff* $x \models^n A$, *for* $\circ \in \{\sim, [m], [n]\}$.

If $\mathcal{F} = \langle M, R\rangle$ *is a Kripke frame, then* $\langle M, R, \models^+, \models^-, \models^m, \models^n\rangle$ *is a* Kripke model *for* **N4mn** *based on* $\mathcal{F}$.

The heredity condition holds for $\models^+$, $\models^-$, $\models^m$, and $\models^n$, i.e., for any $\mathcal{L}$-formula A and any $x, y \in M$, if $x \models^* A$ and xRy, then $y \models^* A$, for $* \in \{+, -, m, n\}$.

As to a motivation of the semantical clauses for $[m]$ and $[n]$, we may note that a state supports the meaningfulness (nonsensicality) of a compound formula iff the state supports the meaningfulness (nonsensicality) of all (some) of its immediate proper subformulas; meaninglessness is 'infectious'. Thus, in particular, $x \models^m [n]A$ iff $x \models^m A$ iff $x \models^- [n]A$, and $x \models^m [n]A$ does not, in general, imply $x \models^+ [n]A$, although it does imply $x \models^- [n]A$. A state supports the meaningfulness of the statement that A is nonsensical iff the state supports the meaningfulness of A, and in this case the falsity of $[n]A$ is supported.

Definition 5 *An $\mathcal{L}$-formula A is said to be* true *in a Kripke model for* **N4mn** $\langle M, R, \models^+, \models^-, \models^m, \models^n\rangle$ *if* $x \models^+ A$ *for any* $x \in M$, *and to be valid on a Kripke frame* $\mathcal{F} = \langle M, R\rangle$ *if it is true for every Kripke model for* **N4mn** *based on* $\mathcal{F}$. *An $\mathcal{L}$-formula A is said to be* **N4mn**-*valid if A is valid on every Kripke frame. Let* $\Gamma \cup \{A\}$ *be a set of $\mathcal{L}$-formulas. Entailment is defined in terms of support-of-truth preservation at each state:* $\Gamma \models^+ A$ *if for all Kripke* *models for* **N4mn** $\langle M, R, \models^+, \models^-, \models^m, \models^n \rangle$ *and for all* $x \in M$*,* $x \models^+ A$ *if* $x \models^+ B$ *for all* $B \in \Gamma$*. We write* $A \models^+ B$ *for* $\{A\} \models^+ B$*. We define the logic* **N4mn** *model-theoretically as the pair* $\langle \mathcal{L}, \{\langle \Gamma, A \rangle \mid \Gamma \models^+ A\} \rangle$ *and* **N4** *is model-theoretically defined as* $\langle \mathcal{L}'', \{\langle \Gamma, A \rangle \mid \Gamma \models^+ A\} \rangle$*.*

Proposition (Wansing & Ayhan, 2023) *Each of the unary connectives* $\circ \in \{\sim, [m], [n]\}$ *is congruentiality-breaking in the sense that there are* $\mathcal{L}$*-formulas* A *and* B *such that* $A \models^+ B$ *and* $B \models^+ A$ *but not:* $\circ A \models^+ \circ B$ *and* $\circ B \models^+ \circ A$*.*

Definition 6 *Given the set* Φ *of propositional variables, we define three more sets of propositional variables, namely* $\Phi^- := \{p^- \mid p \in \Phi\}$*,* $\Phi^m := \{p^m \mid p \in \Phi\}$*, and* $\Phi^n := \{p^n \mid p \in \Phi\}$*. We inductively define a mapping* f *from* $\mathrm{Form}_{\mathcal{L}}(\Phi)$ *to the set of formulas of the language* $\mathcal{L}'$ *of* $\mathbf{IPL}^+$ *defined over* $\Phi \cup \Phi^- \cup \Phi^m \cup \Phi^n$ *as follows:*

1. *for any* $p \in \Phi$, $f(p) = p$, $f(\sim p) = p^-$, $f([m]p) = p^m$, $f([n]p) = p^n$,
2. $f(A \circ B) = f(A) \circ f(B)$*, for* $\circ \in \{\rightarrow, \wedge, \vee\}$*,*
3. $f(\sim(A \wedge B)) = f(\sim A) \vee f(\sim B)$*,*
4. $f(\sim(A \vee B)) = f(\sim A) \wedge f(\sim B)$*,*
5. $f(\sim(A \rightarrow B)) = f(A) \wedge f(\sim B)$*,*
6. $f(\sim\sim A) = f(A)$*,*
7. $f(\sim[m]A) = f([n]A)$*,*
8. $f(\sim[n]A) = f([m]A)$*,*
9. $f([m](A \circ B)) = f([m]A) \wedge f([m]B)$*, for* $\circ \in \{\rightarrow, \wedge, \vee\}$*,*
10. $f([m] \circ A) = f([m]A)$*, for* $\circ \in \{\sim, [m], [n]\}$*,*
11. $f([n](A \circ B)) = f([n]A) \vee f([n]B)$*, for* $\circ \in \{\rightarrow, \wedge, \vee\}$*,*
12. $f([n] \circ A) = f([n]A)$*, for* $\circ \in \{\sim, [m], [n]\}$*.*

We write $f(\Gamma)$ *to denote the result of replacing every formula* A *in* Γ *by* $f(A)$*; thus,* $f(\varnothing) = \varnothing$*.*

Lemma 1 *Let* f *be the function defined in Definition 6. For any Kripke model for* **N4mn** $\langle M, R, \models^+, \models^-, \models^m, \models^n \rangle$*, we can define a Kripke model for* $\mathbf{Int}^+$ $\langle M, R, \models \rangle$ *such that for any* $A \in \mathrm{Form}_{\mathcal{L}}(\Phi)$ *and any* $x \in M$*,*

1. $x \models^+ A$ *iff* $x \models f(A)$*,*

2. $x \models^- A$ *iff* $x \models f(\sim A)$,
3. $x \models^m A$ *iff* $x \models f([m]A)$,
4. $x \models^n A$ *iff* $x \models f([n]A)$.

Lemma 2 *Let f again be the function defined in Definition 6. Then for any Kripke model $\langle M, R, \models\rangle$ for* **IPL**$^+$*, we can construct a Kripke model $\langle M, R, \models^+, \models^-, \models^m, \models^n\rangle$ for* **N4mn** *such that for any $\mathcal{L}$-formula A and any $x \in M$,*

1. $x \models f(A)$ *iff* $x \models^+ A$,
2. $x \models f(\sim A)$ *iff* $x \models^- A$,
3. $x \models f([m]A)$ *iff* $x \models^m A$,
4. $x \models f([n]A)$ *iff* $x \models^n A$.

Theorem 1 (Semantical embedding) (Wansing and Ayhan (2023)) *Let f be the mapping from Definition 6. For any set of $\mathcal{L}$-formulas $\Gamma \cup A$, $\Gamma \models^+ A$ in* **N4mn** *iff* $f(\Gamma) \models f(A)$ *in* **IPL**$^+$.

4 The logic $\mathbb{inf}$ of the information order of the lattice $\mathbb{16}_{inf}$

In this section, I will introduce another logic in a language that contains next to negation, $\sim$, the one-place sentential operators $[m]$ and $[n]$, expressing meaningfulness, respectively nonsensicality. This system, $\mathbb{inf}$, is a many-valued logic that is arrived at by (i) translating the support of truth, support of falsity, support of meaningfulness, and support of nonsensicality conditions for $\sim$, $[m]$, and $[n]$ in **N4mn** into truth tables and (ii) interpreting conjunction and disjunction as the lattice meet and lattice join of a certain lattice of generalized semantical values. As a result, in $\mathbb{inf}$ the connectives $\sim$, $[m]$, $[n]$, conjunction, and disjunction interact differently from how they interact in **N4mn**. I will keep the notation for the meaningfulness and nonsensicality connectives but use the notation for fusion and fission known from, for example, the logic of logical bilattices for conjunction, respectively disjunction. In contrast to the language of **N4mn**, the language of $\mathbb{inf}$ contains no (primitive) conditional.

4.1 The semantics of $\mathbb{inf}$

The propositional language $\mathcal{L}_{inf}$ contains the unary connectives $\sim$, $[m]$, $[n]$ and the binary connectives $\otimes$ (fusion) and $\oplus$ (fission) over a denumerable set

Φ of propositional variables. The set $Form(\mathcal{L}_{inf})$ of $\mathcal{L}_{inf}$-formulas over Φ is defined in the standard way. For the semantics, our starting point is the set $\mathbb{4} = \{\mathbf{t}, \mathbf{f}, \mathbf{m}, \mathbf{n}\}$. We read $\mathbf{t}$ as "true", $\mathbf{f}$ as "false", $\mathbf{m}$ as "meaningful", and $\mathbf{n}$ as "nonsensical". We generalize these basic values to obtain 'told-values' and consider the sixteen elements of $\mathbb{16}$, the powerset $\mathcal{P}(\mathbb{4})$ of $\mathbb{4}$:

1. $\varnothing$ (told neither true nor false nor meaningful nor nonsensical)
2. $\{\mathbf{t}\}$ (told only true)
3. $\{\mathbf{f}\}$ (told only false)
4. $\{\mathbf{m}\}$ (told only meaningful)
5. $\{\mathbf{n}\}$ (told only nonsensical)
6. $\{\mathbf{t}, \mathbf{f}\}$ (told both true and false)
7. $\{\mathbf{t}, \mathbf{m}\}$ (told both true and meaningful)
8. $\{\mathbf{t}, \mathbf{n}\}$ (told both true and nonsensical)
9. $\{\mathbf{f}, \mathbf{m}\}$ (told both false and meaningful)
10. $\{\mathbf{f}, \mathbf{n}\}$ (told both false and nonsensical)
11. $\{\mathbf{m}, \mathbf{n}\}$ (told both meaningful and nonsensical)
12. $\{\mathbf{t}, \mathbf{f}, \mathbf{m}\}$ (told true, false, and meaningful)
13. $\{\mathbf{t}, \mathbf{f}, \mathbf{n}\}$ (told true, false, and nonsensical)
14. $\{\mathbf{t}, \mathbf{m}, \mathbf{n}\}$ (told true, meaningful, and nonsensical)
15. $\{\mathbf{f}, \mathbf{m}, \mathbf{n}\}$ (told false, meaningful, and nonsensical)
16. $\{\mathbf{t}, \mathbf{f}, \mathbf{m}, \mathbf{n}\}$ (told true, false, meaningful, and nonsensical).

If we order $\mathbb{16}$ by set-inclusion, we obtain the distributive complete lattice $\mathbb{16}_{inf} = (\mathbb{16}, \subseteq)$, which is depicted as a Hasse-diagram in Figure 1.

A valuation in $\mathbb{16}$ is a function v^{16}: $\Phi \longrightarrow \mathbb{16}$. Valuation functions v^{16} in $\mathbb{16}$ are extended to functions from $Form(\mathcal{L}_{inf})$ to $\mathbb{16}$ as follows:

$$
\begin{array}{lcl@{\qquad}lcl}
v^{16}(A \otimes B) & = & v^{16}(A) \cap v^{16}(B) & \mathbf{t} \in v^{16}(\sim A) & \text{iff} & \mathbf{f} \in v^{16}(A) \\
v^{16}(A \oplus B) & = & v^{16}(A) \cup v^{16}(B) & \mathbf{f} \in v^{16}(\sim A) & \text{iff} & \mathbf{t} \in v^{16}(A) \\
 & & & \mathbf{m} \in v^{16}(\sim A) & \text{iff} & \mathbf{m} \in v^{16}(A) \\
 & & & \mathbf{n} \in v^{16}(\sim A) & \text{iff} & \mathbf{n} \in v^{16}(A)
\end{array}
$$

$$
\begin{array}{lcl@{\qquad}lcl}
\mathbf{t} \in v^{16}([m]A) & \text{iff} & \mathbf{m} \in v^{16}(A) & \mathbf{t} \in v^{16}([n]A) & \text{iff} & \mathbf{n} \in v^{16}(A) \\
\mathbf{f} \in v^{16}([m]A) & \text{iff} & \mathbf{n} \in v^{16}(A) & \mathbf{f} \in v^{16}([n]A) & \text{iff} & \mathbf{m} \in v^{16}(A) \\
\mathbf{m} \in v^{16}([m]A) & \text{iff} & \mathbf{m} \in v^{16}(A) & \mathbf{m} \in v^{16}([n]A) & \text{iff} & \mathbf{m} \in v^{16}(A) \\
\mathbf{n} \in v^{16}([m]A) & \text{iff} & \mathbf{n} \in v^{16}(A) & \mathbf{n} \in v^{16}([n]A) & \text{iff} & \mathbf{n} \in v^{16}(A).
\end{array}
$$

Definition 7 (Informational entailment) *The entailment relation* $\models_i^{16} \subseteq (Form(\mathcal{L}_{inf}) \times Form(\mathcal{L}_{inf}))$ *is defined by setting*

$$A \models_i^{16} B \text{ iff for every valuation } v^{16} \text{ in } \mathbb{16}, v^{16}(A) \subseteq v^{16}(B).$$

Definition 8 *The logic* $\mathbb{Inf}$ *is presented syntactically as the relation* $\vdash_i \subseteq (Form(\mathcal{L}_{inf}) \times Form(\mathcal{L}_{inf}))$ *defined by the following axiomatic statements and rules, where* $\circ \in \{\sim, [m], [n]\}$, $\sharp \in \{\otimes, \oplus\}$, *and* $A \dashv\vdash_i B$ *is a shorthand for* $A \vdash_i B$ *and* $B \vdash_i A$*:*

Axioms	$[m]{\circ}A \dashv\vdash_i [m]A$	$([m]\circ$ *reduction*$)$
	$[n]{\circ}A \dashv\vdash_i [n]A$	$([n]\circ$ *reduction*$)$
	$[m](A\sharp B) \dashv\vdash_i [m]A\sharp[m]B$	$([m]\sharp$ *distribution*$)$
	$[n](A\sharp B) \dashv\vdash_i [n]A\sharp[n]B$	$([n]\sharp$ *distribution*$)$
	$A \otimes B \vdash_i A,\ A \otimes B \vdash_i B$	$(\otimes$*-elim*$)$
	$A \vdash_i A \oplus B,\ B \vdash_i A \oplus B$	$(\oplus$*-intro*$)$
	$A \otimes (B \oplus C) \vdash_i (A \otimes B) \oplus (A \otimes C)$	*(distribution)*
	$\sim\sim A \dashv\vdash_i A$	$($*double negation*$)$
	$\sim[m]A \dashv\vdash_i [n]A$	$($*negated* $[m])$
	$\sim[n]A \dashv\vdash_i [m]A$	$($*negated* $[n])$
	$\sim(A \otimes B) \dashv\vdash_i \sim A \otimes \sim B$	$($*negated fusion*$)$
	$\sim(A \oplus B) \dashv\vdash_i \sim A \oplus \sim B$	$($*negated fission*$)$

Rules	$A \vdash_i B$ *and* $A \vdash_i C$ *together imply* $A \vdash_i B \otimes C$	$(\otimes$*-intro*$)$
	$A \vdash_i C$ *and* $B \vdash_i C$ *together imply* $A \oplus B \vdash_i C$	$(\oplus$*-elim*$)$
	$A \vdash_i B$ *and* $B \vdash_i C$ *together imply* $A \vdash C$	$($*transitivity*$)$

Note that

1. $$A \vdash_i A \qquad \text{(reflexivity)}$$
   is derivable by (double negation) and (transitivity),
2. from the axioms and rules for $\otimes$ and $\oplus$ it is clear that there is sense in which $A \otimes B$, respectively $A \oplus B$, is a conjunction, respectively disjunction, and
3. $$A \vdash_i B \text{ implies } \sim B \vdash_i \sim A \qquad \text{(contraposition)}$$
   is not validity preserving.

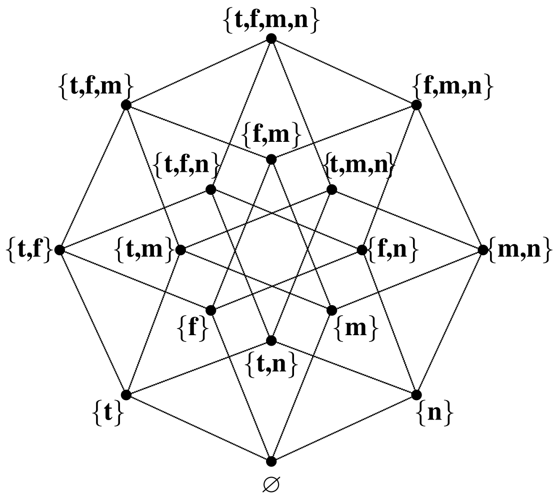

Figure 1: The lattice $\mathbb{16}_{inf}$

The failure of (contraposition) is as it should be if truth and falsity are two independent semantical dimensions in their own right and (contraposition) is not explicitly imposed by definition on a negation connective, as it is usually the case in the study of logics resulting from bi- or tri- or other multilattices.

To prove completeness we will construct a suitable canonical model, see (Shramko & Wansing, 2005; 2011). Let $\alpha \subseteq Form(\mathcal{L}_{inf})$. Then α is a *theory* if

- if $A \in \alpha$ and $A \vdash_i B$, then $B \in \alpha$,
- if $A \in \alpha$ and $B \in \alpha$, then $A \otimes B \in \alpha$.

A theory α is said to be *prime* iff $A \oplus B \in \alpha$ implies that $A \in \alpha$ or $B \in \alpha$. The following fact about prime theories is very well known, a proof is given, for example, in (Dunn, 2000, p. 13):

Lemma 3 *For any A and $B \in Form(\mathcal{L}_{inf})$, if $A \nvdash_i B$, then there exists a prime theory α such that $A \in \alpha$ and $B \notin \alpha$.*

For any prime theory α we define the canonical valuation $v_{\mathcal{T}}$: $\Phi \longrightarrow \mathbb{16}$ as follows:

$$\mathbf{t} \in v_{\mathcal{T}}(p) \text{ iff } p \in \alpha; \qquad \mathbf{f} \in v_{\mathcal{T}}(p) \text{ iff } {\sim}p \in \alpha;$$
$$\mathbf{m} \in v_{\mathcal{T}}(p) \text{ iff } [m]p \in \alpha; \qquad \mathbf{n} \in v_{\mathcal{T}}(p) \text{ iff } [n]p \in \alpha.$$

Canonical valuations can be extended to arbitrary $Form(\mathcal{L}_{inf})$-formulas.

Lemma 4 *Let α be a prime theory and let $v_{\mathcal{T}}$ be defined as above. Then for any $A \in Form(\mathcal{L}_{inf})$:*

$$\mathbf{t} \in v_{\mathcal{T}}(A) \text{ iff } A \in \alpha; \qquad \mathbf{f} \in v_{\mathcal{T}}(A) \text{ iff } {\sim}A \in \alpha;$$
$$\mathbf{m} \in v_{\mathcal{T}}(A) \text{ iff } [m]A \in \alpha; \qquad \mathbf{n} \in v_{\mathcal{T}}(A) \text{ iff } [n]A \in \alpha.$$

Proof. The proof is by induction on the construction of formulas $A \in Form(\mathcal{L}_{inf})$. For propositional variables the claim holds by definition.

If A has the form ${\sim}B$, then we have $\mathbf{t} \in v_{\mathcal{T}}({\sim}B)$ iff $\mathbf{f} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, ${\sim}B \in \alpha$; $\mathbf{m} \in v_{\mathcal{T}}({\sim}B)$ iff $\mathbf{m} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[m]B \in \alpha$ iff $[m]{\sim}B \in \alpha$ by ($[m]{\sim}$ reduction); $\mathbf{f} \in v_{\mathcal{T}}({\sim}B)$ iff $\mathbf{t} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $B \in \alpha$ iff, by (double negation), ${\sim}{\sim}B \in \alpha$; $\mathbf{n} \in v_{\mathcal{T}}({\sim}B)$ iff $\mathbf{n} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[n]B \in \alpha$ iff, by ($[n]{\sim}$ reduction), $[n]{\sim}B \in \alpha$.

If A has the form $[m]B$, then $\mathbf{t} \in v_{\mathcal{T}}([m]B)$ iff $\mathbf{m} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[m]B \in \alpha$; $\mathbf{m} \in v_{\mathcal{T}}([m]B)$ iff $\mathbf{m} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[m]B \in \alpha$ iff $[m][m]B \in \alpha$ by ($[m][m]$ reduction); $\mathbf{f} \in v_{\mathcal{T}}([m]B)$ iff $\mathbf{n} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[n]B \in \alpha$ iff, by (negated $[m]$), ${\sim}[m]B \in \alpha$; $\mathbf{n} \in v_{\mathcal{T}}([m]B)$ iff $\mathbf{n} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[n]B \in \alpha$ iff, by ($[n][m]$ reduction), $[n][m]B \in \alpha$.

If A has the form $[n]B$, then $\mathbf{t} \in v_{\mathcal{T}}([n]B)$ iff $\mathbf{n} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[n]B \in \alpha$; $\mathbf{m} \in v_{\mathcal{T}}([n]B)$ iff $\mathbf{m} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[m]B \in \alpha$ iff $[m][n]B \in \alpha$ by ($[m][n]$ reduction); $\mathbf{f} \in v_{\mathcal{T}}([n]B)$ iff $\mathbf{m} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[m]B \in \alpha$ iff, by (negated $[n]$), ${\sim}[n]B \in \alpha$; $\mathbf{n} \in v_{\mathcal{T}}([n]B)$ iff $\mathbf{n} \in v_{\mathcal{T}}(B)$ iff, by the induction hypothesis, $[n]B \in \alpha$ iff, by ($[n][n]$ reduction), $[n][n]B \in \alpha$.

If A has the form $B \otimes C$, we have $\mathbf{t} \in v_{\mathcal{T}}(B \otimes C)$ iff $\mathbf{t} \in v_{\mathcal{T}}(B) \cap v_{\mathcal{T}}(C)$ iff ($\mathbf{t} \in (v_{\mathcal{T}}(B)$ and $\mathbf{t} \in v_{\mathcal{T}}(C)$) iff, by the induction hypothesis, ($B \in \alpha$ and $C \in \alpha$) iff, by the definition of theories and ($\otimes$-elim), $B \otimes C \in \alpha$; $\mathbf{m} \in v_{\mathcal{T}}(B \otimes C)$ iff ($\mathbf{m} \in (v_{\mathcal{T}}(B)$ and $\mathbf{m} \in v_{\mathcal{T}}(C)$) iff, by the induction hypothesis, ($[m]B \in \alpha$ and $[m]C \in \alpha$) iff, by the definition of theories and ($\otimes$-elim), $[m]B \otimes [m]C \in \alpha$ iff $[m](B \otimes C) \in \alpha$ by ($[m]\otimes$ distribution). Next, $\mathbf{f} \in v_{\mathcal{T}}(B \otimes C)$ iff $\mathbf{f} \in v_{\mathcal{T}}(B) \cap v_{\mathcal{T}}(C)$ iff ($\mathbf{f} \in (v_{\mathcal{T}}(B)$ and $\mathbf{f} \in v_{\mathcal{T}}(C)$) iff, by the induction hypothesis, (${\sim}B \in \alpha$ and ${\sim}C \in \alpha$) iff, by the definition of theories, ($\otimes$-elim), and (negated fusion), ${\sim}(B \otimes C) \in \alpha$; $\mathbf{n} \in v_{\mathcal{T}}(B \otimes C)$ iff ($\mathbf{n} \in (v_{\mathcal{T}}(B)$ and $\mathbf{n} \in v_{\mathcal{T}}(C)$) iff, by the induction hypothesis, ($[n]B \in \alpha$ and $[n]C \in \alpha$) iff, by the definition of theories and ($\otimes$-elim), $[n]B \otimes [n]C \in \alpha$ iff $[n](B \otimes C) \in \alpha$ by ($[n]\otimes$ distribution).

If A has the form $B \oplus C$, the reasoning is analogous to that of the previous case and makes use of (negated fission), ($[m]\oplus$ distribution), ($[n]\oplus$ distribution), ($\oplus$-intro), and the definition of prime theories. $\square$

The proof of the characterization theorem for $\vdash_i$ follows a standard pattern.

Theorem 2 *For any $A, B \in Form(\mathcal{L}_{inf})$: $A \models^{16}_i B$ iff then $A \vdash_i B$.*

Proof. Right-to-left (soundness): It is easy to show that the axioms are valid and that the rules preserve validity. For the (negated $[m]$) axioms, for example, we have

$$\begin{array}{rlcrl} \mathbf{t} \in v^{16}(\sim[m]A) & \text{iff} & \quad & \mathbf{f} \in v^{16}(\sim[m]A) & \text{iff} \\ \mathbf{f} \in v^{16}([m]A) & \text{iff} & & \mathbf{t} \in v^{16}([m]A) & \text{iff} \\ \mathbf{n} \in v^{16}(A) & \text{iff} & & \mathbf{m} \in v^{16}(A) & \text{iff} \\ \mathbf{t} \in v^{16}([n]A) & & & \mathbf{f} \in v^{16}([n]A) & \end{array}$$

$$\begin{array}{rlcrl} \mathbf{m} \in v^{16}(\sim[m]A) & \text{iff} & \quad & \mathbf{n} \in v^{16}(\sim[m]A) & \text{iff} \\ \mathbf{m} \in v^{16}([m]A) & \text{iff} & & \mathbf{n} \in v^{16}([m]A) & \text{iff} \\ \mathbf{m} \in v^{16}(A) & \text{iff} & & \mathbf{n} \in v^{16}(A) & \text{iff} \\ \mathbf{m} \in v^{16}([n]A) & & & \mathbf{n} \in v^{16}([n]A). & \end{array}$$

Left-to-right (completeness): Let $A \models^{16}_i B$ and assume $A \nvdash B$. By Lemma 3, there exists a prime theory α such that $A \in \alpha$ and $B \notin \alpha$. Then, by Lemma 4, **t** $\in v_{\mathcal{T}}(A)$ but **t** $\notin v_{\mathcal{T}}(B)$, and thus $A \not\models^{16}_i B$. (Likewise, we can consider **f** instead of **t**. Let $A \in \alpha$ and $B \notin \alpha$. Then, by (double negation), this is the case iff $\sim\sim A \in \alpha$ and $\sim\sim B \notin \alpha$. By Lemma 4, **f** $\in v_{\mathcal{T}}(\sim A)$ but **f** $\notin v_{\mathcal{T}}(\sim B)$, and thus $A \not\models^{16}_i B$.) $\square$

4.2 From the lattice $\mathbb{16}_{inf}$ to the pentalattice $\mathbb{65536}_5$

It is, of course, possible to define further partial orderings on $\mathbb{16}$ in addition to the subset relation, but I will define additional orderings on $\mathbb{65536} = \mathcal{P}(\mathbb{16})$ instead of $\mathbb{16}$. The reason for this is similar to the reason for considering the trilattice $SIXTEEN_3$ instead of the bilattice $FOUR_2$. Both $FOUR_2$ and $SIXTEEN_3$ give rise to a semantics for the basic propositional paraconsistent relevance logic known as Belnap-Dunn logic, Dunn-Belnap logic, or first-degree entailment logic, $\mathbf{FDE}$, see (Anderson & Belnap, 1975, § 15.2), or, for a survey and additional references, (Omori & Wansing, 2017). The language of $\mathbf{FDE}$ contains the connectives $\sim$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction), and $\mathbf{FDE}$ can be defined as the logic of what is usually said to be the *truth order*, $\leq_t$, of $FOUR_2$. The bilattice $FOUR_2$ is defined on a set of four semantical values, **T**, **F**, **N**, and **B**, which have the following intuitive reading, stated already in Section 1:

T (*told true but not false*)
F (*told false but not true*)
N (*told neither true nor false*)
B (*told both true and false*).

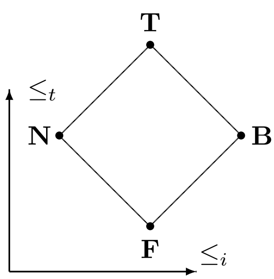

Figure 2: The bilattice $FOUR_2$.

In Figure 2, the bilattice $FOUR_2$ with its two partial orders is depicted as a Hasse diagram. The values **T**, **F**, **N**, and **B** can be represented as the elements of the powerset $\mathcal{P}(\{T, F\})$ = **4** of the set of classical truth values **2** = $\{T, F\}$: $\mathbf{N} = \varnothing$, $\mathbf{T} = \{T\}$, $\mathbf{F} = \{F\}$, $\mathbf{B} = \{T, F\}$, see (Dunn, 1976; 2000). With this representation, the information order $\leq_i$ on **4** is the subset relation.

First-degree entailment logic is semantically determined by interpreting conjunction and disjunction as the lattice meet, respectively lattice join of $\leq_t$. Negation is interpreted by a unary operation, $-$, that inverts the truth order, leaves the information order untouched and satisfies $x = --x$. Alternatively, the semantics can be given by the matrix $\langle \mathbf{4}, \{\mathbf{T}, \mathbf{B}\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$, where the functions f_c are defined by the following truth tables:

$f_\sim$	
T	**F**
B	**B**
N	**N**
F	**T**

$f_\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

$f_\vee$	**T**	**B**	**N**	**F**
T	**T**	**T**	**T**	**T**
B	**T**	**B**	**T**	**B**
N	**T**	**T**	**N**	**N**
F	**T**	**B**	**N**	**F**

The set $\mathcal{D} = \{\mathbf{T}, \mathbf{B}\}$ is the set of designated values. A valuation function v mapping propositional variables into **4** is extended to a valuation of arbitrary formulas by requiring that $v(c(A_1, \ldots, A_m)) = f_c(v(A_1), \ldots, v(A_m))$, and the semantic consequence relation $\models_{\mathbf{FDE}}$ between single formulas A and B is defined as follows:

$A \models_{\mathbf{FDE}} B$ iff for every valuation function v, $v(A) \in \mathcal{D}$ implies $v(B) \in \mathcal{D}$

or, equivalently, by setting

$$A \models_{\mathbf{FDE}} B \text{ iff for every valuation function } v, v(A) \leq_t v(B).$$

The relation $\leq_i$ on **4** can quite convincingly be seen as an information ordering, the idea being that the more elements a semantical value contains the more informative is the assignment of that value to a propositional variable. In (Shramko & Wansing, 2005; 2011) it is argued that it is much less convincing to regard $\leq_t$ as a truth ordering. The reason for viewing $\{T\}$ as 'more true' than $\{T, F\}$ and regarding $\varnothing$ as 'more true' than $\{F\}$ is the absence of the classical value F from $\{T\}$, respectively $\varnothing$. The relation $\leq_t$ is thus not defined only with respect to the presence of T in or the absence of T from elements of **4**. If one moves from **4** to $\mathcal{P}(\mathbf{4}) = \mathbf{16}$, however, it is possible not only to define a pure truth ordering in terms of the presence of T in or the absence of T from elements of elements of **16** but also a pure falsity ordering in terms of the presence of F in or the absence of F from elements of elements of **16**.

In (Shramko & Wansing, 2005; 2011) in addition to the subset relation as an information order on **16**, a truth and a falsity ordering are defined. In a first step, for every x in **16** the sets x^t, x^{-t}, x^f, and x^{-f} are defined by the following equations:

$$x^t := \{y \in x \mid T \in y\}; \qquad x^{-t} := \{y \in x \mid T \notin y\};$$
$$x^f := \{y \in x \mid F \in y\}; \qquad x^{-f} := \{y \in x \mid F \notin y\}.$$

Definition 9 *For every x, y in* **16***:*

- $x \leq_i y$ *iff* $x \subseteq y$*;*
- $x \leq_t y$ *iff* $x^t \subseteq y^t$ *and* $y^{-t} \subseteq x^{-t}$*;*
- $x \leq_f y$ *iff* $x^f \subseteq y^f$ *and* $y^{-f} \subseteq x^{-f}$.

Following that strategy, for every x in $\mathbb{65536}$ we define the sets x^t, x^{-t}, x^f, x^{-f}, x^m, x^{-m}, x^n, and x^{-n} as follows:

$$\begin{array}{rlrl} x^t &:= \{y \in x \mid \mathbf{t} \in y\}\,; & x^{-t} &:= \{y \in x \mid \mathbf{t} \notin y\}\,; \\ x^f &:= \{y \in x \mid \mathbf{f} \in y\}\,; & x^{-f} &:= \{y \in x \mid \mathbf{f} \notin y\}\,; \\ x^m &:= \{y \in x \mid \mathbf{m} \in y\}\,; & x^{-m} &:= \{y \in x \mid \mathbf{m} \notin y\}\,; \\ x^n &:= \{y \in x \mid \mathbf{n} \in y\}\,; & x^{-n} &:= \{y \in x \mid \mathbf{n} \notin y\}\,. \end{array}$$

Definition 10 *For every x, y in $\mathbb{65536}$:*

- $x \leq_i y$ *iff* $x \subseteq y$*;*
- $x \leq_\circ y$ *iff* $x^\circ \subseteq y^\circ$ *and* $y^{-\circ} \subseteq x^{-\circ}$*, for* $\circ \in \{t, f, m, n\}$.

Lattice meet and lattice join operations for all five partial orderings exist, and we will denote them as $\mathsf{X} \sqcap_\circ Y$, respectively $\mathsf{X} \sqcup_\circ Y$ for $\circ \in \{t, f, m, n\}$. With this definition, we obtain the pentalattice

$$\mathbb{65536}_5 = (\mathbb{65536}, \subseteq, \leq_t, \leq_f, \leq_m, \leq_n).$$

The pentalattice $\mathbb{65536}_5$ gives rise to the propositional language $\mathcal{L}(\mathbb{65536}_5)$ based on a denumerable set of propositional variables Φ defined in Backus-Naur form as follows:

$$\begin{array}{ll} & \text{variables } \Phi\text{:} \quad p \in \Phi \\ & \text{formulas:} \quad A \in Form_{(\mathcal{L}_{\mathbb{65536}_5})}(\Phi) \\ A ::= & p \mid (A \otimes A) \mid (A \oplus A) \mid (A \wedge_t A) \mid (A \vee_t A) \mid (A \wedge_f A) \mid (A \vee_f A) \mid \\ & (A \wedge_m A) \mid (A \vee_m A) \mid (A \wedge_n A) \mid (A \vee_n A) \mid {\sim}A \mid [m]A \mid [n]A. \end{array}$$

Let $\mathbf{t}^* := \mathbf{f}$, $\mathbf{f}^* := \mathbf{t}$, $\mathbf{m}^* := \mathbf{m}$, $\mathbf{n}^* := \mathbf{n}$, $X^* := \{x^* \mid x \in X\}$ for $X \in \mathbb{16}$, and $\mathsf{X}^* := \{X^* \mid X \in \mathsf{X}\}$ for $\mathsf{X} \in \mathbb{65536}$. Let $\mathbf{t}^m := \mathbf{m}$, $\mathbf{f}^m := \mathbf{n}$, $\mathbf{m}^m := \mathbf{m}$, $\mathbf{n}^m := \mathbf{n}$, $X^m := \{x^m \mid x \in X\}$ for $X \in \mathbb{16}$, and $\mathsf{X}^m := \{X^m \mid X \in \mathsf{X}\}$ for $\mathsf{X} \in \mathbb{65536}$. Let $\mathbf{t}^n := \mathbf{n}$, $\mathbf{f}^n := \mathbf{m}$, $\mathbf{m}^n := \mathbf{m}$, $\mathbf{n}^n := \mathbf{n}$, $X^n := \{x^n \mid x \in X\}$ for $X \in \mathbb{16}$, and $\mathsf{X}^n := \{X^n \mid X \in \mathsf{X}\}$ for $\mathsf{X} \in \mathbb{65536}$. A valuation in $\mathbb{65536}$ is a function $v^\odot \colon \Phi \longrightarrow \mathbb{65536}$. Valuation functions $v^\odot$ in $\mathbb{65536}$ are extended to functions from $Form_{(\mathcal{L}_{\mathbb{65536}_5})}(\Phi)$ to $\mathbb{65536}$ as follows, where $\circ \in \{t, f, m, n\}$:

$$\begin{array}{lcllcl} v^\odot(A \otimes B) &=& v^\odot(A) \cap v^\odot(B); & v^\odot({\sim}A) &=& (v^\odot(A))^*; \\ v^\odot(A \oplus B) &=& v^\odot(A) \cup v^\odot(B); & v^\odot([m]A) &=& (v^\odot(A))^m; \\ v^\odot(A \wedge_\circ B) &=& v^\odot(A) \sqcap_\circ v^\odot(B); & v^\odot([n]A) &=& (v^\odot(A))^n. \\ v^\odot(A \vee_\circ B) &=& v^\odot(A) \sqcup_\circ v^\odot(B); & & & \end{array}$$

Definition 11 *The relation* $\models_i^{65536} \subseteq (Form(\mathcal{L}_{inf}) \times Form(\mathcal{L}_{inf}))$ *is defined by setting*

$$A \models_i^{65536} B \textit{ iff for every valuation } v^{\odot} \textit{ in } \mathbb{65536}, v^{\odot}(A) \subseteq v^{\odot}(B).$$

Conjecture *For any* $A, B \in Form(\mathcal{L}_{inf})$*:* $A \models_i^{65536} B$ *iff* $A \vdash_i B$*.*

I expect no particular obstacle to verifying the conjecture. An anonymous referee wondered whether it could be proved by some embedding of the lattice $\mathbb{16}$ into the lattice $\mathbb{65536}_5$.

5 Negation inconsistency

It is well known that a certain simple modification of the support of falsity condition for implications in **N4** leads to a non-trivial negation inconsistent connexive logic, namely the system **C**, see (Wansing, 2005), (Omori & Wansing, 2020), (Niki & Wansing, 2023). The same modification brings us from **N4mn** to the connexive logic **Cmn**. The notion of a Kripke model for **Cmn** and the notion of **Cmn**-validity are defined in analogy to the case of **N4mn**, except that the following falsification clause for implications $A \to B$ is used: $x \models^- A \to B$ iff $\forall y \in M$ [xRy and $y \models^+ A$ imply $y \models^- B$].

Assuming this falsification clause, the logic **Cmn** is model-theoretically defined as the pair $\langle \mathcal{L}, \{\langle \Gamma, A \rangle \mid \Gamma \models^+ A\} \rangle$.

Theorem 3 (Semantical embedding) *Let* f' *be the mapping from* $\mathrm{Form}_{\mathcal{L}}(\Phi)$ *to the set of formulas of the language* $\mathcal{L}'$ *defined over* $\Phi \cup \Phi^- \cup \Phi^m \cup \Phi^n$ *that is defined exactly like the function* f *from Definition 6, except that*

$$f'(\sim(A \to B)) = f'(A) \to f'(\sim B).$$

Then, for any set of $\mathcal{L}$*-formulas* $\Gamma \cup A$*,* $\Gamma \models^+ A$ *in* **Cmn** *iff* $f'(\Gamma) \models f'(A)$ *in* **IPL**$^+$.

The following schematic formulas, for instance, are **Cmn**-valid:

$$(A \to (\sim A \to A)) \text{ and } \sim(A \to (\sim A \to A)).$$

The language $\mathcal{L}_{inf}$ of $\mathbb{inf}$ does not contain a genuine conditional, by which I mean an implication connective, $\to$, that satisfies the deduction theorem, i.e., validates implication introduction, and modus ponens. The addition of any such conditional to $\mathbb{inf}$ presented as a relation $\vdash_i$ between sets of $\mathcal{L}_{inf}$-formulas and single $\mathcal{L}_{inf}$-formulas will result in a nontrivial negation inconsistent logic. Given that $\varnothing \vdash_i A \to A$ is provable, we get the following derivations:

$$\begin{array}{lll}
1. & \varnothing \vdash_i A \to A & \\
2. & A \to A \vdash_i {\sim}(A \to A) \oplus (A \to A) & \oplus\text{-introduction} \\
3. & \varnothing \vdash_i {\sim}(A \to A) \oplus (A \to A) & 1., 2., \text{transitivity}
\end{array}$$

$$\begin{array}{lll}
1. & \varnothing \vdash_i A \to A & \\
2. & A \to A \vdash_i {\sim}{\sim}(A \to A) & \text{double negation} \\
3. & \varnothing \vdash_i {\sim}{\sim}(A \to A) & 1., 2., \text{transitivity} \\
4. & {\sim}{\sim}(A \to A) \vdash_i {\sim}{\sim}(A \to A) \oplus {\sim}(A \to A) & \oplus\text{-introduction} \\
5. & \varnothing \vdash_i {\sim}{\sim}(A \to A) \oplus {\sim}(A \to A) & 3., 4., \text{transitivity} \\
6. & {\sim}{\sim}(A \to A) \oplus {\sim}(A \to A) \vdash_i {\sim}({\sim}(A \to A) \oplus (A \to A)) & \text{negated fission} \\
7. & \varnothing \vdash_i {\sim}({\sim}(A \to A) \oplus (A \to A)) & 5., 6., \text{transitivity}
\end{array}$$

6 Open problems and further directions

In addition to deciding the above conjecture, open questions and directions for future research abound. First, in addition to $\mathcal{L}_{inf}$ we can define the following fragments $\mathcal{L}_\circ$ of $\mathcal{L}(\mathbb{65536}_5)$ based on a denumerable set of propositional variables Φ, for $\circ \in \{t, f, m, n\}$:

$$\begin{array}{rl}
\text{variables } \Phi: & p \in \Phi \\
\text{formulas:} & A \in \mathit{Form}_{\mathcal{L}_\circ}(\Phi) \\
A ::= & p \mid (A \wedge_\circ A) \mid (A \vee_\circ A) \mid {\sim}A \mid [m]A \mid [n]A.
\end{array}$$

Naturally, for each of the languages $\mathcal{L}(\mathbb{65536}_5)$ and $\mathcal{L}_\circ$, and not only for $\mathcal{L}_{inf}$, we can define informational entailment:

$$A \models_i^{65536} B \text{ iff for every valuation } v^{\odot} \text{ in } \mathbb{65536}, v^{\odot}(A) \subseteq v^{\odot}(B).$$

Next, for each of the languages $\mathcal{L}(\mathbb{65536}_5)$ and $\mathcal{L}_\circ$, we can define the following entailment relations:

Definition 12 *The entailment relation* $\models_\circ^{65536} \subseteq (\mathit{Form}_{(\mathcal{L}_{\mathbb{65536}_5})}(\Phi) \times \mathit{Form}_{(\mathcal{L}_{\mathbb{65536}_5})}(\Phi))$ *is defined by setting*

$$A \models_\circ^{65536} B \textit{ iff for every valuation } v^{\odot} \textit{ in } \mathbb{65536}, v^{\odot}(A) \leq_\circ v^{\odot}(B).$$

The relation $\models_{\mathcal{L}_\circ}^{65536} \subseteq (\mathit{Form}(\mathcal{L}_\circ)(\Phi) \times \mathit{Form}(\mathcal{L}_\circ)(\Phi))$ *is defined by setting*

$$A \models_{\mathcal{L}_\circ}^{65536} B \textit{ iff for every valuation } v^{\odot} \textit{ in } \mathbb{65536}, v^{\odot}(A) \leq_\circ v^{\odot}(B).$$

Moreover, we can think of combinations of those relations, e.g., by considering intersections. Since all theses languages lack a primitive conditional, it makes a lot of sense to expand the languages with a genuine implication connective. An obvious task then is to define proof systems for the various semantically introduced logics with nice proof-theoretic properties, especially sequent calculi that allow for proof analysis.

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