Chapter 9

Summary and Outlook

In this chapter we summarize and discuss the open problems connected with our study of the geodesics on the two-dimensional torus. In the previous chapters we proved sufficient and necessary conditions on the geodesics for vanishing topological entropy of the geodesic flow. An open question is if there exists a condition on the geodesics which is equivalent to vanishing topological entropy. The sufficient condition on the number of intersections of lifted geodesics with their translates, as formulated in Main Theorem VI, is too strong for an equivalence: Considering the torus of revolution we get an counter-example for the statement that no periodic geodesics intersect its translates. For the necessary conditions it is not clear if for example the existence of rotation directions for all geodesics already implies vanishing topological entropy.

We know from Theorem 5.6 that a contractible closed geodesic implies positive topological entropy and in Chapter 6 we also present sufficient conditions on geodesic segments for the existence of a closed contractible geodesic. Now we want to present a further condition which is inspired by the study of horseshoes in Chapter 8:

**Lemma 9.1.** Let $(T^2, g)$ be a Riemannian torus and let $\Lambda$ be a horseshoe of geodesics on $T^2$. If $\Lambda$ contains three periodic geodesics fulffiling the property that every connected component of $S^1$ containing their forward rotation directions is properly larger than a half-circle, then there exists a closed contractible geodesic on $T^2$.

**Proof.** We will only sketch the arguments of this proof. The main idea is to construct a cloud-geodesic. Then by Lemma 6.22 there exists a closed contractible geodesic:

We denote the periodic geodesics by $c_1, c_2, c_3$ and their forward rotation directions by $\delta_1, \delta_2, \delta_3$. As $c_1, c_2, c_3 \in \Lambda$ there exist six asymptotic geodesics $a_{ij}$ connecting $c_i$ and $c_j$ for $i, j \in \{1, 2, 3\}$, see Figure 9.1. Translating them by adequate translation elements we create a cloud-geodesic as shown in Figure 9.1. \hfill \Box

Starting with a closed contractible geodesic as we have shown in the proof of Theorem 5.7 we can construct for each rational number $\alpha$ an exponentially growing set of periodic geodesics with equal rotation number $\alpha$.

Connected with the result in Lemma 9.1 the question arises if there exist conditions on the rotation directions of geodesics in a horseshoe, e.g. if there exist Riemannian metrics on $T^2$ with positive topological entropy such that there exist horseshoes consisting only of geodesics with the same rotation direction, or the same rotation number.
As described in the previous chapters a still open problem in our approach characterizing Riemannian metrics is to prove that for vanishing topological entropy no geodesic has a contractible self-intersection and that for all geodesics the forward rotation direction is the negative of the backward rotation direction.

We now want to present a necessary result for vanishing topological entropy which holds for monotone twist maps. Then we will explain how an analogous statement would solve our just formulated problems. The statement of this result in the formulation of S. B. Angenent [7] is the following:

**Theorem 9.2** (S. B. Angenent, see [7]). Let $A = S^1 \times [0, 1]$ be an annulus and $f : A \to A$ a monotone twist map. Let the two boundary components of $A$ denoted by $A_0 = S^1 \times \{0\}$ and $A_1 = S^1 \times \{1\}$ be invariant under $f^q$, for some $q \in \mathbb{N}$. We denote the rotation numbers of $f|_{A_0}$ and $f|_{A_1}$ by $\rho_0$ and $\rho_1$, respectively. If the topological entropy of $f$ vanishes, then $f$ must have an invariant circle of rotation number $\omega$, for any $\omega \in (\rho_0, \rho_1)$.

With independent arguments J. Mather [42] proves the same result formulated the other way round. He starts with the assumption that there exists a region of instability. He then concludes the existence of a chaotic orbit. The existence of this chaotic orbit then implies high complexity, i.e. positive topological entropy for the considered twist map.

The analogon to invariant circles, introduced in Definition and Remark 4.12 in the context of geodesics on a two-torus is the foliation of the torus by minimal geodesics with equal rotation numbers. Hence, an analogous statement to Theorem 9.2 would be the following conjecture:

**Conjecture.** Let $(T^2, g)$ be a Riemannian torus. If the topological entropy of the geodesic flow vanishes, then for each rotation number $\alpha \in \mathbb{R} \cup \{\infty\}$ there exists a foliation of $T^2$ by minimal geodesics.

Such a foliation by minimal geodesics for each direction implies very strong conditions on
all geodesics. Each of the following properties of lifts of geodesics prohibits the existence of a foliation by minimals at least for an open interval $I \subset \mathbb{R} \cup \{\infty\}$ of rotation numbers:

a) Existence of a contractible closed geodesic $c$

b) Existence of a $V$-geodesic $c$ with rotation directions $\delta_+(c) \neq -\delta_-(c)$

c) Existence of a geodesic $c$ with a contractible self-intersections and arbitrary forward and backward rotation directions $\delta_+(c), \delta_-(c)$

The central idea in the proof of this statement in the cases a), b), and c) is to choose two minimal geodesics $\gamma_1, \gamma_2$ with an equal rotation number such that $\gamma_1$ does not intersect $c$ and $\gamma_2$ intersects $c$ an even number of times. Then, if for this fixed rotation number the minimals were foliated we would also have a minimal geodesic lying between $\gamma_1$ and $\Gamma_2$ which has a common tangency with $c$, in contradiction to the uniqueness of geodesics. In a) we get $I = \mathbb{R} \cup \{\infty\}$ and in b) and c) the interval is at least the connected component of $S^1 \setminus \{\delta_+(c), \delta_-(c)\}$.

This means, that if the conjecture was true additional to the properties we already know it would completely clarify the behavior of geodesics for vanishing topological entropy: All geodesics are unbounded, have antipodal forward and backward rotation directions and have no self-intersections on the universal covering. Furthermore, by Theorem 7.10 the rotation numbers are continuous functions on the unit tangent bundle. It is not clear if the non-existence of foliations for an interval of rotation numbers already implies that there exist $V$-geodesics or geodesics with contractible self-intersections.

Furthermore by the same arguments it follows that an arbitrary geodesic intersects each minimal geodesic at most once.

The torus of revolution is an example of a Riemannian metric with vanishing topological entropy and a foliation for each rotation number by minimals. Except of one rotation number for all other rational rotation numbers the torus is even foliated by closed minimal geodesics.

As a last remark in this thesis we want to mention a question which is of big interest for geodesic flows on arbitrary Riemannian manifolds. It asks for the growth of the number of closed geodesics. We want to give the formulation of the question for the case of the torus:

**Question.** Let $P(t)$ be a function which counts the number of closed geodesics on $T^2$ with length $\leq t$. For positive topological entropy $P(t)$ grows obviously exponentially with $t$. Is it true that also for vanishing topological entropy

$$\liminf_{t \to \infty} \frac{\log P(t)}{t} > 0 ?$$

As the topological entropy vanishes this would imply that there exists $t_0 > 0$ such that $P(t_0) = \infty$.

The conjecture above gives no information if the foliations are by closed or asymptotic geodesics. But if it held, it would be enough to show that for at least one rational rotation number there exists a strip which is foliated by closed minimal geodesics in order to answer the question positively.