1 Introduction

In the seminal paper [44], Nualart and Peccati discovered a surprising characterization of central limit theorems on a fixed Wiener chaos: A sequence of multiple stochastic integrals of fixed order with variance tending to 1 converges in distribution to a standard Gaussian random variable, if and only if the sequence of its fourth moments converges to 3, the fourth moment of the standard normal distribution. This result, known as the fourth moment theorem, yields a drastic simplification of the classical method of moments. Shortly afterwards, in [48], Peccati and Tudor extended the findings of [44] to the multidimensional case. The next methodological step was taken by Nualart and Ortiz-Latorre in [43] by linking central limit theorems on a fixed Wiener chaos (and their multidimensional extensions) to Malliavin calculus to provide a new characterization in terms of Malliavin operators as well as an alternative proof of the main result in [44].

In [35], Nourdin and Peccati combined Stein’s method and Malliavin calculus for the first time by approaching the method through an integration by parts formula based on Malliavin operators instead through one of the several classical coupling constructions as in, e.g., the local approach, the exchangeable pair approach or the size and zero biasing approaches. This new approach to Stein’s method, also called the Malliavin-Stein method, led to a significant refinement of the findings in [44], [48] and [43] in terms of explicit bounds on the error in the normal (and gamma) approximation of multiple stochastic integrals of fixed order. The theoretical background of [35] reached even further as it treated general functionals of Gaussian fields by using the fact that they admit a chaos representation in form of an infinite sum of multiple stochastic integrals. Along similar lines, the results of [35] were extended to the multidimensional case by Nourdin, Peccati and Réveillac in [42]. These references laid the foundations for a new field of research in probability theory. In [39], Nourdin, Peccati and Reinert developed infinite-dimensional second order Poincaré inequalities for general functionals of Gaussian fields. Such inequalities are bounds on the error in the normal approximation solely in terms of the first and second order gradient operator (instead of further operators) from Malliavin calculus. Such bounds extend the reach of applications to general functionals for which a chaos representation is not explicitly known. In [40], the same authors linked the results of [35] and [42] to invariance principles for multilinear homogeneous sums of independent centered random variables by showing that the normal (and chi-squared) approximation of these sums is completely characterized by their Gaussian counterparts. Other applications concerned, e.g., Breuer-Major central limit theorems (see Chapter 7 in [37]), random fields on the sphere (see [32]) and...
random matrices (see [36]). Optimal upper and lower bounds on the rate of convergence in the quantitative fourth moment theorem were developed, at first, by Biermé, Bonami, Nourdin and Peccati in [4] with respect to a probability metric based on smooth test functions, and later, by Nourdin and Peccati in [38] with respect to the total variation distance.

The Malliavin-Stein method is not restricted to the consideration of functionals of Gaussian fields. In [46], Peccati, Solé, Taqqu and Utzet developed the Malliavin-Stein method for the normal approximation of functionals of Poisson measures with respect to the Wasserstein distance. Their work was extended to the multidimensional case by Peccati and Zheng in [49]. In [28], Lachièze-Rey and Peccati established a fourth moment theorem for functionals of Poisson measures having the form of finite sums of multiple stochastic integrals with non-negative integrands. These findings were refined by Peccati and Zheng in [50] for elements inside a fixed discrete Poisson chaos having the form of multilinear homogeneous sums of independent centered Poisson random variables without resorting to integrands of constant sign as in [28]. Also in [50], the authors showed new universality results in the sense of [40] involving such sums. Other applications concerned, e.g., geometric $U$-statistics (see [58], [28] and [29]), Boolean models (see [20]) and geometric random graphs (see [59] and [65]). Bounds with respect to the Kolmogorov distance (which we call Berry-Esseen bounds in the following) were first developed by Schulte in [64] and further refined by Eichelsbacher and Thäle in [16]. Infinite-dimensional second order Poincaré inequalities for functionals of Poisson measures were developed by Last, Peccati and Schulte in [30].

In [41], Nourdin, Peccati and Reinert laid the foundations for a discrete version of the Malliavin-Stein method to deduce quantitative central limit theorems for functionals of infinite sequences of independent Rademacher random variables. Here, the term Rademacher random variable refers to a symmetric Bernoulli random variable, i.e., a random variable only taking the values $\pm 1$, with success probability $p = 1/2$. Besides a bound for general Rademacher functionals, the findings in [41] contained explicit bounds for the asymptotic normality of discrete multiple stochastic integrals of fixed order and sums of discrete stochastic single and double integrals, all with respect to a probability metric based on smooth test functions. For this, a new product formula for discrete multiple stochastic integrals was proved. Applications concerned discrete multiple stochastic integrals over sparse sets and infinite weighted 2-runs. Moreover, a characterization of central limit theorems for discrete stochastic double integrals was discussed.

With the present thesis, we further develop the discrete Malliavin-Stein method introduced in [41]. In particular, a first Berry-Esseen bound on the error in the normal approximation of general functionals of possibly non-symmetric and non-homogeneous infinite Rademacher sequences is proved in Theorem 3.1.1, thereby not only refining but also extending the main result of [41]. Here, by a non-symmetric Rademacher sequence, we mean a sequence of Rademacher random variables with arbitrary success probability $p \in (0, 1)$ each. By non-homogeneous Rademacher sequence, we mean a sequence of Rademacher random variables with varying success probabilities. A first bound on the distance be-
tween the distribution of a vector of general functionals of possibly non-symmetric and non-homogeneous infinite Rademacher sequences and a multivariate normal distribution is established in Theorem 3.1.4 with respect to a probability metric based on smooth test functions. From these results, we deduce explicit Berry-Esseen bounds for the asymptotic normality of discrete multiple stochastic integrals of fixed order and sums of discrete stochastic single and double integrals in Theorem 3.2.1, 3.2.3 and 3.4.1 (as refinements of the bounds in [41]) as well as a new bound for the multivariate normal approximation of vectors of discrete multiple stochastic integrals in Theorem 3.5.1. Furthermore, we exhibit that the characterization of central limit theorems for discrete double integrals as claimed in [41] does not hold and show that answering the question of a characterization is a much more complex task than in the Gaussian or Poisson case by giving a new necessary condition in Theorem 3.3.2. Moreover, a first second order Poincaré inequality for the normal approximation of general functionals of possibly non-symmetric and non-homogeneous infinite Rademacher sequences is developed in Theorem 3.7.1. For this, some aspects of the foundations of the discrete Malliavin calculus as introduced by Privault in [52] had to be refined (see, in particular, Proposition 2.1.17).

Several applications yield (some of the first) Berry-Esseen bounds in central limit theorems concerning discrete multiple stochastic integrals over sparse sets in Theorem 3.6.2, infinite weighted 2-runs in Theorem 3.6.4 and counting statistics associated with the Erdős-Rényi random graph in Theorem 3.8.1 and 3.8.3. A new bound in a multivariate central limit theorem concerning vectors of traces of powers of Bernoulli random matrices is presented in Theorem 3.6.6. A first quantitative central limit theorem concerning percolation problems on trees is shown in Theorem 3.8.5.

In addition, we introduce a discrete Malliavin-Stein method for the Poisson approximation of general integer valued functionals of possibly non-symmetric and non-homogeneous Rademacher sequences in Chapter 4. In particular, a bound on the total variation distance between the distribution of general integer valued functionals of possibly non-symmetric and non-homogeneous Rademacher sequences and a Poisson distribution is proved in Theorem 4.1.1. Explicit bounds for suitably shifted discrete multiple stochastic integrals of fixed order are deduced in Theorem 4.2.1 and 4.2.4. For this, a generalization of the product formula for discrete multiple stochastic integrals from [41] is proved in Proposition 2.1.7. Moreover, a first second order Poincaré inequality for the Poisson approximation of general integer valued functionals of possibly non-symmetric and non-homogeneous Rademacher sequences is developed in Theorem 4.3.1.

Sections 3.1 to 3.6 build a substantial generalization and refinement of our article [25]. Sections 3.7 and 3.8 are based on our article [26], while Chapter 4 is based on the article [24]. We also have to mention that, in parallel to the works on this thesis (and the articles [25], [26] and [24]), Privault and Torrisi established bounds for the normal and Poisson approximation of general functionals of possibly non-symmetric and non-homogeneous Rademacher sequences, too, in [54]. However, concerning normal approximation, the bounds in [54] are with respect to a probability metric based on smooth test functions as in [41]. The whole theory leading to second order Poincaré inequalities for
normal and Poisson approximation is unique to our work. Concerning Poisson approximation, the few findings of this thesis that have a corresponding counterpart in [54] were worked out independently and differ (see, e.g., Remark 4.2.3).

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