Symmetries of Supermanifolds

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Hannah Bergner
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Introduction

Supermanifolds are generalizations of classical manifolds, whose definition is motivated by concepts stemming from supersymmetry. A supermanifold is a real or complex manifold with an enriched structure sheaf, meaning that locally, in addition to the usual coordinates, there are “odd” coordinates \( \theta_1, \ldots, \theta_n \) which anticommute, i.e. \( \theta_i \theta_j = -\theta_j \theta_i \).

In this monograph, we study symmetries of supermanifolds. We will be concerned with two main questions: The first is about the relation of infinitesimal symmetries and local or global symmetries on supermanifolds. The second question is about the structure of the global symmetries of a compact complex supermanifold and the existence of Lie supergroup of automorphisms. Our motivation are the following classical results.

A basic result in the study of symmetries on manifolds is Palais’ theory (see [Pal57]) on the relation of finite-dimensional Lie algebras of vector fields on manifolds and actions of the corresponding Lie groups. Palais proves that any finite-dimensional Lie algebra of vector fields on a manifold is induced by a local action of the corresponding Lie group and provides necessary and sufficient conditions for the Lie algebra of vector fields to be induced by a global action of the Lie group.

In complex geometry, an important result is the fact that the automorphism group of a compact complex manifold carries the structure of a complex Lie group such that its action on the compact complex manifold is holomorphic (see [BM47]). One fundamental fact is that the Lie algebra \( \text{Vec}(M) \) of holomorphic vector fields on a compact complex manifold \( M \) is finite-dimensional. The results of Palais’ imply that the simply-connected Lie group with Lie algebra \( \text{Vec}(M) \) acts on \( M \). It turns out that this Lie group, is up to ineffectivity, the connected component of the automorphism group of \( M \).

A vector field\(^{(1)}\) on a supermanifold \( \mathcal{M} \), can be written as a sum of an even and an odd vector field on \( \mathcal{M} \). As proven in [MSV93] and [GW13], the flow of the vector field \( X \) defines a local \( \mathbb{R}^{1|1} \)-action on \( \mathcal{M} \) if and only if the even part and the odd part of the vector field \( X \) are contained in a \((1|1)\)-dimensional Lie subsuperalgebra of the Lie superalgebra of vector fields on \( \mathcal{M} \). This generalizes the well known result that the flow of a vector field \( X \) on a classical manifold \( M \) is a local \( \mathbb{R} \)-action; note that in this case \( \mathbb{R}X \) is automatically a Lie subalgebra of \( \text{Vec}(M) \). Just as a (local) \( \mathbb{R} \)-action on a manifold \( M \) induces a vector field, any (local) action of a Lie group \( G \) on the manifold \( M \) induces an infinitesimal action on \( M \), i.e. a homomorphism \( \mathfrak{g} \rightarrow \text{Vec}(M) \) of Lie algebras, where \( \mathfrak{g} \) is the Lie algebra of right-invariant vector fields on \( G \) and \( \text{Vec}(M) \)

\(^{(1)}\)To be more precise, a vector field on a supermanifold should be called a super vector field. But to avoid excessive use of the word “super”, we will simply talk about vector fields on supermanifolds and always mean super vector fields.
denotes the set of vector fields on $M$. Similarly, any (local) action of a Lie supergroup $G$ on a supermanifold $\mathcal{M}$ induces an infinitesimal action $\mathfrak{g} \to \text{Vec}(\mathcal{M})$, where $\mathfrak{g}$ is now the Lie superalgebra of right-invariant vector fields on the Lie supergroup $G$; cf. Section 2.2. It is a natural question to ask in which cases it is possible to go the other way: Let $\mathcal{M}$ be a supermanifold and $\mathfrak{g}$ a finite-dimensional Lie superalgebra. We ask which homomorphisms $\lambda : \mathfrak{g} \to \text{Vec}(\mathcal{M})$ of Lie superalgebras are induced by a local or global action of a Lie supergroup $G$ with Lie superalgebra $\mathfrak{g}$.

In the setting of classical manifolds and Lie groups, Palais studied these questions in [Pal57]. Palais proved that, given a Lie group $G$ with Lie algebra of right-invariant vector fields $\mathfrak{g}$ and an infinitesimal action $\lambda : \mathfrak{g} \to \text{Vec}(M)$ on a manifold $M$, there always exists a local action $\varphi : W \subseteq G \times M \to M$ of the Lie group $G$ which induces $\lambda$.

It is easy to construct examples where the infinitesimal action is not induced by a global action of $G$ on $M$: If $M'$ is a manifold with $G$-action and we take an open subset $M \subset M'$ which is not $G$-invariant, then the restriction of the infinitesimal action to $M$ is not induced by a $G$-action on $M$. Another example is e.g. given by the vector field $-x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$. Its flow $\varphi : W \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\varphi(t,x) = \frac{x}{1+tx}$, is not global. But if we embed $\mathbb{R}$ into the real projective space, $\mathbb{R} \hookrightarrow \mathbb{P}_1(\mathbb{R})$, $x \mapsto [1 : x]$, we have the $\mathbb{R}$-action given by $(t, [x_0 : x_1]) \mapsto [x_0 + tx_1 : x_1]$. In the coordinate chart $\mathbb{R} \to U_0 \subset \mathbb{P}_1(\mathbb{R})$, $x \mapsto [x : 1]$, the action is simply given by addition $(t,x) \mapsto x+t$. The $\mathbb{R}$-action on $\mathbb{P}_1(\mathbb{R})$ extends the flow on $\mathbb{R}$ and induces the vector field $-x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$.

Note that in both cases the infinitesimal action on $M$ is still induced by a $G$-action on the larger manifold $M'$. We call infinitesimal actions which arise in this way globalizable. Necessary and sufficient conditions for an infinitesimal action to be globalizable are found in [Pal57].

We generalize the results of [Pal57] to the case of infinitesimal actions on supermanifolds. The existence of a local action with a given infinitesimal action on a supermanifold is proven and conditions for the globalizability of an infinitesimal actions are found. As in the classical case, a key point in the proof is the study of a certain distribution $D_\lambda$ on the product $G \times \mathcal{M}$ associated with an infinitesimal action $\lambda : \mathfrak{g} \to \text{Vec}(\mathcal{M})$ on the supermanifold $\mathcal{M}$, where $G$ is again a Lie supergroup with Lie superalgebra of right-invariant vector fields $\mathfrak{g}$; see Section 4.1. The distribution $D_\lambda$ is spanned by vector fields of the form

$$X + \lambda(X) \text{ for } X \in \mathfrak{g},$$

considering $X$ and $\lambda(X)$ as vector fields on the product $G \times \mathcal{M}$, so more formally we should write $X \otimes \text{id}_\mathcal{M}^* + \text{id}_G^* \otimes \lambda(X)$ instead of $X + \lambda(X)$.

Regarding the equivalence of infinitesimal and local actions up to shrinking, we have the full analogue of the classical result:

**Theorem 1** (see Section 4.2). Let $G$ be a Lie supergroup with Lie superalgebra of right-invariant vector fields $\mathfrak{g}$, and let $\lambda : \mathfrak{g} \to \text{Vec}(\mathcal{M})$ be an infinitesimal action on a supermanifold $\mathcal{M}$. Then there exists a local $G$-action $\varphi : W \subseteq G \times \mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$ which induces the infinitesimal action $\lambda$.

Moreover, any local action $\varphi : W \to \mathcal{M}$ is uniquely determined by its induced infinitesimal action and domain of definition.

Using the notation of the preceding theorem, we now ask when the infinitesimal action $\lambda$ is globalizable. The notion of univalence of an infinitesimal action $\lambda$, which is related to properties of the distribution $D_\lambda$, is extended to the case of supermanifolds.
A first guess could be that $\lambda$ is globalizable if and only if the underlying infinitesimal action on the underlying manifold $M$ is globalizable to an action of the Lie group $G$ which underlies $\mathcal{G}$. This guess is actually false. It is necessary that the underlying infinitesimal action is globalizable, but this is in general not sufficient. In the case where the underlying infinitesimal action of $\lambda$ is globalizable, the obstruction for $\lambda$ to be globalizable is a holonomy phenomenon of the distribution $\mathcal{D}_\lambda$, which only appears in the setting of supermanifolds.

Let $\mathfrak{g}_0$ be the even part of the Lie superalgebra $\mathfrak{g}$, and $\mathfrak{g}_1$ the odd, $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$. The Lie algebra of the Lie group $G$, which is the underlying group of $\mathcal{G}$, can be identified with $\mathfrak{g}_0$. The main result for the conditions of globalizability is the following:

**Theorem 2** (see Section 4.3). The infinitesimal action $\lambda : \mathfrak{g} \rightarrow \text{Vec}(M)$ is globalizable if and only if one of the following equivalent conditions is satisfied:

(i) The restricted infinitesimal action $\lambda|_{\mathfrak{g}_0} : \mathfrak{g}_0 \rightarrow \text{Vec}(M)$ is globalizable to an action of $G$ on a supermanifold $M'$ which contains $M$ as an open subsupermanifold and the $G$-action induces $\lambda$ on $M$.

(ii) The infinitesimal action $\lambda$ is univalent.

(iii) The underlying infinitesimal action is globalizable, and all leaves $\Sigma \subset G \times M$ of the distribution $\mathcal{D}_\lambda$ are holonomy free.

Condition (iii) of this theorem together with the appropriate classical results of Palais formulated in [Pal57] then yield the following corollaries stated in Section 4.5.

**Corollary 1.** Let $\mathcal{G}$ be a simply-connected Lie supergroup, i.e. a Lie supergroup whose underlying Lie group is simply connected, and $\lambda : \mathfrak{g} \rightarrow \text{Vec}(M)$ an infinitesimal action whose support is relatively compact in $M$. Then there exists a global $\mathcal{G}$-action on $M$ which induces $\lambda$.

In particular, there is a one-to-one correspondence between infinitesimal and global actions of a simply-connected Lie supergroup on a supermanifold with compact underlying manifold.

**Corollary 2.** Let $\lambda : \mathfrak{g} \rightarrow \text{Vec}(M)$ be an infinitesimal action of a simply-connected Lie supergroup $\mathcal{G}$ and let $\{X_i\}_{i \in I}, X_i \in \mathfrak{g}_0$, be a set of generators of $\mathfrak{g}_0$ such that the underlying vector fields of $\lambda(X_i), i \in I$, on $M$ have global flows. Then there exists a global $\mathcal{G}$-action on $M$ which induces $\lambda$.

If $M$ is a compact complex manifold, the Lie algebra $\text{Vec}(M)$ of holomorphic vector fields is finite-dimensional. Therefore, the simply-connected complex Lie group $G$ with Lie algebra $\text{Vec}(M)$ acts holomorphically on $M$ by the version of Corollary 1 for classical manifolds. The Lie group $G$ thus is a candidate for the connected component of the automorphism group $\text{Aut}(M)$ of $M$. Since $\text{Aut}(M)$ is a complex Lie group which acts holomorphically on $M$ by [BM47], the simply-connected Lie group $G$ is, after taking a quotient by the ineffectivity, indeed the connected component of $\text{Aut}(M)$.

We want to obtain similar results on the automorphism group of a compact complex supermanifold, i.e. a complex supermanifold whose underlying manifold is compact. In Section 5.2, we prove that the Lie superalgebra $\text{Vec}(M)$ of holomorphic vector fields on a compact complex supermanifold $M$ is finite-dimensional. The infinitesimal action
\( \lambda : \text{Vec}(\mathcal{M}) \rightarrow \text{Vec}(\mathcal{M}) \) given by the identity map has necessarily compact support since \( \mathcal{M} \) is compact. By applying Corollary 1 to this special situation, we get that the simply-connected Lie supergroup \( \mathbb{G} \) with Lie superalgebra \( \text{Vec}(\mathcal{M}) \) acts on \( \mathcal{M} \) such that the induced infinitesimal action is the identity \( \text{Vec}(\mathcal{M}) \rightarrow \text{Vec}(\mathcal{M}) \). This gives a first idea what the automorphism group of a compact complex supermanifold should be: Up to ineffectivity, the Lie supergroup \( \mathbb{G} \) with Lie superalgebra \( \text{Vec}(\mathcal{M}) \) should be the connected component of the automorphism Lie supergroup of \( \mathcal{M} \). Furthermore, we would like the underlying Lie group to consist of automorphisms of \( \mathcal{M} \).

An approach to the automorphism group of a compact complex supermanifold \( \mathcal{M} \) is to consider the set of automorphisms of \( \mathcal{M} \), which we denote by \( \text{Aut}_{\bar{0}}(\mathcal{M}) \). Since an automorphism \( \varphi \) of the supermanifold \( \mathcal{M} \) (with structure sheaf \( \mathcal{O}_\mathcal{M} \)) is “even” in the sense that its pullback \( \varphi^*: \mathcal{O}_\mathcal{M} \rightarrow \tilde{\varphi}^*(\mathcal{O}_\mathcal{M}) \) is a parity-preserving morphism, we can (at most) expect the set \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) to carry the structure of a classical Lie group if we require its action on \( \mathcal{M} \) to be smooth or holomorphic. Thus \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) cannot be a Lie supergroup of positive odd dimension.

In order to obtain a Lie supergroup as automorphism group we also have to consider the Lie superalgebra of holomorphic vector fields on \( \mathcal{M} \). This is a candidate for the Lie superalgebra of the automorphism group. It is import to establish a connection between the Lie superalgebra of vector fields on \( \mathcal{M} \) and the set of automorphisms \( \text{Aut}_{\bar{0}}(\mathcal{M}) \). For this, we need to prove that \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) carries the structure of a complex Lie group that acts holomorphically on \( \mathcal{M} \), which then implies that its Lie algebra is isomorphic to the Lie algebra of even holomorphic vector fields on \( \mathcal{M} \).

First, an analogue of the compact-open topology is introduced on the set \( \text{Aut}_{\bar{0}}(\mathcal{M}) \). The group \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) endowed with this topology is a topological group. There are two main steps in the proof that \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) carries the structure of a complex Lie group with Lie algebra isomorphic to the Lie algebra of even holomorphic vector fields on \( \mathcal{M} \):

The first is to prove that the topological group \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) carries the structure of a Lie group. The crucial step here is the following proposition on the topological properties of \( \text{Aut}_{\bar{0}}(\mathcal{M}) \).

**Proposition 1** (see Section 5.1). The topological group \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) does not contain small subgroups, i.e. there is a neighbourhood of the identity such that the trivial subgroup is the only subgroup contained in this neighbourhood.

A result on the existence of Lie group structures on locally compact topological groups without small subgroups (see [Yam53]) then implies that \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) carries the structure of real Lie group.

The second step is the proof of a regularity statement for actions of continuous one-parameter subgroups of \( \text{Aut}_{\bar{0}}(\mathcal{M}) \). This is needed to prove that the action of \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) on the supermanifold \( \mathcal{M} \) is holomorphic and to relate even holomorphic vector fields and one-parameter subgroups of \( \text{Aut}_{\bar{0}}(\mathcal{M}) \). We have the following:

**Proposition 2** (see Section 5.1). Let \( \varphi : \mathbb{R} \rightarrow \text{Aut}_{\bar{0}}(\mathcal{M}) \) be a continuous one-parameter subgroup of \( \text{Aut}_{\bar{0}}(\mathcal{M}) \). Then the natural action of \( \varphi \) on \( \mathcal{M} \) is analytic.

A consequence of the two preceding propositions is the desired result on the structure of \( \text{Aut}_{\bar{0}}(\mathcal{M}) \):

**Theorem 3** (see Section 5.1). The topological group \( \text{Aut}_{\bar{0}}(\mathcal{M}) \) carries the structure of a complex Lie group such that its Lie algebra is isomorphic to the Lie algebra
of even holomorphic vector field on $\mathcal{M}$ and the natural action of $\text{Aut}_0(\mathcal{M})$ on the supermanifold $\mathcal{M}$ is holomorphic.

We have a representation $\alpha$ of $\text{Aut}_0(\mathcal{M})$ on $\text{Vec}(\mathcal{M})$ induced by the action of $\text{Aut}_0(\mathcal{M})$ on $\mathcal{M}$. Using the equivalence of Lie supergroups and Harish-Chandra pairs\(^{(2)}\) we can now give the following definition.

**Definition (Automorphism Lie supergroup).** The automorphism group $\text{Aut}(\mathcal{M})$ of a compact complex supermanifold is the unique complex Lie supergroup associated with the Harish-Chandra pair $(\text{Aut}_0(\mathcal{M}), \text{Vec}(\mathcal{M}))$ with representation $\alpha$.

Remark that the simply-connected Lie supergroup $\mathcal{G}$ with Lie superalgebra $\text{Vec}(\mathcal{M})$ is, up to ineffectivity, really the connected component of the automorphism Lie supergroup of $\mathcal{M}$. This was a priori not obvious. Before knowing that $\text{Aut}_0(\mathcal{M})$ is a complex Lie group, which acts holomorphically on $\mathcal{M}$, we did not know that the connected component of $\text{Aut}_0(\mathcal{M})$ is generated by the flows of even holomorphic vector fields on $\mathcal{M}$.

The natural action of the automorphism Lie supergroup $\text{Aut}(\mathcal{M})$ on $\mathcal{M}$ is holomorphic, i.e. we have a morphism $\Psi : \text{Aut}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ of complex supermanifolds. The automorphism Lie supergroup $\text{Aut}(\mathcal{M})$ satisfies the following universal property (see Section 5.3):

**Theorem 4.** If $\mathcal{G}$ is a complex Lie supergroup with a holomorphic action $\Psi_{\mathcal{G}} : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$, then there is a unique morphism $\sigma : \mathcal{G} \to \text{Aut}(\mathcal{M})$ of Lie supergroups such that the diagram

$$
\begin{array}{ccc}
\mathcal{G} \times \mathcal{M} & \xrightarrow{\Psi_{\mathcal{G}}} & \mathcal{M} \\
\sigma \times \text{id}_\mathcal{M} \downarrow & & \downarrow \Psi \\
\text{Aut}(\mathcal{M}) \times \mathcal{M} & & 
\end{array}
$$

is commutative.

Using the “functor of points” approach to supermanifolds, an alternative definition of the automorphism group as a functor in analogy to [SW11] and [Ost15] is possible. We can define this functor from the category of superpoints, i.e. supermanifolds over a point, to the category of sets by the assignment

$$
\mathbb{C}^{0|k} \mapsto \left\{ \varphi : \mathbb{C}^{0|k} \times \mathcal{M} \to \mathbb{C}^{0|k} \times \mathcal{M} \mid \varphi \text{ is invertible, and } \text{pr}_{\mathbb{C}^{0|k}} \circ \varphi = \text{pr}_{\mathbb{C}^{0|k}} \right\},
$$

where $\text{pr}_{\mathbb{C}^{0|k}} : \mathbb{C}^{0|k} \times \mathcal{M} \to \mathbb{C}^{0|k}$ denotes the projection onto the first component. In Section 5.4 we prove that the two approaches to the automorphism group are equivalent and that the constructed automorphism group $\text{Aut}(\mathcal{M})$ represents the just defined functor.

In the classical case, automorphism groups of bounded domains in $\mathbb{C}^m$ are also known to carry the structure of a Lie group (see e.g. [Car79]). This result cannot be generalized to the category of supermanifolds as will be shown by an example in Section 5.5.

\(^{(2)}\)As in the case of vector fields on supermanifolds, it would be more precise to talk about super Harish-Chandra pairs.
Outline

First, we review some basic definitions related to supermanifolds and introduce some notation. This is followed by a chapter on Lie supergroups and their actions which mostly aims at collecting facts which are used later on. After recalling some basic definitions, we explain how (local) actions of Lie supergroups induce infinitesimal actions and we collect important facts about flows. Then we describe the equivalence of Lie supergroups and Harish-Chandra pairs. At the end of the chapter, the notion of an effective action of a Lie supergroup is introduced and an equivalent formulation in terms of the corresponding Harish-Chandra pair is found.

In Chapter 3, distributions on supermanifolds are studied. The local Frobenius’ theorem for distributions on supermanifolds is discussed. We also provide an example which shows that a global Frobenius’ theorem does not hold in the context of supermanifold since integral manifolds do not give enough information about a distribution. This is due to the fact that we do “point evaluation” transverse to integral manifolds when going from the distribution to the integral manifolds.

The relation between infinitesimal, local, and global actions on supermanifolds is analyzed in Chapter 4. In the last chapter, the automorphism group of a compact complex manifold is studied.

The main results of this thesis are published [Ber14] and [BK15].
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Chapter 1

Preliminaries

In this chapter, we recall important definitions and facts in the context of supermanifolds. Throughout, we work with the “Berezin-Leites-Kostant”-approach to supermanifolds (see e.g. [Ber87], [Le˘ı80], or [Kos77], and also [Var04]).

1.1 Supermanifolds

The notion of a supermanifold extends the notion of a usual, in the following often called classical, manifold. Heuristically, a supermanifold is a manifold where we also have anticommuting variables $\theta_1, \ldots, \theta_n$ with $\theta_i \theta_j = -\theta_j \theta_i$ in addition to the usual local coordinates $x_1, \ldots, x_m$. Let $\wedge \mathbb{R}^n$ denote the exterior algebra of the vector space $\mathbb{R}^n$, which is isomorphic to $\mathbb{R}[\theta_1, \ldots, \theta_n]$ with relations $\theta_i \theta_j = -\theta_j \theta_i$. The model space $\mathbb{R}^m|n$ can be defined as the pair $(\mathbb{R}^m, C^\infty_\mathbb{R}^m \otimes \wedge \mathbb{R}^n)$, i.e. the underlying space is $\mathbb{R}^m$ and “functions” of $\mathbb{R}^m|n$ on open subsets of $U \subseteq \mathbb{R}^m$ are elements of $C^\infty_\mathbb{R}^m(U) \otimes \wedge \mathbb{R}^n \cong C^\infty_\mathbb{R}^m(U) \otimes \mathbb{R}[\theta_1, \ldots, \theta_n]$. Thus, any element $f \in C^\infty_\mathbb{R}^m(U) \otimes \wedge \mathbb{R}^n$ can be written as

$$f = \sum_{\nu \in (\mathbb{Z}_2)^n} f_\nu \theta^\nu$$

for smooth functions $f_\nu$ on $U$, where we write $\theta^\nu$ for $\theta_1^{\nu_1} \ldots \theta_n^{\nu_n}$ if $\nu = (\nu_1, \ldots, \nu_2) \in (\mathbb{Z}_2)^n = \{0, 1\}^n$, and set $\theta^0_1 = 1, \theta^1_1 = \theta_j$. Similarly, we define $C^m|n$ by replacing $\wedge \mathbb{R}^n$ by $\wedge C^n$ and smooth by holomorphic functions.

Before giving the definition of a supermanifold, we make a few remarks about the algebraic structure of the structure sheaf $C^\infty_\mathbb{R}^m \otimes \wedge \mathbb{R}^n$ of $\mathbb{R}^m|n$, which also apply to the structure sheaf of $C^m|n$. We have a natural $\mathbb{Z}_2$-grading on $C^\infty_\mathbb{R}^m \otimes \wedge \mathbb{R}^n$ induced by the grading of $\wedge \mathbb{R}^n$. On the exterior algebra $\wedge \mathbb{R}^n$ we have the grading

$$\wedge \mathbb{R}^n = \left(\wedge \mathbb{R}^n\right)_0 \oplus \left(\wedge \mathbb{R}^n\right)_1 = \bigoplus_k \left(\wedge^{2k} \mathbb{R}^n\right) \oplus \bigoplus_k \left(\wedge^{2k+1} \mathbb{R}^n\right),$$

where the first summand is the even part, i.e. $0$-part, of $\wedge \mathbb{R}^n$, and the second is the odd part, i.e. $1$-part. Accordingly, a function $f = \sum f_\nu \theta^\nu \in C^\infty_\mathbb{R}^m(U) \otimes \wedge \mathbb{R}^n$ is even if $f_\nu = 0$ for all $\nu$ with $|\nu| = 1$ and odd if $f_\nu = 0$ for all $\nu$ with $|\nu| = 0$, where $|\nu| = |(\nu_1, \ldots, \nu_n)| = \nu_1 + \ldots + \nu_n \in \mathbb{Z}_2 = \{0, 1\}$. This gives a grading

$$C^\infty_\mathbb{R}^m \otimes \wedge \mathbb{R}^n = \left(C^\infty_\mathbb{R}^m \otimes \wedge \mathbb{R}^n\right)_0 \oplus \left(C^\infty_\mathbb{R}^m \otimes \wedge \mathbb{R}^n\right)_1$$

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on the structure sheaf and the multiplication of two elements respects this grading. Consequently, the structure sheaf $\mathcal{C}_R^\infty \otimes \bigwedge R^n$ is a sheaf of $\mathbb{Z}_2$-graded algebras. We call an element $f \in \mathcal{C}_{R}^\infty (U) \otimes \bigwedge R^n$ homogeneous of parity $|f| \in \mathbb{Z}_2$ if $f \in (\mathcal{C}_{R}^\infty (U) \otimes \bigwedge R^n)_{|f|}$.

A superdomain $U^{m|n}$ in $\mathbb{R}^{m|n}$ (or $\mathbb{C}^{m|n}$) consists of an open subset $U \subseteq \mathbb{R}^m$ (or $U \subseteq \mathbb{C}^m$) and the restriction of the structure sheaf $\mathcal{C}_R^\infty \otimes \bigwedge R^n$ (or $\mathcal{O}_{\mathbb{C}^m} \otimes \bigwedge \mathbb{C}^n$) to $U$.

A supermanifold can now be defined to be a space that is locally isomorphic to superdomains in $\mathbb{R}^{m|n}$ (or $\mathbb{C}^{m|n}$). More precisely:

**Definition 1.1.1.** A real supermanifold $\mathcal{M}$ is a locally ringed space $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$, consisting of a second-countable Hausdorff topological space $\mathcal{M}$ and a sheaf $\mathcal{O}_{\mathcal{M}}$ of $\mathbb{Z}_2$-graded algebras on $\mathcal{M}$, which is locally isomorphic to superdomains in $\mathbb{R}^{m|n}$.

Similarly, $\mathcal{M} = (\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ is a complex supermanifold if it is locally isomorphic to superdomains in $\mathbb{C}^{m|n}$.

The pair $(m|n)$ is defined to be the dimension of the supermanifold $\mathcal{M}$, and we also call $m$ the even and $n$ the odd dimension of $\mathcal{M}$.

The underlying space $\mathcal{M}$ of a real or complex supermanifold $\mathcal{M} = (\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ carries a unique structure of a real or complex manifold with sheaf of smooth or holomorphic functions $\mathcal{O}_{\mathcal{M}}$ such that the evaluation $ev = ev_{\mathcal{M}} : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$, locally given by $f = \sum f_\nu \theta^\nu \mapsto f_0$, is a morphism of sheaves. We also write $\tilde{f}$ for $ev(f)$. In the following, if we consider the underlying space $\mathcal{M}$ of a supermanifold $\mathcal{M}$ as a manifold, we always implicitly mean the just described structure.

Usually, we denote a supermanifold by a calligraphic letter, e.g. $\mathcal{M}$, and then the underlying manifold by the corresponding standard uppercase letters, e.g. $M$. The notation for the structure sheaf of a supermanifold $\mathcal{M}$ is $\mathcal{O}_{\mathcal{M}}$.

**Example 1.1.2.** We have the following examples of supermanifolds:

1. $\mathbb{R}^{m|n}$, $\mathbb{C}^{m|n}$, $U^{m|n}$ for $U \subseteq \mathbb{R}^m$ or $U \subseteq \mathbb{C}^m$ open.

2. $(\mathcal{M}, \Omega_{\mathcal{M}})$ for a manifold $\mathcal{M}$ with sheaf of differential forms $\Omega_{\mathcal{M}}$.

3. $(\mathcal{M}, \bigwedge E)$, where $E$ is the sheaf of sections of a vector bundle $E \to M$ on $\mathcal{M}$.

Supermanifolds which are isomorphic to a supermanifold associated with a vector bundle as in the third example are called split supermanifold. All real supermanifolds are split, given any real supermanifold $\mathcal{M} = (\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ there exists a smooth vector bundle $E \to M$ on $\mathcal{M}$ with sheaf of sections $E$ such that $\mathcal{M}$ is isomorphic to $(\mathcal{M}, \bigwedge E)$ (see [Bat79]). This result is however not true for complex supermanifolds, there are complex supermanifolds which are not split; see e.g. [Gre82].

### 1.2 Morphisms and vector fields on supermanifolds

In this section, the definitions of a morphism between supermanifolds and (super) vector fields on supermanifolds are recalled. Moreover, a useful fact about the relation of invertible morphisms of a supermanifold satisfying a nilpotency condition and nilpotent (super) vector fields on the supermanifold is stated.
Definition 1.2.1. A morphism between supermanifolds \( \mathcal{M} \) and \( \mathcal{N} \) is a morphism \( \varphi = (\tilde{\varphi}, \varphi^*): (\mathcal{M}, \mathcal{O}_\mathcal{M}) \to (\mathcal{N}, \mathcal{O}_\mathcal{N}) \) of ringed spaces, i.e. \( \tilde{\varphi} : \mathcal{M} \to \mathcal{N} \) is a continuous map, \( \varphi^* : \mathcal{O}_\mathcal{N} \to \tilde{\varphi}_* (\mathcal{O}_\mathcal{N}) \) a morphism of sheaves of \( \mathbb{Z}_2 \)-graded algebras, and \( \tilde{\varphi} \) and \( \varphi^* \) are compatible in the sense that \( \text{ev}_\mathcal{M} (\varphi^*(f)) = \text{ev}_\mathcal{N} (f) \circ \tilde{\varphi} \) for any \( f \in \mathcal{O}_\mathcal{N} (U) \), \( U \subseteq N \) open.

For any morphism \( \varphi : \mathcal{M} \to \mathcal{N} \) of supermanifolds we denote the underlying map \( M \to N \) by \( \tilde{\varphi} \), and the morphism \( \mathcal{O}_\mathcal{N} \to \tilde{\varphi}_* (\mathcal{O}_\mathcal{N}) \) on the level of sheaves denotes by \( \varphi^* \) and referred to as the pullback of \( \varphi \).

The underlying map \( \tilde{\varphi} : M \to N \) of a morphism \( \mathcal{M} \to \mathcal{N} \) is smooth if \( \mathcal{M} \) and \( \mathcal{N} \) are real supermanifolds and holomorphic if \( \mathcal{M} \) and \( \mathcal{N} \) are complex.

Definition 1.2.2. An immersed subsupermanifold of a supermanifold \( \mathcal{M} \) is a supermanifold \( \mathcal{N} \) together with an immersion \( \varphi : \mathcal{N} \to \mathcal{M} \) with injective underlying map \( \tilde{\varphi} \) (cf. [Lei80]). We say that \( \mathcal{N} \) is an open subsupermanifold of \( \mathcal{M} \) if there is an open subset \( U \subseteq M \) such that \( \mathcal{N} = (U, \mathcal{O}_\mathcal{M} (U)) \). In this case we also write \( \mathcal{N} \subseteq \mathcal{M} \).

Superdomains are for instance examples of open subsupermanifolds of \( \mathbb{R}^{m|n} \) or \( \mathbb{C}^{m|n} \).

Definition 1.2.3. A vector field (or derivation) \( X \) of parity \( |X| \) on a supermanifold \( \mathcal{M} \) is a derivation of graded sheaves \( X : \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M} \), i.e. \( X \) is linear and for any \( f, g \in \mathcal{O}_\mathcal{M} (U), U \subseteq M \) open, \( f \) homogeneous of parity \( |f| \), we have

\[
X(fg) = X(f)g + (-1)^{|X||f|}fX(g).
\]

The vector field \( X \) of parity \( |X| \) is called even if \( |X| = \overline{0} \) and odd if \( |X| = \overline{1} \).

A vector field (or derivation) on \( \mathcal{M} \) is then a morphism \( \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M} \) which is the sum of an even vector field \( X_0 \) and an odd vector field \( X_1 \), \( X = X_0 + X_1 \).

We denote by \( \text{Der}(\mathcal{O}_\mathcal{M}) = \mathcal{T}_\mathcal{M} \) the sheaf of derivations or tangent sheaf of \( \mathcal{M} \). Moreover, we set \( \text{Vec}(\mathcal{M}) = T_\mathcal{M}(\mathcal{M}) \) for the space of vector fields. This space \( \text{Vec}(\mathcal{M}) \) is a Lie superalgebra, possibly of infinite dimension, with commutator \( [X,Y] = XY - (-1)^{|X||Y|}YX \) for vector fields \( X \) and \( Y \) of parity \( |X| \) and \( |Y| \).

It would be more precise to talk about “super vector fields” on supermanifolds and their “super commutators”. For the convenience of notation, we will however stick to the words “vector field” and “commutator”, and always implicitly mean the corresponding objects on supermanifolds.

Example 1.2.4. Let \( x_1, \ldots, x_m, \theta_1, \ldots, \theta_n \) denote coordinates on \( \mathbb{R}^{m|n} \). Then \( \frac{\partial}{\partial x_j} \)

\[
\frac{\partial}{\partial x_j} (x_j) = 1, \quad \frac{\partial}{\partial x_j} (x_k) = 0 \text{ for } j \neq k, \quad \text{and } \frac{\partial}{\partial x_j} (\theta_k) = 0
\]

is an even vector field on \( \mathbb{R}^{m|n} \). Similarly, \( \frac{\partial}{\partial \theta_j} \)

\[
\frac{\partial}{\partial \theta_j} (x_k) = 0, \quad \frac{\partial}{\partial \theta_j} (\theta_j) = 1, \quad \text{and } \frac{\partial}{\partial \theta_j} (\theta_k) = 0 \text{ for } j \neq k
\]

\footnote{The sheaf \( \tilde{\varphi}_* (\mathcal{O}_\mathcal{M}) \) denotes the direct image sheaf, i.e. \( \tilde{\varphi}_* (\mathcal{O}_\mathcal{M}) (U) = \mathcal{O}_\mathcal{M} (\tilde{\varphi}^{-1} (U)) \) for any \( U \subseteq N \) open.}
is an odd vector field.

Moreover, any vector field $X$ on $\mathbb{R}^{m|n}$ has the form

$$X = \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j} + \sum_{k=1}^{n} b_k \frac{\partial}{\partial \theta_k}$$

for suitable $a_j, b_k \in C^\infty_{\mathbb{R}^m}(\mathbb{R}^m) \otimes \wedge \mathbb{R}^n$.

We have the following relation between nilpotent even vector fields on a supermanifold and morphisms of this supermanifold satisfying a certain nilpotency condition:

**Lemma 1.2.5.** Let $\varphi : \mathcal{M} \to \mathcal{M}$ be a morphism of supermanifolds with underlying map $\tilde{\varphi} = \text{id}_\mathcal{M}$ and such that $\varphi^* - \text{id}_\mathcal{M}^* : \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}$ is nilpotent, i.e. there is $N \in \mathbb{N}$ with $(\varphi^* - \text{id}_\mathcal{M}^*)^N = 0$. Then

$$X = \log(\varphi^*) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} (\varphi^* - \text{id}_\mathcal{M}^*)^n$$

is a nilpotent even vector field on $\mathcal{M}$ and we have

$$\varphi^* = \exp(X) = \sum_{n \geq 0} \frac{1}{n!} X^n.$$ 

Moreover, for any nilpotent even vector field $X$ on $\mathcal{M}$, the (finite) sum $\exp(X)$ defines a map $\mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}$ which is the pullback of an invertible morphism $\mathcal{M} \to \mathcal{M}$ with the identity as underlying map, and the pullback of the inverse is $\exp(-X)$.

This lemma follows directly from the next technical result on the relation of algebra homomorphisms and derivations (cf. [vdE00], Proposition 2.1.3 and Lemma 2.1.4):

**Lemma 1.2.6.** Let $A$ be an algebra (containing $\mathbb{Q}$), and $\chi : A \to A$ an algebra homomorphism such that $(\chi - \text{id}_A)$ is nilpotent. Then the map $D : A \to A$ defined by

$$D(a) = \log(\chi)(a) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} (\chi - \text{id}_A)^n(a) \quad \text{for} \quad a \in A$$

is a derivation and $\chi = \exp(D) = \sum_{n \geq 0} \frac{1}{n!} D^n$.

Moreover, $\exp(D) : A \to A$ is an automorphism with inverse $\exp(-D)$ for any nilpotent derivation $D$. 

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Chapter 2

Lie supergroups and their actions

Lie supergroups and their actions on supermanifolds are important for the study of symmetries of supermanifolds. They generalize the notion of Lie groups and their actions on classical manifolds. In this chapter, Lie supergroups, their actions and related concepts are discussed.

First, we recall the basic definitions of a Lie supergroup and actions on supermanifolds. Then we give the definition of local and infinitesimal actions on supermanifolds and explain how local actions induce infinitesimal actions. In the next section, we describe how to "go from infinitesimal to local" in the case of one vector field on a supermanifold. For this we collect important facts about flows of vector fields on supermanifolds (cf. [MSV93] and [GW13]), and we study a few properties of the flows.

A Lie supergroup is uniquely determined by its underlying classical Lie group, its Lie superalgebra, and a representation of the classical Lie group on the Lie superalgebra, which is formalized by the concept of a Harish-Chandra pair. In Section 2.4, we recall the related definitions and state the equivalence between Lie supergroups and Harish-Chandra pairs (cf. [Kos83], and [Vis11] for the case of complex Lie supergroups).

In the last section of this chapter, the notion of an effective action of a Lie supergroup on a supermanifold is introduced, generalizing the classical definition of an effective action. We also prove an equivalent characterization of the effectiveness of an action in terms of the corresponding action of the associated Harish-Chandra pair.

2.1 Basic definitions

In this section, the basic definitions in the context of Lie supergroups are recalled.

Definition 2.1.1. A Lie supergroup is a supermanifold $\mathcal{G}$ together with morphism $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $\iota : \mathcal{G} \to \mathcal{G}$, and $e_{\mathcal{G}} : \{e\} \hookrightarrow \mathcal{G}$ for multiplication, inversion and the neutral element, so that the usual group axioms are satisfied.

The underlying manifold $G$ of a Lie supergroup $\mathcal{G} = (G, O_G)$ is a classical Lie group. A vector field $X$ on a Lie supergroup $\mathcal{G}$ is called right-invariant if

$$\mu^* \circ X = (X \otimes \text{id}_{\mathcal{O}_G}) \circ \mu^{*(1)}.$$
We define the Lie superalgebra \( g \) of \( G \) to be the set of right-invariant vector fields on \( G \). Its even part \( g_0 \) can be identified with the Lie algebra (of right-invariant vector fields) of \( G \).

Similarly, one can define a complex Lie supergroup as complex supermanifold with holomorphic morphisms for multiplication, inversion, and the neutral element. The underlying manifold then carries the structure of a complex Lie group and the set of right-invariant holomorphic vector fields forms a complex Lie superalgebra.

**Example 2.1.2.** Let \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). Then the supermanifold \( \mathbb{K}^{1|1} \) carries three non-isomorphic structures of a Lie supergroup (cf. [GW13], Lemma 3.1). Let \( t \) and \( \tau \) denote coordinates on \( \mathbb{K}^{1|1} \). Then the multiplication \( \mu = \mu_{a,b} : \mathbb{K}^{1|1} \times \mathbb{K}^{1|1} \to \mathbb{K}^{1|1} \) given by \( \mu^*(t) = t_1 + t_2 + a\tau_1\tau_2, \mu^*(\tau) = \tau_1 + \epsilon^{1b}\tau_2 \) for any \( a, b \in \mathbb{K} \) with \( ab = 0 \) is the multiplication of a Lie supergroup structure on \( \mathbb{K}^{1|1} \), and any Lie supergroup structure on the supermanifold \( \mathbb{K}^{1|1} \) is isomorphic to one of the three Lie supergroups \((\mathbb{K}^{1|1}, \mu_{0,0}), (\mathbb{K}^{1|1}, \mu_{1,0}) \) and \((\mathbb{K}^{1|1}, \mu_{0,1}) \).

The Lie superalgebra of right-invariant vector fields on \((\mathbb{K}^{1|1}, \mu_{a,b})\) is isomorphic to \( \mathbb{K}X_0 \oplus \mathbb{K}X_1 \) with relations \([X_0, X_0] = 0, [X_0, X_1] = -bX_1 \) and \([X_1, X_1] = 2aX_0 \).

As in the classical case, an action of a Lie supergroup on a supermanifold can be defined.

**Definition 2.1.3.** An action of a Lie supergroup \( G \) on a supermanifold \( M \) is a morphism \( \Psi : G \times M \to M \) satisfying the usual action properties, i.e.

(i) \( \Psi \circ \iota_e = \text{id}_M \), where \( \iota_e : M \to \{e\} \times M \subset G \times M \) is the canonical inclusion, and

(ii) \( \Psi \circ (\mu \times \text{id}_M) = \Psi \circ (\text{id}_G \times \Psi) \).

In the same manner, an action of a complex Lie supergroup on a complex supermanifold is defined.

### 2.2 Local and infinitesimal actions of Lie supergroups

A local action of a Lie supergroup on a supermanifold can be defined in complete analogy to the classical case. We give the precise definition here in order to recall the requirements on the domain of definition of a local action.

**Definition 2.2.1.** A local action of a Lie supergroup \( G = (G, O_G) \) on a supermanifold \( M = (M, O_M) \) is a morphism \( \varphi : W \to M, W = (W, O_G \times M|W) \), where \( W \) is an open neighbourhood of \( \{e\} \times M \) in \( G \times M \) such that \( W_p = \{g \in G | (g, p) \in W\} \) is connected for each \( p \in M \), satisfying the action properties:

(i) \( \varphi \circ \iota_e = \text{id}_M \)

(ii) The equation

\[ \varphi \circ (\mu \times \text{id}_M) = \varphi \circ (\text{id}_G \times \varphi) \]

holds on the open subsupermanifold of \( G \times G \times M \) where both sides are defined.

---

(1) The vector field \( (X \otimes \text{id}_G) \) is the extension of the vector field \( X \) on \( G \) to a vector field on the product \( G \times G \) such that \( X = (X \otimes \text{id}_G) \circ \pi_1 \) and \( 0 = (X \otimes \text{id}_G) \circ \pi_2 \) if \( \pi_i : G \times G \to G, i = 1, 2 \), is the projection onto the \( i \)-th component.
Let $G$ be a classical Lie group with Lie algebra $\mathfrak{g}$ of right-invariant vector fields. Any (local) $G$-action $\psi : G \times M \to M$ on a manifold $M$ induces an "infinitesimal action" by mapping a right-invariant vector field $X \in \mathfrak{g}$ on $G$ to the induced vector field $\frac{\partial}{\partial t}|_0 \psi(\exp(tX), -)$ on $M$ and the map

$$\mathfrak{g} \to \text{Vec}(M), \quad X \mapsto \frac{\partial}{\partial t}|_0 \psi(\exp(tX), -)$$

is a homomorphism of Lie algebras.

The same statement can be extended to the case of action of Lie supergroups on supermanifolds. First, we define the notion of an infinitesimal action.

**Definition 2.2.2.** Let $\mathcal{G}$ be a Lie supergroup with Lie superalgebra $\mathfrak{g}$, and $\mathcal{M}$ a supermanifold. An infinitesimal action of $\mathcal{G}$ on $\mathcal{M}$ is a homomorphism $\lambda : \mathfrak{g} \to \text{Vec}(\mathcal{M})$ of Lie superalgebras. If $\mathcal{G}$ and $\mathcal{M}$ are complex, we require $\lambda$ to be a homomorphism of complex Lie superalgebras.

As in the classical case, (local) actions of Lie supergroups induce infinitesimal actions:

**Proposition 2.2.3.** Let $\Psi$ be a local action of a Lie supergroup $\mathcal{G}$, with Lie superalgebra $\mathfrak{g}$, on a supermanifold $\mathcal{M}$. Then $(X(e) \otimes \text{id}_{\mathcal{M}}^*) \circ \Psi^*$ is a vector field on $\mathcal{M}$ for any $X \in \mathfrak{g}$, where $X(e)$ is the evaluation of the vector field in $e$. The map

$$\lambda_\Psi : \mathfrak{g} \to \text{Vec}(\mathcal{M}), \quad X \mapsto (X(e) \otimes \text{id}_{\mathcal{M}}^*) \circ \Psi^*$$

is a homomorphism of Lie superalgebras.

**Proof.** Let $X, Y \in \mathfrak{g}$ be homogeneous. The vector field $X \otimes \text{id}_{\mathcal{M}}^*$ has the parity $|X|$ of $X$. Let $f, g \in \mathcal{O}(\mathcal{M})$ and let $f$ be homogeneous with parity $|f|$. Then, using $\varphi \circ \iota_e = \text{id}_{\mathcal{M}}$, we get

$$((X(e) \otimes \text{id}_{\mathcal{M}}^*) \circ \Psi^*)(fg) = \iota_e^*(X \otimes \text{id}_{\mathcal{M}}^*)(\Psi^*(f)\Psi^*(g))$$

$$= \left( \left( (X(e) \otimes \text{id}_{\mathcal{M}}^*) \circ \Psi^* \right)(f) \right) g + (-1)^{|X||f|} f \left( \left( (X(e) \otimes \text{id}_{\mathcal{M}}^*) \circ \Psi^* \right)(g) \right).$$

Since $\Psi$ is an action, we have $\Psi \circ (\mu \times \text{id}_{\mathcal{M}}) = \Psi \circ (\text{id}_{\mathcal{G}} \times \Psi)$. A calculation using this identity results in $\lambda_\Psi(X)\lambda_\Psi(Y) = (XY(e) \otimes \text{id}_{\mathcal{M}}^*) \circ \Psi^*$. Thus

$$[\lambda_\Psi(X), \lambda_\Psi(Y)] = \left( \left( XY(e) - (-1)^{|X||Y|} YX(e) \right) \otimes \text{id}_{\mathcal{M}}^* \right) \circ \Psi^* = \lambda_\Psi([X, Y]).$$

**Definition 2.2.4.** Let $\mathcal{M}$ be a supermanifold and $\mathcal{G}$ a Lie supergroup with Lie superalgebra $\mathfrak{g}$. The induced infinitesimal action of a $\mathcal{G}$-action $\Psi$ on $\mathcal{M}$ is the homomorphism

$$\lambda_\Psi : \mathfrak{g} \to \text{Vec}(\mathcal{M}), \quad \lambda_\Psi(X) = (X(e) \otimes \text{id}_{\mathcal{M}}^*) \circ \Psi^*.$$

### 2.3 Flows of vector fields

After explaining how local actions induce infinitesimal actions on supermanifolds, we now describe how an infinitesimal symmetry on a supermanifold, given by one vector field, can be integrated to a local symmetry, given by the flow of the vector field.
In analogy to the classical case, we can define the flow of a vector field on a supermanifold. In the case of an odd vector field or a non-homogeneous vector field, i.e. a vector field \( X = X_0 + X_1 \), where \( X_0 \neq 0 \) is an even vector field and \( X_1 \neq 0 \) is an odd vector field, some subtleties appear in the definition of a flow or the existence of flows (see Remark 2.3.9 or e.g. [GW13]). For our purposes, the definition of the flow of an even vector field suffices.

**Definition 2.3.1.** Let \( X \) be an even vector field on a supermanifold \( \mathcal{M} \). A flow of \( X \) (with respect to the initial condition \( \text{id}_\mathcal{M} : \mathcal{M} \to \mathcal{M} \) and \( t_0 = 0 \in \mathbb{R} \)) is a morphism \( \varphi = \varphi^X : \mathcal{W} \to \mathcal{M} \), where \( \mathcal{W} = (W, \mathcal{O}_{\mathcal{W}}|W) \) and \( W \subseteq \mathbb{R} \times \mathcal{M} \) is an open neighbourhood of \( \{0\} \times \mathcal{M} \) such that \( W_p = \{t \in \mathbb{R} | (t, p) \in W \} \subseteq \mathbb{R} \) is connected for each \( p \in \mathcal{M} \), with

(i) \( \varphi \circ \iota_0 = \text{id}_\mathcal{M} \), where \( \iota_0 : \mathcal{M} \to \{0\} \times \mathcal{M} \subseteq \mathcal{W} \) is the canonical inclusion, and

(ii) \( \frac{\partial}{\partial t} \circ \varphi^* = \varphi^* \circ X \).

The flow \( \varphi : \mathcal{W} \to \mathcal{M} \) is called maximal if for any flow \( \varphi' : \mathcal{W}' = (W', \mathcal{O}_{\mathcal{W}'}|W') \to \mathcal{M} \) of \( X \) we have \( W' \subseteq W \) and \( \varphi' = \varphi|_{W'} \).

**Remark 2.3.2.** Any vector field \( X \) on a supermanifold \( \mathcal{M} \) induces a vector field \( \tilde{X} \) on the underlying manifold \( M \). The reduced vector field \( \tilde{X} \) can be defined by \( \tilde{X}(f) = \text{ev}(X(F)) \), if \( F \) is a function on \( \mathcal{M} \) with \( \text{ev}(F) = F = f \), where \( \text{ev} : \mathcal{O}_\mathcal{M} \to C^\infty_{\tilde{M}} \) denotes the evaluation map.

As in the classical case, there exists a unique maximal flow of an even vector field on a real supermanifold.

**Theorem 2.3.3** (see [MSV93], Theorem 3.5/3.6, or [GW13], Theorem 2.3 and Theorem 3.4). Let \( X \) be an even vector field on a real supermanifold \( \mathcal{M} \). Then there exists a unique maximal flow \( \varphi \) of \( X \). The reduced map \( \tilde{\varphi} : \mathcal{W} \to \mathcal{M} \) then is the unique maximal flow of the reduced vector field \( \tilde{X} \) on \( M \). Moreover, the flow \( \varphi \) defines a local \( \mathbb{R} \)-action on \( \mathcal{M} \).

**Remark 2.3.4.** For an even holomorphic vector field \( X \) on a complex supermanifold \( \mathcal{M} \), there also exists a flow map \( \varphi : \mathcal{W} \subseteq \mathbb{C} \times \mathcal{M} \to \mathcal{M} \) (see [GW13], Theorem 5.4). But in contrast to the real case there is in general no unique maximal domain of definition, as already in the classical case, since two flow maps \( \varphi_1 : \mathcal{W}_1 = (W_1, \mathcal{O}_{\mathcal{W}_1}|W_1) \to \mathcal{M} \) and \( \varphi_2 : \mathcal{W}_2 = (W_2, \mathcal{O}_{\mathcal{W}_2}|W_2) \to \mathcal{M} \) of the same holomorphic vector field \( X \) do not necessarily agree on their common domain of definition since for \( p \in \mathcal{M} \) the set \( (W_1 \cap W_2)_p = \{z \in \mathbb{C} | (z, p) \in (W_1 \cap W_2)\} \) might not be connected. Nevertheless, the flow maps \( \varphi_1 \) and \( \varphi_2 \) coincide on the open subsupermanifold of \( \mathbb{C} \times \mathcal{M} \) with underlying set

\[
\bigcup_{p \in \mathcal{M}} ((W_1 \cap W_2)_p)^0 \times \{p\}
\]

where \((W_1 \cap W_2)_p)^0\) is the connected component of 0 in \((W_1 \cap W_2)_p\).

Moreover, it is not always possible to define the flow map of \( X \) on the open supermanifold whose underlying set is the domain of definition for a given flow map of the reduced vector field.
As in the classical case, a flow map $\varphi : W \to M$ of a holomorphic vector field $X$ on $M$ may not satisfy the equation $\varphi \circ (\text{id}_C \times \varphi) = \varphi \circ (\mu_C \times \text{id}_M)$ on the open subsupermanifold of $\mathbb{C} \times \mathbb{C} \times M$ on which both sides of the equation are defined, where $\mu_C$ denotes the multiplication on $C$. Therefore, in the complex case not every flow is a local $C$-action. A flow does however yield a local $C$-action after shrinking its domain of definition.

**Lemma 2.3.5.** Let $\varphi : W = (W, O_{C \times M}|W) \to M$ be the flow of a holomorphic vector field $X$ on a supermanifold $M$. Then the maps $\psi_1 = \varphi \circ (\text{id}_C \times \varphi) : \mathcal{Y}_1 \to M$ and $\psi_2 = \varphi \circ (\mu_C \times \text{id}_M) : \mathcal{Y}_2 \to M$, where $\mathcal{Y}_i = (Y_i, O_{C \times C \times M}|Y_i)$, $i = 1, 2$, are open subsupermanifolds of $\mathbb{C} \times \mathbb{C} \times M$, both satisfy

(a) $\psi_1 \circ i_0^{C \times M} = \varphi$, and

(b) $\frac{\partial}{\partial t} \circ \psi_1^* = \psi_1^* \circ X$,

for $i = 1, 2$, where $i_0^{C \times M}$ is the inclusion

$$i_0^{C \times M} : \mathbb{C} \times M \hookrightarrow \{0\} \times \mathbb{C} \times M \subset \mathbb{C} \times \mathbb{C} \times M$$

and $t$ denotes the coordinate on the first component of the product $\mathbb{C} \times \mathbb{C} \times M$.

**Proof.** Let $i_0^{C} : N \hookrightarrow \{0\} \times N \subset \mathbb{C} \times N$ be the canonical inclusion for $N = \mathbb{C}$, $M$, or $\mathbb{C} \times M$. We have

$$\psi_1 \circ i_0^{C \times M} = (\varphi \circ (\text{id}_C \times \varphi)) \circ i_0^{C \times M} = \varphi \circ \psi_1 \circ \varphi = \varphi \circ \text{id}_M \circ \varphi = \varphi$$

and

$$\psi_2 \circ i_0^{C \times M} = (\varphi \circ (\mu_C \times \text{id}_M)) \circ i_0^{C \times M} = \varphi \circ (\mu_C \times \text{id}_M) \circ (i_0^{C} \times \text{id}_M)$$

$$= \varphi \circ ((\mu_C \circ i_0^{C}) \times \text{id}_M) = \varphi \circ (\text{id}_C \times \text{id}_M) = \varphi.$$ 

Using $\frac{\partial}{\partial t} \circ \varphi^* = \varphi^* \circ X$, we calculate

$$\frac{\partial}{\partial t} \circ \psi_1^* = \frac{\partial}{\partial t} \circ (\text{id}_C \times \varphi^*) \circ \varphi^* = (\text{id}_C^* \otimes \varphi^*) \circ (\frac{\partial}{\partial t} \circ \varphi^*)$$

$$= (\text{id}_C^* \otimes \varphi^*) \circ (\varphi^* \circ X) = \psi_1^* \circ X$$

and

$$\frac{\partial}{\partial t} \circ \psi_2^* = \frac{\partial}{\partial t} \circ (\mu_C^* \otimes \text{id}_M^*) \circ \varphi^* = \left(\left(\frac{\partial}{\partial t} \circ \mu_C^*\right) \otimes \text{id}_M^*\right) \circ \varphi^*$$

$$= \left(\mu_C^* \circ \frac{\partial}{\partial t}\right) \otimes \text{id}_M^* \circ \varphi^* = (\mu_C^* \otimes \text{id}_M^*) \circ \frac{\partial}{\partial t} \circ \varphi^*$$

$$= (\mu_C \otimes \text{id}_M^*) \circ (\varphi^* \circ X) = \psi_2^* \circ X. \qed$$

**Corollary 2.3.6.** Let $\varphi : W = (W, O_{C \times M}|W) \to M$ be the flow of a holomorphic vector field $X$ on a supermanifold $M$ as before. Then $\psi_1 = \varphi \circ (\text{id}_C \times \varphi) : \mathcal{Y}_1 \to M$ and $\psi_2 = \varphi \circ (\mu_C \times \text{id}_M) : \mathcal{Y}_2 \to M$ are both flow maps of $X$, with respect to the initial condition $\varphi$ and $x_0 = 0$, on an open neighbourhood of $(0, 0, p)$ for any $p \in M$.

Therefore, $\psi_1$ and $\psi_2$ agree on their common domain of definition if $(Y_1 \cap Y_2) \cap (\mathbb{C} \times \mathbb{C} \times M_\alpha)$ is connected for each connected component $M_\alpha$ of $M$. In particular, the underlying set $W$ of $W \subseteq \mathbb{C} \times M$ can be shrunk such that $\varphi : W \to M$ is a local $\mathbb{C}$-action.

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Proof. We have $(0,0,p) ∈ Y₁$ and $(0,0,p) ∈ Y₂$ for any $p ∈ M$. The preceding lemma implies now that $ψ₁$ and $ψ₂$ are flow maps of $X$ with respect to the initial condition $φ$ and $z₀ = 0$ on some open neighbourhood of $(0,0,p)$.

Hence, the local uniqueness of such flow maps (cf. [GW13], Theorem 5.4) yields $ψ₁ = ψ₂$ near $(0,0,p)$. Thus, if $(Y₁ ∩ Y₂)(C × C × Mₐ)$ is connected for each connected component $Mₐ$ of $φ$, $ψ₁$ and $ψ₂$ agree on there common domain of definition $(Y₁ ∩ Y₂, O_{C × C × Mₐ})$ by the identity principle for holomorphic maps.

The set $W ⊆ C × M$ can be shrunk such that the intersection $(Y₁ ∩ Y₂)(C × C × Mₐ)$ of underlying set $Y₁ ∩ Y₂$ of the common domain of definition $ψ₁ = φ ∘ (idₐ × φ)$ and $ψ₂ = φ ∘ (μₐ × idₐ)$ and an arbitrary connected component $Mₐ$ of $φ$ is connected. For $(z,p) ∈ W$, we have

$$(Y₁)(z,p) = \{ t ∈ C | (t,z,p) ∈ Y₁ \} = \{ z ∈ C | (t,φ(z,p)) ∈ W \} = W_φ(z,p) \text{ and}$$

$$(Y₂)(z,p) = \{ t ∈ C | (t + z,p) ∈ W \} = \{ t - z ∈ C | t ∈ W_p \} = W_p - z,$$

and $(Y₁)(z,p)$ and $(Y₂)(z,p)$ are both non-empty open connected neighbourhoods of $0 ∈ C$. Therefore, $(Y₁ ∩ Y₂)(C × C × Mₐ)$ is connected if $(Y₁ ∩ Y₂)(z,p) = (Y₁)(z,p) ∩ (Y₂)(z,p)$ is connected for each $(z,p) ∈ W$. For example, this can be accomplished by shrinking $W$ such that $W_p = \{ t ∈ C | (t,p) \} ⊆ C$ is convex for each $p ∈ M$. □

Lemma 2.3.7. Let $X$ and $Y$ be vector fields on $M$, assume that $Y$ is even and let $φ^Y_t$ be the flow of $Y$. If $ψ_t : M → \{ t \} × M ⊂ R × M$ is the canonical inclusion, denote by $φ^X_t$ the composition $φ^X_t ∘ ψ_t$. Then

$$[X,Y] = \frac{∂}{∂t} (φ^Y_t)_* X$$

for $(φ^Y_t)_* X = (φ^Y_t)* ∘ X ∘ (φ^Y_t)*$, where $\frac{∂}{∂t}$ is again considered as a vector field on the product $R × M$ and $\frac{∂}{∂t}|_s = τ^*_s ∘ \frac{∂}{∂t}$ for $s ∈ R$.

Proof. Since vector fields are derivations of sheaves, it is enough to show the statement locally. For $f ∈ O_{M}(V), V ⊆ M$ open, the pullback $(φ^Y_t)^*(f)$ is a function on some subset of $R × V$. In local coordinates $(t,x₁,...,x_k,ξ₁,...,ξ_l)$ on $R × M$, we have

$$(φ^Y_t)^*(f) = \sum_{ν ∈ (Z₂)^l} g_ν(t,x)ξ^ν = \sum_{ν ∈ (Z₂)^l} g_ν(t,x₁,...,x_k)ξ^ν₁,...ξ^ν_l$$

for some smooth functions $g_ν$ on a subset of $R × V$. For each $g_ν$, Taylor expansion yields $g_ν(t,x) = g_ν(0,x) + t\bar{g}_ν(t,x)$ for $\bar{g}_ν(t,x) = \int_0^1 (\frac{∂}{∂t}|_{r=st} g_ν(r,x) ds$. If $g = \sum_ν g_ν(0,x)ξ^ν$ and $\bar{g}_t = \sum_ν \bar{g}_ν(t,x)ξ^ν$, then $(φ^Y_t)^*(f) = g + t\bar{g}_t$. Since $(φ^Y_0)^* = id^*_M$, we have $g = (φ^Y_0)^*(f) = f$ and consequently

$$(φ^Y_t)^*(f) = f + t\bar{g}_t.$$
Under the assumptions of the above lemma, we have:

\[
\left( \frac{\partial}{\partial t} \right)_{t=0} (\varphi_t^Y)_{*} X = (\varphi_t^Y)_{*}X + \frac{\partial}{\partial t} (\varphi_t^Y)_{*}((\varphi_t^Y)^*(X(f + t\hat{g}))) = \frac{\partial}{\partial t} (\varphi_t^Y)_{*}((\varphi_t^Y)^*(X(f))) = \frac{\partial}{\partial t} (\varphi_t^Y)_{*}((\varphi_t^Y)^*(X(t\hat{g}))) = -YX(f) + X(\hat{g})_t = [X,Y](f). \]

\[ \square \]

**Corollary 2.3.8.** Under the assumptions of the above lemma, we have:

(i) If \([X,Y] = 0\), then \((\varphi_t^Y)_{*} X = (\varphi_t^Y)_{*}X\).

(ii) If \([X,Y] = 0\) and \(X\) is also even, then the flows of \(X\) and \(Y\) locally commute, i.e. \(\varphi_t^X \circ \varphi_s^Y = \varphi_s^Y \circ \varphi_t^X\) for small \(s, t\).

Proof. The statements of the corollary can be concluded from the above lemma in the same manner as in the classical case.

Remark first that for any diffeomorphism \(\psi : \mathcal{M} \to \mathcal{M}\) the map

\[ \psi_* (\varphi_t^Y) = \psi \circ \varphi_t^Y \circ \psi^{-1} \]

is the flow of the even vector field \(\psi_* Y\) because

\[
\frac{\partial}{\partial t} (\psi \circ \varphi_t^Y \circ \psi^{-1})_{*}((\varphi_t^Y)^*(\psi_{*}(f))) = (\psi^{-1})_{*} \frac{\partial}{\partial t} (\varphi_t^Y)_{*}((\varphi_t^Y)^*(\psi_{*}(f)))
\]

\[
= (\psi^{-1})_{*}((\varphi_t^Y)_{*}((Y(\psi_{*}(f)))) = (\psi \circ \varphi_t^Y \circ \psi^{-1})_{*}((\psi^{-1})_{*} \circ Y \circ \psi_{*}(f))
\]

for all \(f \in \mathcal{O}_M(M)\) and \(\psi_* (\varphi_t^Y)_{*} X = \psi \circ \varphi_t^Y \circ \psi^{-1} = \psi \circ \text{id}_M \circ \psi^{-1} = \text{id}_M\).

Applying this to \(\psi = \varphi_t^Y\), we get \((\varphi_s^Y)_{*} X = X\) since the flows of the two vector fields coincide because \((\varphi_s^Y)_{*}((\varphi_t^Y)^* Y) = \varphi_t^Y \circ \psi_t^Y \circ \varphi_s^Y_{*} Y = \varphi_{s+t}^Y_{*} Y = \varphi_t^Y\) for all \(t\). Therefore, using preceding lemma we get

\[
\frac{\partial}{\partial t} \mid_{s} (\varphi_t^Y)_{*} X = \frac{\partial}{\partial t} \mid_{0} (\varphi_{t+s}^Y)_{*} X = \frac{\partial}{\partial t} \mid_{0} (\varphi_t^Y)_{*}((\varphi_t^Y)^* X) = [(\varphi_t^Y)_{*} X,Y]
\]

which is part (i).

Assume now \([X,Y] = 0\). Part (i) then yields

\[ \frac{\partial}{\partial t} \mid_{s} (\varphi_t^Y)_{*} X = (\varphi_t^Y)_{*}X = 0 \]

for small \(s\), and thus \((\varphi_t^Y)_{*} X = (\varphi_t^Y)_{*} X = \text{id}_M \circ X = X\) for arbitrary \(t\).

To prove (iii), let \(X\) be even and denote its flow by \(\varphi_t^X\). The flow of \(X = (\varphi_t^Y)_{*} X\) is then also given by \((\varphi_s^Y)_{*} (\varphi_t^X) = \varphi_s^Y \circ \varphi_t^X \circ \varphi_s^{-X}\). Consequently,

\[ \varphi_t^X = \varphi_s^Y \circ \varphi_t^X \circ \varphi_s^{-X} \]

and the flows of \(X\) and \(Y\) commute. \(\square\)
**Remark 2.3.9.** It is also possible to define the flow of a not necessarily homogeneous vector field $X$ on a supermanifold $\mathcal{M}$; cf. [MSV93], [GW13]. Then the flow map is a morphism $\mathcal{W} \subseteq \mathbb{R}^{1|1} \times \mathcal{M} \to \mathcal{M}$. But it does not always define a local $\mathbb{R}^{1|1}$-action on $\mathcal{M}$.

In contrast to the classical case, the span $\mathbb{R} X$ of the vector field $X$ is in general not a Lie subsuperalgebra of $\text{Vec}(\mathcal{M})$. Let $X_0$ be the even part of $X$, and $X_1$ the odd, $X = X_0 + X_1$. Even the span $\mathbb{R} X_0 \oplus \mathbb{R} X_1$ is in general not a Lie superalgebra since we might have $[X_0, X_1] \notin \mathbb{R} X_1$ or $[X_1, X_1] = 2X_1 X_1 \notin \mathbb{R} X_0$.

It turns out that the flow of $X$ defines a local $\mathbb{R}^{1|1}$-action precisely if $X_0$ and $X_1$ are contained in a $(1|1)$-dimensional Lie subsuperalgebra $\mathfrak{g}$ of $\text{Vec}(\mathcal{M})$. Note that the Lie supergroup structure on $\mathbb{R}^{1|1}$ is not unique (see Example 2.1.2), and the flow is an $\mathbb{R}^{1|1}$-action with respect to structure of a Lie supergroup on $\mathbb{R}^{1|1}$ induced by the structure of the Lie superalgebra $\mathfrak{g}$.

In the special case of an odd vector field $X$ which commutes with itself, i.e. $[X, X] = 2XX = 0$, there is an $\mathbb{R}^{0|1}$-action inducing this vector field. The $\mathbb{R}^{0|1}$-action is given by $\psi : \mathbb{R}^{0|1} \times \mathcal{M} \to \mathcal{M}$, $\psi^*(f) = f + \tau X(f)$, where $\tau$ denotes the coordinate on $\mathbb{R}^{0|1} \subseteq \mathbb{R}^{0|1} \times \mathcal{M}$ and $f$ and $X(f)$ are considered as functions on the product $\mathbb{R}^{0|1} \times \mathcal{M}$ on the right-hand side of the equation.

### 2.4 Harish-Chandra pairs

An important tool for the study of Lie supergroups is the equivalence of Lie supergroups and Harish-Chandra pairs. We briefly recall some important definitions and how the isomorphism of the category of Lie supergroups and Harish-Chandra pairs can be realized.

**Definition 2.4.1.** A pair $(G, \mathfrak{g})$ consisting of a (real or complex) Lie group $G$ and a (real or complex) Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ together with a parity-preserving representation $\sigma$ of $G$ on $\mathfrak{g}$, i.e. a (smooth or holomorphic) morphism $\sigma : G \to \text{GL}(\mathfrak{g}_0) \times \text{GL}(\mathfrak{g}_1)$, is called a (real or complex) Harish-Chandra pair if the following holds:

(i) The Lie algebra of $G$ is $\mathfrak{g}_0$ and the restriction of $\sigma$ to $\mathfrak{g}_0$ coincides with the adjoint action of $G$ on its Lie algebra.

(ii) The differential of $\sigma$ at the identity $e \in G$ coincides with the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}$.

Let $(G, \mathfrak{g})$ with representation $\sigma$ and $(H, \mathfrak{h})$ with representation $\tau$ be two Harish-Chandra pairs. A morphism between those consists of a homomorphism $\varphi : G \to H$ of Lie groups and a homomorphism $\lambda : \mathfrak{g} \to \mathfrak{h}$ of Lie superalgebras such that the differential of $\varphi$ at the identity $e \in G$ coincides with $\lambda|_{\mathfrak{g}_0}$ and such that $\tau(\varphi(g)) = \lambda \circ \sigma(g)$ for all $g \in G$.

If $G$ is a real or complex Lie supergroup, then we have an associated Harish-Chandra pair: the Lie group $G$ is the underlying Lie group of $G$, $\mathfrak{g}$ the Lie superalgebra of $G$, and $\sigma$ is the adjoint representation of $G$ on $\mathfrak{g}$. Furthermore, any morphism of Lie supergroups $G \to \mathcal{H}$ induces a morphism of the corresponding Harish-Chandra pairs.

**Theorem 2.4.2.** The categories of Lie supergroups and Harish-Chandra pairs are isomorphic.
For a proof see [Kos83], and [Vis11] in the complex case. For convenience, we briefly describe how a Lie supergroup can be constructed from a Harish-Chandra pair.

Let \((G, g)\) together with the representation \(\sigma\) be a Harish-Chandra pair. Let \(\mathcal{U}(g)\) denote the universal enveloping algebra of \(g\) and \(\mathcal{U}(g_0)\) the universal enveloping algebra of \(g_0\). Furthermore, let \(\mathcal{F}_G\) denote the sheaf of smooth functions on \(G\) if \(G\) is a real Lie group and the sheaf of holomorphic functions if \(G\) is a complex Lie group. Since \(g_0\) is the Lie algebra of \(G\), we have a natural action of \(\mathcal{U}(g_0)\) on the sheaf \(\mathcal{F}_G\). Furthermore, \(\mathcal{U}(g_0)\) acts on \(\mathcal{U}(g)\) by multiplication from the left if \(\mathcal{U}(g_0)\) is considered as a subset of \(\mathcal{U}(g)\). We define a sheaf \(\mathcal{O}\) on the \(G\) by setting

\[
\mathcal{O}(U) = \text{Hom}_{\mathcal{U}(g_0)}(\mathcal{U}(g), \mathcal{F}_G(U))
\]

for open subsets \(U \subseteq G\). The elements of \(\text{Hom}_{\mathcal{U}(g_0)}(\mathcal{U}(g), \mathcal{F}_G(U))\) are \(\mathcal{U}(g_0)\)-linear morphisms \(\mathcal{U}(g) \rightarrow \mathcal{F}_G(U)\). We consider \(\mathcal{F}_G(U)\) as an “even” object, and define a morphism \(\mathcal{U}(g) \rightarrow \mathcal{F}_G(U)\) to be even if it is parity-preserving and odd if it is parity-interchanging. In this way, \(\mathcal{O}\) is a sheaf of \(\mathbb{Z}_2\)-graded vector spaces. Using the Hopf superalgebra structure on the universal enveloping algebra \(\mathcal{U}(g)\), it is possible to define the multiplication of two elements in \(\mathcal{O}(U)\), and \(G = (G, \mathcal{O})\) carries the structure of a (real or complex) supermanifold. Moreover, using the explicit realization of the sheaf \(\mathcal{O}\), the pullbacks of morphisms for multiplication, inversion, and the neutral element on \(G\) can be defined and thus \(G\) is a Lie supergroup. For details of this construction see e.g. [Vis11].

Let \(\Psi : G \times M \rightarrow M\) be the action of a Lie supergroup on the supermanifold \(M\). Then by composing with the canonical inclusion \(G \hookrightarrow G\) we get an action \(\psi : G \times M \rightarrow M\) of the underlying Lie group \(G\) on the supermanifold \(M\). Moreover, we have the induced infinitesimal action \(\lambda_\Psi : g \rightarrow \text{Vec}(M)\), \(X \mapsto (X(e) \otimes \text{id}_M^\ast) \circ \Psi^\ast\) as in Proposition 2.2.3. Let \((G, g)\) be the Harish-Chandra pair associated with \(G\) with adjoint representation \(\sigma\). The maps \(\psi\) and \(\lambda_\Psi\) satisfy

\[
\begin{align*}
\lambda_\Psi(X) &= (X(e) \otimes \text{id}_M^\ast) \circ \psi^\ast & \text{for all } X \in g_0, \text{ and} \\
\lambda_\Psi(\sigma(g)(Y)) &= (\psi_g^{-1})^\ast \lambda_\Psi(Y)(\psi_g)^\ast & \text{for all } g \in G, Y \in g,
\end{align*}
\]

where \(\psi_g : M \rightarrow M\) is the automorphism of \(M\) given by the composition of \(g\) and the canonical inclusion \(\{g\} \times M \hookrightarrow G \times M\).

**Definition 2.4.3.** An action of a Harish-Chandra pair \((G, g)\) with representation \(\sigma\) on a supermanifold \(M\) is an action \(\psi : G \times M \rightarrow M\) and a homomorphism \(\lambda : g \rightarrow \text{Vec}(M)\) of Lie superalgebras satisfying the compatibility conditions as in (2.1) and (2.2).

There is a one-to-one correspondence between actions of Lie supergroups and actions of Harish-Chandra pairs (see e.g. [BCC09]). Above, we described how we can associate an action of a Harish-Chandra pair with an action of the corresponding Lie supergroup. Conversely, if \((G, g)\) is a Harish-Chandra pair and an action of \((G, g)\) on a supermanifold \(M\) is given by \(\psi : G \times M \rightarrow M\) and \(\lambda : g \rightarrow \text{Vec}(M)\), we can reconstruct the action of the corresponding Lie supergroup \(G\) using the explicit realization of the structure sheaf of \(G\) by \(\mathcal{O}_G(U) = \text{Hom}_{\mathcal{U}(g_0)}(\mathcal{U}(g), \mathcal{F}_G(U))\) for \(U \subseteq G\) open. The structure sheaf on the product manifold \(G \times M\) is then given by

\[
\mathcal{O}_G \otimes \mathcal{O}_M \cong \text{Hom}_{\mathcal{U}(g_0)}(\mathcal{U}(g), \mathcal{F}_G \otimes \mathcal{O}_M),
\]
where $\hat{\otimes}$ denotes the completed tensor product in the sense of [Gro55], and the action of $G$ has the following form:

**Proposition 2.4.4.** Let $\Psi$ be the action of the Lie supergroup $G$ on $M$, corresponding to the action of the corresponding Harish-Chandra pair given by $\psi$ and $\lambda$. Then the pullback of $\Psi$ has the form

$$O_M(V) \to \text{Hom}_{U(g)}(\mathfrak{U}(g), (F_G \hat{\otimes} O_M)(\psi^{-1}(V))),$$

$$f \mapsto \left( X \mapsto (-1)^{|X||f|}(\psi^* \circ \lambda(X))(f) \right)$$

for any $f \in O_M(V), V \subseteq M$ open, where we extend $\lambda : g \to \text{Vec}(M)$ to a map

$$\lambda : \mathfrak{U}(g) \to \mathfrak{U}(\text{Vec}(M))$$

by setting

$$\lambda(X_1 \ldots X_r) = ((X_1 \ldots X_r)(e) \otimes \text{id}_M) \circ \psi^* = \lambda(X_1) \ldots \lambda(X_r) \text{ for } X_1, \ldots, X_r \in \mathfrak{g}.$$ 

In [BCC09], Proposition 4.3, the corresponding formula is given if working with left-invariant vector fields.

### 2.5 Effective actions of Lie supergroups

In this section, the notion of an effective action of a Lie supergroup on a supermanifold is defined, and an equivalent characterization of an effective action in terms of the Harish-Chandra pair is given.

As a generalization of an effective action of a classical Lie group, we define:

**Definition 2.5.1.** Let $\Psi : G \times M \to M$ be the action of a Lie supergroup $G$ on a supermanifold $M$. The action $\Psi$ is called effective if it satisfies the following property: Let $N$ be a supermanifold and $\chi_1, \chi_2 : N \to G$ be arbitrary morphisms. Then the equality

$$\Psi \circ (\chi_1 \times \text{id}_M) = \Psi \circ (\chi_2 \times \text{id}_M)$$

implies $\chi_1 = \chi_2$.

We can characterize the effectiveness of an action of a Lie supergroup by properties of the action of the corresponding Harish-Chandra pair:

**Proposition 2.5.2.** Let $G$ be a Lie supergroup and $(G, \mathfrak{g})$ with representation $\sigma$ the associated Harish-Chandra pair. Let $\Psi : G \times M \to M$ be an action on the supermanifold $M$, and suppose the corresponding action of the Harish-Chandra pair $(G, \mathfrak{g})$ is given by $\psi : G \times M \to M$ and $\lambda : \mathfrak{g} \to \text{Vec}(M)$. The action $\Psi$ is effective if and only if the $G$-action $\psi$ is effective and $\lambda$ is injective.

**Remark 2.5.3.** The action $\psi$ of the classical Lie group $G$ on the supermanifold $M$ is effective if and only if

$$\psi_g = \psi_{g'} \text{ for } g, g' \in G \implies g = g',$$

where $\psi_h : M \to M$ denotes again the automorphism of $M$ given by the composition of $\psi$ and the inclusion $M \hookrightarrow \{h\} \times M$ for $h \in G$. 

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Proof of Proposition 2.5.2. First, suppose that $\Psi$ is effective. Let $g, g' \in G$ such that $\psi_g = \psi_{g'}$. Define $N = \{0\}$ and let $\chi_1, \chi_2 : N \to \mathcal{G}$ be given by the inclusions $\{0\} \hookrightarrow \{g\} \subset \mathcal{G}$ and $\{0\} \hookrightarrow \{g'\} \subset \mathcal{G}$. Identifying $N \times M$ with $\mathcal{M}$, we have $\Psi \circ (\chi_1 \times \text{id}_M) = \psi_g$ and $\Psi \circ (\chi_2 \times \text{id}_M) = \psi_{g'}$. Therefore, $\Psi \circ (\chi_1 \times \text{id}_M) = \Psi \circ (\chi_2 \times \text{id}_M)$ and it follows $\chi_1 = \chi_2$ since $\Psi$ is supposed to be effective. Consequently, we have $g = g'$.

Now, let $X \in \mathfrak{g}$ such that $\lambda(X) = (X(e) \otimes \text{id}^*) \circ \Psi^* = 0$, where $X(e)$ denotes again the evaluation of the right-invariant vector field $X$ in the identity element $e$ of $\mathcal{G}$. We first consider the case where $X$ is even, i.e. $X$ is contained in $\mathfrak{g}_0$, which is the Lie algebra of $G$. Let $\exp(tX)$ denote the corresponding one-parameter subgroup of $G$, and define $\chi_1 : \mathbb{K} \to \mathcal{G}$ by the pullback $\mathcal{O}_G \to \mathcal{O}_X$, $f \mapsto f(\exp(tX))$, and $\chi_2 : \mathbb{K} \to \mathcal{G}$ as the composition $\mathbb{C} \to \{e\} \hookrightarrow \mathcal{G}$ for the identity element $e$ of $\mathcal{G}$, where $\mathbb{K} = \mathbb{R}$ if $G$ is a real Lie supergroup and $\mathbb{K} = \mathbb{C}$ if $\mathcal{G}$ is a complex Lie supergroup. For any element $f \in \mathcal{O}_G(U)$, $U \subseteq G$ open, we have

$$
\frac{\partial}{\partial t} \bigg|_{s} (\chi_1^* \otimes \text{id}^*) \circ \Psi^* (f) = \frac{\partial}{\partial t} \bigg|_{0} \exp(tX)^*(f) = \frac{\partial}{\partial t} \bigg|_{0} \exp((t + s)tX)^*(f)
$$

where $t$ denotes the coefficient on $\mathbb{K}$ and $s$ is contained in some neighbourhood $\Omega$ of $0$ in $\mathbb{K}$. Thus we have $\frac{\partial}{\partial t} \bigg|_{0} (\chi_1^* \otimes \text{id}^*) \circ \Psi^* = 0 = \frac{\partial}{\partial t} \bigg|_{s} (\chi_2^* \otimes \text{id}^*) \circ \Psi^*$ for $s \in \Omega$. Since we also have $\text{ev}_{t=0}(\chi_1^* \otimes \text{id}^*)(\Psi^*(f)) = f = \text{ev}_{t=0}(\chi_2^* \otimes \text{id}^*)(\Psi^*(f))$ for any $f \in \mathcal{O}_G(U)$ if $\text{ev}_{t=0}$ denotes the evaluation of $f$ in the first component in $t = 0$, we deduce

$$
\Psi \circ (\chi_1|\Omega \times \text{id}) = \Psi \circ (\chi_2|\Omega \times \text{id})
$$

This implies $\chi_1|\Omega = \chi_2|\Omega$ due to the effectiveness of $\Psi$ and thus $\exp(tX) = 0$ for all $t \in \Omega$, i.e. $X = 0$. Consequently, $\lambda|\mathfrak{g}_0$ is injective.

Consider now the case $X \in \mathfrak{g}_1$ and suppose again $\lambda(X) = (X(e) \otimes \text{id}^*) \circ \Psi^* = 0$. Since $X$ is an odd vector field on $\mathcal{G}$, we have $[X, X] = 2X^2 \in \mathfrak{g}_0$. As $\lambda|\mathfrak{g}_0$ is injective and $\lambda([X, X]) = [\lambda(X), \lambda(X)] = 0$, the vector field $X$ commutes with itself, i.e. $[X, X] = 0$. Therefore, the morphism $\gamma : \mathbb{K}^{0|1} \times \mathcal{M} \to \mathcal{M}$ given by the pullback $\gamma^*(f) = f + \tau X(f)$, if $\tau$ is the coordinate on $\mathbb{K}^{0|1}$, defines a flow of $X$ with $\text{ev}_{\tau=0} \gamma^*(f) = f$ and $\frac{\partial}{\partial \tau} \gamma^*(f) = X(f)$; see also [GW13], § 4. Define $\chi_1 : \mathbb{K}^{0|1} \to \mathcal{G}$ by $\chi_1^*(f) = f(e) + \tau(X(f))(e)$, where $f(e)$ and $X(f)(e)$ are the evaluations of $f$ and $X(f)$ in $e$. Moreover, let $\chi_2 : \mathbb{K}^{0|1} \to \mathcal{G}$ be given by $\chi_2^*(f) = f(e)$. We have $\chi_1 = \chi_2$ precisely if $X = 0$. A calculation yields

$$
\frac{\partial}{\partial \tau} \bigg|_{0} (\chi_1^* \otimes \text{id}^*) = (\chi_2^* \otimes \text{id}^*) \circ \Psi^* = \lambda(X) = 0
$$

Thus we have

$$
(\frac{\partial}{\partial \tau} \bigg|_{0} \otimes \text{id}^*) (\chi_1^* \otimes \text{id}^*) \circ \Psi^* = (X(e) \otimes \text{id}^*) \circ \Psi^* = \lambda(X) = 0
$$

Since also

$$
(\text{ev}_{\tau=0} \otimes \text{id}^*) (\chi_1^* \otimes \text{id}^*) \circ \Psi^* = \text{id}^* = (\text{ev}_{\tau=0} \otimes \text{id}^*) (\chi_2^* \otimes \text{id}^*) \circ \Psi^*,
$$

we have $\Psi \circ (\chi_1 \times \text{id}) = \Psi \circ (\chi_2 \times \text{id})$ and thus by assumption $\chi_1 = \chi_2$, which implies $X = 0$.

Suppose now that the action $\psi$ of the classical Lie group $G$ on the supermanifold $\mathcal{M}$ is effective and that $\lambda$ is injective. Let $\chi_j : \mathcal{N} \to \mathcal{G}$, $j = 1, 2$, be morphisms with $\Psi \circ (\chi_1 \times \text{id}) = \Psi \circ (\chi_2 \times \text{id})$. We need to show $\chi_1 = \chi_2$. 27
First, we consider the case where $N = \mathbb{K}^{0|k}$ and $\chi_1(0) = e = \chi_2(0)$. Let $\tau_1, \ldots, \tau_k$ denote coordinates on $\mathbb{K}^{0|k}$. Then there are morphisms $\alpha_\nu, \beta_\nu : G \to \mathbb{C}$ such that

$$\chi_1^\nu(f) = \sum_{\nu \in (\mathbb{Z}_2)^k} \tau^\nu \alpha_\nu(f) \quad \text{and} \quad \chi_2^\nu(f) = \sum_{\nu \in (\mathbb{Z}_2)^k} \tau^\nu \beta_\nu(f)$$

for all $f$, and $\alpha_\nu, \beta_\nu$ are of parity $|\nu| = |(\nu_1, \ldots, \nu_k)| = \nu_1 + \ldots + \nu_k \in \mathbb{Z}_2$. Since $\chi_j(0) = e$, we have $\alpha_0(f) = f(e) = \beta_0(f)$, where $f(e)$ denotes the evaluation of $f$ in the identity $e$. Assume now $\chi_1 \neq \chi_2$, and let $\nu_0 \in (\mathbb{Z}_2)^k$ such that $\alpha_{\nu_0} \neq \beta_{\nu_0}$ and such that there is no $\mu$ with $\alpha_\mu \neq \beta_\mu$ and $||\mu|| < ||\nu_0||$ for $||\mu|| = ||(\mu_1, \ldots, \mu_k)|| = \mu_1 + \ldots + \mu_k \in \mathbb{Z}$.

(In the definition of $||\mu||$ for $\mu \in \mathbb{Z}_2$, each $\mu_j$ is considered as an element of $\{0, 1\}(\cong \mathbb{Z}_2)$ and the sum $\mu_1 + \ldots + \mu_k$ is understood as a sum in $\mathbb{Z}$.) Define $c_\nu : G \to \mathbb{C}$ as $c_\nu = \alpha_\nu - \beta_\nu$. By the choice of $\nu_0$, we have $c_{\nu_0} = 0$ if $||\nu|| < ||\nu_0||$. For any homogeneous $f, g$ we have

$$\sum_{\nu, ||\nu|| \geq ||\nu_0||} \tau^\nu c_\nu(fg) = (\chi_1^\nu - \chi_2^\nu)(fg) = \chi_1^\nu(f)\chi_1^\nu(g) - \chi_2^\nu(f)\chi_2^\nu(g)$$

$$= \chi_1^\nu(f)(\chi_1^\nu - \chi_2^\nu)(g) + (\chi_1^\nu - \chi_2^\nu)(f)\chi_2^\nu(g)$$

$$= \left(\sum_{\nu} \tau^\nu \alpha_\nu(f)\right) \left(\sum_{\mu, ||\mu|| \geq ||\nu_0||} \tau^\mu c_\mu(f)\right) - \left(\sum_{\nu, ||\nu|| \geq ||\nu_0||} \tau^\nu \beta_\nu(f)\right).$$

A comparison of the coefficient of $\tau^{\nu_0}$ on both sides of this equation yields

$$c_{\nu_0}(fg) = c_{\nu_0}(f)\beta_0(g) + (-1)^{|f||\nu_0|} \alpha_0(f)c_{\nu_0}(g) = c_{\nu_0}(f)g(e) + (-1)^{|f||\nu_0|} f(e)c_{\nu_0}(g).$$

Thus, $c_{\nu_0}$ defines a point-derivation $O_{\mathbb{G}, e} \to \mathbb{C}$ of parity $|\nu_0|$ on $\mathbb{G}$ in the identity $e$. Let $Y$ denote the corresponding right-invariant vector field on $\mathbb{G}$, i.e. $Y = (c_{\nu_0} \otimes \text{id}) \circ \mu^*$ if $\mu$ denotes the multiplication of $\mathbb{G}$, and $Y(e) = c_{\nu_0}$. Since $\Psi \circ (\chi_1 \times \text{id}) = \Psi \circ (\chi_2 \times \text{id})$, we have $0 = ((\chi_1^\nu - \chi_2^\nu) \otimes \text{id}^*) \circ \Psi^* = ((\sum_{\nu} \tau^\nu c_\nu) \otimes \text{id}^*) \circ \Psi$ and thus $0 = (c_{\nu_0} \otimes \text{id}^*) \circ \Psi^* = (Y(e) \otimes \text{id}^*) \circ \Psi^*$. This implies $Y = 0$ since we supposed that the infinitesimal action $\lambda$ is injective. Consequently, we have $c_{\nu_0} = Y(e) = 0$ which is a contradiction to the choice of $\nu_0$ with $c_{\nu_0} = \alpha_{\nu_0} - \beta_{\nu_0} \neq 0$. Therefore, the morphisms $\chi_1$ and $\chi_2$ have to be identical.

The general case can be reduced to the just described special case. In order to do so, consider now arbitrary $N$ and $\chi_j : N \to G$, $j = 1, 2$, with $\Psi \circ (\chi_1 \times \text{id}) = \Psi \circ (\chi_2 \times \text{id})$. If $\chi_1 \neq \chi_2$ then there is a morphism $\varphi : \mathbb{K}^{0|k} \to N$ such that $\chi_1 \circ \varphi \neq \chi_2 \circ \varphi$. Therefore, we may assume $N = \mathbb{K}^{0|k}$. Let $g = \hat{\chi}_1(0)$ and $g' = \hat{\chi}_2(0)$. We have $\psi_g = \psi_{g'}$ since $\Psi \circ (\chi_1 \times \text{id}) = \Psi \circ (\chi_2 \times \text{id})$. This implies $g = g'$ by assumption. Let $\mu_g : G \to G$ denote the left-multiplication with $g$, i.e. $\mu_g = (ev_g \otimes \text{id}^*) \circ \mu^*$. Define $\chi_1' = \mu_g \circ \chi_j$, $j = 1, 2$. Since $\mu_g$ is invertible we have $\chi_1' = \chi_2$ and only if $\chi_1' = \chi_2$. By construction we have $\chi_1'(0) = e = \chi_2'(0)$ and $\Psi \circ (\chi_1' \times \text{id}) = \Psi \circ (\chi_2' \times \text{id})$. By the preceding considerations we get $\chi_1' = \chi_2'$ and thus $\chi_1 = \chi_2$. Hence, the action $\Psi$ is effective. \qed
Chapter 3

Distributions on supermanifolds and Frobenius’ theorem

In this chapter, we study distributions, and in particular involutive distributions, on supermanifolds. Distributions play an important role when studying how infinitesimal symmetries can be integrated to local or global symmetries. Thus, the results of this chapter are an important tool in the next chapter where we construct local actions from infinitesimal actions and find conditions for the existence of global actions.

We start by recalling the definition of a distribution, which is a straightforward generalization of the classical definition in a sheaf-theoretic formulation. As a technical preparation for the study of the local structure of an involutive distribution, commuting vector fields on supermanifolds and their flows are studied in Section 3.1.

Then the structure of involutive distributions on supermanifolds is examined. A local version of Frobenius’ theorem for involutive distributions is proven; a different proof of this theorem can also be found in [Var04]. Contrary to the classical case however, there is no global version of Frobenius’ theorem on supermanifolds since integral subsupermanifold do not contain enough information about the distribution. We give an example of a non-involutive distribution on a supermanifold which has integral subsupermanifolds through each point.

The results of this chapter are included in [Ber14].

Definition 3.0.4. A (smooth or holomorphic) distribution \( \mathcal{D} \) on a (real or complex) supermanifold \( \mathcal{M} \) is a graded \( \mathcal{O}_\mathcal{M} \)-subsheaf of the tangent sheaf \( \mathcal{T}_\mathcal{M} = \text{Der} \mathcal{O}_\mathcal{M} \) of \( \mathcal{M} \) which is locally a direct factor, i.e. for each point \( p \in \mathcal{M} \) there exists an open neighbourhood \( U \) of \( p \) in \( \mathcal{M} \) and a subsheaf \( \mathcal{E} \) of \( \text{Der} \mathcal{O}_\mathcal{M}|_U \) on \( U \) such that \( \mathcal{D}|_U \oplus \mathcal{E} = \text{Der} \mathcal{O}_\mathcal{M}|_U \).

Remark 3.0.5 (cf. [Var04], Section 4.7). Locally there exist independent vector fields \( X_1, \ldots, X_r, Y_1, \ldots, Y_s \) spanning a given distribution \( \mathcal{D} \), where the \( X_i \) are even and the \( Y_j \) are odd. Moreover, \( (r|s) = \dim \mathcal{D}(p) \) for any \( p \in \mathcal{M} \) (if \( \mathcal{M} \) is connected) and \( (r|s) \) is called the rank of the distribution \( \mathcal{D} \).

If \( \psi : \mathcal{M} \to \mathcal{N} \) is a diffeomorphism and \( \mathcal{D} \) is a distribution on \( \mathcal{M} \), then there is a distribution \( \psi_*(\mathcal{D}) \) on \( \mathcal{N} \) which is spanned by vector fields of the form \( \psi_*(X) = (\psi^{-1})^* \circ X \circ \psi^* \) for vector fields \( X \) on \( \mathcal{M} \) belonging to \( \mathcal{D} \).
Remark 3.0.6. A distribution $\mathcal{D}$ of rank $(r|s)$ on a supermanifold $\mathcal{M}$ induces a distribution $\tilde{\mathcal{D}}$ of rank $r$ on the underlying classical manifold $\mathcal{M}$ by defining $\tilde{\mathcal{D}}(p) = \{\tilde{X}(p) | X \in \mathcal{D}\} \subseteq T_p\mathcal{M}$ for $p \in \mathcal{M}$. A vector field on $\mathcal{M}$ belongs to the reduced distribution $\tilde{\mathcal{D}}$ if and only if it is the reduced vector field $\tilde{X}$ of some vector field $X$ on $\mathcal{M}$ belonging to $\mathcal{D}$.

3.1 Commuting vector fields

In this section, commuting vector fields on a supermanifold $\mathcal{M}$ are studied. The results are used in the next section for a local characterization of involutive distributions.

Let $\mu : \mathbb{R}^{m|n} \times \mathbb{R}^{m|n} \rightarrow \mathbb{R}^{m|n}$ denote the addition on $\mathbb{R}^{m|n}$ which is given by $((s,\sigma), (t,\tau)) \mapsto (s + t, \sigma + \tau)$ in coordinates. The goal of this section is the proof of the following theorem.

Theorem 3.1.1. Let $\mathcal{M} = (\mathcal{M}, \mathcal{O}_\mathcal{M})$ be a supermanifold, let $X_1, \ldots, X_m$ be even and let $Y_1, \ldots, Y_n$ be odd vector fields on $\mathcal{M}$ which all commute. Then, for any $p \in \mathcal{M}$ there exists an open connected neighbourhood $U$ of 0 in $\mathbb{R}^m$, an open neighbourhood $V$ of $p$ in $\mathcal{M}$ and a morphism $\varphi = (\tilde{\varphi}, \varphi^*) : U \times V \rightarrow \mathcal{M}$, where $U = (U, \mathcal{O}_{\mathbb{R}^{m|n}}|U)$ and $V = (V, \mathcal{O}_\mathcal{M}|V)$, such that:

(i) The map $\varphi$ has the action property, i.e. we have $\varphi \circ \iota_0 = \text{id}_\mathcal{M}$ if $\iota_0 : \mathcal{M} \hookrightarrow \{0\} \times \mathcal{M} \subset \mathbb{R}^{m|n} \times \mathcal{M}$ denotes the canonical inclusion, and the equality

$$\varphi \circ (\text{id}_{\mathbb{R}^{m|n}} \times \varphi) = \varphi \circ (\mu \times \text{id}_\mathcal{M})$$

holds on the open subsupermanifold of $\mathbb{R}^{m|n} \times \mathbb{R}^{m|n} \times \mathcal{M}$ on which both sides are defined.

(ii) If $(t, \tau)$ are coordinates on $\mathbb{R}^{m|n}$, then for all $i = 1, \ldots, m$, $j = 1, \ldots, n$ we have

$$\frac{\partial}{\partial t_i} \circ \varphi^* = \varphi^* \circ X_i \quad \text{and} \quad \frac{\partial}{\partial \tau_j} \circ \varphi^* = \varphi^* \circ Y_j. \tag{1}$$

By replacing $\mathbb{R}^{m|n}$ by $\mathbb{C}^{m|n}$ an analogous result also holds true for a complex supermanifold $\mathcal{M}$ and holomorphic vector fields $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$.

Remark 3.1.2. Note that $\varphi$ locally defines a local action of $\mathbb{R}^{m|n}$ on $\mathcal{M}$ in the sense that $\varphi$ would be a local action if the assumption that $\{0\} \times \mathcal{M}$ is contained in the domain of definition was dropped.

Proof of Theorem 3.1.1. Let $X_1, \ldots, X_m$ be even and $Y_1, \ldots, Y_n$ be odd vector fields on $\mathcal{M}$ which all commute. Furthermore, let $\varphi^X_i : W_i \rightarrow \mathcal{M}$ denote the flow of $X_i$ for any $i$. By Corollary 2.3.8 (iii) these flows all commute. Given $p \in \mathcal{M}$, there exist an open neighbourhood $V \subseteq \mathcal{M}$ of $p$ and an open connected neighbourhood $U \subseteq \mathbb{R}^m$ of 0 such that

$$\beta : U \times V \rightarrow \mathcal{M}, \beta_t = \varphi^{X_{t_1}}_{t_1} \circ \ldots \circ \varphi^{X_{t_m}}_{t_m},$$

is defined, where $V = (V, \mathcal{O}_\mathcal{M}|V)$, $t = (t_1, \ldots, t_m)$ coordinates on $U \subseteq \mathbb{R}^m$, and $\beta_t$ is the evaluation of $\beta$ in the first component in $t$. Since the flows $\varphi^{X_i}$ commute and each

(1) Considering $\frac{\partial}{\partial t_i}$ and $\frac{\partial}{\partial \tau_j}$ as vector fields on the product $\mathbb{R}^{m|n} \times \mathcal{M}$.  

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flow defines a local $\mathbb{R}$-action on $\mathcal{M}$, the map $\beta$ has the action property, i.e. $\beta \circ \iota_0 = \text{id}_\mathcal{M}$ and $\beta_s \circ \beta_t = \beta_{s+t}$ for all $s, t$ for which both sides of the equation are defined.

Let $\tau_1, \ldots, \tau_n$ be coordinates on $\mathbb{R}^{0|n}$ and define

$$\alpha : \mathbb{R}^{0|n} \times \mathcal{M} \to \mathcal{M} \quad \text{by} \quad \alpha^*(f) = \exp \left( \sum_{j=1}^{n} \tau_j Y_j \right)(f). \quad (2)$$

The underlying map is $\tilde{\alpha} = \text{id}_\mathcal{M}$. The sum

$$\exp \left( \sum_{j=1}^{n} \tau_j Y_j \right)(f) = \sum_{k=0}^{\infty} \frac{1}{k!} (\sum_{j=1}^{n} \tau_j Y_j)^k(f)$$

is finite because $(\sum_{j=1}^{n} \tau_j Y_j)^{n+1}(f) = 0$. Since the odd vector fields $Y_j$ all commute, i.e.

$$[Y_i, Y_j] = Y_i Y_j + Y_j Y_i = 0,$$

we get $(\sigma_i Y_i)(\tau_j Y_j) = (\tau_j Y_j)(\sigma_i Y_i)$, and thus $\exp(\sum_i \sigma_i Y_i) \exp(\sum_j \tau_j Y_j) = \exp(\sum_k (\sigma_k + \tau_k Y_k))$ if $(\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n)$ are coordinates on $\mathbb{R}^{0|n} \times \mathbb{R}^{0|n}$. Consequently, the map $\alpha$ defines an $\mathbb{R}^{0|n}$-action on $\mathcal{M}$.

Now, let $W = (U \times V, \mathcal{O}_{\mathbb{R}^{m|n} \times \mathcal{M}}|_{U \times V})$ and define

$$\varphi : W \to \mathcal{M}, \varphi = \beta \circ (\text{id}_U \times \alpha) \quad \text{on} \quad W.$$ 

Then $\varphi^*(f) = \exp(\sum_j \tau_j Y_j)(\varphi^X_1)^* \circ \ldots \circ (\varphi^X_m)^*(f)$. The map $\varphi$ satisfies $\varphi \circ \iota_0 = \text{id}_\mathcal{M}$ and $\frac{\partial}{\partial \tau_i} \circ \varphi^* = X_i \circ \varphi^*$, $\frac{\partial}{\partial \tau_i} \circ \varphi^* = Y_j \circ \varphi^*$ for all $i, j$, making use of the commutativity of the vector fields and Corollary 2.3.8. A calculation using the action properties of $\alpha$ and $\beta$ and again Corollary 2.3.8 shows that $\varphi \circ (\text{id}_{\mathbb{R}^{m|n}} \times \varphi) = \varphi \circ (\mu \times \text{id}_\mathcal{M})$ on the open subsupermanifold of $\mathbb{R}^{m|n} \times \mathbb{R}^{m|n} \times \mathcal{M}$ on which both $\varphi \circ (\text{id}_{\mathbb{R}^{m|n}} \times \varphi)$ and $\varphi \circ (\mu \times \text{id}_\mathcal{M})$ are defined. 

\textbf{Remark 3.1.3.} The complex version of the result can be proven along the lines of the real case using holomorphic flows and an analogue of Corollary 2.3.8 for holomorphic vector fields.

### 3.2 Involutive distributions and Frobenius’ theorem

As in the classical case, the notion of an involutive distribution can be defined.

\textbf{Definition 3.2.1.} A distribution $\mathcal{D}$ is called involutive if $\mathcal{D}_p$ is a Lie subalgebra of $(\text{Der } \mathcal{O}_\mathcal{M})_p$ for each $p \in \mathcal{M}$, i.e. if for any two vector fields $X$ and $Y$ on $\mathcal{M}$ belonging to $\mathcal{D}$ their commutator $[X, Y]$ also belongs to $\mathcal{D}$.

In the following involutive distributions on supermanifolds will be studied further. The goal is the proof of the local version of Frobenius’ theorem.

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\textsuperscript{2}The sum $\exp(\sum_{j=1}^{n} \tau_j Y_j)(f) = \sum_{k=0}^{\infty} \frac{1}{k!} (\sum_{j=1}^{n} \tau_j Y_j)^k(f)$ has to be understood in the following way: The vector fields $Y_j$, which are a priori vector fields on $\mathcal{M}$, are considered as vector fields on the product $\mathbb{R}^{0|n} \times \mathcal{M}$ and similarly $f$ is considered as a function on this product. Hence, $\tau_j Y_j(f)$ here in fact means $\tau_j \cdot (\text{id}_{\mathbb{R}^{0|n}} \otimes Y_j)(\text{id}_{\mathcal{M}}(f))$, where $\tau_j$ is now considered as a coordinate on $\mathbb{R}^{0|n} \times \mathcal{M}$, $\tau_j : \mathbb{R}^{0|n} \times \mathcal{M} \to \mathcal{M}$ is the canonical projection onto $\mathcal{M}$, and $\text{id}_{\mathbb{R}^{0|n}} \otimes Y_j$ is the extension of $Y_j$ to a vector field on $\mathbb{R}^{0|n} \times \mathcal{M}$. 31
Theorem 3.2.2 (Local version of Frobenius’ Theorem, cf. [Var04] Theorem 4.7.1). A distribution \( D \) on a real (complex) supermanifold \( M \) is involutive if and only if there are local coordinates \((x, \theta) = (x_1, \ldots, x_m, \theta_1, \ldots, \theta_n)\) around each point \( p \in M \) such that \( D \) is locally spanned by \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_s} \), i.e. there exist supermanifolds \( \mathcal{U} \) and \( S \) and a diffeomorphism (resp. biholomorphism) onto its image \( \psi : \mathcal{U} \times S \to M \) such that \( \psi_*(\mathcal{D}_U) = D \) for the distribution \( \mathcal{D}_U \) on \( \mathcal{U} \times S \) in \( \mathcal{U} \)-direction, i.e. the distribution spanned by vector fields of the form \( X \otimes 1 \) on \( \mathcal{U} \times S \) where \( X \) is a vector field on \( \mathcal{U} \).

If the distribution \( D \) is locally spanned by \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_s} \), it follows directly that \( D \) is involutive. Consequently, only the existence of such coordinates for an involutive distribution remains to be proven. The proof in [Var04] discusses the even and odd part of the distribution \( D \) separately. Here, first the existence of \( r \) even and \( s \) odd commuting vector fields that locally span \( D \) will be shown, similarly as in [Var04]. Then Theorem 3.1.1 is used to obtain the local coordinates \((x, \theta)\) with the required property.

Lemma 3.2.3. Let \( D \) be an involutive distribution of rank \((r|s)\) on a real (resp. complex) supermanifold \( M \). Then, there locally exist smooth (resp. holomorphic) even vector fields \( X_1, \ldots, X_r \) and odd vector fields \( Y_1, \ldots, Y_s \) which locally span \( D \) and which all commute.

The proof of this lemma can be carried out as in the classical case.

Proof. Let \( X'_1, \ldots, X'_r, Y'_1, \ldots, Y'_s \) be \( r \) even and \( s \) odd independent vector fields on some open subset \( U \) of \( M \) that span the distribution \( D \) on \( U \). If \( x_1, \ldots, x_m, \theta_1, \ldots, \theta_n \) are local coordinates on \( U \), then

\[
X'_i = \sum_{k=1}^{m} a'_{ik} \frac{\partial}{\partial x_k} + \sum_{l=1}^{n} b'_{il} \frac{\partial}{\partial \theta_l} \quad \text{and} \quad Y'_i = \sum_{k=1}^{m} c'_{jk} \frac{\partial}{\partial x_k} + \sum_{l=1}^{n} b'_j \frac{\partial}{\partial \theta_l}
\]

for some functions \( a'_{ik}, d'_{ij} \in \mathcal{O}_M(U) \) and \( b'_{il}, c'_{jk} \in \mathcal{O}_M(U) \). Defining the matrices \( A', B', C' \) and \( D' \) by \( A'_{ij} = a'_{ij} \) etc., we get

\[
\begin{pmatrix}
X'_1 \\
\vdots \\
X'_r \\
Y'_1 \\
\vdots \\
Y'_s
\end{pmatrix} = \begin{pmatrix}
A' & B' \\
C' & D'
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\vdots \\
\frac{\partial}{\partial x_m} \\
\frac{\partial}{\partial \theta_1} \\
\vdots \\
\frac{\partial}{\partial \theta_n}
\end{pmatrix}.
\]

The matrix \( \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \) is an \((r + s) \times (m + n)\)-matrix and its rank is maximal since the vector fields \( X'_1, \ldots, X'_r, Y'_1, \ldots, Y'_s \) were assumed to be independent. Hence, after interchanging coordinates, the matrices \( A', B', C' \) and \( D' \) may be written as \( A' = (A_1 A_2) \), \( B' = (B_1 B_2) \), \( C' = (C_1 C_2) \) and \( D' = (D_1 D_2) \) so that \( \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \) is an
we get independent vector fields $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ locally spanning the distribution $\mathcal{D}$ such that

$$X_i = \frac{\partial}{\partial x_i} + \sum_{k=r+1}^{m} a_{ik} \frac{\partial}{\partial x_k} + \sum_{l=s+1}^{n} b_{il} \frac{\partial}{\partial \theta_l}$$

and similarly

$$Y_j = \frac{\partial}{\partial \theta_j} + \sum_{k=r+1}^{m} c_{jk} \frac{\partial}{\partial x_k} + \sum_{l=s+1}^{n} d_{jl} \frac{\partial}{\partial \theta_l}$$

for some $a_{ik}, d_{jl} \in O_M(U)_0$ and $b_{il}, c_{jk} \in O_M(U)_{-1}$. Since $\mathcal{D}$ is involutive, the commutator

$$[X_i, Y_j] = \left[ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \theta_j} \right] + \sum_{k=r+1}^{m} \left[ a_{ik} \frac{\partial}{\partial x_k}, c_{jk} \frac{\partial}{\partial x_k} \right] + \sum_{k=r+1}^{m} [a_{ik}, d_{jl} \frac{\partial}{\partial \theta_j}] + \ldots$$

also belongs to $\mathcal{D}$, where $\lambda_k \in O_M(U)_{-1}$ and $\mu_l \in O_M(U)_0$. Hence, this commutator may be written as a combination of the vector fields $X_1, \ldots, X_r, Y_1, \ldots, Y_s$. The special form of those vector fields now yields $[X_i, Y_j] = 0$. A similar argument shows $[X_i, r] = 0$ and $[Y_i, Y_j] = 0$ for all $i, j$. Therefore, the vector fields $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ all commute. \hfill \square

**Proof of Theorem 3.2.2.** Let $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ be $r$ even and $s$ odd independent and commuting vector fields, as constructed in the above lemma, which locally span the distribution $\mathcal{D}$.

By Theorem 3.1.1 there exist an open connected neighbourhood $U \subseteq \mathbb{R}^{\tau}$ (resp. $\mathbb{C}^{\tau}$) of $0 \in \mathbb{R}^{\tau}$ and an open neighbourhood $V \subseteq M$ of a given point $p \in M$ such that there is a map $\varphi : U \times V \rightarrow \mathcal{M}$, where $U = (U, O_{\mathbb{R}^{\tau+|s|}})$ (resp. $(U, O_{\mathbb{C}^{\tau+|s|}})$) and $V = (V, O_M)$, satisfying the properties mentioned in Theorem 3.1.1.

Consider now the restricted map $\varphi|_{U \times \{p\}} : U \times \{p\} \rightarrow \mathcal{M}$. For the derivative $d\varphi|_{U \times \{p\}}(v) = v \circ (\varphi|_{U \times \{p\}})^*$ for $v \in T_{(0,p)}(U \times \{p\})$ we have

$$d\varphi|_{U \times \{p\}} \left( \frac{\partial}{\partial t_i} \bigg|_0 \right) = \frac{\partial}{\partial t_i} \bigg|_0 \circ (\varphi|_{U \times \{p\}})^* = X_i(p)$$

and

$$d\varphi|_{U \times \{p\}} \left( \frac{\partial}{\partial \tau_j} \bigg|_0 \right) = \frac{\partial}{\partial \tau_j} \bigg|_0 \circ (\varphi|_{U \times \{p\}})^* = Y_j(p),$$

where $(t, \tau)$ are coordinates on $\mathbb{R}^{r+s}$ (resp. $\mathbb{C}^{r+s}$).
Since the vector fields $X_1,\ldots,X_r,Y_1,\ldots,Y_s$ are independent, the restricted map $\varphi|_{U \times \{p\}}$ is an immersion in $(0,p)$. Hence, after possibly shrinking $U$, $\varphi|_{U \times \{p\}}$ is an immersion and $U \times \{p\} \cong \varphi(U \times \{p\})$ is a subsupermanifold of $\mathcal{M}$. Let $S$ be a subsupermanifold of $\mathcal{M}$ which is transverse to $\varphi(U \times \{p\})$ in $\tilde{\varphi}(0,p) = p$. Moreover, define $\psi : U \times S \to \mathcal{M}$ by $\psi = \varphi|_{U \times S}$.

For the derivative of $\psi$ in $(0,p)$ we have $d\psi(T_{(0,p)}(U \times \{p\})) = T_{(0,p)}(\varphi(U \times \{p\}))$ and $d\psi(T_{(0,p)}(0 \times S)) = d(\varphi|_{\{0\} \times S})(0 \times T_p S) = T_p S$. Therefore, making use of the inverse function theorem on supermanifolds (see e.g. Theorem 2.3.1 in [Leı80]), the map $\psi$ is a local diffeomorphism (resp. biholomorphism) in $(0,p)$ as $S$ was chosen to be transversal to $\varphi(U \times \{p\})$ and the dimensions of $U \times S$ and $M$ coincide. After shrinking the domain of definition, the map $\psi$ is a diffeomorphism (resp. biholomorphism) onto its image. Furthermore, $\psi_* \left( \frac{\partial}{\partial t_i} \right) = (\psi^{-1})^* \circ \frac{\partial}{\partial t_i} \circ \psi^* = (\psi^{-1})^* \circ \psi^* \circ X_i = X_i$
and similarly $\psi_* \left( \frac{\partial}{\partial \tau_j} \right) = Y_j$, which now yields the existence of appropriate local coordinates. Therefore, the distribution $\mathcal{D}$ can be written in the required form with respect to these coordinates. \hfill \Box

As in the classical case, the notion of an integral manifold of a distribution can be defined.

**Definition 3.2.4.** Let $\mathcal{D}$ be a distribution of rank $(r|s)$ on a supermanifold $M = (M,\mathcal{O}_M)$. An $(r|s)$-dimensional subsupermanifold $j : N \to M$, $N = (N,\mathcal{O}_N)$, of $M$ is called an integral manifold of $\mathcal{D}$ through $p \in M$ if

(i) the point $p$ is contained in $j(N)$, and

(ii) for any vector field $X$ on $M$ belonging to $\mathcal{D}$ there exists a vector field $\tilde{X}$ on $N$ such that $\tilde{X} \circ j^* = j^* \circ X$, or equivalently $X(\ker j^*) \subseteq \ker j^*$, and all vector fields on $N$ are of the form $\tilde{X}$ for some vector field $X$ on $M$ which belongs to $\mathcal{D}$.

**Remark 3.2.5.** In the case of supermanifolds, integral manifolds of a distribution do not provide as much information about the distribution as in the classical case. This is illustrated in the subsequent example.

Contrary to the classical case, there is no global version of Frobenius’ theorem for the above defined notion of an integral manifold. The existence of integral manifolds through each point does not imply that the distribution is involutive. Nevertheless, the local version of Frobenius’ theorem still guarantees the existence of integral manifolds through each point for an involutive distribution.

There exist non-involutive distributions which have integral manifolds through each point.

**Example 3.2.6 (A non-involutive distribution with integral manifolds).** Let $\mathcal{M} = \mathbb{R}^{0|2}$, with coordinates $\theta_1$ and $\theta_2$, and let $\mathcal{D}$ be the distribution on $\mathcal{M}$ spanned by the odd vector field $X = \frac{\partial}{\partial \theta_1} + \theta_1 \theta_2 \frac{\partial}{\partial \theta_2}$. Since the distribution $\mathcal{D}$ is a sheaf on $\mathbb{R}^0 = \{0\}$, it is
uniquely determined by its stalk $D_0$ in 0 and thus $D$ will be identified with $D_0$ in this example.

The distribution $D$ is not involutive since

$$[X, X] = 2XX = 2\left(\frac{\partial}{\partial \theta_1} + \theta_1 \theta_2 \frac{\partial}{\partial \theta_2}\right)\left(\frac{\partial}{\partial \theta_1} + \theta_1 \theta_2 \frac{\partial}{\partial \theta_2}\right) = 2\theta_2 \frac{\partial}{\partial \theta_2} \notin D.$$ 

Let $N = \mathbb{R}^{01}$, with coordinate $\theta$, and $j : N = \mathbb{R}^{01} \hookrightarrow \mathbb{R}^{00} \times \{0\} \subset \mathbb{R}^{01} \times \mathbb{R}^{01} \cong \mathbb{R}^{02} = M$ be the natural inclusion with pullback $j^*(\theta_1, \theta_2) = (\theta, 0)$ and kernel $\ker j^* = \mathcal{O}(M) \cdot \theta_2$. The map $j^*$ is surjective, hence $N$ is a closed submanifold of $M$.

The vector field $X = \frac{\partial}{\partial \theta_1} + \theta_1 \theta_2 \frac{\partial}{\partial \theta_2}$ is tangent to $N$ because $X(\theta_2) = \theta_1 \theta_2 \in \ker j^*$. Therefore, we have $Y(\ker j^*) \subseteq \ker j^*$ for every element $Y \in D$ and consequently $Y$ defines a vector field $Y : \mathcal{O}(M) / \ker j^* \cong \mathcal{O}(N) \rightarrow \mathcal{O}(M) / \ker j^* \cong \mathcal{O}(N)$ on $N$. The vector field $\tilde{X}$ on $N$ induced by $X$ equals $\frac{\partial}{\partial \theta_1}$ since $\tilde{X}(\theta) = \tilde{X}(j^*(\theta_1)) = j^*(X(\theta_1)) = j^*(1) = 1$. Hence, the distribution $D$ is not involutive, but there exist integral manifolds of $D$ through each point (which is only 0).

Moreover, remark that the involutive distribution spanned by $\frac{\partial}{\partial \theta_1}$ has the same integral manifolds as $D$. 

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Chapter 4

Globalizations of infinitesimal actions on supermanifolds

Let \( \mathfrak{g} \) be a finite-dimensional Lie superalgebra, \( \mathcal{G} \) a Lie supergroup with \( \mathfrak{g} \) as its Lie superalgebra of right-invariant vector fields, \( \mathcal{M} \) a supermanifold and denote the Lie superalgebra of vector fields on \( \mathcal{M} \) by \( \text{Vec}(\mathcal{M}) \). Any action \( \Psi : \mathcal{G} \times \mathcal{M} \to \mathcal{M} \), or local action, of the Lie supergroup \( \mathcal{G} \) on \( \mathcal{M} \) induces an infinitesimal action \( \mathfrak{g} \to \text{Vec}(\mathcal{M}) \) on \( \mathcal{M} \) as described in Proposition 2.2.3.

Starting with an infinitesimal action \( \lambda : \mathfrak{g} \to \text{Vec}(\mathcal{M}) \) of \( \mathcal{G} \) on \( \mathcal{M} \), it is a natural question to ask in which cases this infinitesimal action is induced by a local or global action of \( \mathcal{G} \) on \( \mathcal{M} \), or some larger supermanifold \( \mathcal{M}^* \) containing \( \mathcal{M} \) as an open subsupermanifold.

In the case of a smooth manifold \( M \) and a Lie group \( G \), Palais studied these questions in detail ([Pal57]). Concerning the existence of local actions, he proved that every infinitesimal action \( \lambda \) is induced by a local action of \( G \) on \( M \). This generalizes the fact that the flow of any vector field on \( M \) defines a local \( \mathbb{R} \)-action on \( M \).

In [Pal57], Palais also found necessary and sufficient conditions for the existence of a globalization, i.e. a (possibly non-Hausdorff) manifold \( M^* \), containing \( M \) as an open submanifold, with a \( G \)-action \( \varphi : G \times M^* \to M^* \) on \( M^* \) that induces the infinitesimal action \( \lambda \) on \( M \) and satisfies \( G \cdot M := \varphi(G \times M) = M^* \).

In this chapter, we extend these results to the case of (real or complex) supermanifolds and (real or complex) Lie supergroups. The existence of a local action with a given infinitesimal action and conditions for the existence of a globalization are proven. A key point in the proof is, as in the classical case in [Pal57], the study of the distribution \( \mathcal{D} = \mathcal{D}_\lambda \) on the product \( \mathcal{G} \times \mathcal{M} \) spanned by vector fields of the form

\[ X + \lambda(X) \text{ for } X \in \mathfrak{g}, \]

considering \( X \) and \( \lambda(X) \) as vector fields on the product \( \mathcal{G} \times \mathcal{M} \). Also, the fact that the flow of one even vector field on a supermanifold defines a local \( \mathbb{R} \)-action, or \( \mathbb{C} \)-action in the complex case, is used (see [MSV93] and [GW13]).

The results of this chapter are also published in [Ber14].
4.1 Distributions associated with infinitesimal actions

Given an infinitesimal action \( \lambda : \mathfrak{g} \to \text{Vec}(\mathcal{M}) \), define the distribution \( \mathcal{D} = \mathcal{D}_\lambda \) on \( \mathcal{G} \times \mathcal{M} \) as the distribution spanned by vector fields of the form \( X + \lambda(X) \) for \( X \in \mathfrak{g} \) (cf. [Pal57], Chapter II, Definition 7). The rank of the distribution \( \mathcal{D} \) equals the dimension of the Lie superalgebra \( \mathfrak{g} \).

If the homomorphism \( \lambda : \mathfrak{g} \to \text{Vec}(\mathcal{M}) \) is given, we can take \( \mathcal{G} \) to be any Lie supergroup with Lie superalgebra \( \mathfrak{g} \). One choice is for example the unique Lie supergroup with simply-connected underlying Lie group and Lie superalgebra \( \mathfrak{g} \) (for the existence of such \( \mathcal{G} \) see e.g. [Kos83], and in the complex case [Vis11]).

In the following, properties of the distribution associated with an infinitesimal action are studied.

**Lemma 4.1.1.** The distribution \( \mathcal{D} = \mathcal{D}_\lambda \) associated with an infinitesimal action \( \lambda \) is involutive.

**Proof.** Since \( \mathcal{D} \) is spanned by vector fields of the form \( X + \lambda(X) \) for \( X \in \mathfrak{g} \), it is enough to check that the bracket of two such vector fields belongs again to \( \mathcal{D} \). Using that \( \lambda \) is a homomorphism of Lie superalgebras we get

\[
[X + \lambda(X), Y + \lambda(Y)] = [X, Y] + [\lambda(X), \lambda(Y)] = [X, Y] + \lambda([X, Y])
\]

for any \( X, Y \in \mathfrak{g} \) and thus \( [X + \lambda(X), Y + \lambda(Y)] = [X, Y] + \lambda([X, Y]) \) also belongs to the distribution \( \mathcal{D} \).

The local version of Frobenius’ theorem now yields that the distribution \( \mathcal{D} \) associated with an infinitesimal action locally looks like a standard distribution on a product. In the following, local charts for the distribution \( \mathcal{D} \) which locally transform \( \mathcal{D} \) to a standard distribution and satisfy a few more properties with respect to the product structure of \( \mathcal{G} \times \mathcal{M} \) are of special interest.

**Definition 4.1.2** (flat chart). Let \( \mathcal{G} \) be a Lie supergroup, \( \mathcal{M} \) a supermanifold and \( \mathcal{D} \) the distribution on \( \mathcal{G} \times \mathcal{M} \) associated with an infinitesimal action on \( \mathcal{M} \). Let \( U \subseteq \mathcal{G} \) be an open connected neighbourhood of a point \( g \in \mathcal{G} \), \( U = (U, \mathcal{O}_\mathcal{G}|_U) \), and denote by \( t_g : \mathcal{M} \hookrightarrow \{g\} \times \mathcal{M} \subset \mathcal{G} \times \mathcal{M} \) the canonical inclusion. Moreover, let \( V \subseteq \mathcal{M} \) be open, \( \mathcal{V} = (V, \mathcal{O}_\mathcal{M}|_V) \), and let \( \rho : \mathcal{V} \to \mathcal{M} \) be a diffeomorphism onto its image. Denote by \( \mathcal{D}_\mathcal{G} \) the standard distribution on \( \mathcal{G} \times \mathcal{M} \) in \( \mathcal{G} \)-direction, which is spanned by vector fields \( X \otimes \text{id}_\mathcal{M} \) on \( \mathcal{G} \times \mathcal{M} \), where \( X \) is an arbitrary vector field on \( \mathcal{G} \).

A diffeomorphism onto its image \( \psi : U \times \mathcal{V} \to \mathcal{G} \times \mathcal{M} \) is called a flat chart with respect to \((\mathcal{D}, U, V, g, \rho)\), or simply a flat chart (in \( g \)), if the following conditions are satisfied:

(i) \( \psi_* (\mathcal{D}_\mathcal{G}|_W) = \mathcal{D}|_{\psi(W)} \) for each open subset \( W \subseteq U \times V \),

(ii) \( \pi_\mathcal{G}|_{U \times V} = \pi_\mathcal{G} \circ \psi \) for the projection \( \pi_\mathcal{G} : \mathcal{G} \times \mathcal{M} \to \mathcal{G} \),

(iii) \( \psi \circ t_g|_V = t_g|_{\psi(V)} \circ \rho \).

(1) The vector field \( X + \lambda(X) \) is here considered as a vector field on \( \mathcal{G} \times \mathcal{M} \), so more formally one should write \( X \otimes \text{id}_\mathcal{M} + \text{id}_\mathcal{G} \otimes \lambda(X) \) for \( X + \lambda(X) \).
Remark 4.1.3. If \( \psi : U \times V_1 \to \mathcal{G} \times M \) is a flat chart with respect to \((\mathcal{D}, U, V_1, g, \rho_1)\) and \( \rho_2 : V_2 \to M \) is a diffeomorphism onto its image with \( \hat{\rho}_2(V_2) \subseteq V_1 \), then the map \( \psi' = \psi \circ (\text{id}_U \times \hat{\rho}_2) \) is a flat chart with respect to \((\mathcal{D}, U, V_2, g, \rho_1 \circ \hat{\rho}_2)\).

Lemma 4.1.4. Let \( \mathcal{D} \) be the distribution on \( \mathcal{G} \times M \) associated with the infinitesimal action \( \lambda : g \to \text{Vec}(\mathcal{M}) \), \( \lambda_0 = \lambda|_{g_0} : g_0 \to \text{Vec}(\mathcal{M}) \) the restriction of \( \lambda \) to the even part \( g_0 \) of \( g \), and \( \mathcal{D}_0 \) the distribution on \( \mathcal{G} \times M \) associated with \( \lambda_0 \). Let \( \psi : U \times V \to \mathcal{G} \times M \) be a flat chart with respect to \((\mathcal{D}, U, V, g, \rho)\) and define \( \psi_0 : U \times V \to G \times M \) by \( \psi_0 = (\text{id}_G \times \varphi_0) \circ (\text{diag} \times \text{id}_V) \), where \( \text{diag} : U \to U \times U \) denotes the diagonal map and \( \varphi_0 \) is the composition of \( \pi_M \circ \psi \) and the canonical inclusion \( U \times V \to U \times V \). Then \( \psi_0 \) is a flat chart with respect to \((\mathcal{D}_0, U, V, g, \rho)\).

Proof. It can be checked by direct calculations that \( \psi_0 \) is a flat chart, using that the even right-invariant vector fields on \( G \) can be identified with the right-invariant vector fields on \( G \), i.e. \( \text{Lie}(G) \cong g_0 \) if \( g_0 \) denotes the even part of the Lie superalgebra \( g = g_0 + g_1 \) of \( G \).

Proposition 4.1.5 (Local existence of flat charts). Let \( \mathcal{D} \) be the distribution associated with the infinitesimal action \( \lambda : g \to \text{Vec}(\mathcal{M}) \) on the supermanifold \( \mathcal{M} \) and let \( \mathcal{G} \) be a Lie supergroup with \( g \) as its Lie superalgebra of right-invariant vector fields. For any point \( (g, p) \in G \times M \) there are an open connected neighbourhood \( U \) of \( g \) in \( G \) and an open neighbourhood \( V \) of \( p \) in \( M \) such that there exists a flat chart \( \psi : U \times V \to \mathcal{G} \times M \) with respect to \((\mathcal{D}, U, V, g, \rho)\) for arbitrary \( \rho \).

Proof. Let \( X_1, \ldots, X_k, Y_1, \ldots, Y_l \) be a basis of \( g \) such that \( X_1, \ldots, X_k \) are even and \( Y_1, \ldots, Y_l \) odd vector fields. Then the tangent vectors

\[
X_1(g'), \ldots, X_k(g'), Y_1(g'), \ldots, Y_l(g') \in T_g\mathcal{G} = \{X(g') | X \in g\}
\]

are linearly independent for all \( g' \in G \). Since the distribution \( \mathcal{D} \) is spanned by vector fields of the form \( X + \lambda(X) \) for \( X \in g \), there exist local coordinates \((t, \tau)\) for \( \mathcal{G} \) on an open connected neighbourhood \( U \subseteq G \) of \( g \) and local coordinates \((x, \theta)\) for \( \mathcal{M} \) on an open neighbourhood \( V \subseteq M \) of \( p \) so that \( \mathcal{D} \) is locally spanned by the commuting vector fields

\[
A_i = \frac{\partial}{\partial t_i} + \sum_{u=1}^{m} a_{iu} \frac{\partial}{\partial x_u} + \sum_{v=1}^{n} b_{iv} \frac{\partial}{\partial \theta_v} \quad \text{and} \quad B_j = \frac{\partial}{\partial \tau_j} + \sum_{u=1}^{m} c_{ju} \frac{\partial}{\partial x_u} + \sum_{v=1}^{n} d_{jv} \frac{\partial}{\partial \theta_v}
\]

for \( i = 1, \ldots, k \) and \( j = 1, \ldots, l \), where \( a_{iu}, b_{iv}, c_{ju}, d_{jv} \in \mathcal{O}_G \times \mathcal{M}(U \times V) \).

After shrinking \( U \) and \( V \), \( U = (U, \mathcal{O}_G|_U) \) can be assumed to be an open subsupermanifold of \( \mathbb{R}^{k|l} \) with \( g = 0 \) and there is a morphism \( \varphi : U \times (U \times V) \to \mathcal{G} \times M \) associated with the above defined commuting vector fields, satisfying the properties \( \varphi \circ t_0 = \text{id} \) and \( \varphi \circ (\text{id}_{\mathbb{R}^{k|l}} \times \varphi) = \varphi \circ (\mu_{\mathbb{R}^{k|l}} \times \text{id}_M) \) (cf. Theorem 3.1.1). Since

\[
\frac{\partial}{\partial t_i} \circ \varphi^* = \varphi^* \circ A_i \quad \text{and} \quad \frac{\partial}{\partial t_j} \circ \varphi^* = \varphi^* \circ B_j,
\]

the subsupermanifold \( \mathcal{V} \cong \{0\} \times V \subset U \times V \) is transverse to \( \varphi(U \times \{(0, p)\}) \) in \( \varphi(0, 0, p) = (0, p) \). The map \( \psi : U \times V \to \mathcal{G} \times M \), \( \psi = \varphi|_{U \times \{0\} \times V} \), identifies the standard distribution.

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\[ \mathcal{D}_\mathcal{U} \text{ on } \mathcal{U} \times \mathcal{V} \text{ with } \mathcal{D}, \text{ i.e. } \psi_* (\mathcal{D}_G|_W) = \psi_* (\mathcal{D}_U|_W) = \mathcal{D}|_{\tilde{\psi}(W)} \text{ for all open subsets } W \subseteq U \times V. \] The action property of the map \( \varphi \) moreover implies that

\[ \psi \circ t_g|_V = \psi \circ \iota_0 |_V = \varphi|_{\mathcal{U} \times \{0\} \times V} = \varphi|_{\mathcal{U} \times \{0\} \times V} \circ \iota_0 |_V = \text{id}_{\mathcal{U} \times \{0\} \times V} = \iota_0 |_V = t_g|_V. \]

Hence, it only remains to show that \( \pi_G|_{U \times V} = \pi_G \circ \psi \) in order to prove that \( \psi \) is a flat chart. This is equivalent to showing \( \psi^*(t_i) = t_i \) and \( \psi^*(\tau_j) = \tau_j \) for all \( i, j \), where \( t_i \) and \( \tau_j \) are now considered as local coordinate functions on \( G \times M \). Since \( \frac{\partial}{\partial x^r}(\psi^*(t_i)) = \psi^*(\frac{\partial}{\partial x^r}(t_i)) = \psi^*(0) = 0 \) for all \( r \neq i \), \( \frac{\partial}{\partial x^r}(\psi^*(t_i)) = 1 \) and \( \frac{\partial}{\partial x^r}(\psi^*(t_i)) = \psi^*(\psi^*(\frac{\partial}{\partial x^r}(t_i))) = \psi^*(0) = 0 \) for all \( s \), the function \( \psi^*(t_i) \in \mathcal{O}_{G \times M}(U \times V) \) is of the form

\[ \psi^*(t_i) = (t_i + c_i(x)) + \sum_{\nu \neq 0} c_\nu(x) \theta^\nu \]

for some smooth functions \( c_i, c_\nu(\nu \neq 0) \) on \( V \). The property \( \psi \circ \iota_0 = \iota_0 \) now implies \( 0 = t_0^\nu(t_i) = t_0^\nu \circ \psi^*(t_i) = (0 + c_i(x)) + \sum_{\nu \neq 0} c_\nu(x) \theta^\nu \) and therefore \( c_i = c_\nu = 0 \). Hence \( \psi^*(t_i) = t_i \) as required. A similar argument yields \( \psi^*(\tau_j) = \tau_j \).

**Lemma 4.1.6.** Let \( \psi : \mathcal{U} \times \mathcal{V} \to G \times M \) be a diffeomorphism onto its image such that

(i) \( \psi_* (\mathcal{D}_G|_W) = \mathcal{D}|_{\tilde{\psi}(W)} \) for a distribution \( \mathcal{D} \) associated with an infinitesimal action
and any open subset \( W \subseteq U \times V \), and

(ii) \( \pi_G|_{U \times V} = \pi_G \circ \psi \).

Then given any element \( g \in U \) there exists a diffeomorphism onto its image \( \rho : \mathcal{V} \to M \) such that \( \psi \circ t_g|_V = t_g|_{\tilde{\rho}(\mathcal{V})} \circ \rho \). Hence \( \psi \) is a flat chart with respect to \( (\mathcal{D}, U, V, \rho) \).

**Proof.** The property \( \pi_G \circ \psi = \pi_G|_{U \times V} \) implies \( (\psi \circ t_g)(V) \subseteq \{g\} \times M \). To show that \( \psi \circ t_g|_V : \mathcal{V} \to G \times M \) induces a map \( \rho : \mathcal{V} \to M \cong \{g\} \times M \), it is enough to check that \( (\psi \circ t_g)^*(\pi_G^* \circ \psi) = \text{ev}_g \), where \( \text{ev}_g : \mathcal{O}_G(G) \to \mathbb{R} \) denotes the evaluation in \( g \).

The fact \( \pi_G \circ \psi = \pi_G|_{U \times V} \) implies \( (\psi \circ t_g)^*(\pi_G^* \circ \psi) = (\pi_G \circ \psi \circ t_g)|_V )^* = (\pi_G|_{U \times V} \circ t_g|_V )^* = \text{ev}_g \) since \( (\pi_G \circ t_g) \) is the unique map \( M \to \{g\} \subset G \).

The map \( \rho \) satisfies \( \psi \circ t_g|_V = t_g|_{\tilde{\rho}(\mathcal{V})} \circ \rho \) by definition and is a diffeomorphism onto its image since \( \psi \) is a diffeomorphism onto its image and \( \mathcal{V} \) and \( \mathcal{M} \) have the same dimension. \( \square \)

**Proposition 4.1.7 (Uniqueness of flat charts).** If \( \mathcal{D} \) is a distribution on \( G \times M \) associated to an infinitesimal action \( \lambda : \mathfrak{g} \to \text{Vec}(M) \), then a flat chart \( \psi : \mathcal{U} \times \mathcal{V} \to G \times M \) with respect to \( (\mathcal{D}, U, V, g, \rho) \) is unique.

The proof of this proposition is carried out in two steps:

(i) First, it is shown that two flat charts \( \psi_1 \) and \( \psi_2 \) with respect to \( (\mathcal{D}, U, V, g, \rho) \)
coincide on an open neighbourhood of \( \{g\} \times V \) in \( U \times V \).

(ii) Second, the local statement and the connectedness of \( U \) are used to globally get \( \psi_1 = \psi_2 \).
Proof. (i) Consider first the case where \( r = \text{id} \) and \( D = D_G \), i.e. the infinitesimal action \( \lambda \) is the zero map. Now, let \( \psi \) be a flat chart with respect to \((D_G, U, V, g, \text{id})\). Since the identity \( \text{id}_{G \times M(U \times V)} \) is also a flat chart with respect to \((D_G, U, V, g, \text{id})\), the equality of \( \psi \) and the identity in a neighbourhood of any point \((g, p)\) for \( p \in V \) needs to be shown. Let \((t, \tau)\) be local coordinates on a connected neighbourhood \( U' \) of \( g \in G \) such that \( g = 0 \) in these coordinates and let \((x, \theta)\) be local coordinates on a neighbourhood \( V' \) of \( p \in V \). Then \((t, x, \theta)\) are local coordinates for \( G \times M \) in a neighbourhood of \((0, p) = \hat{\psi}(0, p)\) and therefore on a neighbourhood of \( \hat{\psi}(U' \times V') \subseteq U' \times V' \) for appropriate subsets \( U'' \subseteq U' \) and \( V'' \subseteq V' \). Since \( \pi_G|_{U \times V} = \pi_G \circ \psi \), we have

\[
\hat{\psi}^*(t, \tau) = \hat{\psi}^*(\pi_G^*(t, \tau)) = \pi_G^*(t, \tau) = (t, \tau).
\]

Moreover, \( \psi_*(D_G) = D_G \) implies

\[
\psi_*(\frac{\partial}{\partial t_r}), \psi_*(\frac{\partial}{\partial \tau_s}) \in \text{span}\left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_k}, \frac{\partial}{\partial \tau_1}, \ldots, \frac{\partial}{\partial \tau_l}\right\}
\]

for all \( r, s \). Hence

\[
\psi_*(\frac{\partial}{\partial t_r})(x, \theta) = \psi_*(\frac{\partial}{\partial \tau_s})(x, \theta) = (0, 0),
\]

and then \( \frac{\partial}{\partial t_r} \psi^*(x, \theta) = \psi^*(\psi_*(\frac{\partial}{\partial t_r})(x, \theta)) = \psi^*(0, 0) = (0, 0) \), and similarly we get \( \frac{\partial}{\partial \tau_s} \psi^*(x, \theta) = (0, 0) \). Consequently, we have

\[
\psi^*(x, \theta) = \iota_0 \circ \psi^*(x, \theta) = \iota_0^*(x, \theta) = (x, \theta)
\]

and thus \( \psi = \text{id} \) on \( U'' \times V'' \).

Let \( \psi_1 \) and \( \psi_2 \) now be flat charts with respect to \((D, U, V, g, \rho)\) for a distribution \( D \) associated with an arbitrary infinitesimal action and arbitrary \( \rho \). For any \( p \in V \), we have

\[
\hat{\psi}_1(g, p) = \hat{\psi}_1 \circ \hat{\iota}_g(p) = \hat{\iota}_g(\hat{\rho}(p)) = \hat{\psi}_2 \circ \hat{\iota}_g(p) = \hat{\psi}_2(g, p).
\]

Now let \( U' \) and \( U'' \) be open connected neighbourhoods of \( g \) and \( V' \) and \( V'' \) open neighbourhoods of \( p \) such that \((\hat{\psi}_1(U' \times V'), \hat{\psi}_2(U'' \times V'')) \) is contained in \( \hat{\psi}_2(U' \times V') \). The composition \( \psi_2^{-1} \circ \psi_1 \) is defined on \( U'' \times V'' \) and a flat chart with respect to \((D_G, U'', V'', g, \text{id})\). By the above argumentation, \( \psi_2^{-1} \circ \psi_1 = \text{id} \) and thus \( \psi_1 = \psi_2 \) on \( U'' \times V'' \).

(ii) Let \( \psi_1 \) and \( \psi_2 \) be two flat charts with respect to \((D, U, V, g, \rho)\). For each \( p \in V \) define the subset \( W_p \subseteq U \) containing the points \( t \in U \) such that \( \psi_1 = \psi_2 \) on an open neighbourhood of \((t, p)\) in \( U \times V \). The sets \( W_p \) are open by definition and contain \( g \) as a consequence of (i). To prove \( \psi_1 = \psi_2 \), i.e. \( W_p = U \) for each \( p \), it is therefore enough to show that each \( W_p \) is also closed in \( U \) due to the connectedness of \( U \).

If \( W_p \) is not closed, then there is a point \( t_0 \in U \setminus W_p \) such that \( W_p \cap \Omega \neq \emptyset \) for every open neighbourhood \( \Omega \) of \( t_0 \). The continuity of the underlying maps implies \( \hat{\psi}_1(t_0, p) = \hat{\psi}_2(t_0, p) \). Let \( U' \) be an open neighbourhood of \( t_0 \) in \( U \) and \( V' \) an open neighbourhood of \( p \) in \( V \) such that the associated open subsupermanifolds \( U' \) and \( V' \) are isomorphic to superdomains. Now let \( U'' \subseteq U' \) be an open connected neighbourhood
of $t_0$ and $V'' \subseteq V'$ an open neighbourhood of $p$ such that $\tilde{\psi}_1(U'' \times V'') \subseteq \tilde{\psi}_2(U' \times V')$. Moreover, let $s_0$ be an element of $U'' \cap W_p$ which exists by the choice of $t_0$. After shrinking $V'$ and $V''$, the maps $\psi_1$ and $\psi_2$ coincide on an open neighbourhood of $\{s_0\} \times V''$. By Lemma 4.1.6 there exists a diffeomorphism onto its image $\rho_0 : V' \rightarrow \mathcal{M}$ such that the restrictions of $\psi_1$ and $\psi_2$ to $U' \times V'$ are flat charts with respect to $(\mathcal{D},U',V',s_0,\rho_0)$. The same argument as given in (i) then shows that $\psi_1$ and $\psi_2$ coincide on $U'' \times V''$ which is a contradiction to the assumption $t_0 \notin W_p$.

**Remark 4.1.8.** Let $\psi$ and $\psi'$ be flat charts and denote by $\psi_0$ and $\psi'_0$ the associated flat charts for the even part (cf. Lemma 4.1.4). Then $\psi = \psi'$ if and only if $\psi_0 = \psi'_0$.

Moreover, if $\psi_0 : U \times \mathcal{V} \rightarrow G \times \mathcal{M}$ is a flat chart with respect to $(\mathcal{D},U,V,g,\rho)$ then the uniqueness and local existence of flat charts imply that there is a flat chart $\psi : U \times \mathcal{V} \rightarrow G \times \mathcal{M}$ with respect to $(\mathcal{D},U,V,g,\rho)$.

### 4.2 Existence of local actions

In the classical case, the flow $\varphi^X : \Omega \subseteq \mathbb{R} \times M \rightarrow M$ of a vector field $X$ on a manifold $M$ defines a local $\mathbb{R}$-action on $M$. More generally, as proven in [Pal57], Chapter II, if we have a Lie algebra homomorphism $\lambda : g \rightarrow \text{Vec}(M)$ of a finite dimensional Lie algebra $g$ into the Lie algebra of vector fields on $M$, there is a local action $\varphi : \Omega \subseteq G \times M \rightarrow M$ of a Lie group $G$, with Lie algebra of right-invariant vector fields $g$, such that its induced infinitesimal action

$$g \rightarrow \text{Vec}(M), \quad X \mapsto \left. \frac{\partial}{\partial t} \right|_{t=0} \varphi(\exp(tX),-) = (X(e) \otimes \text{id}_M^*) \circ \varphi^*,$$

coincides with $\lambda$. A typical example is the case where $X_1, \ldots, X_k$ are vector fields on $M$ whose $\mathbb{R}$-span $g = \text{span}_{\mathbb{R}}\{X_1, \ldots, X_k\}$ is a Lie subalgebra of $\text{Vec}(M)$ and $\lambda$ is the inclusion $g \hookrightarrow \text{Vec}(M)$.

The goal in this section is the proof of an analogous theorem on supermanifolds. Important for the proof is the study of the distribution associated with an infinitesimal action, as in the classical case in [Pal57].

This also generalizes the result in [MSV93] and [GW13] saying that the flow of one vector field $X$ on a supermanifold $\mathcal{M}$ is a local $\mathbb{R}^{1|1}$-action if and only if $X$ is contained in a $(1|1)$-dimensional Lie subsuperalgebra $g \subseteq \text{Vec}(\mathcal{M})$.

The results in this section are formulated for the real case, but are equally true in the complex case, i.e. for complex supermanifolds $\mathcal{M}$, complex Lie supergroups $\mathcal{G}$ and morphisms $\lambda : g \rightarrow \text{Vec}(\mathcal{M})$ of complex Lie superalgebras.

As in the classical case, there is an equivalence of infinitesimal actions and local actions up to shrinking. This is the content of the following theorem whose proof is carried out in the remainder of this section.

**Theorem 4.2.1.** Let $\lambda : g \rightarrow \text{Vec}(\mathcal{M})$ be an infinitesimal action. Then there exists a local $\mathcal{G}$-action $\varphi : \mathcal{W} \subseteq \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$ on $\mathcal{M}$ such that its induced infinitesimal action $\lambda_\varphi$ equals $\lambda$.

Moreover, any local action $\varphi : \mathcal{W} \rightarrow \mathcal{M}$ is uniquely determined by its induced infinitesimal action and domain of definition.
First, the uniqueness of a local action up to shrinking is proven. Then we describe how a local action can be constructed from an infinitesimal action using the flat charts of the distribution studied in the preceding section.

### 4.2.1 Uniqueness of local actions

We need the following lemma to prove the uniqueness of local actions with a prescribed infinitesimal action up to shrinking of the domain of definition.

**Lemma 4.2.2.** Let $\varphi : W \subseteq G \times M \to M$ be the local action of the Lie supergroup $G$ on a supermanifold $M$ with induced infinitesimal action $\lambda_\varphi$. Let $D$ denote the distribution associated with $\lambda_\varphi$. Define

$$\psi = (\text{id}_G \times \varphi) \circ (\text{diag} \times \text{id}_M) : W \to G \times M$$

where $\text{diag} : G \to G \times G$ denotes the diagonal map.

Let $p \in M$ and $U \subset W_p = \{ g \in G | (g,p) \in W \}$ be a relatively compact connected open neighbourhood of $e \in W_p \subset G$. Then there exists an open neighbourhood $V$ of $p \in M$ with $U \times V \subset W$. The restriction $\psi|_{U \times V}$ is a flat chart with respect to $(D,U,V,e,\text{id})$.

**Proof.** Since $\overline{U} \times \{p\} \subset W$ is compact, we can find an open neighbourhood $V$ of $p$ with $U \times V \subset W$. By definition of $\psi$, we have $\pi_G \circ \psi = \pi_G$. Moreover, $\psi \circ i_e = i_e$ since $\varphi$ is a local action.

Let $X \in \mathfrak{g}$, then $X \circ \text{diag}^* = \text{diag}^* \circ (X \otimes \text{id}_G^* + \text{id}_G^* \otimes X)$ since $X$ is a derivation. If $\iota_G^G : G \to \{e\} \times G \subset G \times G$ is the inclusion, then $\mu \circ \iota_G^G = \text{id}_G$ for the multiplication $\mu$ on $G$. Since $X$ is right-invariant, we have $X = (X(e) \otimes \iota_G^G) \circ \mu^*$. By a calculation, using these facts and that $\varphi$ is a local action, we obtain

$$\psi_*(X \otimes \text{id}^*) = X \otimes \text{id}^* + \text{id}^* \otimes \lambda_\varphi(X),$$

which yields $\psi_*(D_G) = D$. \hfill \fbox

The lemma implies that every local action of a Lie supergroup is uniquely determined by its domain of definition and its induced infinitesimal action:

**Corollary 4.2.3.** The domain of definition of a local action and the induced infinitesimal action uniquely determine the local action, i.e. if $\varphi_1 : W \to G \times M$ and $\varphi_2 : W \to G \times M$ are local actions of the Lie supergroup $G$ on the supermanifold $M$ with the same induced infinitesimal action $\lambda = \lambda_{\varphi_1} = \lambda_{\varphi_2}$, then $\varphi_1 = \varphi_2$.

**Proof.** By the preceding lemma and the uniqueness of flat charts we have $\psi_1 = (\text{id}_G \times \varphi_1) \circ (\text{diag} \times \text{id}_M)$ and $\psi_2 = (\text{id}_G \times \varphi_2) \circ (\text{diag} \times \text{id}_M)$ and hence $\varphi_1 = \pi_M \circ \psi_1 = \pi_M \circ \psi_2 = \varphi_2$. \hfill \fbox

### 4.2.2 Construction of local actions

In the following, let $\lambda : \mathfrak{g} \to \text{Vec}(M)$ be a fixed infinitesimal action and $G$ a Lie supergroup with multiplication $\mu : G \times G \to G$ and Lie superalgebra of right-invariant vector fields $\mathfrak{g}$. 43
The goal is to find a local $\mathcal{G}$-action on $M$ with induced infinitesimal action $\lambda$. Such a local action of $\mathcal{G}$ on $M$ is constructed using flat charts for the distribution $\mathcal{D}$ associated with $\lambda$. The domain of definition of the constructed action depends, in general, on some choices. After restricting two local actions with the same infinitesimal action to a neighbourhood of $\{e\} \times M$ in $G \times M$, the local actions coincide as proven in the previous paragraph. Nevertheless, in general there is, as in the classical case (cf. [Pal57] Chapter III.4), no unique maximal domain of definition on which the local action can be defined.

**Definition of a local action:**

Choose a neighbourhood basis $\{U_\alpha\}_{\alpha \in A}$ of the identity $e \in G$ such that (cf. [Pal57], Chapter II, §7):

(i) Each $U_\alpha$ is connected and $U_\alpha = (U_\alpha)^{-1} = \{g \in G | g^{-1} \in U_\alpha\}$.

(ii) For $\alpha, \beta \in A$ either $U_\alpha \subseteq U_\beta$ or $U_\beta \subseteq U_\alpha$ holds.

Note that the two conditions guarantee the connectedness of $U_\alpha \cap U_\beta$ for arbitrary $\alpha, \beta \in A$.

For each $p \in M$ choose $\alpha(p) \in A$ and a neighbourhood $V_p \subseteq M$ of $p$ such that there is a flat chart

$$
\psi_p : U^2_{\alpha(p)} \times V_p \to \mathcal{G} \times M
$$

with respect to $(\mathcal{D}, U^2_{\alpha(p)}, V_p, e, id)$, where $U^2_{\alpha(p)}$ and $V_p$ denote again the open subsupermanifolds of $\mathcal{G}$ and $M$ with underlying sets $U^2_{\alpha(p)} = \{gh : g, h \in U_\alpha\}$ and $V_p$. For two elements $p, q \in M$, we may assume $U_{\alpha(p)} \subseteq U_{\alpha(q)}$. Therefore, if the intersection

$$(U^2_{\alpha(p)} \times V_p) \cap (U^2_{\alpha(q)} \times V_q) = U^2_{\alpha(p)} \times (V_p \cap V_q)$$

is non-empty, then the restrictions of $\psi_p$ and $\psi_q$ to their common domain of definition are both flat charts with respect to $(\mathcal{D}, U^2_{\alpha(p)}, (V_p \cap V_q), e, id)$ and hence coincide. Let

$$W = \bigcup_{p \in M} (U_{\alpha(p)} \times V_p) \subseteq G \times M.$$ 

The set $W$ is open by definition and contains $\{e\} \times M$. Furthermore, for each $p \in M$ the subset $W_p = \{g \in G | (g, p) \in W\} \subseteq G$ is connected since all $U_{\alpha(q)}$ are connected. Let $\mathcal{W} = (W, C_{\mathcal{G} \times M}|W)$ and define a morphism $\varphi : \mathcal{W} \to \mathcal{G} \times M$ by $\psi|_{U_{\alpha(p)} \times V_p} = \psi_p$ for each $p \in M$. Let

$$\varphi : \mathcal{W} \to M, \varphi = \pi_M \circ \psi.$$ 

We now show that $\varphi$ defines a local group action with induced infinitesimal action $\lambda$.

**Proposition 4.2.4.** The map $\varphi$ defines a local action of $\mathcal{G}$ on the supermanifold $M$.

**Proof.** Since the map $\psi$ defining $\varphi = \pi_M \circ \psi$ is locally given by flat charts with respect to $(\mathcal{D}, U^2_{\alpha(p)}, V_p, e, id)$, we have $\psi \circ \iota_e = \iota_e$ and therefore $\varphi \circ \iota_e = \pi_M \circ \psi \circ \iota_e = \pi_M \circ \iota_e = id_M$. Thus it remains to show that

$$\varphi \circ (\mu \times id_M) = \varphi \circ (id_G \times \varphi) \quad (*)$$
on the open subsupermanifold of $G \times G \times M$ on which both sides are defined. To prove ($\ast$) the special form of the distribution associated with an infinitesimal action with respect to the group structure of $G$ is used. The commutativity of the following diagram will be shown:

\[
\begin{array}{ccc}
G \times G \times M & \xrightarrow{\chi \times \text{id}_M} & G \times G \times M \\
\downarrow \text{id}_G \times \psi & & \downarrow (\chi^{-1} \times \text{id}_M) \circ (\text{id}_G \times \psi) \\
G \times G \times M & \xrightarrow{\tau \times \text{id}_M \circ (\text{id}_G \times \psi) \circ (\tau \times \text{id}_M)} & G \times G \times M
\end{array}
\]

In the above diagram all maps are only defined on appropriate open subsupermanifolds of $G \times G \times M$. The map $\tau : G \times G \to G \times G$ denotes the map which interchanges the two components. Moreover, the map $\chi : G \times G \to G \times G$ is defined by

\[
\chi = (\text{id}_G \times \mu) \circ (\text{diag} \times \text{id}_G),
\]

such that $\mu = \pi_2 \circ \chi$ and $\pi_i = \pi_1 \circ \chi$, where $\pi_i$, $i = 1, 2$, is the projection onto the $i$-th factor. The underlying map is given by $\tilde{\chi}(g, h) = (g, gh)$.

Note that if $G = M$ and the infinitesimal action $\lambda$ is the canonical inclusion $\mathfrak{g} \hookrightarrow \text{Vec}(G)$ of the right-invariant vector fields, then $\chi$ is a flat chart for the distribution $\mathcal{D}$ with respect to $(\mathcal{D}, G, M, e, \text{id})$, $\chi$ and $\psi$ coincide (on their common domain of definition) and ($\ast$) is equivalent to the associativity of the multiplication $\mu$.

Let

\[
\Psi_1 = (\tau \times \text{id}_M) \circ (\text{id}_G \times \psi) \circ (\tau \times \text{id}_M) \circ (\text{id}_G \times \psi)
\]

and

\[
\Psi_2 = (\chi^{-1} \times \text{id}_M) \circ (\text{id}_G \times \psi) \circ (\chi \times \text{id}_M).
\]

The underlying maps are given by $\tilde{\Psi}_1(g, h, p) = (g, h, \tilde{\varphi}(g, \tilde{\varphi}(h, p)))$ and $\tilde{\Psi}_2(g, h, p) = (g, h, \tilde{\varphi}(gh, p))$. The open subsupermanifold of $G \times G \times M$ on which both morphisms $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are defined is exactly the open subsupermanifold which is the common domain of definition of $\varphi \circ (\mu \times \text{id}_M)$ and $\varphi \circ (\text{id}_G \times \varphi)$. A calculation shows $\pi_M \circ \tilde{\Psi}_1 = \varphi \circ (\text{id}_G \times \varphi)$ and $\pi_M \circ \tilde{\Psi}_2 = \varphi \circ (\mu \times \text{id}_M)$ and the commutativity of the above diagram directly implies ($\ast$).

To show the equality of $\Psi_1$ and $\Psi_2$, we consider the distribution $\mathcal{D}_1 \otimes \lambda$ on $G \times (G \times M)$ associated with the infinitesimal action $1 \otimes \lambda : g \to \text{Vec}(G \times M)$, $X \mapsto (\text{id}_G^\ast \otimes \lambda(X))$, of $G$ on $G \times M$ and show that $\Psi_1$ and $\Psi_2$ are both locally flat charts for this distribution.

Let $\mathcal{D}_G$ be the distribution on $G \times G \times M$ which is spanned by vector fields of the form $X \otimes \text{id}_G^\ast \otimes \text{id}_M^\ast$ for $X \in \mathfrak{g}$. The distribution $\mathcal{D}_1 \otimes \lambda$ is spanned by vector fields of the form $X \otimes \text{id}_G^\ast \otimes \text{id}_M^\ast + \text{id}_G^\ast \otimes \text{id}_G^\ast \otimes \lambda(X)$. For $X \in \mathfrak{g}$ and writing $\mathbb{1}$ for $\text{id}_G^\ast$ or $\text{id}_M^\ast$, we have

\[
\begin{align*}
(\Psi_1)_*(X \otimes \mathbb{1} \otimes \mathbb{1}) &= (\tau \times \text{id}_M)_*(\text{id}_G \times \psi)_*(\tau \times \text{id}_M)_*(\text{id}_G \times \psi)_*(X \otimes \mathbb{1} \otimes \mathbb{1}) \\
&= (\tau \times \text{id}_M)_*(\text{id}_G \times \psi)_*(\mathbb{1} \otimes X \otimes \mathbb{1}) \\
&= (\tau \times \text{id}_M)_*(\mathbb{1} \otimes X \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \lambda(X)) \\
&= X \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \lambda(X),
\end{align*}
\]
where the fact that $\psi$ is a flat chart for the distribution $\mathcal{D}$ associated with $\lambda$ is used. Similarly, using the fact that $\chi_* (X \otimes 1) = (X \otimes 1 \otimes 1 \otimes X)$, we get $(\Psi_2)_* (X \otimes 1 \otimes 1 \otimes X) = X \otimes 1 \otimes 1 \otimes 1 \otimes \lambda (X)$. Hence, $\Psi_1$ and $\Psi_2$ both transform $\mathcal{D}_G$ into $\mathcal{D}_{1 \otimes \lambda}$, i.e. $(\Psi_1)_* (\mathcal{D}_G) = \mathcal{D}_{1 \otimes \lambda}$.

Remark that $\pi_G \circ \Psi_1 = \pi_G = \pi_G \circ \Psi_2$ if $\pi_G : \mathcal{G} \times \mathcal{G} \times \mathcal{M} \to \mathcal{G}$ denotes the projection onto the first component. Now, let $\iota_e^{G \times \mathcal{M}} : \mathcal{G} \times \mathcal{M} \mapsto \{e\} \times \mathcal{G} \times \mathcal{M} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{M}$ and $\iota_e^\mathcal{M} : \mathcal{M} \mapsto \{e\} \times \mathcal{M} \subset \mathcal{G} \times \mathcal{M}$ denote the inclusions. Then a calculation shows

$$
\Psi_1 \circ \iota_e^{G \times \mathcal{M}} = \iota_e^{G \times \mathcal{M}} \circ \psi \quad \text{and} \quad \Psi_2 \circ \iota_e^{G \times \mathcal{M}} = \iota_e^{G \times \mathcal{M}} \circ \psi
$$

since $\psi \circ \iota_e^\mathcal{M} = \iota_e^\mathcal{M}$ and $(\chi \times \text{id}_\mathcal{M}) \circ \iota_e^{G \times \mathcal{M}} = \iota_e^{G \times \mathcal{M}}$ using that $e$ is the identity element of $\mathcal{G}$ and the definition of $\chi$. This shows that $\Psi_1$ and $\Psi_2$ are both locally flat charts in $e$.

In order to check that $\Psi_1$ and $\Psi_2$ coincide everywhere, the special form of the sets in $\{U_\alpha\}_{\alpha \in A}$ is important. Let $(g, g', p) \in \mathcal{G} \times \mathcal{G} \times \mathcal{M}$ such that $\Psi_1$ and $\Psi_2$ are defined on a neighbourhood of $(g, g', p)$, i.e. such that $(g', p), (g, \varphi(g, p)) \in W$. Then by definition of $\psi$, there exists $U_\alpha \in \{U_\gamma\}_{\gamma \in A}$ containing $g$, a neighbourhood $V_\alpha$ of $q = \varphi(g', p)$ in $M$ and a flat chart $\psi_\alpha : U_\alpha^2 \times V_\alpha \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U_\alpha^2, V_\alpha, e, \text{id})$ and with $\psi|_{U_\alpha \times V_\alpha} = \psi_\alpha|_{U_\alpha \times V_\alpha}$. Furthermore, choose $U_\beta \in \{U_\gamma\}_{\gamma \in A}$ containing $g'$ and $gg'$ and a neighbourhood $V_\beta$ of $p$ in $M$ such that there exists a flat chart $\psi_\beta : U_\beta^2 \times V_\beta \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U_\beta^2, V_\beta, e, \text{id})$ and with $\psi|_{U_\beta \times V_\beta} = \psi_\beta|_{U_\beta \times V_\beta}$. Shrink $V_\alpha$ and choose a neighbourhood $U' \subset \mathcal{G}$ with $U \subseteq U_\beta$ such that $\varphi(U \times V_\beta) \subseteq V_\alpha$. By the special choice of the neighbourhood basis $\{U_\gamma\}_{\gamma \in A}$ of $e$ in $G$ either $U_\alpha \subseteq U_\beta$ or $U_\beta \subseteq U_\alpha$ is true.

First, suppose that $U_\alpha \subseteq U_\beta$. The map $\Psi_1$ is defined on $U_\alpha \times U \times V_\beta$ and the restriction of $\Psi_1$ to $U_\alpha \times U \times V_\beta$ is a flat chart with respect to $(\mathcal{D}_{1 \otimes \lambda}, U_\alpha, U \times V_\beta, e, \psi)$. Moreover, we have $\tilde{\mu}(U_\alpha \times U) = U_\alpha \cdot U \subseteq U_\alpha^2$ and thus $\Psi_{2, \beta} = (\chi^{-1} \times \text{id}_\mathcal{M}) \circ (\text{id}_\mathcal{G} \times \psi_\beta) \circ (\chi \times \text{id}_\mathcal{M})$ is also defined on $U_\alpha \times U \times V_\beta$ and a flat chart with respect to $(\mathcal{D}_{1 \otimes \lambda}, U_\alpha, U \times V_\beta, e, \psi)$. By the uniqueness of flat charts, $\Psi_1$ and $\Psi_{2, \beta}$ coincide on $U_\alpha \times U \times V_\beta$ so that $\Psi_1 = \Psi_2$ near $(g, g', p) \in U_\alpha \times U \times V_\beta$.

Consider now the case $U_\beta \subseteq U_\alpha$. The map $\Psi_{1, \alpha} = (\tau \times \text{id}_\mathcal{M}) \circ (\text{id}_\mathcal{G} \times \psi_\alpha) \circ (\tau \times \text{id}_\mathcal{M}) \circ (\text{id}_\mathcal{G} \times \psi)$ is defined on $U_\alpha^2 \times U \times V_\beta$ and is a flat chart with respect to $(\mathcal{D}_{1 \otimes \lambda}, U_\alpha^2, U \times V_\beta, e, \psi)$. The set $U_\beta(g')^{-1}$ contains $e$ and $g$. Since $U_\beta(g')^{-1} \subset U_\beta \cdot U_\beta^{-1} = U_\beta^2 \subseteq U_\alpha^2$, the map $\Psi_{1, \alpha}$ may be restricted to $U_\beta(g')^{-1} \times U \times V_\beta$ and is then a flat chart with respect to $(\mathcal{D}_{1 \otimes \lambda}, U_\beta(g')^{-1}, U \times V_\beta, e, \psi)$. After possibly shrinking $U$, we may assume $\tilde{\mu}(U_\beta(g')^{-1}) \cdot U = U_\beta(g')^{-1} \cdot U \subseteq U_\alpha^2$. Then $\Psi_2$ is defined on $U_\beta(g')^{-1} \times U \times V_\beta$ and is a flat chart with respect to $(\mathcal{D}_{1 \otimes \lambda}, U_\beta(g')^{-1}, U \times V_\beta, e, \psi)$. Again, the uniqueness of flat charts implies $\Psi_1 = \Psi_2$ near $(g, g', p) \in U_\beta(g')^{-1} \times U \times V_\beta$.

\begin{proposition}
\begin{proposition}
\end{proposition}

\textit{The infinitesimal action}

$$
\lambda_\varphi : \mathfrak{g} \to \text{Vec}(\mathcal{M}), \ X \mapsto (X(e) \otimes \text{id}_\mathcal{M}) \circ \varphi^*,
$$

\textit{induced by the local $\mathcal{G}$-action $\varphi$ is $\lambda$.}

\textbf{Proof.}

Since the map $\psi$ is locally a flat chart, $\psi$ is a local diffeomorphism, $\pi_G \circ \psi = \pi_G$ and $\psi \circ \iota_e = \iota_e$. Moreover, we have locally $\psi_* (\mathcal{D}_G) = \mathcal{D}$. Therefore, the vector field
\( \psi_\ast (X \otimes \text{id}_M^\ast) \) on \( \mathcal{G} \times \mathcal{M} \) belongs to the distribution \( \mathcal{D} \) for each vector field \( X \in \mathfrak{g} \). We have
\[
\psi_\ast (X \otimes \text{id}_M^\ast) \circ \pi_G^\ast = (\psi^{-1})^\ast \circ (X \otimes \text{id}_M^\ast) \circ \psi^\ast \circ \pi_G^\ast = (\psi^{-1})^\ast \circ (X \otimes \text{id}_M^\ast) \circ \pi_G^\ast \\
= (\psi^{-1})^\ast \circ \pi_G^\ast \circ X = \pi_G^\ast \circ X = (X \otimes \text{id}_M^\ast) \circ \pi_G^\ast.
\]
Let \( X_1, \ldots, X_k \) be a basis of \( \mathfrak{g} \) and \( a_1, \ldots, a_k \) local functions on \( \mathcal{G} \times \mathcal{M} \) such that
\[
\psi_\ast (X \otimes \text{id}_M^\ast) \circ \pi_G^\ast = \sum_{i=1}^{k+l} a_i (X_i \otimes \text{id}_M^\ast + \text{id}_G^\ast \otimes \lambda(X_i)),
\]
which is possible since \( \psi_\ast (X \otimes \text{id}_M^\ast) \) belongs to \( \mathcal{D} \). Then, combining the above, we have
\[
(X \otimes \text{id}_M^\ast) \circ \pi_G^\ast = \left( \sum_{i=1}^{k+l} a_i (X_i \otimes \text{id}_M^\ast + \text{id}_G^\ast \otimes \lambda(X_i)) \right) \circ \pi_G^\ast \\
= \left( \sum_{i=1}^{k+l} a_i X_i \right) \otimes \text{id}_M^\ast \circ \pi_G^\ast,
\]
which implies \( X = \sum_{i=1}^{k+l} a_i X_i \). Since \( X \in \mathfrak{g} \) and \( X_1, \ldots, X_k \) is a basis of \( \mathfrak{g} \), the \( a_i \)'s are all constants and hence
\[
\psi_\ast (X \otimes \text{id}_M^\ast) = X \otimes \text{id}_M^\ast + \text{id}_G^\ast \otimes \lambda(X).
\]
Therefore,
\[
\lambda_\varphi(X) = (X(\epsilon) \otimes \text{id}_M^\ast) \circ \psi^\ast = \iota_\epsilon^\ast \circ (X \otimes \text{id}_M^\ast) \circ \psi^\ast \circ \pi_M^\ast \\
= \iota_\epsilon^\ast \circ \psi^\ast \circ \psi_\ast (X \otimes \text{id}_M^\ast) \circ \pi_M^\ast = (\psi \circ \iota_\epsilon)^\ast \circ (X \otimes \text{id}_M^\ast + \text{id}_G^\ast \otimes \lambda(X)) \circ \pi_M^\ast \\
= \iota_\epsilon^\ast \circ (0 + \pi_M^\ast \circ \lambda(X)) = \lambda(X).
\]

### 4.3 Globalizations of infinitesimal actions

After we proved the existence of a local action of a Lie supergroup \( \mathcal{G} \) on a supermanifold \( \mathcal{M} \) with a given induced infinitesimal action \( \lambda : \mathfrak{g} \to \text{Vec}(\mathcal{M}) \), it is natural to ask in which cases this extends to a global \( \mathcal{G} \)-action on \( \mathcal{M} \).

A simple way to obtain examples of a local action which is not global is to start with an action on a supermanifold \( \mathcal{M}' \). This action then induces a local action, which is not global, on every non-invariant open subsupermanifold \( \mathcal{M} \subset \mathcal{M}' \).

The aim of this section is to characterize all infinitesimal action which “arise” in the just described way from a global action. These infinitesimal actions are called globalizable.

In the classical case (see [Pal57], Chapter III), Palais found necessary and sufficient conditions for an infinitesimal action to be globalizable, allowing the larger manifold \( \mathcal{M}' \), a globalization of the infinitesimal action, to be a possibly non-Hausdorff manifold.

In this section, similar conditions for the existence of globalizations of infinitesimal actions of Lie supergroups on supermanifolds are proven, and differences to the classical case are pointed out. It is also shown by an example that an infinitesimal action on a supermanifold may not be globalizable even if its underlying infinitesimal action is.

In analogy to the classical case (cf. [Pal57], Chapter III, Definition II), we define the notion of a globalization of an infinitesimal action.
**Definition 4.3.1.** A globalization of an infinitesimal action $\lambda : \mathfrak{g} \to \text{Vec}(M)$ of a Lie supergroup $\mathcal{G}$ on a supermanifold $\mathcal{M}$ is a pair $(\mathcal{M}', \varphi')$ with the following properties:

(i) $\mathcal{M}'$ is a supermanifold, whose underlying manifold $M'$ is allowed to be a non-Hausdorff manifold, and $\mathcal{M}$ is an open subsupermanifold of $\mathcal{M}'$, and

(ii) $\varphi' : \mathcal{G} \times \mathcal{M}' \to \mathcal{M}'$ is an action of the Lie supergroup $\mathcal{G}$ on $\mathcal{M}'$ such that its infinitesimal action restricted to $\mathcal{M}$ coincides with $\lambda$, and

(iii) $\tilde{\varphi}'(G \times M) = M'$.

If there is no chance of confusion the supermanifold $\mathcal{M}'$ is also called a globalization. The infinitesimal action $\lambda$ is called globalizable if there exists a globalization $(\mathcal{M}', \varphi')$ of $\lambda$.

**Definition 4.3.2.** The domain of definition $W = (W, O_W)$ of a local $\mathcal{G}$-action $\varphi : W \to M$ is called maximally balanced if $W_p = W_{\tilde{\varphi}(h,p)}h = \{gh | g \in W_{\tilde{\varphi}(h,p)}\}$ for all $(h, p) \in W$, where again $W_q = \{g \in G | (g, q) \in W\}$ for any $q \in M$.

**Remark 4.3.3.** In [Pal57] a maximally balanced domain of definition of a local action is called maximum (see Definition VII in Chapter III, [Pal57]). Here, the term maximally balanced is used instead of maximum in order to avoid any confusion with maximal domains of definition.

In the classical case, the following theorem states necessary and sufficient conditions for the existence of a globalization of an infinitesimal action.

**Theorem 4.3.4** (see [Pal57], Chapter III, Theorem X). Let $\lambda : \mathfrak{g} \to \text{Vec}(M)$ be an infinitesimal action of a Lie group $G$ on a manifold $M$. Then the following statements are equivalent:

(i) The infinitesimal action $\lambda$ is globalizable.

(ii) There exists a local action $\varphi : W \to M$ of $G$ on $M$ with infinitesimal action $\lambda$ whose domain of definition $W$ is maximally balanced; this local action with a maximally balanced domain of definition is then unique and any other local action with the same infinitesimal action $\lambda$ is a restriction of $\varphi$.

(iii) Let $\mathcal{D}$ be the distribution on $G \times M$ associated with the infinitesimal action $\lambda$ and $\Sigma \subset G \times M$ any leaf of $\mathcal{D}$, i.e. a maximal connected integral manifold. Then the map $\pi_G|\Sigma : \Sigma \to M$ is injective, where $\pi_G : G \times M \to G$ denotes the projection onto $G$.

**Remark 4.3.5.** If $\varphi' : \mathcal{G} \times \mathcal{M}' \to \mathcal{M}'$ is a globalization of the infinitesimal action $\lambda$, then the underlying action $\tilde{\varphi}' : G \times M' \to M'$ is a globalization of the reduced infinitesimal action $\tilde{\lambda} : \mathfrak{g}_0 \to \text{Vec}(M)$. Therefore, a necessary condition for an infinitesimal action to be globalizable is the existence of a globalization $M'$ of the reduced infinitesimal action.
4.3.1 Univalent leaves and holonomy

In this paragraph, the notion of univalent leaves of the distribution associated with an infinitesimal action is introduced and holonomy phenomena are discussed. These notions play an important role in the characterization of globalizable infinitesimal actions on supermanifolds.

Throughout, let $\lambda : g \to \text{Vec}(M)$ be a fixed infinitesimal action of a Lie supergroup $G$ and let $D$ be the associated distribution on $G \times M$.

Trying to generalize the classical result (Theorem 4.3.4) to supermanifolds, the question of an appropriate formulation of condition $(iii)$ arises since integral manifolds do not uniquely determine a distribution on a supermanifold (cf. Example 3.2.6). For that purpose the notion of a univalent leaf $\Sigma \subset G \times M$, extending the classical notion, is introduced.

Then holonomy phenomena of the distribution $D$ are studied and a connection between the absence of such phenomena and the notion of univalent leaves is established.

Remark 4.3.6. The infinitesimal action $\lambda : g \to \text{Vec}(M)$ induces an infinitesimal action $\tilde{\lambda} : g_0 \to \text{Vec}(M)$ of the classical Lie group $G$ on the manifold $M$, where the Lie algebra of $G$ is identified with the even part $g_0$ of $g$. For any vector field $X \in g_0(\subset g)$ we define $\tilde{\lambda}(X)$ to be the reduced vector field $\tilde{Y}$ of $Y = \lambda(X)$.

Lemma 4.3.7. Let $\tilde{\lambda} : g_0 \to \text{Vec}(M)$ denote the induced infinitesimal action of $G$ on $M$ and $D\tilde{\lambda}$ the associated distribution on $G \times M$. Then we have $\tilde{D} = D\tilde{\lambda}$, where $\tilde{D}$ denotes the distribution on $G \times M$ induced by $\mathcal{D}$ (cf. Remark 3.0.6).

Proof. For any odd vector field $Y$ on a supermanifold we have $\tilde{Y} = 0$. Since $X + \lambda(X)$ has the same parity as $X$ if $X$ is homogeneous, the reduced distribution $\tilde{D}$ is spanned by vector fields of the form $\tilde{X} + \tilde{\lambda}(X)$ for $X \in g_0$. These vector fields also generate $D\tilde{\lambda}$ and thus $\tilde{D} = D\tilde{\lambda}$. \hfill $\Box$

As a corollary of the identity $\tilde{D} = D\tilde{\lambda}$, we get the following relation between integral manifolds of the involutive distributions $\mathcal{D}$ on $G \times M$ and $D\tilde{\lambda}$ on $G \times M$.

Corollary 4.3.8. Every integral manifold $N \subset G \times M$ of the distribution $D\tilde{\lambda}$ is the underlying manifold of some integral manifold $\tilde{N} \subset G \times M$ of the distribution $\mathcal{D}$ and conversely the underlying manifold of every integral manifold of $\mathcal{D}$ is an integral manifold of $D\tilde{\lambda}$.

In the following, by a leaf $\Sigma \subset G \times M$ a leaf, i.e. a maximal connected integral manifold, of the distribution $D\tilde{\lambda} = \tilde{D}$ is meant. By the preceding corollary every leaf $\Sigma$ is as well the underlying manifold of an integral manifold of the distribution $\mathcal{D}$.

The fact that the distribution $D\tilde{\lambda}$ is involutive guarantees the existence of a leaf through each point $(g,p) \in G \times M$. This leaf is denoted by $\Sigma_{(g,p)}$.

Definition 4.3.9. A leaf $\Sigma \subset G \times M$ is called univalent (with respect to $\mathcal{D}$) if for every path $\gamma : [0,1] \to \Sigma \subset G \times M$ there exists a flat chart $\psi : U \times V \to G \times M$ with respect to $(\mathcal{D},U,V,\gamma(0),\text{id})$, for $\gamma := \pi_G \circ \gamma : [0,1] \to G$, such that $\tilde{\psi}(U \times V)$ contains $\gamma([0,1])$.

The infinitesimal action $\lambda$ is called univalent if all leaves $\Sigma \subset G \times M$ are univalent.
Remark 4.3.10. By Remark 4.1.3, a leaf $\Sigma$ is univalent if for every path $\gamma : [0, 1] \to \Sigma$ there exists a flat chart $\psi$ with respect to $(\mathcal{D}, U, V, \gamma_0(0), \rho)$ for some $\rho$ or equivalently, after shrinking, for any $\rho$, with $\gamma([0, 1]) \subset \tilde{\psi}(U \times V)$.

Remark 4.3.11. In the definition of a univalent leaf $\Sigma$ it is enough for the defining property to hold true for closed paths $\gamma : [0, 1] \to \Sigma$ because for any path $\gamma'$ the composition $(\gamma')^{-1} \circ \gamma'$ of $\gamma'$ and $(\gamma')^{-1}$, $(\gamma')^{-1}(t) = \gamma'(1 - t)$, is a closed path with $\gamma'([0, 1]) = ((\gamma')^{-1} \circ \gamma')([0, 1])$.

Remark 4.3.12. If $\lambda$ is univalent, then so is the induced infinitesimal action $\tilde{\lambda} : g_0 \to \text{Vec}(\mathcal{M})$.

In [Pal57], an infinitesimal action (in the classical case) is called univalent if the restriction of the projection $\pi_G : \mathcal{G} \times \mathcal{M} \to \mathcal{G}$ to an arbitrary leaf $\Sigma \subset \mathcal{G} \times \mathcal{M}$ is injective. The above defined notion of univalent infinitesimal actions on supermanifolds extends this definition:

Proposition 4.3.13. In the case of classical manifolds $\mathcal{G} = \mathcal{G}$ and $\mathcal{M} = \mathcal{M}$, an infinitesimal action is univalent if and only if the projection $\pi_G|_{\Sigma} : \Sigma \to \mathcal{G}$ is injective for each leaf $\Sigma \subset \mathcal{G} \times \mathcal{M}$.

Proof. Let $\Sigma \subset \mathcal{G} \times \mathcal{M}$ be any leaf, $x = (g, p), y = (g, q) \in \Sigma$, and $\gamma : [0, 1] \to \Sigma$ a path from $x$ to $y$. Since $\lambda$ is univalent, there is a flat chart $\psi : U \times V \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U, V, g, id)$ such that the intersection $\psi(\Sigma) \subset \psi(U \times \{p\})$. We have $\psi_0 = \psi(\Sigma)$ is connected for all $\Sigma \subset \mathcal{G} \times \mathcal{M}$.

Assume now that $\pi_G|_{\Sigma}$ is injective for each leaf $\Sigma \subset \mathcal{G} \times \mathcal{M}$. Let $\Sigma \subset \mathcal{G} \times \mathcal{M}$ be a leaf and $\gamma : [0, 1] \to \Sigma$ a path. Using the compactness of $\gamma([0, 1])$ there are $0 = t_0 < \ldots < t_k = 1$ and flat charts $\psi_i : U_i \times V_i \to \mathcal{G} \times \mathcal{M}, i = 0, \ldots, k - 1$, with respect to $(\mathcal{D}, U_i, V_i, \gamma(t_i), id)$ such that the intersection $U_i \cap U_j$ is connected for all $i, j$ and $\gamma([t_i, t_{i+1}]) \subset \psi(U_i \times V_i)$.

Set $\psi_0 = \psi_0$ and, after possibly shrinking $V_0$, inductively define flat charts $\psi_i : U_i \times V_i \to \mathcal{G} \times \mathcal{M}$ for $i \geq 1$ by composing $\psi_i$ and a local diffeomorphism of the form $(id \times \rho_i)$ such that $\psi_i$ and $\psi_{i+1}$ coincide on $(U_i \cap U_{i+1}) \times V_i$ for $i = 0, \ldots, k - 2$ and $\gamma([t_i, t_{i+1}]) \subset \psi_i(U_i \times V_i)$ holds. We have $\psi_0(\mathcal{D}, U_0 \times \{p\}) \subset \Sigma_{(g,p)}$ for $g := \gamma_0(0)$ and any $p \in V_0$, and then by induction $\psi_i(U_i \times \{p\}) \subset \Sigma_{(g,p)}$ for any $i$. Therefore, $\psi_i = \psi_j$ on $(U_i \cap U_j) \times V_i$ since $\pi_G|_{\Sigma_{(g,p)}}$ is injective and $\pi_G \circ \psi = \pi_G$ for any flat chart $\psi$.

Consequently, we can define a flat chart $\psi : U \times V \to \mathcal{G} \times \mathcal{M}, U := \bigcup_{i=0}^{k-1} U_i, V := V_0$, with respect to $(\mathcal{D}, U, V, g = \gamma_0(0), id)$ by setting $\psi|_{U_i \times V_0} = \psi_i$. The map $\psi$ satisfies $\gamma([0, 1]) \subset \psi(U \times V)$ by construction. □

Proposition 4.3.14. The infinitesimal action $\lambda$ is univalent if and only if for any two flat charts $\psi_i : \mathcal{U}_i \times V_i \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U_i, V_i, g, \rho_i), i = 1, 2,$ and $\rho_1 = \rho_2$ on $V_1 \cap V_2$ we have $\psi_1 = \psi_2$ on their common domain of definition.

Remark that by Proposition 4.1.7 any two flat charts coincide on their common domain of definition $(U_1 \cap U_2) \times (V_1 \cap V_2)$ if $U_1 \cap U_2$ is connected.

Proof. Let $\lambda$ be univalent and $\psi_i$ flat charts with respect to $(\mathcal{D}, U_i, V_i, g, \rho_i)$ and $\rho_1 = \rho_2$ on $V_1 \cap V_2$. Let $(h, p) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$. Since $\psi_1$ and $\psi_2$ are flat charts, $\psi_1(U_1 \times \{p\})$ and $\psi_2(U_2 \times \{p\})$ are both contained in the leaf $\Sigma = \Sigma_{(g, \rho_1(p))} = \Sigma_{(g, \rho_2(p))}$. 50
The univalence of the reduced infinitesimal action $\tilde{\lambda}$ implies that $\pi_G|\Sigma : \Sigma \to G$ is injective and thus $\tilde{\psi}(h,p) = \psi_2(h,p)$. Let $\gamma : [0,1] \to U_1 \cup U_2$ be a closed path with $\gamma(0) = \gamma(1) = g$, $\gamma([0, \frac{1}{2}]) \subseteq U_1$, $\gamma([\frac{1}{2}, 1]) \subseteq U_2$ and $\gamma(\frac{1}{2}) = h$. Then

$$
\gamma' : [0,1] \to \Sigma, \gamma'(t) = \begin{cases} 
\psi_1(\gamma(t), p), & t \leq \frac{1}{2} \\
\psi_2(\gamma(t), p), & t > \frac{1}{2}
\end{cases}
$$

is a closed path. As $\lambda$ is univalent, there is a flat chart $\psi : U \times V \to G \times M$ with respect to $(D, U, V, g, id)$ with $\gamma'([0,1]) = \psi_1(\gamma([0, \frac{1}{2}]) \times \{p\}) \cup \psi_2(\gamma([\frac{1}{2}, 1]) \times \{p\}) \subseteq \psi(U \times V)$. Then, after possibly shrinking $V_1$, $\psi \circ (id \times \rho_1)$ is a flat chart with respect to $(D, U, V_1, g, \rho_1)$. By Proposition 4.1.7 the flat charts $\psi \circ (id \times \rho_1)$ and $\psi_1$ coincide on a neighbourhood of $\gamma([0, \frac{1}{2}]) \times \{p\}$. Moreover, $\psi \circ (id \times \rho_1)$ and $\psi_2$ coincide on a neighbourhood of $\gamma([\frac{1}{2}, 1]) \times \{p\}$ since $\rho_1 = \rho_2$ on $V_1 \cap V_2$. In particular, we get $\psi_1 = \psi \circ (id \times \rho_1) = \psi_2$ near $(h, p) = (\gamma(\frac{1}{2}), p)$.

Suppose now that any two flat charts $\psi_i$ with respect to $(D, U_i, V_i, g, \rho_i)$, $i = 1, 2$, with $\rho_1 = \rho_2$ already coincide on their common domain of definition. Assume that there is a leaf $\Sigma \subseteq G \times M$ which is not univalent and let $\gamma : [0,1] \to \Sigma$ be a path for which there is no flat chart $\psi : U \times V \to G \times M$ with $\gamma([0,1]) \subseteq \psi(U \times V)$. Define $I$ to be the set of points $t \in [0,1]$ such that there exists a flat chart $\psi : U \times V \to G \times M$ with $\gamma([0, t]) \subseteq \psi(U \times V)$. The set $I$ is open and $I \neq [0,1]$ by assumption. Let $s$ be the minimum of $[0,1] \setminus I$. There is a flat chart $\psi_1 : U_1 \times V_1 \to G \times M$ with respect to $(D, U_1, V_1, \pi_G(\gamma(s)), id)$ with $\gamma(s) \in \psi_1(U_1 \times V_1)$. By the choice of $s$ there is $t \in [0, s)$ with $\gamma([t, s]) \subseteq \psi_1(U_1 \times V_1)$ such that there is a flat chart $\psi_2 : U_2 \times V_2 \to G \times M$ with respect to $(D, U_2, V_2, \pi_G(\gamma(t)), id)$ with $\gamma([0, t]) \subseteq \psi_2(U_2 \times V_2)$. Then $h = \pi_G(\gamma(t)) \in U_1 \cap U_2$ and after possibly shrinking $V_2$ there exists a diffeomorphism $\rho$ such that $\psi_1 \circ (id \times \rho)$ is a flat chart with respect to $(D, U_1, V_1, h, \rho')$ and $\psi_2$ with respect to $(D, U_2, V_2, h, \rho')$ for some $\rho' : V_2 \to M$. Therefore, $\psi_1 \circ (id \times \rho)$ and $\psi_2$ agree on their common domain of definition and they define a flat chart $\psi : (U_1 \cup U_2) \times V_2 \to G \times M$ with $\gamma([0, s]) \subseteq \psi(U_1 \cup U_2) \times V_2)$, contradicting the definition of $s$.

The preceding proposition allows us to glue together flat charts in the case of a univalent infinitesimal action. This also implies the next corollary.

**Corollary 4.3.15.** The infinitesimal action $\lambda$ is univalent if and only if for any compact subset $\Sigma'$ of a leaf $\Sigma \subseteq G \times M$ there exists a flat chart $\psi : U \times V \to G \times M$ with $\Sigma' \subseteq \psi(U \times V)$.

In the following the structure of the distribution $D$ associated with $\lambda$ is investigated further. Let $\Sigma \subseteq G \times M$ be a leaf, $(g,p) \in \Sigma$ and $\gamma : [0,1] \to \Sigma$ a closed path with $\gamma(0) = \gamma(1) = (g,p)$ and let $\gamma_G = \pi_G \circ \gamma$. We now want to associate a germ of a local diffeomorphism of $M$ around $p$ measuring the holonomy along the path $\gamma$.

To do so, let $0 = t_0 < \ldots < t_k = 1$ be a partition of $[0,1]$ such that there are flat charts $\psi_i : U_i \times V \to G \times M, i = 0, \ldots, k-1$, with $\gamma([t_i, t_{i+1}]) \subseteq \psi_i(U_i \times \{p\})$, such that $\psi_i$ and $\psi_{i+1}$ coincide on their common domain of definition and such that $\psi_0$ is a flat chart with respect to $(D, U_0, V, g, id)$. We have $\psi_i(U_i \times \{p\}) \subseteq \Sigma$ for any $i$. By Lemma 4.1.6 there is a diffeomorphism $\rho : V \to M$ onto its image such that $\psi_{k-1}$ is a flat chart with respect to $(D, U_{k-1}, V, g = \gamma_G(1), \rho)$. Since $(g,p) = \gamma(1) \in \psi_{k-1}(U_{k-1} \times \{p\})$, we have $(g, \hat{\rho}(p)) = \psi_{k-1}(g,p) = (g,p)$ and thus $\hat{\rho}(p) = p$. 

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Define \( \Phi(\gamma) \) to be the germ of the local diffeomorphism \( \rho \) in \( p \). The local uniqueness of flat chart implies that \( \Phi(\gamma) \) does not depend on the actual choice of the flat charts \( \psi_1: U_1 \times V_1 \to G \times M \). Let \( \text{Diff}_p(M) \) denote the set of germs of local diffeomorphisms \( \chi: V_1 \to V_2 \) in \( p \in M \), where \( V_i = (V_i, \mathcal{O}_M) \), \( i = 1, 2 \), are open subsupermanifolds of \( M \) with \( p \in V_i \). Then \( \Phi(\gamma) \) is an element of \( \text{Diff}_p(M) \) for each closed path \( \gamma: [0,1] \to \Sigma \) with \( \gamma(0) = \gamma(1) = (g, p) \).

In the case of complex supermanifolds and holomorphic maps \( \text{Diff}_p(M) \) should be replaced by \( \text{Hol}_p(M) \), the set of germs of local biholomorphisms \( \chi: V_1 \to V_2 \).

**Proposition 4.3.16.** The germ \( \Phi(\gamma) \) only depends on the homotopy class \( [\gamma] \) of the closed path \( \gamma \) with \( \gamma(0) = \gamma(1) = (g, p) \). Therefore, the assignment \( \gamma \mapsto \Phi([\gamma]) = \Phi(\gamma) \) defines a map

\[
\Phi = \Phi_\Sigma = \Phi_{\Sigma,(g,p)} : \pi_1(\Sigma, (g,p)) \to \text{Diff}_p(M).
\]

**Proof.** Let \( \gamma_s : [0,1] \to \Sigma, s \in [0,1] \), be a continuous family of closed paths with \( \gamma_s(0) = \gamma_s(1) = (g, p) \) and \( \gamma_0 = \gamma \). Let \( s_0 \in [0,1] \), \( 0 = t_0 < \ldots < t_k = 1 \), and flat charts \( \psi_0 : U_0 \times V_0 \to G \times M \) such that \( \psi_0 \) is a flat chart with respect to \( (D, U_0, V_0, g, \text{id}) \) and \( \psi_t \) and \( \psi_{t+1} \) coincide on their common domain of definition. Then \( \gamma_s \) is a flat chart with respect to \( (D, U_s, V_s, g, \rho) \) for some local diffeomorphism \( \rho \) around \( p \) and \( \Phi(\gamma_s) \) is the germ of \( \rho \) in \( p \).

Since all intervals \( [t_i, t_{i+1}] \) are compact there exists an open neighbourhood \( J \subseteq [0,1] \) of \( s_0 \) such that \( \gamma_s([t_i, t_{i+1}]) \subseteq \psi_i(U_i \times V_i) \) for \( i = 0, \ldots, k-1 \) and all \( s \in J \). Therefore, \( \Phi(\gamma_s) = \Phi(\gamma_s) \) for all \( s \in J \) and the set \( \{ s \in [0,1] \mid \Phi(\gamma_s) = \Phi(\gamma_s) \} \) is open and closed. Since \( \gamma = \gamma_0 \) we get \( \Phi(\gamma_s) = \Phi(\gamma) \) for all \( s \in [0,1] \).

**Remark 4.3.17.** The definition of \( \Phi \) implies that it is a group homomorphism: If \( \gamma \) is a constant path, then \( \Phi([\gamma]) \) is the germ of the identity id : \( M \to M \) and we have \( \Phi([\gamma_1] \cdot [\gamma_2]) = \Phi([\gamma_1]) \circ \Phi([\gamma_2]) \) for the composition \( \gamma_1 \cdot \gamma_2 \) of two closed paths \( \gamma_1 \) and \( \gamma_2 \).

**Remark 4.3.18.** By Lemma 4.1.4 and Remark 4.1.8 the morphism \( \Phi \) does not depend on whether its construction is done with respect to the distribution \( D \) on \( G \times M \) associated with \( \lambda \) or to the distribution \( D_0 \) on \( G \times M \) associated with the infinitesimal action \( \lambda_0 = \lambda|_{\Sigma} \).

**Remark 4.3.19.** Due to the connectedness of the leaves \( \Sigma \) the map \( \Phi_{\Sigma,(g,p)} \) is trivial for some \( (g,p) \in \Sigma \) if and only if \( \Phi_{\Sigma,(h,q)} \) is trivial for all \( (h,q) \in \Sigma \).

The triviality of the map \( \Phi = \Phi_{\Sigma} = \Phi_{\Sigma,(g,p)} \) can be viewed as an absence of holonomy for the leaf \( \Sigma \).

**Example 4.3.20.** Let \( G = S^1 \), with coordinate \( \phi \), and \( M = \mathbb{R}^{0|2} \), with coordinates \( \theta_1, \theta_2 \). Let \( X = \theta_1 \frac{\partial}{\partial \theta_1} \) and consider the infinitesimal \( S^1 \)-action \( \lambda : \text{Lie}(S^1) \cong \mathbb{R} \to \text{Vec}(M) \), \( \lambda(t) = tX \). The unique leaf of the distribution \( D \) on \( S^1 \times M \), spanned by \( \frac{\partial}{\partial \phi} + X \), is \( \Sigma = S^1 \times \{0\} = S^1 \times M \). Let \( \phi_0 \in S^1 \) and \( r : \Omega \to \mathbb{R} \) be a local inverse around \( 1 \in \Omega \subset S^1 \) of \( r : \mathbb{R} \to S^1 \), \( t \mapsto e^{it} \). Then

\[
\psi^*(\phi, \theta_1, \theta_2) = (\phi, \rho^*(\theta_1), \rho^*(\theta_2)) + (0, 0, r(\phi_0)^{-1}\rho^*(\theta_1))
\]
defines the pullback of a flat chart \( \psi \) with respect to \( (D, U, V, \phi_0, \rho) \) for \( V = \{0\} = M \), \( U \subseteq S^1 \) with \( \phi_0 \in U \) and \( U \phi_0^{-1} \subseteq \Omega \) and a diffeomorphism \( \rho : \mathbb{R}^{0|2} \to \mathbb{R}^{0|2} \). For any
\( \phi'_0 \in U \) the map \( \psi \) is also a flat chart with respect to \((D, U, \phi'_0, \rho')\) for \( \rho' : \mathbb{R}^{0|2} \to \mathbb{R}^{0|2} \) with pullback
\[
(r')^*(\theta_1, \theta_2) = r^*(\theta_1, \theta_2) + (0, r(\phi'_0\phi_0^{-1})r^*(\theta_1)) = r^*(\theta_1, \theta_2) + \left(0, \int_{\phi_0} 1d\phi \right) r^*(\theta_1),
\]
where the integral might be taken along any path in \( \Omega \). In particular \((r')^*(\theta_1) = r^*(\theta_1)\).

A calculation shows that the map \( \Phi : \pi_1(\Sigma, (\phi_0, 0)) \to \text{Diff}(\mathbb{R}^{0|2}) = \text{Diff}(\mathbb{R}^{0|2}) \) is given by
\[
\Phi([\gamma])^*(\theta_1, \theta_2) + \int_{\gamma} 1d\phi \theta_1 = (\theta_1, \theta_2 + \int_{\gamma} 1d\phi \theta_1)
\]
for any \( \gamma : [0, 1] \to S^1 \cong \Sigma, \gamma(0) = \gamma(1) = \phi_0 \). Thus, identifying \( \pi_1(\Sigma, (\phi_0, 0)) \) and \( \mathbb{Z} \), we get
\[
\Phi : \mathbb{Z} \to \text{Diff}(\mathbb{R}^{0|2}), \Phi(k)^*(\theta_1, \theta_2) = (\theta_1, \theta_2 + 2\pi k \theta_1).
\]

We now establish an equivalence between univalent infinitesimal actions and infinitesimal actions with univalent reduced infinitesimal action and leaves without holonomy.

**Proposition 4.3.21.** The infinitesimal action \( \lambda \) is univalent if and only if the reduced infinitesimal action \( \bar{\lambda} : g_0 \to \text{Vec}(M) \) is univalent and for all leaves \( \Sigma \) the map \( \Phi_{\Sigma} \) is trivial.

**Proof.** If the infinitesimal action \( \lambda \) is univalent, then the reduced infinitesimal action \( \bar{\lambda} \) is univalent as a direct consequence. Moreover, for any closed path \( \gamma : [0, 1] \to \Sigma \) there is a flat chart \( \psi : U \times V \to G \times M \) with \( \gamma([0, 1]) \subset \psi(U \times V) \) and thus \( \Phi([\gamma]) = \text{id} \).

Now, suppose that \( \bar{\lambda} \) is univalent and \( \Phi_{\Sigma} \) is trivial for all leaves \( \Sigma \subset G \times M \). We need to show that any two flat charts \( \psi_i : U_i \times V_i \to G \times M \), \( i = 1, 2 \), with respect to \((D, U_i, V_i, g, \rho_i)\) with \( \rho_1 = \rho_2 \) coincide on their common domain of definition (cf. Proposition 4.3.14). By replacing \( \psi_i \) by \( \psi_i \circ (\text{id} \times \rho_i^{-1}) \) it is enough to show \( \psi_1 = \psi_2 \) in the case of \( \rho_1 = \rho_2 = \text{id} \).

Let \((h, p) \in (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \) be arbitrary. We have \( \tilde{\psi}_1(h, p) \subset \psi_1(U_1 \times \{p\}) \subset \Sigma_{(g, p)} \) and thus \( \tilde{\psi}_1(h, p) = \tilde{\psi}_2(h, p) \) since \( \pi_G|\Sigma_{(g, p)} \) is injective. After shrinking \( V_2 \) there is a diffeomorphism \( \rho : V_2 \to M \) such that \( \psi_2 \circ (\text{id} \times \rho) \) is defined and coincides with \( \psi_1 \) near \((h, p)\). Note that \( \rho(p) = p \) since \( \psi_1(h, p) = \psi_2(h, p) \). The composition \( \psi_2 \circ (\text{id} \times \rho) \) is a flat chart with respect to \((D, U_2, V_2, g, \rho)\). Let \( \alpha : [0, 1] \to \Sigma_{(g, p)} \) be a closed path with \( \alpha(0) = \alpha(1) = (g, p) \), \( \alpha\left(\frac{1}{2}\right) = \tilde{\psi}_1(h, p) = \tilde{\psi}_2(h, \rho(p)) = \tilde{\psi}_2(h, p) \) and \( \alpha([0, 1]) \subset \psi_1(U_1 \times \{p\}) \) and \( \alpha([\frac{1}{2}, 1]) \subset \psi_2(U_2 \times \{p\}) = \tilde{\psi}_2((\text{id} \times \rho)(U_2 \times \{p\})) \). By the definition of \( \Phi = \Phi_{\Sigma_{(g, p)}} \) we have \( \Phi([\alpha]) = \rho \). The triviality of \( \Phi \) then gives \( \rho = \text{id} \) so that \( \psi_1 \) and \( \psi_2 \) agree near \((h, p)\).

**Corollary 4.3.22.** The infinitesimal action \( \lambda \) is univalent if and only if its restriction \( \lambda_0 : g_0 \to \text{vec}(M) \) is univalent.

**Proof.** The statement follows from the preceding proposition and the observation formulated in Remark 4.3.18.
4.3.2 The action on $G \times M$ from the right and invariant functions

In this paragraph, the action $R : G \times (G \times M) \to G \times M$ of the Lie supergroup $G$ on the product $G \times M$ from the right, which is in the classical case given by $(g, (h, x)) \mapsto (hg^{-1}, x)$, is introduced. Its behaviour with respect to the distribution $D$ associated with $\lambda : g \to \text{Vec}(M)$ and in particular $D$-invariant functions is studied.

Definition 4.3.23. Let $\mathcal{O}^D_{G \times M}$ be the sheaf of $D$-invariant functions on $G \times M$, i.e.

$$\mathcal{O}^D_{G \times M}(\Omega) = \{ f \in \mathcal{O}_{G \times M}(\Omega) | D \cdot f = 0 \}$$

for any open subset $\Omega \subset G \times M$.

Remark 4.3.24. For any $f$ the form $D$ for any open subset $\Omega \subset G \times M$.

Moreover, if $\psi : U \times V \to G \times M$ is a flat chart and $f$ is $D$-invariant, then $\psi^*(f)$ is $D_G$-invariant since $\psi_* (D_G) = D$. The $D_G$-invariant functions on $U \times V \subseteq G \times M$ are of the form $f_M = 1 \otimes f_M$ for $f_M \in \mathcal{O}_M(V)$ so that $\psi^*(f) = 1 \otimes f_M$ for an appropriate $\mathcal{O}_{G \times M}$.

We have the following identity principle for $D$-invariant functions on $G \times M$:

Lemma 4.3.25. Let $W \subseteq G \times M$ be open, $\Sigma \subseteq G \times M$ be a leaf with $\Sigma \subset W$ and let $f_1, f_2 \in \mathcal{O}^D_{G \times M}(W)$. If $f_1 = f_2$ on an open neighbourhood of some $x \in \Sigma$, then $f_1$ and $f_2$ coincide on an open neighbourhood of the leaf $\Sigma$.

Proof. Define $\Sigma' \subseteq \Sigma$ to be the subset of points $y \in \Sigma$ such $f_1$ and $f_2$ are equal on some open neighbourhood of $y$. The set $\Sigma'$ is open and contains $x$ by assumption.

For any flat chart $\psi : U \times V \to G \times M$ with $\psi(U \times V) \subset W$ we have $\psi^*(f_i) = 1 \otimes f_{M,i}$ for appropriate $f_{M,i}$ by Remark 4.3.24. Therefore, $f_1 = f_2$ near $\psi(U \times \{q\})$ if and only if $f_{M,1} = f_{M,2}$ near $q \in M$. It follows that the set $\Sigma'$ is also closed and thus equal to $\Sigma$.

Definition 4.3.26. Let $\mu : G \times G \to G$ denote the multiplication of $G$, $\iota : G \to G$ the inversion and $\tau : G \times G \to G \times G$ the morphism which interchanges the two components. Then define the action of $G$ on itself from the right as $r : G \times G \to G$, $r = \mu \circ \iota \circ (\iota \times \text{id})$. The underlying action is given by $(g, h) \mapsto hg^{-1}$. Define now a $G$-action on $G \times M$ by

$$R : G \times (G \times M) \to G \times M, \quad R = r \times \text{id}.$$ 

Lemma 4.3.27. For every right-invariant vector field $X$ on $G$ we have

$$(\text{id}^* \otimes (X \otimes \text{id}^* + \text{id}^* \otimes \lambda(X))) \circ R^* = R^* \circ (X \otimes \text{id}^* + \text{id}^* \otimes \lambda(X)).$$

Proof. The right-invariance of $X$ is equivalent to $\mu^* \circ X = (X \otimes \text{id}^*) \circ \mu^*$ and a short calculation yields $(\text{id}^* \otimes X) \circ r^* = r^* \circ X$, which directly implies the desired equality.

Corollary 4.3.28. We have

$$R^*(\mathcal{O}^D_{G \times M}) \subseteq \mathcal{R}_\lambda(\mathcal{O}^\otimes D_{G \times G \times M}),$$

where $1 \otimes D$ is the distribution on $G \times (G \times M)$ spanned by vector fields of the form $\text{id}^* \otimes Y$ for vector fields $Y$ belonging to $D$ and $\mathcal{O}^\otimes D_{G \times G \times M}$ denotes the sheaf of $1 \otimes D$-invariant functions.
Proof. Using the preceding lemma, we have
\[(\text{id}^* \otimes (X \otimes \text{id}^* + \text{id}^* \otimes \lambda(X))) (R^*(f)) = R^* \circ (X \otimes \text{id}^* + \text{id}^* \otimes \lambda(X))(f) = R^*(0) = 0\]
for any $D$-invariant function $f$ on $G \times M$ and $X \in \mathfrak{g}$. Thus, $R^*(f)$ is $1 \otimes D$-invariant. \qed

**Definition 4.3.29.** For $g \in G$, let $r_g : G \to G$ denote the composition of the action $r$ and the inclusion $G \to \{g\} \times G \subset G \times G$, and define
\[R_g : G \times M \to G \times M, \quad R_g = (r_g \times \text{id}).\]
Since $r$ is an action, $r_g$ and $R_g$ are diffeomorphisms, $(r_g)^{-1} = r_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$.

**Lemma 4.3.30.** For each $g \in G$ the map $R_g : G \times M \to G \times M$ satisfies

(i) $(R_g)_*(D_G) = D_G,$ and

(ii) $(R_g)_*(D) = D.$

**Proof.** Property (i) can be directly obtained from the definition of $R_g$. Property (ii) follows from the fact that $(r_g)_*(X) = X$ for every right-invariant vector field $X$, and therefore $(R_g)_*(X + \lambda(X)) = (r_g)_*(X) + \lambda(X) = X + \lambda(X)$ so that $(R_g)_*(D) = D$. \qed

The composition of flat charts and maps of the form $R_g$ exhibits a special behaviour as specified in the following lemma.

**Lemma 4.3.31.** Let $g \in G$ and let $\psi : U \times V \to G \times M$ be a flat chart with respect to $(D, U, V, h, \rho)$. Then the composition
\[\psi' = R_{g^{-1}} \circ \psi \circ (R_g)_{|Ug \times V'} : Ug \times V' \to G \times M\]
is a flat chart with respect to $(D, Ug, V, hg, \rho)$ where $Ug = \{ug \mid u \in U\}$ and $Ug = (Ug, \mathcal{O}_G|_{Ug})$.

**Proof.** We have $(\psi')_*(D_G) = D$ using $(R_g)_*(D_G) = D_G$ and $(R_g)_*(D) = D$. Moreover, direct calculations show $\pi_G \circ \psi' = \pi_G$ and $\psi' \circ t_{hg} = t_{hg} \circ \rho$. \qed

**Corollary 4.3.32.** The underlying classical Lie group $G$ acts on the space of leaves $\Sigma \subset G \times M$ by $(g, \Sigma) \mapsto \tilde{R}_g(\Sigma)$. For $(h, p) \in G \times M$ we have $\tilde{R}_g(\Sigma(h, p)) = \Sigma(hg^{-1}, p)$.

**Proof.** Since $R_g$ preserves the distribution $D$, $\tilde{R}_g$ maps leaves diffeomorphically onto leaves. We have $\tilde{R}_g(\Sigma(h, p)) = \Sigma(hg^{-1}, p)$ because $(hg^{-1}, p) = \tilde{R}_g(h, p) \in \tilde{R}_g(\Sigma(h, p))$. \qed

**Remark 4.3.33.** If there exists an action $\varphi : G \times M \to M$ with infinitesimal action $\lambda$, then $\tilde{R}_g(\Sigma_{(e,p)}) = \Sigma(g^{-1}, p)$ for any $p \in M$.

Let $\psi = (\text{id} \times \varphi) \circ (\text{diag} \times \text{id}) : G \times M \to G \times M$. The map $\psi$ is a flat chart (cf. Lemma 4.2.2) and $\tilde{\psi}(G \times \{p\}) = \Sigma_{(e,p)}$. We have $\tilde{\psi}(g, \varphi(g, p)) = \tilde{R}_g(g, \varphi(g, p)) = \tilde{R}_g(\tilde{\psi}(G \times \{p\}))$ and thus $\tilde{R}_g(\Sigma_{(e,p)}) = \Sigma(g^{-1}, p) = \Sigma(e, \varphi(g, p))$.

**Proposition 4.3.34.** The infinitesimal action $\lambda : \mathfrak{g} \to \text{Vec}(M)$ is univalent if and only if every leaf of the form $\Sigma_{(e,p)}$ for $p \in M$ is univalent.
Proof. If $\lambda$ is univalent, then all leaves are univalent, in particular each leaf of the form $\Sigma_{(e,p)}$. Assume now that all leaves $\Sigma_{(e,p)}$ are univalent. Let $\Sigma$ be an arbitrary leaf and let $(g,p) \in \Sigma$. We have $\Sigma_{(e,p)} = \tilde{R}_g(\Sigma_{(g,p)}) = \tilde{R}_g(\Sigma)$. If $\Omega \subset \Sigma = \Sigma_{(g,p)}$ is a relatively compact subset, then $\tilde{R}_g(\Omega) \subset \Sigma_{(e,p)}$ is relatively compact and the univalence of $\Sigma_{(e,p)}$ yields the existence of a flat chart $\psi : U \times V \to G \times M$ with $\tilde{R}_g(\Omega) \subset \tilde{\psi}(U \times V)$. By Lemma 4.3.31 the map

$$R_{g^{-1}} \circ \psi \circ R_g : Ug \times V \to G \times M$$

is a flat chart and

$$\Omega = \tilde{R}_{g^{-1}}(\tilde{R}_g(\Omega)) \subset \tilde{R}_{g^{-1}}(\tilde{\psi}(U \times V)) = (\tilde{R}_{g^{-1}} \circ \tilde{\psi} \circ \tilde{R}_g)(Ug \times V).$$

\[\square\]

4.3.3 Characterization of globalizable infinitesimal actions

We now study conditions for the existence of globalizations. The main result is the following:

**Theorem 4.3.35.** Let $\lambda : g \to \text{Vec}(M)$ be an infinitesimal action. Then the following statements are equivalent:

(i) The infinitesimal action $\lambda$ is globalizable.

(ii) The restricted infinitesimal action $\lambda_0 = \lambda|_{G_0}$ is globalizable.

(iii) The infinitesimal action $\lambda$ is univalent.

(iv) The reduced infinitesimal action $\tilde{\lambda}$ is univalent, i.e. $\pi_G|_{\Sigma}$ is injective for each leaf $\Sigma$, and all leaves $\Sigma \subset G \times M$ are holonomy free, i.e. the morphism $\Phi_{\Sigma}$ is trivial.

(v) There exists a local action $\varphi : W \to M$ with induced infinitesimal action $\lambda$ whose domain of definition is maximally balanced.

The equivalence of (iii) and (iv) is the content of Proposition 4.3.21 and once the equivalence of (i) and (iii) is established the equivalence of (i) and (ii) is a consequence of Corollary 4.3.22. The other equivalences are proven in the following: The implication (iii) $\Rightarrow$ (i) is the content of Proposition 4.3.44, (i) $\Rightarrow$ (v) follows from Proposition 4.3.47, and (v) $\Rightarrow$ (iii) is proven in Proposition 4.3.48.

**Remark 4.3.36** (see [Pal57], Chapter III, Theorem IV and Theorem V). In the classical case, Palais shows that for a univalent infinitesimal action the space of leaves $\Sigma \subset G \times M$ of the distribution $D$, which is denoted by $M^* = (G \times M)/\sim$, carries in a natural way the structure of a possibly non-Hausdorff manifold, the action of $G$ on $G \times M$ by $g \cdot (h,p) = (hg^{-1}, p)$ induces an action on the quotient space $M^*$ and $M \to M^*$, $p \mapsto \Sigma_{(e,p)}$ is an injective embedding.

A similar construction is also important in the case of supermanifolds.

**Definition 4.3.37.** Let $\lambda$ be an infinitesimal action of $G$ on $M$ and $D$ the associated distribution on $G \times M$. Define

$$M^* = (G \times M)/\sim$$

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to be the space of leaves $\Sigma \subset G \times M$ and denote by $\tilde{\pi} : G \times M \to M^*$, $\tilde{\pi}(g, p) = \Sigma_{(g, p)}$, the projection. Now endow $M^*$ with the quotient topology and define the sheaf $\mathcal{O}_{M^*}$ of $\mathbb{Z}_2$-graded algebras on $M^*$ by setting

$$\mathcal{O}_{M^*} = \tilde{\pi}^*(\mathcal{O}_{G \times M})$$

i.e. $\mathcal{O}_{M^*}(\Omega) = \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega))$ for any open subset $\Omega \subset M^*$, where $\mathcal{O}_{G \times M}$ denotes again the sheaf of $\mathcal{D}$-invariant functions on $G \times M$. Define the ringed space

$$M^* = (M^*, \mathcal{O}_{M^*})$$

and let $\pi = (\pi^*, \tilde{\pi}) : G \times M \to M^*$, where $\pi^* : \mathcal{O}_{M^*} \to \tilde{\pi}^*(\mathcal{O}_{G \times M})$ is given by the canonical inclusion $\mathcal{O}_{M^*}(\Omega) = \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega)) \to \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega))$ for any open subset $\Omega \subset M^*$.

**Definition 4.3.38.** For any open subset $V \subseteq M$, $V = (V, \mathcal{O}_M|_V)$, and $g \in G$ we define a morphism of ringed spaces

$$\iota_{V,g} : V \to M^*$$

by $\iota_{V,g} = \pi \circ \iota_g$, where $\iota_g : V \to \{g\} \times V \subset G \times M$ denotes again the inclusion. The reduced map of $\iota_{V,g}$ is given by $p \mapsto \Sigma_{(g, p)}$. For $g = e$ and $V = M$, let

$$\iota_M = \iota_e : M \to M^*$$

**Remark 4.3.39.** Let $\psi : U \times V \to G \times M$ be a flat chart with respect to $(\mathcal{D}, U, V, g, \text{id})$ and let $\pi_M = \pi_M|_{U \times V} : U \times V \to V$ denote the projection onto the second component. As $\iota_g : V \to U \times V$ is a section of $\pi_M$ and $\psi \circ \iota_g = \iota_g$, the diagram

$$\begin{array}{ccc}
U \times V & \xrightarrow{\psi} & G \times M \\
\downarrow \pi_M & & \downarrow \pi \\
V & \xrightarrow{\iota_{V,g}} & M^*
\end{array}$$

is commutative.

We will see later on that $M^*$ carries the structure of a supermanifold and $\iota_M : M \to M^*$ is an open embedding if and only if the infinitesimal action $\lambda$ is globalizable. In this case $M^*$ is itself a globalization of $\lambda$ and the $G$-action on $M^*$ is induced by the $G$-action $R$ on $G \times M$.

**Remark 4.3.40.** The topological space $M^*$ only depends on the underlying infinitesimal action $\tilde{\lambda} : \mathfrak{g}_0 \to \text{Vec}(M)$ since $\mathcal{D} = \mathcal{D}_{\tilde{\lambda}}$ (cf. Lemma 4.3.7). Hence, the map $\tilde{\pi}$ is an open map as in the classical case (cf. [Pal57], Chapter I, Theorem III, or [HI97], § 2, Proposition 2).

The topological space $M^*$ fulfills the second axiom of countability because $\tilde{\pi}$ is an open quotient map and $G \times M$ a manifold.

**Lemma 4.3.41.** The space $\mathcal{M}^* = (M^*, \mathcal{O}_{M^*})$ is a locally ringed space.
Proof. For \( f \in \mathcal{O}_{M^*}(\Omega) = \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega)) \) with \( \Sigma \in \Omega \subseteq M^* \) denote by \([f]_\Sigma\) the germ of \( f \) in the stalk \((\mathcal{O}_{M^*})_\Sigma\). We can define the ideal
\[
\mathfrak{m}_\Sigma = \{[f]_\Sigma | \tilde{f}(\Sigma) = 0\} \triangleleft (\mathcal{O}_{M^*})_\Sigma
\]
for any leaf \( \Sigma \in M^* \). Assume that \( \mathfrak{m}_\Sigma \) is not a maximal ideal in the stalk \((\mathcal{O}_{M^*})_\Sigma\). Then there is a proper ideal \( I \triangleleft (\mathcal{O}_{M^*})_\Sigma \) which is not contained in \( \mathfrak{m}_\Sigma \). Let \([f]_\Sigma \in I \setminus \mathfrak{m}_\Sigma\). Then we have \( \tilde{f}(\Sigma) \neq 0 \) and the continuity of \( \tilde{f} \) gives the existence of an open neighbourhood \( \Omega \) of \( \Sigma \in M^* \) such that \( f \) is defined on \( \tilde{\pi}^{-1}(\Omega) \), i.e.
\[
f \in \mathcal{O}_{M^*}(\Omega) = \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega)) \subset \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega)),
\]
and \( \tilde{f}(x) \neq 0 \) for all \( x \in \tilde{\pi}^{-1}(\Omega) \). Consequently, there exists \( g \in \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega)) \) with \( gf = fg = 1 \). The function \( g \) is also \( D \)-invariant since
\[
0 = Y(1) = Y(gf) = Y(g)f + (-1)^{|g||Y|}gY(f) = Y(g)f + 0 = Y(g)f
\]
for all vector fields \( Y \) belonging to \( D \) and thus \( Y(g) = 0 \). Therefore, \( g \) defines an element in \( \mathcal{O}_{M^*}(\Omega) \) and \( |g|_\Sigma[f]_\Sigma = [g]_\Sigma = 1 \in I \) which contradicts the assumption \( I \neq (\mathcal{O}_{M^*})_\Sigma \). \( \square \)

Lemma 4.3.42. The action of \( G \) on \( G \times M \) from the right \( R : G \times (G \times M) \to G \times M \) induces an action \( \chi \) of \( G \) on the space \( M^* \), i.e. a morphism of ringed spaces satisfying the usual action properties, such that
\[
G \times (G \times M) \xrightarrow{\chi} G \times M \xrightarrow{\pi} \mathcal{M}^*
\]
commutes. In particular, if \( M^* \) is a supermanifold, then \( \chi \) is an action of the Lie supergroup \( G \) on \( M^* \).

Proof. The underlying action \( \tilde{\chi} \) of \( G \) on \( M^* \) is given by the formula
\[
\tilde{\chi} : G \times M^* \to M^*, \quad (g, \Sigma_{h,p}) \mapsto \tilde{R}g(\Sigma_{h,p}) = \Sigma_{(h\gamma^{-1},p)}
\]
and continuous since \( \tilde{\pi} \circ \tilde{R} = \tilde{\chi} \circ (\text{id}_G \times \tilde{\pi}) \).

If \( f \) is any \( D \)-invariant function on \( G \times M \), then \( R^*(f) \) is \((1 \otimes D)\)-invariant (see Corollary 4.3.28). Therefore,
\[
(\pi \circ R)^*(f) \in \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega)) \cong \mathcal{O}_{G \times G \times M}(\tilde{\pi}^{-1}(\Omega)) \quad \text{for any } f \in \mathcal{O}_{M^*}(\Omega) = \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega)), \Omega \subseteq M^* , \text{ and } (\pi \circ R)^* \text{ induces a morphism } \chi^* : \mathcal{O}_{M^*} \to \mathcal{O}_{G \times M^*} \text{ with } R^* \circ \pi^* = (\text{id} \times \pi)^* \circ \chi^*. \quad \text{The map } \chi \text{ defines an action of } G \text{ on } M^* \text{ since } R \text{ is an action and } \chi \text{ inherits the respective properties.} \quad \square
\]

The infinitesimal action
\[
\lambda_{\chi} : \mathfrak{g} \to \text{Vec}(M^*), \quad \lambda_{\chi}(X) = (X(e) \otimes \text{id}^*) \circ \chi^*
\]
on the ringed space \( M^* \) extends the infinitesimal action \( \lambda \) on \( M \) in the following sense:
Proof. Let Ω ⊆ M* be open and f ∈ O_M∗(Ω). Then π∗(f) is D-invariant on π−1(Ω) ⊆ G × M and thus (id∗ ⊗ λ(X))(π∗(f)) = −(X ⊗ id∗)(π∗(f)) for X ∈ g. Consequently, we have

\[ (λ(X) ⊙ i_M)(f) = (λ(X) ⊙ i_e)(π∗(f)) = i_e((id∗ ⊗ λ(X))(π∗(f))) = −((X(e) ⊗ id∗)(π∗(f))). \]

A calculation using the identities π ◦ R = χ ◦ (id × π) and R ◦ (id × i) = (i × id) gives i_M ⊗ λ(X) = ((−X(e)) ⊗ id∗) ◦ π∗ so that λ(X) ⊙ i_M = i_M ⊗ λ(X). □

Proposition 4.3.44. Let λ be univalent. Then M* is a supermanifold, with a possibly non-Hausdorff underlying manifold M*.

Moreover, the morphism i_M : M → M* is an open embedding, the infinitesimal action λX induced by χ extends the infinitesimal action λ on M ≃ (i_M(M), O_M∗|i_M(M)), and we have G · i_M(M) = χ(G × i_M(M)) = M*.

Thus M* is a globalization of λ and the infinitesimal action λ is globalizable.

The proof of the proposition makes use of the next lemma.

Lemma 4.3.45. Let λ be univalent and W ⊆ G × M an open connected subset. For any f ∈ O_g×M(W) there exists a unique extension

\[ \hat{f} \in O_g×M(π−1(π(W))) \]

with \( \hat{f}|_W = f \). Proof. Note that the open set

\[ π−1(π(W)) = \bigcup_{\Sigma \text{ leaf } \Sigma \cap W \neq \emptyset} Σ \]

is again connected and by Lemma 4.3.25 a D-invariant extension \( \hat{f} \) of f is unique.

Let Σ be a leaf with Σ ∩ W ≠ ∅, i.e. Σ ∈ π(W), and let Σ' ⊂ Σ be relatively compact. Since λ is univalent, there exists a flat chart ψ : U × V → G × M with Σ' ⊂ ψ(U × V) and ψ(U × V) ⊂ W for some open subset U' ⊂ U. The function \( (ψ′|_U)^*(f) \) is Dg-invariant and thus of the form \( (ψ′|_U)^*(f) = 1 ⊗ f_M \) on U' × V ⊂ U′ × V (V') for some f_M ∈ O_M(V). Now, 1 ⊗ f_M is already defined on U × V and we define \( \hat{f} \) on \( ψ(U × V) \) by

\[ \hat{f}|_ψ(U × V) = (ψ−1)^*(1 ⊗ f_M). \]

This yields a well-defined function \( \hat{f} \) on π−1(π(W)) due to the uniqueness of flat charts following from the univalence of λ (see Proposition 4.3.14). Moreover, \( \hat{f} \) is D-invariant by construction and \( \hat{f}|_W = f \). □

Proof of Proposition 4.3.44. We prove that in the case of a univalent infinitesimal action the morphism \( \nu_{V,g} : V → M^* \) defines a chart for M* if there is a flat chart \( ψ : U × V → G × M \) with respect to (D, U, V, g, id). Due to the local existence of flat charts this implies that M* is a supermanifold.
By Proposition 4.3.21, the restriction $\tilde{\pi}_G|_\Sigma : \Sigma \to G$ of the projection $\pi_G : G \times M \to G$ is injective for all leaves $\Sigma \subset G \times M$. Therefore, we have $i_{\Sigma G}(p) = \Sigma_{(g,p)} = \tilde{\pi}_G(q)$ if and only if $p = q$. The map $i_{\Sigma G}$ is open because for any open subset $V' \subset V$ the set $i_{\Sigma G}(V') = i_{\Sigma G}(\tilde{\pi}_G(U \times V')) = \tilde{\pi}(\psi(U \times V))$ is open in $M^*$ using that $\psi$ is a local diffeomorphism and $\tilde{\pi}$ an open map. Consequently, $i_{\Sigma G}$ is a homeomorphism onto its image.

To show that $i_{\Sigma G}^*\pi$ is injective, let $\Omega \subseteq M^*$ be open and $f_1, f_2 \in \mathcal{O}_{M^*}(\Omega)$. If $i_{\Sigma G}^*(f_1) = i_{\Sigma G}^*(f_2)$, then also

$$\psi^*(\pi^*(f_1)) = \pi_{M^*}(i_{\Sigma G}^*(f_1)) = \pi_{M^*}(i_{\Sigma G}^*(f_2)) = \psi^*(\pi^*(f_2)).$$

Since $\pi^* : \mathcal{O}_{M^*} = \tilde{\pi}_*(\mathcal{O}_{G \times M}) \hookrightarrow \tilde{\pi}_*(\mathcal{O}_{G \times M})$ is the canonical inclusion, $\pi^*(f_1)$ and $f_1$ can be identified. If $\psi^*(f_1) = \psi^*(f_2)$, then $f_1 = f_2$ on $\tilde{\psi}\tilde{\psi}^{-1}(\tilde{\pi}^{-1}(\Omega))$ and thus $f_1 = f_2$ by Lemma 4.3.25.

For any $f_V \in \mathcal{O}_{\tilde{M}|V}(i_{\Sigma G}^{-1}(\Omega))$ we have

$$1 \otimes f_V = \pi_{M^*}(f_V) \in \mathcal{O}_{G \times M}(U \times i_{\Sigma G}(\Omega)) = \mathcal{O}_{G \times M}(\tilde{\psi}^{-1}(\tilde{\pi}^{-1}(\Omega))).$$

Thus $(\psi^{-1})^*(1 \otimes f_V)$ is a $D$-invariant function on $\tilde{\psi}(U \times V) \cap \tilde{\pi}^{-1}(\Omega) = \tilde{\psi}(\psi^{-1}(\tilde{\pi}^{-1}(\Omega)))$. By Lemma 4.3.45 there is a $D$-invariant extension $\tilde{f} \in \mathcal{O}_{G \times M}(\tilde{\pi}^{-1}(\Omega)) = \mathcal{O}_{M^*}(\Omega)$ of $(\psi^{-1})^*(1 \otimes f_V)$ and we have $i_{\Sigma G}^*(\tilde{f}) = f_V$ since $\psi^*(\pi^*(\tilde{f})) = \psi^*((\psi^{-1})^*(1 \otimes f_V)) = 1 \otimes f_V$.

Consequently, $i_{\Sigma G}^*$ is also surjective and thus $i_{\Sigma G}$ is a chart for $M^*$.

The morphism $i_{\Sigma M} = i_e$ is an open embedding since the univalence of $\lambda$ implies that $\tilde{\iota}_M$ is injective with the same argument as for $i_{\Sigma G}$ and locally $i_{\Sigma M}$ is of the form $i_{\Sigma V,e}$ such that there exists a flat chart $\psi$ with respect to $(D, \iota, U, V, e, i_d)$.

By Lemma 4.3.43, the infinitesimal action $\lambda_\chi$ induced by $\chi$ extends the infinitesimal action of $\lambda$ on $M \cong (\tilde{i}_M(M), \mathcal{O}_{M^*|\tilde{i}_M(M)})$. Since

$$g : \Sigma_{(e,p)} = \tilde{\chi}(g, \Sigma_{(e,p)}) = \tilde{R}_g(\Sigma_{(e,p)}) = \Sigma_{(g^{-1}p)}$$

for any $p \in M$ and $\tilde{i}_M(M)$ consists exactly of those leaves which intersect $\{e\} \times M$, we have $G : \tilde{i}_M(M) = \tilde{\chi}(G \times \tilde{i}_M(M)) = M^*$.

**Lemma 4.3.46.** Let $\varphi : \mathcal{W} \to \mathcal{M}$ be a local action with induced infinitesimal action $\lambda$ and maximally balanced domain of definition. Then we have $\varphi(W \times \{p\}) = \Sigma_{(e,p)}$ for any $p \in M$ and $\psi = (id \times \varphi) \circ (\text{diag} \times id) : \mathcal{W} \to G \times \mathcal{M}$, which is locally a flat chart.

**Proof.** The lemma follows from the analogous classical result (see [Pal57], Chapter II, Theorem VI) since $\varphi : \mathcal{W} \to \mathcal{M}$ has a maximally balanced domain of definition if and only if the domain of definition $W$ of the reduced local action is maximally balanced.

**Proposition 4.3.47.** Let $\varphi' : G \times \mathcal{M}' \to \mathcal{M}'$ be a globalization of $\lambda$. Then there is a local action $\varphi : \mathcal{W} \to \mathcal{M}$ with maximally balanced domain of definition and infinitesimal action $\lambda$.

Moreover, $\varphi : \mathcal{W} \to \mathcal{M}$ is the unique maximal local action with infinitesimal action $\lambda$. Any two local actions $\varphi_i : \mathcal{W}_i \to \mathcal{M}$, $i = 1, 2$, with infinitesimal action $\lambda$ coincide on their common domain of definition and define a local action $\chi : \mathcal{W}_1 \cup \mathcal{W}_2 \to \mathcal{M}$ with $\chi|_{\mathcal{W}_1} = \varphi_1$ and $\chi|_{\mathcal{W}_2} = \varphi_2$. 

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Proof. The set \((\varphi')^{-1}(M) \cap (G \times M)\) is open in \(G \times M\) and contains \(\{e\} \times M\). Let \(W_p\) be the connected component of \(e\) in \(\{g \in G | \varphi'(g, p) \in M\}\) for \(p \in M\) and define

\[
W = \bigcup_{p \in M} W_p \times \{p\}.
\]

The set \(W \subseteq G \times M\) is open and the largest domain of definition of a local action included in \((\varphi')^{-1}(M) \cap (G \times M)\) (cf. [Pal57], Chapter II, Theorem I). Let \(W = (W, \mathcal{O}_W)\) and define \(\varphi = \varphi'|_W : W \to \mathcal{M}\). The map \(\varphi\) is a local action of \(\mathcal{G}\) on \(\mathcal{M}\). A direct calculation shows that the domain of definition \(W\) of \(\varphi\) is maximally balanced. By the preceding lemma we have \(\tilde{\psi}(W_p \times \{p\}) = \Sigma_{(e,p)}\) for any \(p \in M\), where \(\tilde{\psi} = (\text{id} \times \varphi) \circ (\text{diag} \times \text{id}) : W \to G \times M\).

Let \(\chi : W_\chi \to \mathcal{M}\) be any local action with induced infinitesimal action \(\lambda\) and set \(\psi_\chi = (\text{id} \times \chi) \circ (\text{diag} \times \text{id}) : W_\chi \to G \times M\). For any \(p \in M\), \(\psi_\chi(W_{\chi,p} \times \{p\})\) is contained in the leaf \(\Sigma_{(e,p)}\). Since \(\Sigma_{(e,p)} = \tilde{\psi}(W_p \times \{p\})\), we get

\[
W_{\chi,p} = \pi_G(\psi_\chi(W_{\chi,p} \times \{p\})) \subseteq \pi_G(\Sigma_{(e,p)}) = \pi_G(\tilde{\psi}(W_p \times \{p\})) = W_p,
\]

which implies \(W_\chi \subseteq W\). The uniqueness of local actions with a given domain of definition (see Corollary 4.2.3) yields \(\varphi = \chi\) on \(W_\chi\). \(\square\)

Proposition 4.3.48. Let \(\varphi : W \to \mathcal{M}\) be a local action with maximally balanced domain of definition. Then its induced infinitesimal action is univalent.

Proof. Let \(\psi = (\text{id} \times \varphi) \circ (\text{diag} \times \text{id}) : W \to G \times M\) be the locally flat chart associated with the local action \(\varphi\). By Lemma 4.3.46 we have \(\tilde{\psi}(W_p \times \{p\}) = \Sigma_{(e,p)}\) for any \(p \in M\).

Let \(\Omega \subset \Sigma_{(e,p)}\) be a relatively compact connected subset. By Lemma 4.2.2 there are subsets \(U \subset G\) and \(V \subset M\), \(p \in V\), such that \(\tilde{\psi}|_{U \times V}\) is a flat chart and \(\Omega \subset \tilde{\psi}(U \times V)\). Consequently, \(\Sigma_{(e,p)}\) is univalent for any \(p \in M\) and hence \(\lambda\) is univalent by Proposition 4.3.34. \(\square\)

4.4 Examples

In this section, a family of examples of infinitesimal actions on the supermanifold \(\mathcal{M} = (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{0|2}\) is provided. In all cases the underlying action is globalizable, but the infinitesimal action on the supermanifold is in general not globalizable.

Example 4.4.1. Let \(\mathcal{M} = (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{0|2}\), with coordinates \(z, \theta_1, \theta_2\), and let \(\alpha : \mathbb{C} \setminus \{0\} \to \mathbb{C}\) be a holomorphic function. Consider the even holomorphic vector field

\[
X_\alpha = (1 + \alpha(z)\theta_1\theta_2) \frac{\partial}{\partial z}
\]
on \(\mathcal{M}\). We now examine for which \(\alpha\) the infinitesimal \(\mathbb{C}\)-action \(\lambda_\alpha\) on \(\mathcal{M}\), generated by \(X_\alpha\), is globalizable.

First, note that for any \(\alpha\) the leaves \(\Sigma \subset \mathbb{C} \times M\) of the distribution \(\mathcal{D}_\alpha\) on \(\mathbb{C} \times M\) are of the form

\[
\Sigma = \Sigma_{(t,z)} = \{(t + s, z + s) | s \in \mathbb{C} \setminus \{-z\}\}
\]
for some \((t, z) \in \mathbb{C} \times (\mathbb{C} \setminus \{0\})\), where \(\mathcal{D}_\alpha\) is spanned by the vector field \(\frac{\partial}{\partial t} + X_\alpha\) if \(t\) denotes the coordinate on \(\mathbb{C}\). Each leaf \(\Sigma\) is therefore biholomorphic to \(\mathbb{C} \setminus \{0\}\).
The reduced vector field $\tilde{X}_\alpha = \frac{\partial}{\partial t}$ always generates a globalizable infinitesimal action and a globalization of $M = \mathbb{C} \setminus \{0\}$ is $M^*_\alpha = \mathbb{C}$ with the usual addition as $\mathbb{C}$-action. If $\lambda_\alpha$ is globalizable, then the globalization $\mathcal{M}^*_\alpha = (M^*_\alpha, \tilde{\pi}_*, \mathcal{O}^D_{\mathcal{C}_M})$ is a complex supermanifold of dimension $(1|2)$. Every complex supermanifold $\mathcal{N}$ with underlying manifold $\mathbb{C}$ is split since $\mathbb{C}$ is Stein (see [Oni98], Theorem 3.4). Moreover, $\mathcal{N}$ is isomorphic to $\mathbb{C}^{1|n}$ for some $n \in \mathbb{N}$ since all holomorphic vector bundles on $\mathbb{C}$ are trivial. This implies that $\mathcal{M}^*_\alpha$ is isomorphic to $\mathbb{C}^{1|2}$ if it is a supermanifold.

Let $f \in \mathcal{O}_{\mathbb{C} \times M}(\mathbb{C} \times M)$. There are holomorphic functions $f_0, f_1, f_2, f_{12}$ on $\mathbb{C} \times M = \mathbb{C} \times (\mathbb{C} \setminus \{0\})$ with $f = f_0 + f_1 \theta_1 + f_2 \theta_2 + f_{12} \theta_1 \theta_2$. Then

$$\left( \frac{\partial}{\partial t} + X_\alpha \right)(f) = \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right)(f_0 + f_1 \theta_1 + f_2 \theta_2) + (\alpha \frac{\partial}{\partial z} f_0 + \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) f_{12}) \theta_1 \theta_2.$$ 

Therefore, if $f$ is $\mathcal{D}_\alpha$-invariant, i.e. $(\frac{\partial}{\partial t} + X_\alpha)(f) = 0$, then the functions $f_0, f_1$ and $f_2$ are constant along the leaves $\Sigma_{t, z} = \{(t + s, z + s) | s \in \mathbb{C} \setminus \{z\}\}$ and thus there exist holomorphic functions $g_i : \mathbb{C} \to \mathbb{C}$, $i = 1, 2, 3$, with

$$f_i(t, z) = g_i(z - t).$$

Moreover, the $\mathcal{D}_\alpha$-invariance of $f$ implies $(\alpha \frac{\partial}{\partial z} f_0 + \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \right) f_{12}) = 0$ and consequently $f_{12}$ is locally of the form

$$f_{12}(t, z) = -A(z) \left( \frac{\partial}{\partial z} f_0 \right)(t, z) + g_{12}(z - t),$$

where $A$ is a (local) primitive of $\alpha$, i.e. $A'(z) = \alpha(z)$, and $g_{12}$ a holomorphic function on (some open subset of) $\mathbb{C}$. There are two different cases:

(i) If $\alpha$ has a global primitive $A : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ we have

$$\mathcal{O}_{\mathcal{M}^*_\alpha}(M^*) = \mathcal{O}^D_{\mathcal{C} \times M}(\mathbb{C} \times M) = \left\{ g_0(z - t) + g_1(z - t) \theta_1 + g_2(z - t) \theta_2 
+ \left( g_{12}(z - t) - A(z) \frac{\partial}{\partial z} g_0(z - t) \right) \theta_1 \theta_2 \mid g_0, g_1, g_2, g_{12} \text{ holomorphic} \right\} \cong \mathcal{O}_{\mathbb{C}^{1|2}}(\mathbb{C}).$$

It turns out that the infinitesimal action $\lambda_\alpha$ is globalizable in this case with globalization $\mathcal{M}^*_\alpha = \mathbb{C}^{1|2}$ and $\mathbb{C}$ acts on $\mathbb{C}^{1|2}$ by the usual addition on $\mathbb{C}$ extended to $\mathbb{C}^{1|2}$, i.e. the action $\chi : \mathbb{C} \times \mathbb{C}^{1|2} \to \mathbb{C}^{1|2}$ is given by

$$\chi^*(z, \theta_1, \theta_2) = (z + t, \theta_1, \theta_2).$$

Moreover, the open embedding $\iota_\mathcal{M} : \mathcal{M} \to \mathcal{M}^*_\alpha$ can be realized as

$$\iota_\mathcal{M} : \mathbb{C} \setminus \{0\} \times \mathbb{C}^{0|2} \hookrightarrow \mathbb{C}^{1|2}, \iota_\mathcal{M}^*(z, \theta_1, \theta_2) = (z - A(z) \theta_1 \theta_2, \theta_1, \theta_2).$$
(ii) In the case where \( \alpha \) does not have a global primitive \( A \) on \( \mathbb{C} \setminus \{0\} \), we get

\[
\mathcal{O}_{\mathcal{M}_\alpha^*}(M^*) = \mathcal{O}^D_{\mathcal{C} \times \mathcal{M}}(\mathbb{C} \times M)
\]

\[
= \left\{ \lambda + g_1(z - t)\theta_1 + g_2(z - t)\theta_2 + g_{12}(z - t)\theta_1\theta_2 \middle| \lambda \in \mathbb{C}, g_1, g_2, g_{12} \text{ holomorphic} \right\}
\]

which is not isomorphic to \( \mathcal{O}_{\mathcal{C} \cup \mathcal{L}}(\mathbb{C}) \). Therefore, the ringed space \( \mathcal{M}_\alpha^* \) is not a supermanifold and \( \lambda_\alpha \) cannot be globalizable.

A calculation shows that for arbitrary \( \alpha \) a flat chart \( \psi \) with respect to \( (\mathcal{D}, U, V, t_0, \rho) \) is given by the pullback

\[
\psi^*(t, z, \theta_1, \theta_2)
\]

\[
= \left( t, \rho^*(z, \theta_1, \theta_2) \right) + \left( 0, (t - t_0) + (A(\rho(z) + (t - t_0)) - A(\rho(z)))\rho^*(\theta_1\theta_2), 0, 0 \right)
\]

\[
= \left( t, \rho^*(z) + (t - t_0) + (A(\rho(z) + (t - t_0)) - A(\rho(z)))\rho^*(\theta_1\theta_2), \rho^*(\theta_1), \rho^*(\theta_2) \right),
\]

where \( U \subseteq \mathbb{C} \) and \( V \subseteq \mathbb{C} \setminus \{0\} \) need to be open subsets such that there exists a primitive \( A \) of \( \alpha \) on \( U + \rho(V) = \{ t + z | t \in U, z \in \rho(V) \} \subset \mathbb{C} \setminus \{0\} \).

Let \( \Sigma \subset \mathbb{C} \times M \) be a leaf and \( (s, z) \in \Sigma \). We shall now explicitly describe the morphism

\[
\Phi = \Phi_{\Sigma, (s, z)} : \pi_1(\Sigma, (s, z)) \rightarrow \text{Hol}_z(\mathbb{C} \setminus \{0\} \times \mathbb{C}^{00})
\]

First remark that if \( \psi \) is a flat chart with respect to \( (\mathcal{D}, U, V, t_0, \rho) \) and \( t'_0 \in U \), then \( \psi \) is also a flat chart with respect to \( (\mathcal{D}, U, V, t'_0, \rho') \) for

\[
(\rho')^*(z, \theta_1, \theta_2)
\]

\[
= \rho^*(z, \theta_1, \theta_2) + \left( \int_{t_0}^{t'_0} 1 dt + \left( \int_{t_0}^{t'_0} \alpha(\rho(z) + (t - t_0)) dt \right) \rho^*(\theta_1\theta_2), 0, 0 \right)
\]

where the integrals do not depend on the path, as long as it is contained in \( U \), since \( \alpha \) has a primitive on \( U + \rho(V) \). If \( \psi' \) is another flat chart with respect to \( (\mathcal{D}, U', V', t'_0, \rho') \) with \( V \cap V' \neq \emptyset \) and \( t''_0 \in U' \), then \( \psi' \) is also a flat chart with respect to \( (\mathcal{D}, U', V', t''_0, \rho'') \) for

\[
(\rho'')^*(z, \theta_1, \theta_2)
\]

\[
= (\rho')^*(z, \theta_1, \theta_2) + \left( \int_{t'_0}^{t''_0} 1 dt + \left( \int_{t'_0}^{t''_0} \alpha(\rho'(z) + (t - t'_0)) dt \right) (\rho')^*(\theta_1\theta_2), 0, 0 \right)
\]

\[
= \rho^*(z, \theta_1, \theta_2) + \left( \int_{t_0}^{t'_0} 1 dt + \left( \int_{t_0}^{t'_0} \alpha(\rho(z) + (t - t_0)) dt \right) \rho^*(\theta_1\theta_2), 0, 0 \right)
\]

\[
+ \left( \int_{t'_0}^{t''_0} 1 dt + \left( \int_{t'_0}^{t''_0} \alpha(\rho(z) + (t'_0 - t_0) + (t - t'_0)) dt \right) \rho^*(\theta_1\theta_2), 0, 0 \right)
\]

\[
= \rho^*(z, \theta_1, \theta_2) + \left( \int_{t_0}^{t''_0} 1 dt + \left( \int_{t_0}^{t''_0} \alpha(\rho(z) + (t - t_0)) dt \right) \rho^*(\theta_1\theta_2), 0, 0 \right)
\]

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since $\tilde{\rho}'(z) = \tilde{\rho}(z) + (t_0' - t_0)$ and $(\rho')^*(\theta_i) = \rho^*(\theta_i)$, $i = 1, 2$, where the integrals are taken along appropriate paths.

For any closed path $\gamma : [0, 1] \to \Sigma \subset \mathbb{C} \times M$, $\gamma(r) = (\gamma_1(r), \gamma_2(r)) \in \Sigma \subset \mathbb{C} \times M$, $\gamma(0) = \gamma(1) = (s, z)$, we consequently get that the pullback of the local biholomorphism $\Phi([\gamma]) = \Phi_{\Sigma,(s,z)}([\gamma])$ is given by

$$
\Phi([\gamma])^*(z, \theta_1, \theta_2) = id^*(z, \theta_1, \theta_2) + \left( \int_{\gamma_1} 1 dt + \left( \int_{\gamma_1} \alpha_{s,z} dt \right) \theta_1 \theta_2, 0, 0 \right)
$$

$$
= (z, \theta_1, \theta_2) + \left( \int_{\gamma_1} \alpha_{s,z} dt \right) \theta_1 \theta_2, 0, 0
$$

for $\alpha_{s,z}(t) = \alpha((z-s) + t)$. Thus, the morphism $\Phi$ is trivial if and only if $\alpha$ has a global primitive on $\mathbb{C} \setminus \{0\}$. Using Theorem 4.3.35 this shows again that the infinitesimal action $\lambda_\alpha$ is globalizable if and only if $\alpha$ has a global primitive.

In the special case of $\alpha(z) = z^{-1}$, we have for example

$$
\Phi : \mathbb{Z} \to \text{Hol}_p(\mathbb{C} \setminus \{0\} \times \mathbb{C}^{0|2}), \quad \Phi(k)^*(z, \theta_1, \theta_2) = (z + 2\pi k \theta_1 \theta_2, \theta_1, \theta_2)
$$

for any leaf $\Sigma$, identifying the fundamental group of $\Sigma \cong \mathbb{C} \setminus \{0\}$ with $\mathbb{Z}$.

### 4.5 Actions of simply-connected Lie supergroups

In this section, a few consequences of the characterization of globalizable infinitesimal actions of a simply-connected Lie supergroup, i.e. a Lie supergroup $G$ whose underlying Lie group $\bar{G}$ is simply-connected, are given.

If $G$ is simply-connected and acts on the classical manifold $M$, the leaves $\Sigma \subset G \times M$ of the distribution associated with the infinitesimal action are all isomorphic to $G$ and thus simply-connected. This yields consequences for the existence of globalizations of infinitesimal actions of simply-connected Lie supergroups since there are no holonomy phenomena, i.e. the morphisms $\Phi_{\Sigma} : \pi_1(\Sigma) \to \text{Diff}_p(M)$ are all trivial, if the underlying infinitesimal action is global.

**Definition 4.5.1.** An infinitesimal action $\lambda : g \to \text{Vec}(M)$ is called global if there is a $G$-action on $M$ which induces $\lambda$.

**Remark 4.5.2** (cf. [Pal57], Chapter II, Section 4). Let $G$ be a Lie group acting on a manifold $M$. Then the leaves $\Sigma \subset G \times M$ of the distribution $\mathcal{D}_{\lambda}$ associated with the infinitesimal action $\lambda$ induced by the $G$-action are all isomorphic to $G$.

As a consequence we obtain the following theorem.

**Theorem 4.5.3.** Let $G$ be a simply-connected Lie supergroup and $\lambda : g \to \text{Vec}(M)$ an infinitesimal action of $G$ such that its reduced action $\tilde{\lambda} : g_0 \to \text{Vec}(M)$ is global. Then the infinitesimal action $\lambda$ is globalizable, and $M$ is the unique globalization.

**Proof.** Let $\mathcal{D}$ denote again the distribution on $G \times M$ associated with $\lambda$. By Lemma 4.3.7 we have $\mathcal{D} = \mathcal{D}_{\lambda}$ and by definition the leaves of $\mathcal{D}$ are the leaves of $\mathcal{D}_{\lambda}$. By the preceding remark all leaves $\Sigma \subset G \times M$ are isomorphic to $G$ and consequently simply-connected. Therefore, the morphisms $\Phi : \pi_1(\Sigma, (g, p)) \cong \{1\} \to \text{Diff}_p(M)$ are all trivial. Since
\(\tilde{\lambda}\) is global and thus in particular globalizable, \(\tilde{\lambda}\) is univalent. This implies that \(\lambda\) is globalizable using the equivalent characterizations of globalizability formulated in Theorem 4.3.35. Let \(M'\) be a globalization of \(\lambda\) and \(i_M : M \to M'\) the open embedding. Then \(M'\) is a globalization of \(\tilde{\lambda}\) and because \(M'\) is the unique globalization of \(\tilde{\lambda}\) (cf. \cite{Pal57}, Chapter III, Theorem XII,(4)) we have \(M = M'\) and \(i_M = \text{id}_M\). Since \(i_M\) is an open embedding, this implies \(i_M = \text{id}_M\) and \(M' = M\).

If the assumption on the simply-connectedness of \(G\) is dropped in the above theorem, there exist counterexamples to the statement, see e.g. Example 4.3.20. Also, as illustrated in Example 4.4.1, it is not enough for \(\tilde{\lambda}\) to be globalizable. We really need that \(\tilde{\lambda}\) is global.

**Definition 4.5.4.** Let \(\lambda : g \to \text{Vec}(M)\) be an infinitesimal action. The set of points \(p \in M\) such that there exists an even vector field \(X \in g_{\bar{0}}\) with \(\lambda(X)(p) \neq 0\) is called the support of \(\lambda\).

**Remark 4.5.5.** The definition of the support of an infinitesimal action \(\lambda\) implies that the support of \(\lambda\) coincides with the support of the underlying infinitesimal action \(\tilde{\lambda}\).

In the classical case, we have the following two theorems on actions of simply-connected Lie groups.

**Theorem 4.5.6** (see \cite{Pal57}, Chapter III, Theorem XVIII). Let \(G\) denote a simply-connected Lie group with Lie algebra \(g_{\bar{0}}\) and let \(\tilde{\lambda} : g_{\bar{0}} \to \text{Vec}(M)\) be an infinitesimal action of \(G\) on a manifold \(M\). If the support of \(\tilde{\lambda}\) is relatively compact in \(M\), then \(\tilde{\lambda}\) is global.

In particular, any infinitesimal action of a simply-connected Lie group \(G\) on a compact manifold \(M\) is global.

**Theorem 4.5.7** (see \cite{Pal57}, Chapter IV, Theorem III). Let \(\tilde{\lambda} : g_{\bar{0}} \to \text{Vec}(M)\) be an infinitesimal action of a simply-connected Lie group \(G\) on \(M\). Suppose there exists a set of generators \(\{X_i\}_{i \in I}, X_i \in g_{\bar{0}}\), of the Lie algebra \(g_{\bar{0}}\) such that the flow of each vector field \(\tilde{\lambda}(X_i)\) is global. Then the infinitesimal action \(\tilde{\lambda}\) is global.

Applying Theorem 4.5.3, these results in the classical case can be directly carried over to the case of infinitesimal actions of simply-connected Lie supergroups on supermanifolds.

**Corollary 4.5.8.** Let \(G\) be a simply-connected Lie supergroup and \(\lambda : g \to \text{Vec}(M)\) an infinitesimal action whose support is relatively compact in \(M\). Then the infinitesimal action \(\lambda\) is global.

In particular, any infinitesimal action of a simply-connected Lie supergroup on a supermanifold with compact underlying manifold is global.

**Corollary 4.5.9.** Let \(\lambda : g \to \text{Vec}(M)\) be an infinitesimal action of a simply-connected Lie supergroup \(G\) such that there exists a set of generators \(\{X_i\}_{i \in I}, X_i \in g_{\bar{0}}\), of \(g_{\bar{0}}\) such that each vector field \(\tilde{\lambda}(X_i)\) has a global flow. Then the infinitesimal action \(\lambda\) is global.

A slightly weaker version of this corollary, in a formulation for DeWitt supermanifolds, has been proven, in a different way, in \cite{Tuy13}. The assumption there is that all even vector fields have global flows, and not only a set of generators.
Chapter 5

Automorphism groups of compact complex supermanifolds

The automorphism group of a compact complex manifold $M$ carries the structure of a complex Lie group which acts holomorphically on $M$ and whose Lie algebra consists of the holomorphic vector fields on $M$ (see [BM47]). In this chapter, we investigate how this result can be extended to the category of compact complex supermanifolds.

Let $\mathcal{M}$ be a compact complex supermanifold, i.e. a complex supermanifold whose underlying manifold is compact. An automorphism of $\mathcal{M}$ is a biholomorphic morphism $\mathcal{M} \to \mathcal{M}$. A first candidate for the automorphism group of such a supermanifold is the set of automorphisms, which we denote by $\text{Aut}_{0}(\mathcal{M})$. However, every automorphism $\varphi$ of a supermanifold $\mathcal{M}$ (with structure sheaf $\mathcal{O}_{\mathcal{M}}$) is “even” in the sense that its pullback $\varphi^{*} : \mathcal{O}_{\mathcal{M}} \to \tilde{\varphi}_{*}(\mathcal{O}_{\mathcal{M}})$ is a parity-preserving morphism. Therefore, we can (at most) expect this set of automorphisms of $\mathcal{M}$ to carry the structure of a classical Lie group if we require its action on $\mathcal{M}$ to be smooth or holomorphic. This way we do not obtain a Lie supergroup of positive odd dimension. We prove that the group $\text{Aut}_{0}(\mathcal{M})$, endowed with an analogue of the compact-open topology, carries the structure of a complex Lie group such that the action on $\mathcal{M}$ is holomorphic and its Lie algebra is the Lie algebra of even holomorphic vector fields on $\mathcal{M}$. It should be noted that the group $\text{Aut}_{0}(\mathcal{M})$ is in general different from the group $\text{Aut}(\mathcal{M})$ of automorphisms of the underlying manifold $\mathcal{M}$. There is a group homomorphism $\text{Aut}_{0}(\mathcal{M}) \to \text{Aut}(\mathcal{M})$ given by assigning the underlying map to an automorphism of the supermanifold; this group homomorphism is in general neither injective nor surjective.

We will define the automorphism group of a compact complex supermanifold $\mathcal{M}$ to be a complex Lie supergroup which acts holomorphically on $\mathcal{M}$ and satisfies a universal property. In analogy to the classical case, its Lie superalgebra is the Lie superalgebra of holomorphic vector fields on $\mathcal{M}$, and the underlying Lie group is $\text{Aut}_{0}(\mathcal{M})$, the group of automorphisms of $\mathcal{M}$. Using the equivalence of complex Harish-Chandra pairs and complex Lie supergroups (see [Vis11]), we construct the appropriate automorphism Lie supergroup of $\mathcal{M}$.

Taking a different approach to supermanifolds via functors of points, an alternative definition of the automorphism group as a functor in analogy to [SW11] could be made. In Section 5.4 it is proven that the constructed automorphism group $\text{Aut}(\mathcal{M})$ represents this functor.
In the classical case, another class of complex manifolds where the automorphism group carries the structure of a Lie group is given by the bounded domains in \( \mathbb{C}^m \) (see [Car79]). An analogue statement is false in the case of supermanifolds as shown by an example in Section 5.5.

The results of this chapter, except those formulated in Section 5.4 related to the approach via the functor of points, are contained in [BK15].

### 5.1 The group of automorphisms \( \text{Aut}_0(\mathcal{M}) \)

Let \( \mathcal{M} \) be a compact complex supermanifold. An automorphism of \( \mathcal{M} \) is a biholomorphic morphism \( \varphi : \mathcal{M} \to \mathcal{M} \). Denote by \( \text{Aut}_0(\mathcal{M}) \) the set of automorphisms of \( \mathcal{M} \).

The map \( \text{Aut}_0(\mathcal{M}) \to \text{Aut}(\mathcal{M}) \) which assigns to an automorphism \( \varphi : \mathcal{M} \to \mathcal{M} \) the automorphism \( \tilde{\varphi} : \mathcal{M} \to \mathcal{M} \) of the underlying complex manifold is a group homomorphism. Since \( \mathcal{M} \) is compact, \( \text{Aut}(\mathcal{M}) \) carries the structure of a complex Lie group (see [BM47]). Note that this group homomorphism \( \text{Aut}_0(\mathcal{M}) \to \text{Aut}(\mathcal{M}) \) is in general neither surjective nor injective as illustrated in the next two examples.

**Example 5.1.1.** Let \( \mathcal{M} = \mathbb{C}^{0|1} \). Then the underlying manifold \( M \) is simply a point and we have \( \text{Aut}(M) = \text{Aut}\{\{0\}\} = \{id_M\} \). Let \( \xi \) denote the coordinate on \( \mathcal{M} = \mathbb{C}^{0|1} \). For each automorphism \( \varphi \) of \( \mathcal{M} \) there exists \( c \in \mathbb{C}^* \) such that \( \varphi^*(\xi) = c \cdot \xi \). Thus we have \( \text{Aut}_0(\mathcal{M}) \cong \mathbb{C}^* \) and the homomorphism \( \text{Aut}_0(\mathcal{M}) \to \text{Aut}(\mathcal{M}) \) corresponds to the unique map \( \mathbb{C}^* \to \{id_M\} \).

**Example 5.1.2.** If \( \mathcal{M} \) is a split complex supermanifold associated with a holomorphic vector bundle \( E \to M \), then any automorphism \( \varphi : M \to M \) induces an automorphism \( E \to E \) of the vector bundle \( E \) over the underlying automorphism \( \tilde{\varphi} : M \to M \). In particular, if the homomorphism \( \text{Aut}_0(\mathcal{M}) \to \text{Aut}(\mathcal{M}) \), \( \varphi \mapsto \tilde{\varphi} \), is surjective, each automorphism of \( M \) necessarily lifts to an automorphism of the bundle \( E \) (over the given automorphism of \( M \)). Therefore, in order to give an example of a complex supermanifold \( \mathcal{M} \) such that \( \text{Aut}_0(\mathcal{M}) \to \text{Aut}(M) \) is not surjective, it suffices to find a compact complex manifold \( M \) with holomorphic vector bundle \( E \) with the property that not all automorphisms of \( M \) lift to automorphisms of the vector bundle \( E \).

If the underlying manifold \( M \) is allowed to be disconnected, we could take \( M \) to be the disjoint union of two copies of the same compact complex manifold, i.e. \( M = N \cup N \) for a compact complex manifold \( N \). Then we can for example define a line bundle \( L \) on \( M \) by taking two non-isomorphic line bundles on the two copies of \( N \). In this case the automorphism of \( M \) which interchanges the two copies of \( N \) does not lift to an automorphism of the line bundle \( L \).

For an example where \( M \) is connected, consider a connected compact complex manifold \( N \), and let \( L_j \to N \), \( j = 1, 2 \), be two non-isomorphic line bundles. Define \( M \) to be the product manifold \( M = N \times N \), and let \( E = L_1 \times L_2 \) the product vector bundle of rank 2. Let \( p \in N \). The restriction of the vector bundle \( E \) to the submanifold \( N \times \{p\} \) is isomorphic to the Whitney sum of \( L_1 \) and the trivial line bundle \( L = N \times \mathbb{C} \), i.e. \( E|_{N \times \{p\}} \cong L_1 \oplus L \). Similarly we have \( E|_{\{p\} \times N} \cong L \oplus L_2 \). The automorphisms \( \chi : N \times N \to M = N \times N \), \( \chi(p_1, p_2) = (p_2, p_1) \) for \( p_j \in N \), which interchanges the two factors of \( N \) does not lift to an automorphism of the bundle \( E \).
since \( \chi(N \times \{p\}) = \{p\} \times N \) and the restriction of \( E \) to \( N \times \{p\} \) is not isomorphic to the restriction of \( E \) to \( \{p\} \times N \).

### 5.1.1 The topology on \( \text{Aut}_0(\mathcal{M}) \)

In this paragraph, a topology on \( \text{Aut}_0(\mathcal{M}) \) is introduced, which generalizes the compact-open topology and topology of compact convergence of the classical case. We prove that \( \text{Aut}_0(\mathcal{M}) \) is a locally compact topological group with respect to this topology.

Let \( K \subseteq M \) be a compact subset such that there are local odd coordinates \( \theta_1, \ldots, \theta_n \) for \( \mathcal{M} \) on an open neighbourhood of \( K \). Moreover, let \( U \subseteq M \) be open and \( f \in \mathcal{O}_M(U) \), and let \( U_{\nu} \) be open subsets of \( \mathbb{C} \) for \( \nu \in (\mathbb{Z}_2)^n \). Let \( \varphi: \mathcal{M} \to \mathcal{M} \) be an automorphism with \( \tilde{\varphi}(K) \subseteq U \). Then there are holomorphic functions \( \varphi_{f,\nu} \) on a neighbourhood of \( K \) such that

\[
\varphi^*(f) = \sum_{\nu \in (\mathbb{Z}_2)^n} \varphi_{f,\nu} \theta^n.
\]

Let

\[
\Delta(K, U, f, \theta_j, U_{\nu}) = \{ \varphi \in \text{Aut}_0(\mathcal{M}) | \tilde{\varphi}(K) \subseteq U, \varphi_{f,\nu}(K) \subseteq U_{\nu} \},
\]

and endow \( \text{Aut}_0(\mathcal{M}) \) with the topology generated by sets of this form, i.e. the sets of the form \( \Delta(K, U, f, \theta_j, U_{\nu}) \) form a subbase of the topology.

**Remark 5.1.3.** In particular, the subsets of the form

\[
\Delta(K, U) = \{ \varphi \in \text{Aut}_0(\mathcal{M}) | \tilde{\varphi}(K) \subseteq U \}
\]

are open for \( K \subseteq M \) compact and \( U \subseteq M \) open. Hence the map \( \text{Aut}_0(\mathcal{M}) \to \text{Aut}(\mathcal{M}) \), associating to an automorphism \( \varphi \) of \( \mathcal{M} \) the underlying automorphism \( \tilde{\varphi} \) of \( M \), is continuous.

**Remark 5.1.4.** The group \( \text{Aut}_0(\mathcal{M}) \) endowed with the above topology is a second-countable Hausdorff space since \( M \) is second-countable.

Let \( U \subseteq M \) be open. Then we can define a topology on \( \mathcal{O}_M(U) \) as follows: If \( K \subseteq U \) is compact such that there exist odd coordinates \( \theta_1, \ldots, \theta_n \) on a neighbourhood of \( K \), write \( f \in \mathcal{O}_M(U) \) on \( K \) as \( f = \sum_{\nu} f_{\nu} \theta^n \). Let \( U_{\nu} \subseteq \mathbb{C} \) be open subsets. Then define a topology on \( \mathcal{O}_M(U) \) by requiring that the sets of the form \( \{ f \in \mathcal{O}_M(U) | f_{\nu}(K) \subseteq U_{\nu} \} \) are a subbase of the topology. A sequence of functions \( f_k \) converges to \( f \) if and only if in all local coordinate domains with odd coordinates \( \theta_1, \ldots, \theta_n \) and \( f_k = \sum_{\nu} f_{k,\nu} \theta^n \), \( f = \sum_{\nu} f_{\nu} \theta^n \), the coefficient functions \( f_{k,\nu} \) converge uniformly to \( f_{\nu} \) on compact subsets. Note that for any open subsets \( U_1, U_2 \subseteq M \) with \( U_1 \subseteq U_2 \) the restriction map \( \mathcal{O}_M(U_2) \to \mathcal{O}_M(U_1) \), \( f \mapsto f|_{U_1} \), is continuous.

Using Taylor expansion (in local coordinates) of automorphisms of \( \mathcal{M} \) we can deduce the following lemma:

**Lemma 5.1.5.** A sequence of automorphisms \( \varphi_k : \mathcal{M} \to \mathcal{M} \) converges to an automorphism \( \varphi : \mathcal{M} \to \mathcal{M} \) with respect to the topology of \( \text{Aut}_0(\mathcal{M}) \) if and only if the following condition is satisfied: For all \( U, V \subseteq M \) open subsets such that \( V \) contains the closure of \( \tilde{\varphi}(U) \), there is an \( N \in \mathbb{N} \) such that \( \tilde{\varphi}_k(U) \subseteq V \) for all \( k \geq N \). Furthermore, for any \( f \in \mathcal{O}_M(V) \) the sequence \( (\varphi_k)^*(f) \) converges to \( \varphi^*(f) \) on \( U \) in the topology of \( \mathcal{O}_M(U) \).
Lemma 5.1.6. If \( U, V \subseteq M \) are open subsets, \( K \subseteq M \) is compact with \( V \subseteq K \), then the map
\[
\Delta(K,U) \times \mathcal{O}_M(U) \to \mathcal{O}_M(V), \quad (\varphi, f) \mapsto \varphi^*(f),
\]
is continuous.

Proof. Let \( \varphi_k \in \Delta(K,U) \) be a sequence of automorphisms of \( M \) converging to \( \varphi \in \Delta(K,U) \), and \( f_j \in \mathcal{O}_M(U) \) a sequence converging to \( f \in \mathcal{O}_M(U) \). Choosing appropriate local coordinates and using Taylor expansion of the pullbacks \( (\varphi_k)^*(f_j) \), it can be shown that \( (\varphi_k)^*(f_j) \) converges to \( \varphi^*(f) \) as \( k, l \to \infty \). For this, it is used that the derivatives of a sequence of uniformly converging holomorphic functions also uniformly converge.

Lemma 5.1.7. The topological space \( \text{Aut}_0(M) \) is locally compact.

Proof. Let \( \psi \in \text{Aut}_0(M) \). For each fixed \( x \in M \) there are open neighbourhoods \( V_x \) and \( U_x \) of \( x \) and \( \psi(x) \) respectively such that \( \psi(K_x) \subseteq U_x \) for \( K_x := V_x \). We may additionally assume that there are local odd coordinates \( \xi_1, \ldots, \xi_n \) for \( M \) on \( U_x \), and \( \theta_1, \ldots, \theta_n \) local odd coordinates on an open neighbourhood of \( K_x \). For any automorphism \( \varphi : M \to M \) with \( \tilde{\varphi}(K_x) \subseteq U_x \), let \( \varphi_{j,k}, \varphi_{j,\nu} \) for \( ||\nu|| = ||(\nu_1, \ldots, \nu_n)|| = \nu_1 + \cdots + \nu_n \geq 3 \), the sum again understood as a sum in \( \mathbb{Z} \) be local holomorphic functions such that
\[
\varphi^*(\xi_j) = \sum_{k=1}^n \varphi_{j,k} \theta_k + \sum_{||\nu|| \geq 3} \varphi_{j,\nu} \theta^\nu.
\]
Choose bounded open subsets \( U_{j,k}, U_{j,\nu} \subset C \), such that \( \psi_{j,k}(x) \in U_{j,k} \) and \( \psi_{j,\nu}(x) \in U_{j,\nu} \). Since \( \psi \) is an automorphism, we have
\[
det((\psi_{j,k}(y))_{1 \leq j,k \leq n} \neq 0
\]
for all \( y \in K_x \). For later considerations shrink \( U_{j,k} \) such that \( \det(C) \neq 0 \) for all \( C = (c_{j,k})_{1 \leq j,k \leq n} \) with \( c_{j,k} \in U_{j,k} \). After shrinking \( V_x \) we may assume \( \psi_{j,k}(K_x) \subseteq U_{j,k} \) and \( \psi_{j,\nu}(K_x) \subseteq U_{j,\nu} \). Hence \( \psi \) is contained in the set \( \Theta(x) = \{ \varphi \in \text{Aut}_0(M) \mid \tilde{\varphi}(K_x) \subseteq U_x, \varphi_{j,k}(K_x) \subseteq U_{j,k}, \varphi_{j,\nu}(K_x) \subseteq U_{j,\nu} \} \), which contains an open neighbourhood of \( \psi \).

Since \( M \) is compact, \( M \) is covered by finitely many of the sets \( V_x \), say \( V_{x_1}, \ldots, V_{x_l} \). Then \( \psi \) is contained in \( \Theta = \Theta(x_1) \cap \cdots \cap \Theta(x_l) \). We will now prove that \( \Theta \) is sequentially compact:

Let \( \varphi_k \) be any sequence of automorphisms contained in \( \Theta \). Then, using Montel’s theorem and passing to a subsequence, the sequence \( \varphi_k \) converges to a morphism \( \varphi : M \to M \). It remains to show that \( \varphi \) is an automorphism of \( M \).

The underlying map \( \tilde{\varphi} : M \to M \) is surjective since if \( p \notin \tilde{\varphi}(M) \), then \( \varphi \in \Delta(M, M \setminus \{ p \}) \) and therefore \( \varphi_k \in \Delta(M, M \setminus \{ p \}) \) for \( k \) large enough which contradicts the assumption that \( \varphi_k \) is an automorphism. This also implies that there is an \( x \in M \) such that the differential \( D\tilde{\varphi}(x) \) is invertible. Using Hurwitz’s theorem (see e.g. [Nar71], p. 80) it follows \( \det(D\tilde{\varphi}(x)) \neq 0 \) for all \( x \in M \). Thus \( \tilde{\varphi} \) is locally biholomorphic.

Moreover, \( \varphi \) is locally invertible due to the special form of the sets \( \Theta(x_i) \).

In order to check that \( \tilde{\varphi} \) is injective, let \( p_1, p_2 \in M, p_1 \neq p_2 \), such that \( q = \tilde{\varphi}(p_1) = \tilde{\varphi}(p_2) \). Let \( \Omega_j, j = 1, 2 \), be open neighbourhoods of \( p_j \) with \( \Omega_1 \cap \Omega_2 = \emptyset \). By [Nar71], p. 79, Proposition 5, there exists \( k_0 \) with the property that \( q \in \tilde{\varphi}_k(\Omega_1) \) and \( q \in \tilde{\varphi}_k(\Omega_2) \) for all \( k \geq k_0 \). The fact that the \( \varphi_k \)'s are bijective yields a contradiction to \( \Omega_1 \cap \Omega_2 = \emptyset \).
Proposition 5.1.8. The set $\text{Aut}_0(\mathcal{M})$ is a topological group with composition of automorphisms as multiplication and inversion of automorphisms as the inverse.

**Proof.** Let $\varphi_1$ and $\psi_1$ be two sequences of automorphisms of $\mathcal{M}$ converging to $\varphi$ and $\psi$ respectively. By the classical theory, $\varphi_k \circ \psi_l$ converges to $\varphi \circ \psi$. Let $U, V, W \subseteq M$ be open subsets with $\varphi(V) \subseteq V$, $\varphi_k(V) \subseteq W$, $\psi(U) \subseteq V$, $\psi_l(U) \subseteq V$, for $k$ and $l$ sufficiently large and let $f \in \mathcal{O}_M(W)$. Then the sequence $(\varphi_k)^*(f) \in \mathcal{O}_M(V)$ converges to $\varphi^*(f)$ on $V$, and by Lemma 5.1.6 $(\varphi_k \circ \psi_l)^*(f) = (\psi_l)^*((\varphi_k)^*(f))$ converges to $\psi^*(\varphi^*(f)) = (\varphi \circ \psi)^*(f)$ on $U$ as $k, l \to \infty$, which shows that the multiplication is continuous.

Consider now the inversion map $\text{Aut}_0(\mathcal{M}) \to \text{Aut}_0(\mathcal{M})$, $\varphi \mapsto \varphi^{-1}$. Let $\varphi_k$ be a sequence in $\text{Aut}_0(\mathcal{M})$ converging to $\varphi \in \text{Aut}_0(\mathcal{M})$. We will verify that $\varphi_k^{-1}$ converges to $\varphi^{-1}$ by considerations in local coordinates. In a first step, we make a reduction to the case where the underlying maps of $\varphi_k$ and $\varphi$ are the identity in this local setting. Secondly, we indicate how we can further reduce to the case where the maps induced by $\varphi_k$ and $\varphi$ on the vector bundle are the identity in this local setting.

Since the automorphism group $\text{Aut}(\mathcal{M})$ of the underlying manifold $\mathcal{M}$ is a topological group, the inversion map $\text{Aut}(\mathcal{M}) \to \text{Aut}(\mathcal{M})$ is continuous and $\varphi_k^{-1}$ converges to $\varphi^{-1}$.

Let $U, V \subseteq \mathcal{M}$ be open subsets such that $\varphi(U) \subseteq V$ and such that there are coordinates $z_1, \ldots, z_m, \theta_1, \ldots, \theta_n$ on $U$ and $w_1, \ldots, w_m, \xi_1, \ldots, \xi_n$ on $V$ for $\mathcal{M}$. Let $V'$ and $V''$ be open relatively compact subsets of $\varphi(U)$ such that $\overline{V''} \subset V'$. Since $\varphi_k$ converges to $\varphi$ and $\varphi_k^{-1}$ converges to $\varphi^{-1}$ we can find an open relatively compact subset $U' \subset U$ such that $\overline{V''} \subset \varphi(U') \subset U'$ in $\overline{\varphi(U)}$ and also $V'' \subset \varphi(U') \subset V' \subset \varphi(U)$ for $k$ sufficiently large.

We have to prove that $(\varphi_k^{-1})^*(f)$ converges to $(\varphi^{-1})^*(f)$ in $\mathcal{O}_M(V'')$ for any $f \in \mathcal{O}_M(U')$. Define morphisms $\psi, \psi_k : (U, \mathcal{O}_M|_U) \to (V, \mathcal{O}_M|_V)$ by setting $\psi^*(w) = \varphi(z)$ and $\psi_k^*(w) = \varphi_k(z)$ and similarly $\psi_k^*(\xi_j) = \theta_j$. These morphisms have inverses $\psi^{-1}, \psi_k^{-1} : (V', \mathcal{O}_M|_{V'}) \to (U, \mathcal{O}_M|_U)$ on $V'$. Due to the convergence of $\varphi_k$ to $\varphi$ and $\varphi_k^{-1}$ to $\varphi^{-1}$ it is enough to prove the convergence of $(\varphi_k^{-1} \circ \psi_k)^*(f) = \psi_k^*((\varphi_k^{-1})^*(f))$ to $(\varphi^{-1} \circ \psi)^*(f)$ in $\mathcal{O}_M(U')$ for any $f \in \mathcal{O}_M(U')$ since this implies that $(\varphi_k^{-1})^*(f)$ converges to $(\varphi^{-1})^*(f)$ in $\mathcal{O}_M(U'')$. Therefore, we may assume $\varphi = \text{id}$, $\varphi_k = \text{id}$ on $U$ without loss of generality.

In local coordinates $z_1, \ldots, z_m, \theta_1, \ldots, \theta_n$ we then have

$$\varphi^*(z_j) = z_j + \sum_{|\nu| = 0, ||\nu|| > 0} a_{j,\nu}(z)\theta^\nu$$

and

$$\varphi^*(\theta_j) = \sum_{l=1}^n b_{j,l}(z)\theta_l + \sum_{|\nu| = 1, ||\nu|| > 1} b_{j,\nu}(z)\theta^\nu$$

for appropriate holomorphic functions $a_{j,\nu}$, $b_{j,l}$, $b_{j,\nu}$ on $U$, where again $||\nu|| = \nu_1 + \ldots + \nu_n \in \mathbb{Z}$ considering each $\nu_l \in \mathbb{Z}_2$ as 0 or 1 and taking the sum in $\mathbb{Z}$. We define $B(z)$ to be the matrix with entries $b_{j,l}(z)$ at position $(j,l)$ and set $b_{\nu} = (b_1,\ldots,b_n,\nu)$, and shortly write $\varphi^*(\theta) = B(z)\theta + \sum_{|\nu| = 1, ||\nu|| > 1} b_{\nu}(z)\theta^\nu$, $\varphi_k^*(\theta) = \sum_{l=1}^n b_{j,l}(z)\theta_l + \sum_{|\nu| = 1, ||\nu|| > 1} b_{j,\nu}(z)\theta^\nu$, $j = 1, \ldots, n$. Since $\varphi$ is invertible, $B(z)$ is an invertible $n \times n$-matrix for each $z \in U$, and thus $B$ defines a holomorphic map $B : U \to \text{GL}_n(\mathbb{C})$. Similarly, we have

$$\varphi_k^*(z_j) = z_j + \sum_{|\nu| = 0, ||\nu|| > 0} a_{j,\nu,k}(z)\theta^\nu$$

and

$$\varphi_k^*(\theta) = B_k(z)\theta + \sum_{|\nu| = 1, ||\nu|| > 1} b_{\nu,k}(z)\theta^\nu$$

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for appropriate holomorphic functions $a_{j,\nu} : k$, $b_{j,\nu} : k$ on $U$ and holomorphic maps $B_k : U \to \text{GL}_n(\mathbb{C})$. Define morphisms $\chi, \chi_k : (U, \mathcal{O}_M(U)) \to (U, \mathcal{O}_M(U))$ by

$$\chi^*(z_j) = z_j, \quad \chi^*(\theta) = B(z)\theta$$

and similarly

$$\chi_k^*(z_j) = z_j, \quad \chi_k^*(\theta) = B_k(z)\theta.$$ 

These morphisms are invertible and we have $(\chi^{-1})^*(z_j) = z_j$, $(\chi^{-1})^*(\theta) = (B(z))^{-1}\theta$ and $(\chi_k^{-1})^*(z_j) = z_j$, $(\chi_k^{-1})^*(\theta) = (B_k(z))^{-1}\theta$. Since $\varphi_k$ converges to $\varphi$, $B_k$ uniformly converges to $B$ on compact subset of $U$ and thus $\chi_k$ converges to $\chi$. Moreover, since the inversion in $\text{GL}_n(\mathbb{C})$ is continuous, $(B_k)^{-1}$ uniformly converges to $B^{-1}$ on compact subsets of $U$ which implies that $\chi_k^{-1}$ converges to $\chi^{-1}$. Consequently, $\varphi_k$ converges to $\varphi^{-1}$ precisely if $\varphi_k^{-1} \circ \chi_k$ converges to $\varphi^{-1} \circ \chi$. It is hence enough to consider the case where $B_k(z) = B(z) = E_n, E_n$ the identity matrix of size $n$, because

$$(\varphi \circ \chi)^*(\theta) = \chi^* \left(B(z)\theta + \sum_{|\nu|=1, |||\nu|||>1} b_{\nu}(z)\theta^\nu \right) = B(z)(B(z))^{-1}\theta + \sum_{|\nu|=1, |||\nu|||>1} c_{\nu}(z)\theta^\nu$$

for appropriate holomorphic functions $c_{j,\nu}$ on $U$, $c_{\nu} = (c_{1,\nu}, \ldots c_{n,\nu})$, and similarly

$$(\varphi_k \circ \chi_k)^*(\theta) = \theta + \sum_{|\nu|=1, |||\nu|||>1} c_{\nu,k}(z)\theta^\nu$$

for appropriate holomorphic functions $c_{j,\nu} : k$ on $U$.

Suppose now $B(z) = B_k(z) = E_n$. For the pullback $\varphi^*$ we have

$$(\varphi^* - \text{id}^*)(z_j) = \sum_{|\nu|=0, |||\nu|||>0} a_{j,\nu}(z)\theta^\nu \quad \text{and} \quad (\varphi^* - \text{id}^*)(\theta) = \sum_{|\nu|=1, |||\nu|||>1} c_{\nu}(z)\theta^\nu,$$

and we conclude that $\varphi^* - \text{id}^*$ is nilpotent. Moreover, by an analogous argument $\varphi_k^* - \text{id}^*$ is nilpotent for each $k$. Therefore, by Lemma 1.2.5

$$Y = \log(\varphi^*) = \sum_{l \geq 1} \frac{(-1)^{l+1}}{l} (\varphi^* - \text{id}^*)^l$$

and

$$Y_k = \log(\varphi_k^*) = \sum_{l \geq 1} \frac{(-1)^{l+1}}{l} (\varphi_k^* - \text{id}^*)^l$$

define nilpotent even vector fields on $U$ and $\varphi^* = \exp(Y) = \sum_{l \geq 0} \frac{1}{l!} Y^l$ and $\varphi_k^* = \exp(Y_k)$. Here, the sums for exp and log are finite sums due to the nilpotency of $Y$, $Y_k$ and $(\varphi^* - \text{id}^*$, $(\varphi_k^* - \text{id}^*$). Therefore, the sequence of vector fields $Y_k$ converges to $Y$ in the sense that for any $f \in \mathcal{O}_M(U)$ the sequence $Y_k(f)$ converges to $Y(f)$ in $\mathcal{O}_M(U)$, i.e. if $f_{\nu}, f_{\nu,k}$ are holomorphic functions on $U$ with $Y(f) = \sum_{\nu} f_{\nu}(z)\theta^\nu$ and $Y_k(f) = \sum_{\nu} f_{\nu,k}(z)\theta^\nu$, then $f_{\nu,k}$ uniformly converges to $f_{\nu}$ on compact subsets of $U$. Therefore, $-Y_k$ converges to $-Y$ and this implies that $\varphi_k^{-1}$ converges to $\varphi^{-1}$ since the pullbacks are given by $(\varphi^{-1})^* = \exp(-Y)$ and $(\varphi_k^{-1})^* = \exp(-Y_k)$. \qed
Remark 5.1.9. Let $\mathcal{M}$ be a split supermanifold and let $E \to M$ be a vector bundle with associated sheaf of sections $\mathcal{E}$ such that the structure sheaf $\mathcal{O}_\mathcal{M}$ is isomorphic to $\bigwedge \mathcal{E}$. By [Mor58] the group of automorphisms $\text{Aut}(E)$ of the vector bundle $E$ is a complex Lie group. Each automorphism $\varphi$ of the supermanifold $\mathcal{M}$ induces an automorphism $\varphi_E$ of the vector bundle $E$ over the underlying map $\tilde{\varphi}$ of $\varphi$, and the map $\pi : \text{Aut}_0(\mathcal{M}) \to \text{Aut}(E)$, $\varphi \mapsto \varphi_E$, is continuous. The splitting $\mathcal{O}_\mathcal{M} \cong \bigwedge \mathcal{E}$ yields a section $\sigma$ of $\pi$. If $\chi : E \to E$ is an automorphism of the bundle $E$ with pullback $\chi^*$, we define the automorphism $\sigma(\chi) : M \to M$ by the pullback 

$$f_1 \wedge \ldots \wedge f_k \mapsto \chi^* (f_1) \wedge \ldots \wedge \chi^* (f_k)$$

for $f_1 \wedge \ldots \wedge f_k \in \bigwedge^k E$. In particular, $\pi$ is surjective and we have an exact sequence

$$0 \to \ker \pi \to \text{Aut}_0(\mathcal{M}) \to \text{Aut}(E) \to 0,$$

which splits. Consequently, the topological group $\text{Aut}_0(\mathcal{M})$ is a semidirect product

$$\text{Aut}_0(\mathcal{M}) \cong \ker \pi \rtimes \text{Aut}(E).$$

The kernel of $\pi$ consists of those automorphisms of $\mathcal{M}$ which induce the identity on the vector bundle $E$. If $\varphi : \mathcal{M} \to \mathcal{M}$ is an automorphism, we have $\pi(\varphi) = \text{id}_E$ precisely if the underlying map $\tilde{\varphi}$ is the identity on $M$ and the pullback of $\varphi$ satisfies

$$(\varphi^* - \text{id}^*)(\mathcal{E}) \subseteq \bigoplus_{k>1} \left( \bigwedge^k \mathcal{E} \right).$$

In this case $(\varphi^* - \text{id}^*)$ is nilpotent and there is an even vector field $X$ on $\mathcal{M}$ with $\exp(X) = \varphi^*$ by Lemma 1.2.5. The vector field $X$ is nilpotent and fulfills

$$X \left( \bigwedge^k \mathcal{E} \right) \subseteq \bigoplus_{l \geq k+2} \left( \bigwedge^l \mathcal{E} \right)$$

for all $k$. More generally, let

$$\text{Vec}^{(2)}(\mathcal{M}) = \left\{ X \in \text{Vec}_0(\mathcal{M}) \left| X \left( \bigwedge^k \mathcal{E} \right) \subseteq \bigoplus_{l \geq k+2} \left( \bigwedge^l \mathcal{E} \right) \text{ for all } k \right. \right\}$$

denote the set of even nilpotent vector fields on $\mathcal{M}$ which are increasing with respect to the $\mathbb{Z}$-grading on $\mathcal{O}_\mathcal{M} \cong \bigwedge \mathcal{E}$. In particular, the reduction of vector fields in $\text{Vec}^{(2)}(\mathcal{M})$ to vector fields on $M$ is $0$. The exponential map $\text{Vec}^{(2)}(\mathcal{M}) \to \ker \pi$ assigning to a vector field $X$ the automorphism with pullback $\exp(X)$ is bijective. In Section 5.2, we will prove that the Lie superalgebra $\text{Vec}(\mathcal{M})$ of vector fields on $\mathcal{M}$ is finite-dimensional, which implies that $\text{Vec}^{(2)}(\mathcal{M})$ is also finite-dimensional. This implies that $\text{Aut}_0(\mathcal{M}) \cong \ker \pi \rtimes \text{Aut}(E)$ carries the structure of a complex Lie group.

In the general case of a not necessarily split supermanifold $\mathcal{M}$ the proof that $\text{Aut}_0(\mathcal{M})$ can be endowed with the structure of a complex Lie group requires more work, which is carried out in the next paragraphs.

5.1.2 Non-existence of small subgroups of $\text{Aut}_0(\mathcal{M})$

In this section, we prove that $\text{Aut}_0(\mathcal{M})$ does not contain small subgroups, which means that there exists an open neighbourhood of the identity in $\text{Aut}_0(\mathcal{M})$ such that each
subgroup contained in this neighbourhood consists only of the identity. As a consequence, the topological group $\text{Aut}_0(M)$ carries the structure of a real Lie group by a result of Yamabe (cf. [Yam53]).

Before proving the non-existence of small subgroups, a few technical preparations are needed. Consider $\mathbb{C}^{m|n}$ and let $z_1, \ldots, z_m, \xi_1, \ldots, \xi_n$ denote coordinates on $\mathbb{C}^{m|n}$. Let $U \subseteq \mathbb{C}^m$ be an open subset. For $f = \sum_{\nu} f_{\nu} \xi^\nu \in \mathcal{O}_{\mathbb{C}^{m|n}}(U)$ define

$$||f||_U = \left\| \sum_{\nu} f_{\nu} \xi^\nu \right\|_U := \sum_{\nu} ||f_{\nu}||_U,$$

where $||f_{\nu}||_U$ denotes the supremum norm of the holomorphic function $f_{\nu}$ on $U$. For any morphism $\varphi : U = (U, \mathcal{O}_{\mathbb{C}^{m|n}}(U)) \to \mathbb{C}^{m|n}$ define

$$||\varphi||_U := \sum_{i=1}^m ||\varphi^*(z_i)||_U + \sum_{j=1}^n ||\varphi^*(\xi_j)||_U.$$

**Lemma 5.1.10.** Let $U = (U, \mathcal{O}_{\mathbb{C}^{m|n}}(U))$ be a superdomain in $\mathbb{C}^{m|n}$. For any relatively compact open subset $U'$ of $U$ there exists $\varepsilon > 0$ such that any morphism $\psi : U \to \mathbb{C}^{m|n}$ with the property $||\psi - \text{id}||_U < \varepsilon$ is biholomorphic as a morphism from $U' = (U', \mathcal{O}_{\mathbb{C}^{m|n}}(U'))$ onto its image.

**Proof.** Let $r > 0$ be such that the closure of the polydisc

$$\Delta^m_r(z) = \{(w_1, \ldots, w_m) \mid |w_j - z_j| < r\}$$

is contained in $U$ for any $z = (z_1, \ldots, z_m) \in U'$. Let $v \in \mathbb{C}^m$ be any non-zero vector. Then we have $z + \zeta v \in U$ for any $z \in U'$ and $\zeta$ in the closure of $\Delta^m_r(0) = \{t \in \mathbb{C} \mid |t| < \frac{r}{||v||}\}$. If $||\psi - \text{id}||_U < \varepsilon$ for a given $\varepsilon > 0$, then we have in particular $||\tilde{\psi} - \text{id}||_U < \varepsilon$ for the supremum norm of the underlying maps $\tilde{\psi} : U \to \mathbb{C}^m$. Then, for the differential $D\tilde{\psi}$ of $\tilde{\psi}$ and any non-zero vector $v \in \mathbb{C}^m$ and any $z \in U'$ we have

$$\left\| D\tilde{\psi}(z)(v) - v \right\| = \left\| \frac{d}{dt} \left( \tilde{\psi}(z + tv) - (z + tv) \right) \right\|$$

$$= \frac{1}{2\pi} \left\| \int_{\partial \Delta^m_r(0)} \frac{\tilde{\psi}(z + \zeta v) - (z + \zeta v)}{\zeta^2} d\zeta \right\|$$

$$\leq \frac{1}{2\pi} \int_{\partial \Delta^m_r(0)} \left\| \frac{\tilde{\psi}(z + \zeta v) - (z + \zeta v)}{\zeta^2} \right\| d\zeta$$

$$< \frac{\varepsilon ||v||}{r}.$$

This implies $||D\tilde{\psi}(z) - \text{id}|| < \frac{\varepsilon}{r}$ with respect to the operator norm, for any $z \in U'$. Thus $\tilde{\psi}$ is locally biholomorphic on $U'$ if $\varepsilon$ is small enough. Moreover, $\varepsilon$ may be chosen such that $\tilde{\psi}$ is injective (see e.g. [Hir76], Chapter 2, Lemma 1.3).

Let $\psi_{j,k}, \psi_{j,\nu}$ be holomorphic functions on $U$ such that

$$\psi_j(\xi) = \sum_{k=1}^n \psi_{j,k} \xi_k + \sum_{||\nu|| \geq 3} \psi_{j,\nu} \xi^\nu.$$
It is now enough to show

\[ \det((\psi_{j,k})_{1 \leq j,k \leq n}(z)) \neq 0 \]

for all \( z \in U' \) and \( \varepsilon \) small enough in order to prove that \( \psi \) is a biholomorphism form \( U' \) onto its image. This follows from the fact that we assumed \( \|\psi - \text{id}\|_U < \varepsilon \) which implies \( \|\psi_{j,k}\|_U < \varepsilon \) if \( j \neq k \) and \( \|\psi_{j,k} - 1\|/U < \varepsilon \).

This lemma now allows us to prove that \( \text{Aut}_0(\mathcal{M}) \) contains no small subgroups; for a similar result in the classical case see [BM46], Theorem 1.

**Proposition 5.1.11.** The topological group \( \text{Aut}_0(\mathcal{M}) \) has no small subgroups, i.e. there is a neighbourhood of the identity which contains no non-trivial subgroup.

**Proof.** Let \( U \subset V \subset W \) be open subsets of \( M \) such that \( U \) is relatively compact in \( V \) and \( V \) is relatively compact in \( W \). Moreover, suppose that \( \mathcal{W} = (W, \mathcal{O}_M|_W) \) is isomorphic to a superdomain in \( \mathbb{C}^{m|n} \) and let \( z_1, \ldots, z_m, \xi_1, \ldots, \xi_n \) be local coordinates on \( \mathcal{W} \).

By definition \( \Delta(\mathcal{V}, \mathcal{W}) = \{ \varphi \in \text{Aut}_0(\mathcal{M}) \} \) and \( \Delta(V, \mathcal{V}) \) are open neighbourhoods of the identity in \( \text{Aut}_0(\mathcal{M}) \). Choose \( \varepsilon > 0 \) as in the preceding lemma such that any morphism \( \chi : \mathcal{V} \to \mathbb{C}^{m|n} \) with \( \|\chi - \text{id}\|_V < \varepsilon \) is biholomorphic as a morphism from \( \mathcal{U} \) onto its image. Let \( \Omega \subseteq \Delta(\mathcal{V}, \mathcal{W}) \cap \Delta(\mathcal{U}, \mathcal{V}) \) be the subset whose elements \( \varphi \) satisfy \( \|\varphi - \text{id}\|_V < \varepsilon \). The set \( \Omega \) is open and contains the identity. Since \( \text{Aut}_0(\mathcal{M}) \) is locally compact by Lemma 5.1.7, it is enough to show that each compact subgroup \( Q \subseteq \Omega \) is trivial. Otherwise for non-compact \( Q \), let \( \Omega' \) be an open neighbourhood of the identity with compact closure \( \overline{\Omega} \) which is contained in \( \Omega \), and suppose \( Q \subseteq \Omega' \).

Then \( \overline{Q} \subset \overline{\Omega} \subseteq \Omega \) is a compact subgroup and \( Q \) is trivial if \( \overline{Q} \) is trivial.

Define a morphism \( \psi : \mathcal{V} \to \mathbb{C}^{m|n} \) by setting

\[ \psi^*(z_i) = \int_Q q^*(z_i) \, dq \quad \text{and} \quad \psi^*(\xi_j) = \int_Q q^*(\xi_j) \, dq, \]

where the integral is taken with respect to the normalized Haar measure on \( Q \). This yields a holomorphic morphism \( \psi : \mathcal{V} \to \mathbb{C}^{m|n} \) since each \( q \in Q \) defines a holomorphic morphism \( \mathcal{V} \to \mathcal{W} \subseteq \mathbb{C}^{m|n} \). Its underlying map is \( \tilde{\psi}(z) = \int_Q \tilde{q}(z) \, dq \). The morphism \( \psi \) satisfies

\[ \|\psi^*(z_i) - z_i\|_V = \left\| \int_Q (q^*(z_i) - z_i) \, dq \right\|_V \leq \int_Q \|q^*(z_i) - z_i\|_V \, dq \]

and similarly

\[ \|\psi^*(\xi_j) - \xi_j\|_V \leq \int_Q \|q^*(\xi_j) - \xi_j\|_V \, dq. \]

Consequently, we have

\[ \|\psi - \text{id}\|_V = \sum_{i=1}^m \|\psi^*(z_i) - z_i\|_V + \sum_{j=1}^n \|\psi^*(\xi_j) - \xi_j\|_V \]
\[ \leq \int_Q \left( \sum_{i=1}^m \|q^*(z_i) - z_i\|_V + \sum_{j=1}^n \|q^*(\xi_j) - \xi_j\|_V \right) \, dq \]
\[ = \int_Q \|q - \text{id}\|_V \, dq < \varepsilon. \]
Thus by the preceding lemma, \( \psi|_{U} \) is a biholomorphic morphism onto its image. Furthermore, on \( U \) we have \( \psi \circ q' = \psi \) for any \( q' \in Q \) since

\[
(\psi \circ q')^*(z_i) = (q')^*(\psi^*(z_i)) = (q')^* \left( \int_{Q} q^*(z_i) \, dq \right) = \int_{Q} (q')^*(q^*(z_i)) \, dq
\]

\[
= \int_{Q} (q \circ q')^*(z_i) \, dq = \int_{Q} q^*(z_i) \, dq = \psi^*(z_i)
\]
due to the invariance of the Haar measure, and also

\[
(\psi \circ q')^*(\xi_j) = \psi^*(\xi_j).
\]

The equality \( \psi \circ q' = \psi \) on \( U \) implies \( q'|_{U} = \text{id}_{U} \) because of the invertibility of \( \psi \). By the identity principle it follows that \( q' = \text{id}_{M} \) if \( M \) is connected, and hence \( Q = \{ \text{id}_{M} \} \).

In general, \( M \) has only finitely many connected components since \( M \) is compact. Therefore, a repetition of the preceding argument yields the existence of a neighbourhood of the identity of \( \text{Aut}_{\bar{0}}(M) \) without non-trivial subgroups. \( \square \)

By Theorem 3 in [Yam53], the preceding proposition implies the following:

**Corollary 5.1.12.** The topological group \( \text{Aut}_{\bar{0}}(M) \) can be endowed with the structure of a real Lie group.

### 5.1.3 One-parameter subgroups of \( \text{Aut}_{\bar{0}}(M) \)

In order to obtain results on the regularity of the action of \( \text{Aut}_{\bar{0}}(M) \) on the compact complex supermanifold \( M \) and to characterize the Lie algebra of \( \text{Aut}_{\bar{0}}(M) \), we study continuous one-parameter subgroups of \( \text{Aut}_{\bar{0}}(M) \). Each continuous one-parameter subgroup \( \mathbb{R} \to \text{Aut}_{\bar{0}}(M) \) is an analytic map between the Lie groups \( \mathbb{R} \) and \( \text{Aut}_{\bar{0}}(M) \).

We prove that the action of each continuous one-parameter subgroup of \( \text{Aut}_{\bar{0}}(M) \) on \( M \) is analytic and induces an even holomorphic vector field on \( M \). Consequently, the Lie algebra of \( \text{Aut}_{\bar{0}}(M) \) may be identified with the Lie algebra \( \text{Vec}_{\mathbb{C}}(M) \) of even holomorphic vector fields on \( M \), and \( \text{Aut}_{\bar{0}}(M) \) carries the structure of a complex Lie group whose action on the supermanifold \( M \) is holomorphic.

**Definition 5.1.13.** A continuous one-parameter subgroup \( \varphi \) of automorphisms of \( M \) is a family of automorphisms \( \varphi_{t} : M \to M, \ t \in \mathbb{R} \), such that the map \( \varphi : \mathbb{R} \to \text{Aut}_{\bar{0}}(M), \ t \mapsto \varphi_{t} \), is a continuous group homomorphism.

**Remark 5.1.14.** Let \( \varphi_{t} : M \to M, \ t \in \mathbb{R} \), be a family of automorphisms satisfying \( \varphi_{s+t} = \varphi_{s} \circ \varphi_{t} \) for all \( s, t \in \mathbb{R} \), and such that \( \hat{\varphi} : \mathbb{R} \times M \to M, \ \hat{\varphi}(t, p) = \varphi_{t}(p) \) is continuous. Then \( \varphi_{t} \) is a continuous one-parameter subgroup if and only if the following condition is satisfied: Let \( U, V \subset M \) be open subsets, and \( [a, b] \subset \mathbb{R} \) such that \( \hat{\varphi}([a, b] \times U) \subseteq V \). Assume moreover that there are local coordinates \( z_{1}, \ldots, z_{m}, \xi_{1}, \ldots, \xi_{n} \) for \( M \) on \( U \). Then for any \( f \in \mathcal{O}_{M}(V) \) there are continuous functions \( f_{\nu} : [a, b] \times U \to \mathbb{C} \) with \( (f_{\nu})_{t} = f_{\nu}(t, \cdot) \in \mathcal{O}_{M}(U) \) for fixed \( t \in [a, b] \) such that

\[
(\varphi_{t})^*(f) = \sum_{\nu} f_{\nu}(t, z)\xi_{\nu}.
\]

We say that the action of the one-parameter subgroup \( \varphi \) on \( M \) is analytic if each \( f_{\nu}(t, z) \) is analytic in both components.

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Proposition 5.1.15. Let \( \varphi \) be a continuous one-parameter subgroup of automorphisms of \( \mathcal{M} \). Then the action of \( \varphi \) on \( \mathcal{M} \) is analytic.

Remark 5.1.16. Defining a continuous one-parameter subgroup as in Remark 5.1.14, the statement of Proposition 5.1.15 also holds true for complex supermanifolds \( \mathcal{M} \) with non-compact underlying manifold \( M \) since the compactness of \( M \) is not needed for the proof.

For the proof of the proposition the following technical lemma is needed:

Lemma 5.1.17. Let \( U \subseteq V \subseteq \mathbb{C}^n \) be open subsets, \( p \in U \), \( \Omega \subseteq \mathbb{R} \) an open connected neighbourhood of 0, and let \( \alpha : \Omega \times U \to V \) be a continuous map satisfying

\[
\alpha(t, z) = \alpha(t + s, z) - f(t, s, z)
\]

for \( (t, s, z) \) in a neighbourhood of \((0, 0, p)\) and for some continuous function \( f \) which is analytic in \( (t, z) \). If \( \alpha \) is holomorphic in the second component, then it is analytic on a neighbourhood of \((0, p)\).

Proof. For small \( t, h > 0 \), \( z \) near \( p \), we have

\[
h \cdot \alpha(t, z) = \int_0^h \alpha(t + s, z)ds - \int_0^h f(t, s, z)ds
\]

\[
= \int_t^{t+h} \alpha(s, z)ds - \int_0^h \alpha(s, z)ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds
\]

\[
= \int_0^t (\alpha(s + h) - \alpha(s))ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds
\]

\[
= \int_0^t f(s, h, z)ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds.
\]

The assumption that \( f \) is a continuous function which is analytic in the first and third component therefore implies that \( \alpha \) is analytic.

Proof of Proposition 5.1.15. Due to the action property \( \varphi_{s+t} = \varphi_s \circ \varphi_t \) it is enough to show the statement for the restriction of \( \varphi \) to \((-\varepsilon, \varepsilon) \times \mathcal{M} \) for some \( \varepsilon > 0 \). Let \( U, V \subseteq \mathcal{M} \) be open subsets such that \( U \) is relatively compact in \( V \), and such that there are local coordinates \( z_1, \ldots, z_m, \xi_1, \ldots, \xi_n \) on \( V \) for \( \mathcal{M} \). Choose \( \varepsilon > 0 \) such that \( \hat{\varphi}_t(U) \subseteq V \) for any \( t \in (-\varepsilon, \varepsilon) \). Let \( \alpha_{i,\nu}, \beta_{j,\nu} \) be continuous functions on \((-\varepsilon, \varepsilon) \times U \) with \( (\varphi_t)^*(z_i) = \sum_{|\nu|=0} \alpha_{i,\nu}(t, z)\xi^\nu \) and \( (\varphi_t)^*(\xi_j) = \sum_{|\nu|=1} \beta_{j,\nu}(t, z)\xi^\nu \), where \( |\nu| = |(v_1, \ldots, v_n)| = v_1 + \ldots + v_n \in \mathbb{Z}_2 \). We have to show that \( \alpha \) and \( \beta \) are analytic in \((t, z)\).

The induced map \( \psi' : (-\varepsilon, \varepsilon) \times U \times \mathbb{C}^n \to V \times \mathbb{C}^n \) on the underlying vector bundle is given by

\[
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_m \\
  v_1 \\
  \vdots \\
  v_n
\end{pmatrix}
\quad \mapsto \quad
\begin{pmatrix}
  \alpha_{1,0}(t, z) \\
  \vdots \\
  \alpha_{m,0}(t, z) \\
  \sum_{k=1}^n \beta_{1,k}(t, z)v_k \\
  \vdots \\
  \sum_{k=1}^n \beta_{n,k}(t, z)v_k
\end{pmatrix},
\]
where \( \beta_{j,k} = \beta_{j,e_k} \) if \( e_k = (\bar{0}, \ldots, 0, 1, 0, \ldots, \bar{0}) \) denotes the \( k \)-th unit vector. The map \( \psi' \) is a local continuous one-parameter subgroup on \( U \times \mathbb{C}^n \) because \( \varphi \) is a continuous one-parameter subgroup. By a result of Bochner and Montgomery the map \( \psi' \) is analytic in \( (t, z, v) \) (see [BM45], Theorem 4). Hence, the map \( \psi : (-\varepsilon, \varepsilon) \times U \to V \) given by 
\[
(\psi_t)^*(z_i) = \alpha_i(t, z), 
(\psi_t)^*(\xi_j) = \sum_{k=1}^n \beta_{j,k}(t, z) \xi_k
\]
is analytic. Let \( X \) be the local vector field on \( U \) induced by \( \psi \), i.e.
\[
X(f) = \frac{\partial}{\partial t} \bigg|_0 (\psi_t)^*(f).
\]
We may assume that \( X \) is non-degenerate, i.e. the evaluation \( X(p) \) of \( X \) in \( p \) does not vanish for all \( p \in U \). Otherwise, consider instead of \( \varphi \) the diagonal action on \( \mathbb{C} \times \mathcal{M} \) acting by addition of \( p \in U \) in the first component and \( \varphi_t \) in the second, and note that this action is analytic precisely if \( \varphi \) is analytic. For the differential \( d\psi \) of \( \psi \) in \( (0, p) \) we have
\[
d\psi \left( \frac{\partial}{\partial t} \bigg|_{(0,p)} \right) = \frac{\partial}{\partial t} \bigg|_{(0,p)} \circ \psi^* = X(p) \neq 0.
\]
Therefore, the restricted map \( \psi|_{(-\varepsilon, \varepsilon) \times \{p\}} \) is an immersion and its image \( \psi((-\varepsilon, \varepsilon) \times \{p\}) \) is a submanifold of \( V \). Let \( S \) be a submanifold of \( U \) transverse to \( \psi((-\varepsilon, \varepsilon) \times \{p\}) \). The map \( \psi|_{(-\varepsilon, \varepsilon) \times S} \) is a submersion in \((0, p)\) since \( d\psi(T_{(0,p)}((-\varepsilon, \varepsilon) \times \{p\})) = T_p \psi((-\varepsilon, \varepsilon) \times \{p\}) \) and \( d\psi(T_{(0,p)}\{0\} \times S) = T_p S \) because \( \psi|_{\{0\} \times U} = \text{id} \). Hence \( \chi := \psi|_{(-\varepsilon, \varepsilon) \times \mathcal{S}} \) is locally invertible around \((0, p)\), and thus invertible as a map onto its image after possibly shrinking \( U \) and \( \varepsilon \), and satisfies
\[
\chi^* \left( \frac{\partial}{\partial t} \right) = (\chi^{-1})^* \circ \frac{\partial}{\partial t} \circ \chi^* = (\chi^{-1})^* \circ \chi^* \circ X = X.
\]
Therefore, after defining new coordinates \( w_1, \ldots, w_m, \theta_1, \ldots, \theta_n \) for \( \mathcal{M} \) on \( U \) via \( \chi \), we have \( X = \frac{\partial}{\partial \alpha_{1,i}} \) and \((\varphi_t)^*\) is of the form
\[
(\varphi_t)^*(w_1) = w_1 + t + \sum_{|\nu| = 0, ||\nu|| > 0} \alpha_{1,\nu}(t, w) \theta^\nu,
(\varphi_t)^*(w_i) = w_i + \sum_{|\nu| = 0, ||\nu|| > 0} \alpha_{i,\nu}(t, w) \theta^\nu \quad \text{for} \ i \neq 1,
(\varphi_t)^*(\theta_j) = \theta_j + \sum_{|\nu| = 1, ||\nu|| > 1} \beta_{j,\nu}(t, w) \theta^\nu,
\]
for appropriate \( \alpha_{i,\nu}, \beta_{j,\nu} \), where again \( ||\nu|| = \| (\nu_1, \ldots, \nu_n) \| \in \mathbb{Z} \). For small \( s \) and \( t \) we have
\[
\varphi_t^* (\varphi_s^*(w_1)) = \varphi_t^* \left( w_1 + \delta_{1,1}s + \sum_{|\nu| = 0, ||\nu|| > 0} \alpha_{1,\nu}(s, w) \theta^\nu \right)
= w_1 + \delta_{1,1} (t + s) + \sum_{|\nu| = 0, ||\nu|| > 0} \alpha_{1,\nu}(t, w) \theta^\nu + \sum_{|\nu| = 0, ||\nu|| > 0} \varphi_t^*(\alpha_{1,\nu}(s, w) \theta^\nu). \quad (5.1)
\]
Let \( f_{i,\nu}(t, s, w) \) be such that
\[
\sum_{|\nu| = 0, ||\nu|| > 0} \varphi_t^*(\alpha_{i,\nu}(s, w) \theta^\nu) = \sum_{|\nu| = 0, ||\nu|| > 0} f_{i,\nu}(t, s, w) \theta^\nu. \quad (5.2)
\]
For fixed $\nu_0$ the coefficient $f_{i,\nu_0}(t,s,w)$ of $\theta^{i\nu_0}$ depends only on $\alpha_{i,\nu_0}(s,w+te_1)$ and $\beta_{j,\mu}(t,w)$ for $\mu$ with $||\mu|| \leq ||\nu_0|| - 1$ as well as $\alpha_{i,\nu}(t,w)$ and its partial derivatives in the second component for $\nu$ with $||\nu|| \leq ||\nu_0|| - 2$. This can be shown by a calculation using the special form of $\varphi_1^*(w_j)$ and $\varphi_1^*(\theta_j)$ and general properties of the pullback of a morphism of supermanifolds. Assume now that the analyticity near $(0,p)$ of $\alpha_{i,\nu}$, $\beta_{j,\mu}$ is shown for $||\nu||, ||\mu|| < 2k$ and all $i,j$. Let $\nu_0$ be such that $||\nu_0|| = 2k$. Then $f_{i,\nu_0}(t,s,w)$ is a continuous function which is analytic in $(t,w)$ near $(0,p)$ for fixed $s$. Since $\varphi_1^*(\varphi_1^*(w_i)) = \varphi_1^*(t+s)(w_i)$, using (1) and (2) we get

$$\alpha_{i,\nu_0}(t,w) + f_{i,\nu_0}(t,s,w) = \alpha_{i,\nu_0}(t+s,w),$$

and thus $\alpha_{i,\nu_0}(t,w)$ is analytic near $(0,p)$ by Lemma 5.1.17. Similarly, it can be proven that $\beta_{j,\mu_0}$ is analytic for $||\mu_0|| = 2k + 1$ if $\alpha_{i,\nu}$ and $\beta_{j,\mu}$ are analytic for $||\nu||, ||\mu|| < 2k + 1.$

**Corollary 5.1.18.** The Lie algebra of $\text{Aut}_0(M)$ is isomorphic to the Lie algebra $\text{Vec}_0(M)$ of even vector fields on $M$, and $\text{Aut}_0(M)$ is a complex Lie group.

*Proof.* If $\varphi : \mathbb{R} \to \text{Aut}_0(M)$, $t \mapsto \varphi_t$, is a continuous one-parameter subgroup, then by Proposition 5.1.15 the action of $\varphi$ on $M$ is analytic. Therefore, $\varphi$ induces an even holomorphic vector field $X(\varphi)$ on $M$ by setting

$$X(\varphi) = \left. \frac{\partial}{\partial t} \right|_0 (\varphi_t)^*,$$

and $\varphi$ is the flow map of $X(\varphi)$. On the other hand, each $X \in \text{Vec}_0(M)$ is globally integrable since $M$ is compact (cf. [GW13], Theorem 5.4). Its flow $\varphi_X$ defines a one-parameter subgroup of $\text{Aut}_0(M)$ which is continuous. This yields an isomorphism of Lie algebras

$$\text{Lie}(\text{Aut}_0(M)) \to \text{Vec}_0(M).$$

Consequently, we have $\text{Lie}(\text{Aut}_0(M)) \cong \text{Vec}_0(M)$ and since $\text{Vec}_0(M)$ is a complex Lie algebra, $\text{Aut}_0(M)$ carries the structure of a complex Lie group. \hfill \Box

The Lie group $\text{Aut}_0(M)$ naturally acts on $M$; this action $\psi : \text{Aut}_0(M) \times M \to M$ is given by $(\text{ev}_g \otimes \text{id}_M^*)^* = g^*$ where $\text{ev}_g$ denotes the evaluation in $g \in \text{Aut}_0(M)$.

**Corollary 5.1.19.** The natural action of $\text{Aut}_0(M)$ on $M$ defines a holomorphic morphism of supermanifolds $\psi : \text{Aut}_0(M) \times M \to M$.

*Proof.* Since the action of each continuous one-parameter subgroup of $\text{Aut}_0(M)$ on $M$ is holomorphic by the preceding considerations and each $g \in \text{Aut}_0(M)$ is an automorphism $g : M \to M$, the action $\psi$ is a holomorphic. \hfill \Box

If a Lie supergroup $\mathcal{G}$ (with Lie superalgebra $\mathfrak{g}$ of right-invariant vector fields) acts on a supermanifold $M$ via $\psi : \mathcal{G} \times M \to M$, this action $\psi$ induces an infinitesimal action $d\psi : \mathfrak{g} \to \text{Vec}(M)$ defined by $d\psi(X) = (X(e) \otimes \text{id}_M^*)^* \circ \psi^*$ for any $X \in \mathfrak{g}$, where $X \otimes \text{id}_M^*$ denotes the canonical extension of the vector field $X$ on $\mathcal{G}$ to a vector field on $\mathcal{G} \times M$, and $(X(e) \otimes \text{id}_M^*)$ is its evaluation in the neutral element $e$ of $\mathcal{G}$.

**Corollary 5.1.20.** Identifying the Lie algebra of $\text{Aut}_0(M)$ with $\text{Vec}_0(M)$ as in Corollary 5.1.18, the induced infinitesimal action of the action $\psi : \text{Aut}_0(M) \times M \to M$ in Corollary 5.1.19 is the inclusion $\text{Vec}_0(M) \hookrightarrow \text{Vec}(M)$. 

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5.2 The Lie superalgebra of vector fields

In this section, we prove that the Lie superalgebra Vec(M) of holomorphic super vector fields on a compact complex supermanifold M is finite-dimensional.

First, we prove that Vec(M) is finite-dimensional if M is a split supermanifold using that its tangent sheaf $T_M$ is a coherent sheaf of $O_M$-modules, where $O_M$ denotes again the sheaf of holomorphic functions on the underlying manifold M. Then the statement in the general case is deduced using a filtration of the tangent sheaf.

Remark that since Aut$_0(M)$ is a complex Lie group with Lie algebra Isom to the Lie algebra Vec$_0(M)$ of even holomorphic vector fields on M (see Corollary 5.1.18), we already know that the even part of Vec(M) = Vec$_0(M) \oplus$ Vec$_1(M)$ is finite-dimensional.

Let $\mathcal{M}$ be a complex supermanifold of dimension $(m|n)$, and let $\mathcal{I}_M$ be the ideal subsheaf generated by the odd elements in the structure sheaf $O_M$. As described in [Oni98], we have the filtration

$$O_M = (\mathcal{I}_M)^0 \supset (\mathcal{I}_M)^1 \supset (\mathcal{I}_M)^2 \supset \ldots \supset (\mathcal{I}_M)^{n+1} = 0$$

of the structure sheaf $O_M$ by the powers of $\mathcal{I}_M$. Define the quotient sheaves $\text{gr}_k(O_M) = (\mathcal{I}_M)^k/(\mathcal{I}_M)^{k+1}$. This gives rise to the $\mathbb{Z}$-graded sheaf $\text{gr} O_M = \bigoplus_k \text{gr}_k(O_M)$. Furthermore, $\text{gr} M = (M, \text{gr} O_M)$ is a split complex supermanifold of the same dimension as $\mathcal{M}$.

Lemma 5.2.1. Let $\mathcal{M}$ be a split complex supermanifold. Then its tangent sheaf $T_M$ is a coherent sheaf of $O_M$-modules.

Proof. Since $\mathcal{M}$ is split, its structure sheaf $O_M$ is isomorphic to $\wedge \mathcal{E}$ as an $O_M$-module, where $\mathcal{E}$ is the sheaf of sections of a holomorphic vector bundle on the underlying manifold $M$. Thus, the structure sheaf $O_M$, and hence also the tangent sheaf $T_M$, carry the structure of a sheaf of $O_M$-modules. Let $U \subset M$ be an open subset such that there exist even coordinates $z_1, \ldots, z_m$ and odd coordinates $\xi_1, \ldots, \xi_n$ on $U$. Any derivation $D \in T_M(U)$ on $U$ can uniquely be written as

$$D = \sum_{\nu \in (\mathbb{Z}_2)^n} \left( \sum_{i=1}^m f_{i,\nu}(z) \xi^\nu \frac{\partial}{\partial z_i} + \sum_{j=1}^n g_{j,\nu}(z) \xi^\nu \frac{\partial}{\partial \xi^j} \right)$$

where $f_{i,\nu}, g_{j,\nu}$ are holomorphic functions on $U$. Therefore, the restricted sheaf $T_M|_U$ is isomorphic to $(O_M|_U)^{2n(m+n)}$ and $T_M$ is coherent over $O_M$. \qed

Proposition 5.2.2. The Lie superalgebra Vec(M) of holomorphic vector fields on a compact complex supermanifold $\mathcal{M}$ is finite-dimensional.

Proof. First, assume that $\mathcal{M}$ is split. Then the tangent sheaf $T_M$ is a coherent sheaf of $O_M$-modules. Thus, the space of global sections of $T_M$, Vec(M) = $T_M(M)$, is finite-dimensional since $M$ is compact (cf. [CS53]).

Now, let $\mathcal{M}$ be an arbitrary compact complex supermanifold. We associate the split complex supermanifold $\text{gr} M = (M, \text{gr} O_M)$. As before, let $\mathcal{I}_M$ denote the ideal subsheaf in $O_M$ generated by the odd elements. Define the filtration of sheaves of Lie superalgebras

$$T_M =: (T_M)(-1) \supset (T_M)(0) \supset (T_M)(1) \supset \ldots \supset (T_M)(n+1) = 0$$

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of the tangent sheaf $\mathcal{T}_M$ by setting

$$(\mathcal{T}_M)^{(k)} = \{ D \in \mathcal{T}_M | D(\mathcal{O}_M) \subset (\mathcal{I}_M)^k, D(\mathcal{I}_M) \subset (\mathcal{I}_M)^{k+1} \}$$

for $k \geq 0$. Moreover, define $\text{gr}_k(\mathcal{T}_M) = (\mathcal{T}_M)^{(k)}/(\mathcal{T}_M)^{(k+1)}$ and set

$$\text{gr}(\mathcal{T}_M) = \bigoplus_{k \geq -1} \text{gr}_k(\mathcal{T}_M).$$

By Proposition 1 in [Oni98] the sheaf $\text{gr}(\mathcal{T}_M)$ is isomorphic to the tangent sheaf of the associated split supermanifold $\text{gr} M$. By the preceding considerations, the space of holomorphic vector fields on $\text{gr} M$,

$$\text{Vec}(\text{gr} M) = \text{gr}(\mathcal{T}_M)(M) = \bigoplus_{k \geq -1} \text{gr}_k(\mathcal{T}_M)(M),$$

is of finite dimension. The projection onto the quotient yields

$$\dim(\mathcal{T}_M)^{(k)}(M) - \dim(\mathcal{T}_M)^{(k+1)}(M) \leq \dim(\text{gr}_k(\mathcal{T}_M)(M))$$

and $\dim(\mathcal{T}_M)^{(0)}(M) = \dim(\text{gr}_0(\mathcal{T}_M)(M))$ and hence by induction

$$\dim(\mathcal{T}_M)^{(k)}(M) \leq \sum_{j \geq k} \dim(\text{gr}_j(\mathcal{T}_M)(M)),$$

which gives

$$\dim(\mathcal{T}_M)(M) = \dim((\mathcal{T}_M)^{(-1)}(M)) \leq \dim(\text{gr}(\mathcal{T}_M)(M)).$$

In particular, $\dim(\mathcal{T}_M)(M)$ is finite. \hfill \Box

**Remark 5.2.3.** The proof of the preceding proposition also proves the following inequality:

$$\dim(\text{Vec}(M)) \leq \dim(\text{Vec}(\text{gr} M))$$

### 5.3 The automorphism group

In this section, the automorphism group of a compact complex supermanifold is defined. This is done via the formalism of Harish-Chandra pairs for complex Lie supergroups. The underlying classical Lie group is $\text{Aut}_0(\mathcal{M})$ and the Lie superalgebra is $\text{Vec}(\mathcal{M})$, the Lie superalgebra of vector fields on $\mathcal{M}$. Moreover, we prove that the automorphism group satisfies a universal property, which justifies our terminology.

Consider the representation $\alpha$ of $\text{Aut}_0(\mathcal{M})$ on $\text{Vec}(\mathcal{M})$ given by

$$\alpha(g)(X) = g_* (X) = (g^{-1})^* \circ X \circ g^* \quad \text{for } g \in \text{Aut}_0(\mathcal{M}), X \in \text{Vec}(\mathcal{M}).$$

This representation $\alpha$ preserves the parity on $\text{Vec}(\mathcal{M})$, and its restriction to $\text{Vec}_0(\mathcal{M})$ coincides with the adjoint action of $\text{Aut}_0(\mathcal{M})$ on its Lie algebra $\text{Lie}(\text{Aut}_0(\mathcal{M})) \cong \text{Vec}_0(\mathcal{M})$. Moreover, the differential $(d\alpha)_{id}$ at the identity $id \in \text{Aut}_0(\mathcal{M})$ is the adjoint representation of $\text{Vec}_0(\mathcal{M})$ on $\text{Vec}(\mathcal{M})$:  

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Let $X$ and $Y$ be vector fields on $\mathcal{M}$. Assume that $X$ is even and let $\varphi^X$ denote the corresponding one-parameter subgroup. Then we have

$$(d\alpha)_{id}(X)(Y) = \frac{\partial}{\partial t} \bigg|_{t=0} (\varphi^X_t)_*(Y) = [X,Y]$$

by Corollary 2.3.8. Therefore, the pair $(\text{Aut}_\mathbb{C}(\mathcal{M}), \text{Vec}(\mathcal{M}))$ together with the representation $\alpha$ is a complex Harish-Chandra pair, and using the equivalence between the category of complex Harish-Chandra pairs and complex Lie supergroups (cf. [Vis11], § 2), we can define the automorphism group of a compact complex supermanifold $\mathcal{M}$ as follows:

**Definition 5.3.1.** The automorphism group $\text{Aut}(\mathcal{M})$ of a compact complex supermanifold is the unique complex Lie supergroup whose associated Harish-Chandra pair is

$$(\text{Aut}_\mathbb{C}(\mathcal{M}), \text{Vec}(\mathcal{M}))$$

with $\alpha$ as the adjoint representation.

Since the action $\psi : \text{Aut}_\mathbb{C}(\mathcal{M}) \times \mathcal{M} \rightarrow \mathcal{M}$ induces the inclusion $\text{Vec}_\mathbb{C}(\mathcal{M}) \hookrightarrow \text{Vec}(\mathcal{M})$ as infinitesimal action (see Corollary 5.1.20), there exists a Lie supergroup action $\Psi : \text{Aut}(\mathcal{M}) \times \mathcal{M} \rightarrow \mathcal{M}$ with the identity $\text{Vec}(\mathcal{M}) \rightarrow \text{Vec}(\mathcal{M})$ as induced infinitesimal action and we have $\Psi|_{\text{Aut}_\mathbb{C}(\mathcal{M}) \times \mathcal{M}} = \psi$ (cf. Theorem 4.3.35).

The automorphism group together with $\Psi$ satisfies a universal property.

**Theorem 5.3.2.** Let $\mathcal{G}$ be a complex Lie supergroup with a holomorphic action $\Psi_{\mathcal{G}} : \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$. Then there is a unique morphism $\sigma : \mathcal{G} \rightarrow \text{Aut}(\mathcal{M})$ of Lie supergroups such that the diagram

$$\begin{array}{ccc}
\mathcal{G} \times \mathcal{M} & \xrightarrow{\Psi_{\mathcal{G}}} & \mathcal{M} \\
\sigma \times \text{id}_\mathcal{M} \downarrow & & \downarrow \Psi \\
\text{Aut}(\mathcal{M}) \times \mathcal{M} & & \\
\end{array}$$

is commutative.

**Proof.** Let $G$ be the underlying Lie group of $\mathcal{G}$. For each $g \in G$, we have a morphism $\Psi_{\mathcal{G}}(g) : \mathcal{M} \rightarrow \mathcal{M}$ by setting $(\Psi_{\mathcal{G}}(g))^* = (\text{ev}_g \otimes \text{id}_\mathcal{M})^* \circ (\Psi_{\mathcal{G}})^*$. This morphism $\Psi_{\mathcal{G}}(g)$ is an automorphism of $\mathcal{M}$ with inverse $\Psi_{\mathcal{G}}(g^{-1})$ and gives rise to a group homomorphism $\tilde{\sigma} : G \rightarrow \text{Aut}_\mathbb{C}(\mathcal{M})$, $g \mapsto \Psi_{\mathcal{G}}(g)$.

Let $\mathfrak{g}$ denote the Lie superalgebra (of right-invariant super vector fields) of $\mathcal{G}$, and $d\Psi_{\mathcal{G}} : \mathfrak{g} \rightarrow \text{Vec}(\mathcal{M})$ the infinitesimal action induced by $\Psi_{\mathcal{G}}$. The restriction of $d\Psi_{\mathcal{G}}$ to the even part $\mathfrak{g}_0 = \text{Lie}(G)$ of $\mathfrak{g}$ coincides with the differential $(d\tilde{\sigma})_e$ of $\tilde{\sigma}$ at the identity $e \in G$.

Moreover, if $\alpha_G$ denotes the adjoint action of $G$ on $\mathfrak{g}$, and $\alpha$ denotes, as before, the adjoint action of $\text{Aut}_\mathbb{C}(\mathcal{M})$ on $\text{Vec}(\mathcal{M})$, we have

$$d\Psi_{\mathcal{G}}(\alpha_G(g)(X)) = (\Psi_{\mathcal{G}}(g^{-1}))^* \circ d\Psi_{\mathcal{G}}(X) \circ (\Psi_{\mathcal{G}}(g))^* = (\tilde{\sigma}(g^{-1}))^* \circ d\Psi_{\mathcal{G}}(X) \circ (\tilde{\sigma}(g))^*$$

$$= \alpha(\tilde{\sigma}(g))(d\Psi_{\mathcal{G}}(X))$$

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for any $g \in G$, $X \in \mathfrak{g}$. Using the correspondence between Lie supergroups and Harish-Chandra pairs, it follows that there is a unique morphism $\sigma : G \to \text{Aut}(\mathcal{M})$ of Lie supergroups with underlying map $\bar{\sigma}$ and derivative $d\Psi_G : \mathfrak{g} \to \text{Vec}(\mathcal{M})$ (see e.g. [Vis11], § 2), and $\sigma$ satisfies $\Psi \circ (\sigma \times \text{id}_\mathcal{M}) = \Psi_G$.

The uniqueness of $\sigma$ follows from the fact that each morphism $\tau : G \to \text{Aut}(\mathcal{M})$ of Lie supergroups fulfilling the same properties as $\sigma$ necessarily induces the map $d\Psi_G : \mathfrak{g} \to \text{Vec}(\mathcal{M})$ on the level of Lie superalgebras and its underlying map $\bar{\tau}$ has to satisfy $\bar{\tau}(g) = \Psi_G(g) = \bar{\sigma}(g)$. 

**Remark 5.3.3.** Since the morphism $\sigma$ in Theorem 5.3.2 is unique, the automorphism group of a compact complex supermanifold $\mathcal{M}$ is the unique Lie supergroup satisfying the universal property formulated in Theorem 5.3.2.

**Remark 5.3.4.** We say that a real Lie supergroup $G$ acts on $\mathcal{M}$ by holomorphic transformations if the underlying Lie group $G$ acts on the complex manifold $\mathcal{M}$ by holomorphic transformations and if there is a homomorphism of Lie superalgebras $\mathfrak{g} \to \text{Vec}(\mathcal{M})$ which is compatible with the action of $G$ on $\mathcal{M}$. Using the theory of Harish-Chandra pairs, we also have the Lie supergroup $G^C$, the universal complexification of $G$; see [Kal15]. The underlying Lie group of $G^C$ is the universal complexification $G^C$ of the Lie group $G$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ denote the Lie superalgebra of $G$, $\mathfrak{g}_0$ the Lie algebra of $G$. Then the Lie algebra $\mathfrak{g}_0^C$ of $G^C$ is a quotient of $\mathfrak{g}_0 \otimes \mathbb{C}$, and the Lie superalgebra of $G^C$ can be realized as $\mathfrak{g}_0^C \oplus (\mathfrak{g}_1 \otimes \mathbb{C})$. The action of $G$ on $\mathcal{M}$ extends to a holomorphic $G^C$-action on $\mathcal{M}$, and the homomorphism $\mathfrak{g} \to \text{Vec}(\mathcal{M})$ extends to a homomorphism $\mathfrak{g}_0^C \oplus (\mathfrak{g}_1 \otimes \mathbb{C}) \to \text{Vec}(\mathcal{M})$ of complex Lie superalgebras, which is compatible with the $G^C$-action on $\mathcal{M}$. Thus, we have a holomorphic $G^C$-action on $\mathcal{M}$ extending the $G$-action. Moreover, there is a morphism $\sigma : G^C \to \text{Aut}(\mathcal{M})$ of Lie supergroups as in Theorem 5.3.2.

**Example 5.3.5.** Let $\mathcal{M} = \mathbb{C}^{0|1}$. Denoting the odd coordinate on $\mathbb{C}^{0|1}$ by $\xi$, each vector field on $\mathbb{C}^{0|1}$ is of the form $X = a\xi \frac{\partial}{\partial \xi} + b \frac{\partial}{\partial \bar{\xi}}$ for $a, b \in \mathbb{C}$. The flow $\varphi : \mathbb{C} \times \mathcal{M} \to \mathcal{M}$ of $a\xi \frac{\partial}{\partial \xi}$ is given by $(\varphi_t)^\ast(\xi) = e^{at}\xi$, and the flow $\psi : \mathbb{C}^{0|1} \times \mathcal{M} \to \mathcal{M}$ of $b \frac{\partial}{\partial \bar{\xi}}$ by $\psi_t^\ast(\xi) = b\tau + \xi$. Let $X_0 = \xi \frac{\partial}{\partial \xi}$ and $X_1 = \frac{\partial}{\partial \bar{\xi}}$. Then $\text{Vec}(\mathbb{C}^{0|1}) = \mathbb{C}X_0 \oplus \mathbb{C}X_1 = \mathbb{C}^{1|1}$, where the Lie algebra structure on $\mathbb{C}^{1|1}$ is given by $[X_0, X_1] = -X_1$ and $[X_1, X_1] = 0$. Note that this Lie superalgebra is isomorphic to the Lie superalgebra of right-invariant vector fields on the Lie supergroup $(\mathbb{C}^{1|1}, \mu_{0,1})$, where the multiplication $\mu = \mu_{0,1}$ is given by $\mu^\ast(t) = t_1 + t_2$ and $\mu^\ast(\tau) = \tau_1 + e^{\tau_2}$; for the Lie supergroup structures on $\mathbb{C}^{1|1}$ see Example 2.1.2. In particular, the Lie superalgebra $\text{Vec}(\mathbb{C}^{0|1})$ is not abelian.

Since each automorphism $\varphi$ of $\mathbb{C}^{0|1}$ is given by $\varphi^\ast(\xi) = c \cdot \xi$ for some $c \in \mathbb{C}$, $c \neq 0$, we have $\text{Aut}_0(\mathbb{C}^{0|1}) \cong \mathbb{C}^\ast$.

**Proposition 5.3.6.** The action of the automorphism group $\text{Aut}(\mathcal{M})$ on $\mathcal{M}$ is effective.

**Proof.** Using Proposition 2.5.2, the effectiveness of the $\text{Aut}(\mathcal{M})$-action follows directly from the fact that the underlying Lie group $\text{Aut}_0(\mathcal{M})$ consists of the set of automorphisms and that the induced infinitesimal action is the identity map $\text{Vec}(\mathcal{M}) \to \text{Vec}(\mathcal{M})$. 

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5.4  The functor of points of the automorphism group

The diffeomorphism supergroup of a real supermanifold is studied in [SW11] and is proven to be a Fréchet Lie supergroup in the case where the supermanifold is compact. In [SW11], the “functor of points” approach to supermanifolds is used, i.e. a supermanifold is a representable contravariant functor from the category of superpoints to the category of sets. Similarly, a Lie supergroup is a representable functor from the category of superpoints to the category of groups. A superpoint is a supermanifold whose underlying manifold consists of only one point, i.e. in the case of real supermanifolds these are precisely the supermanifolds $\mathbb{R}^{0|k}$ for $k \in \mathbb{N}$. Starting with a supermanifold $M$ we define the corresponding functor $\text{Hom}(-, M)$ by the assignment $\mathbb{R}^{0|k} \mapsto \text{Hom}(\mathbb{R}^{0|k}, M)$, where $\text{Hom}(\mathbb{R}^{0|k}, M)$ denotes the set of morphisms of supermanifolds $\mathbb{R}^{0|k} \to M$, and for morphisms $\alpha : \mathbb{R}^{0|k} \to \mathbb{R}^{0|l}$ we define $\text{Hom}(-, M)(\alpha) : \text{Hom}(\mathbb{R}^{0|l}, M) \to \text{Hom}(\mathbb{R}^{0|k}, M)$ by $\varphi \mapsto \varphi \circ \alpha$.

In analogy to the definition in [SW11] for the diffeomorphism supergroup of a real supermanifold, we define a functor $\overline{\text{Aut}}(M)$ associated with a complex supermanifold $M$. The goal of this section is to prove that in the case of a compact complex supermanifold $M$, the automorphism Lie supergroup as defined in Section 5.3 represents the functor $\overline{\text{Aut}}(M)$, i.e. the functors $\text{Hom}(-, \text{Aut}(M))$ and $\text{Hom}(-, \overline{\text{Aut}}(M))$ are isomorphic.

**Definition 5.4.1.** Let $M$ be a complex supermanifold. We define the functor $\overline{\text{Aut}}(M)$ from the category of superpoints to the category of groups as follows:

On objects, we define the functor $\overline{\text{Aut}}(M)$ by the assignment

$$C^{0|k} \mapsto \{ \varphi : C^{0|k} \times M \to C^{0|k} \times M \mid \varphi \text{ is invertible, and } \text{pr}_{C^{0|k}} \circ \varphi = \text{pr}_{C^{0|k}} \},$$

where $\text{pr}_{C^{0|k}} : C^{0|k} \times M \to C^{0|k}$ is the projection. For morphisms $\alpha : C^{0|k} \to C^{0|l}$, we set $\overline{\text{Aut}}(M)(\alpha) : \overline{\text{Aut}}(M)(C^{0|l}) \to \overline{\text{Aut}}(M)(C^{0|k})$,

$$\varphi \mapsto (\text{id}_{C^{0|k}} \times (\text{pr}_{M} \circ \varphi \circ (\alpha \times \text{id}_{M}))) \circ (\text{diag} \times \text{id}_{M}),$$

denoting by $\text{diag} : C^{0|k} \to C^{0|k} \times C^{0|k}$ the diagonal map and by $\text{pr}_{M}$ the projection onto $M$. That means that $\overline{\text{Aut}}(M)(\alpha)(\varphi)$ is the unique automorphism $\psi : C^{0|k} \times M \to C^{0|k} \times M$ with $\text{pr}_{C^{0|k}} \circ \psi = \text{pr}_{C^{0|k}}$ and $\text{pr}_{M} \circ \psi = \text{pr}_{M} \circ \varphi \circ (\alpha \times \text{id}_{M})$.

The group structure on $\overline{\text{Aut}}(M)(C^{0|k})$ is defined by the composition and inversion of automorphisms $C^{0|k} \times M \to C^{0|k} \times M$, and the neutral element is the identity map $C^{0|k} \times M \to C^{0|k} \times M$.

Let $\chi : C^{0|k} \to \text{Aut}(M)$ be an arbitrary morphism of complex supermanifolds. Let $\text{diag} : C^{0|k} \to C^{0|k} \times C^{0|k}$ denote the diagonal map and let $\Psi : \text{Aut}(M) \times M \to M$ denote the natural action of $\text{Aut}(M)$ on $M$. Then the composition

$$\varphi_{\chi} = (\text{id}_{C^{0|k}} \times (\Psi \circ (\chi \times \text{id}_{M}))) \circ (\text{diag} \times \text{id}_{M})$$

is an invertible map $C^{0|k} \times M \to C^{0|k} \times M$ with $\text{pr}_{C^{0|k}} = \text{pr}_{C^{0|k}} \circ \varphi_{\chi}$.

**Lemma 5.4.2.** The assignments $\text{Hom}(C^{0|k}, \text{Aut}(M)) \to \overline{\text{Aut}}(M)(C^{0|k})$, $\chi \mapsto \varphi_{\chi}$, define a natural transformation $\text{Hom}(-, \text{Aut}(M)) \to \overline{\text{Aut}}(M)$.
Lemma 5.4.3. Let $\varphi$ be uniquely determined by $\varphi$ in inclusion. The composition morphism with $\text{pr}_C$ follows from the effectivity of the $\text{Aut}(\mathcal{M})$-action on $\mathcal{M}$. The injectivity of the assignment $\chi \mapsto \varphi$, is a homomorphism of groups, as can be verified by direct calculations. For example, for the multiplication $\chi_1 \cdot \chi_2$ of two morphisms $\chi_1, \chi_2 : \mathbb{C}^{0|k} \to \text{Aut}(\mathcal{M})$ we have

$$
\text{pr}_M \circ \varphi_{(\chi_1 \cdot \chi_2)} = \Psi \circ ((\chi_1 \cdot \chi_2) \times \text{id}_M) = \Psi \circ ((\mu \circ (\chi_1 \cdot \chi_2) \circ \text{diag}) \times \text{id}_M)
$$

where $\mu : \text{Aut}(\mathcal{M}) \times \text{Aut}(\mathcal{M})$ denotes the multiplication on $\text{Aut}(\mathcal{M})$ and we used the action property $\Psi \circ (\mu \times \text{id}_M) = \Psi \circ (\text{id}_\text{Aut}(\mathcal{M}) \times \Psi)$ of the action $\Psi$ on $\mathcal{M}$. □

The just defined natural transformation is actually an isomorphism of functors. The proof of this fact is the content of the remainder of this section. The injectivity of the assignment $\chi \mapsto \varphi$ follows from the effectivity of the $\text{Aut}(\mathcal{M})$-action on $\mathcal{M}$. In the proof of the surjectivity a “normal form” of the pullback of automorphisms $\varphi : \mathbb{C}^{0|k} \times \mathcal{M} \to \mathbb{C}^{0|k} \times \mathcal{M}$ with $\text{pr}_{\mathbb{C}^{0|k}} \circ \varphi = \text{pr}_{\mathbb{C}^{0|k}}$ is used. Thus, we start with some technical preparations.

Let $\mathcal{M}$ be a complex supermanifold and $\varphi : \mathbb{C}^{0|k} \times \mathcal{M} \to \mathbb{C}^{0|k} \times \mathcal{M}$ be an invertible morphism with $\text{pr}_{\mathbb{C}^{0|k}} \circ \varphi = \text{pr}_{\mathbb{C}^{0|k}}$. Let $\iota : \mathcal{M} \hookrightarrow \mathbb{C}^{0|k} \times \mathcal{M}$ denote the canonical inclusion. The composition $\varphi_0 = \text{pr}_M \circ \varphi \circ \iota$ is an automorphism of $\mathcal{M}$. Then $\varphi$ is uniquely determined by $\varphi_0$ and a set of vector fields on $\mathcal{M}$.

Lemma 5.4.3. Let $\varphi : \mathbb{C}^{0|k} \times \mathcal{M} \to \mathbb{C}^{0|k} \times \mathcal{M}$ be an invertible morphism with $\text{pr}_{\mathbb{C}^{0|k}} \circ \varphi = \text{pr}_{\mathbb{C}^{0|k}}$. Then there are vector fields $X_\nu$ of parity $|\nu|$ for $\nu \in (\mathbb{Z}_2)^k$, $\nu \neq 0$, on $\mathcal{M}$ such that

$$
\varphi^* = (\text{id} \times \varphi_0)^* \exp \left( \sum_{\nu \neq 0} \tau^\nu X_\nu \right),
$$

if $\tau_1, \ldots, \tau_k$ denote coordinates on $\mathbb{C}^{0|k} \subset \mathbb{C}^{0|k} \times \mathcal{M}$. By $\tau^\nu X_\nu$ we mean the vector field on $\mathbb{C}^{0|k} \times \mathcal{M}$ which is induced by the extension of the vector field $X_\nu$ on $\mathcal{M}$ to a vector field on the product $\mathbb{C}^{0|k} \times \mathcal{M}$ followed by the multiplication with $\tau^\nu = \tau_1^\nu \ldots \tau_k^\nu$. In other words for $U \subseteq \mathcal{M}$ open we have $\tau^\nu X_\nu(f) = 0$ for $f \in \mathcal{O}_{\mathbb{C}^{0|k}}(\{0\}) \subset \mathcal{O}_{\mathbb{C}^{0|k} \times \mathcal{M}}(\{0\} \times U)$ and $(\tau^\nu X_\nu)(g) = \tau^\nu X_\nu(g)$ for $g \in \mathcal{O}_\mathcal{M}(U) \subset \mathcal{O}_{\mathbb{C}^{0|k} \times \mathcal{M}}(\{0\} \times U)$ considering $X_\nu(g)$ as a function on the product.
Moreover,
\[
\exp \left( \sum_{\nu \neq 0} \tau^\nu X_\nu \right) = \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{\nu \neq 0} \tau^\nu X_\nu \right)^n
\]
is a finite sum since \( \left( \sum_{\nu \neq 0} \tau^\nu X_\nu \right)^{k+1} = 0 \).

A version of this lemma is also proven in [SW11], Theorem 5.1. Here, we give a different proof which makes use of the technical result formulated in Lemma 1.2.5.

Proof. It is enough to prove the statement for those morphisms \( \varphi \) with \( \varphi_0 = \text{id}_M \) since we may otherwise consider the morphism \( \varphi \circ (\text{id}_{\mathbb{C}^0 \times k} \times \varphi_0)^{-1} \).

Assume now \( \varphi_0 = \text{id}_M \). Since \( \text{pr}_{\mathbb{C}^0 \times k} \times \varphi = \text{pr}_{\mathbb{C}^0 \times k} \circ \varphi = \text{id}_{\mathbb{C}^0 \times k} \times M \), and \( \mathcal{O}_{\mathbb{C}^0 \times k} \times M \cong \mathbb{C}^k \otimes \mathcal{O}_M \), there are morphisms \( a_\nu : \mathcal{O}_M \to \mathcal{O}_M \) for \( \nu \in (\mathbb{Z}_2)^k, \nu \neq 0 \), with \( a_\nu(1) = 0 \) such that
\[
\varphi^* = \text{id}^* + \sum_{\nu \neq 0} \tau^\nu a_\nu,
\]i.e. for \( f \in \bigwedge \mathbb{C}^k = \mathcal{O}_{\mathbb{C}^0 \times k}(\{0\}) \) considered as a function \( f \otimes 1 \) on the product \( \mathbb{C}^0 \times k \times M \) we have \( \varphi^*(f) = f \), and for \( g \in \mathcal{O}_M(U), U \subseteq M \) open, considered as a function \( 1 \otimes g \) on the product \( \mathbb{C}^0 \times k \times M \) we have \( \varphi^*(g) = g + \sum_{\nu \neq 0} \tau^\nu a_\nu(g) \).

The difference \( \varphi^*-\text{id}^* = \sum_{\nu \neq 0} \tau^\nu a_\nu \) is a nilpotent morphism \( \mathcal{O}_{\mathbb{C}^0 \times k} \times M \to \mathcal{O}_{\mathbb{C}^0 \times k} \times M \).

By Lemma 1.2.5
\[
X = \sum_{n \geq 1} \frac{(-1)^n}{n!} \left( \sum_{\nu \neq 0} \tau^\nu a_\nu \right)^n
\]
defines an even vector field on \( \mathbb{C}^0 \times k \times M \), and \( \varphi^* = \exp(X) \). Let \( X_\nu : \mathcal{O}_M \to \mathcal{O}_M \) be morphisms such that \( X = \sum_{\nu \neq 0} \tau^\nu X_\nu \). Since \( X \) is an even vector field we have
\[
\sum_{\nu \neq 0} \tau^\nu X_\nu(fg) = X(fg) = X(f)g + fX(g) = \sum_{\nu \neq 0} \tau^\nu X_\nu(f)g + f \sum_{\nu \neq 0} \tau^\nu X_\nu(g)
\]
\[
= \sum_{\nu \neq 0} \tau^\nu \left( X_\nu(f)g + (-1)^{\nu ||f||/\nu} fX_\nu(g) \right)
\]
for any \( f \) and \( g \) with \( f \) homogeneous of parity \( |f| \). Therefore, we have
\[
X_\nu(fg) = \left( X_\nu(f)g + (-1)^{\nu ||f||/\nu} fX_\nu(g) \right),
\]
which proves that \( X_\nu \) is a vector field of parity \( |\nu| \) on \( M \).

Proposition 5.4.4. If \( \varphi : \mathbb{C}^0 \times k \times M \to \mathbb{C}^0 \times k \times M \) is an invertible morphism with \( \text{pr}_{\mathbb{C}^0 \times k} = \text{pr}_{\mathbb{C}^0 \times k} \circ \varphi \), then there is a unique morphism \( \chi : \mathbb{C}^0 \times k \to \text{Aut}(M) \) with \( \varphi = \varphi \chi \).

Proof. If \( \chi_1, \chi_2 : \mathbb{C}^0 \times k \to \text{Aut}(M) \) are morphisms with \( \varphi = \varphi \chi_1 = \varphi \chi_2 \), then we necessarily have \( \chi_1 = \chi_2 \) since the Aut(\( M \))-action on \( M \) is effective (cf. Proposition 5.3.6).

Let \( X_\nu \) be vector fields on \( M \) of parity \( |\nu| \), \( \nu \in (\mathbb{Z}_2)^k, \nu \neq 0 \), and \( \varphi_0 : M \to M \) an automorphism such that \( \varphi^* = (\text{id} \times \varphi_0)^* \exp \left( \sum_{\nu \neq 0} \tau^\nu X_\nu \right) \) as in Lemma 5.4.3. Since \( \varphi_0 \) is an automorphism of \( M \), it is an element of \( \text{Aut}_0(M) \) by definition. Let
ev\_φ_0 denote the evaluation in φ_0, i.e. ev\_φ_0 is the pullback of the canonical inclusion \{φ_0\} \hookrightarrow Aut(M), and let pr\_Aut(M) : C^{0|k} × Aut(M) \rightarrow Aut(M) be the projection. We define \( χ : C^{0|k} \rightarrow Aut(M) \) as the morphism given by the pullback

\[
χ^* = (id^* \otimes ev\_φ_0) \circ \exp \left( \sum_{ν\neq 0} τ^ν(X_ν)_R \right) \circ pr\_Aut^*(M),
\]

where \((X_ν)_R\) denotes the right-invariant vector field on Aut(M) corresponding to the vector field \(X_ν\) on \(M\) which is an element of the Lie superalgebra Vec(M) of Aut(M). The vector fields \((X_ν)_R\) on Aut(M) and \(X_ν\) on \(M\) fulfill the relation

\[
((X_ν)_R \otimes id^*) \circ Ψ^* = Ψ^* \circ X_ν
\]

since \((X_ν)_R\) is right-invariant, Ψ is an action, and the induced infinitesimal action of Ψ maps \((X_ν)_R\), considered as an element of the Lie superalgebra of Aut(M), to \(X_ν\). The sum \(Y = \sum_{ν\neq 0} τ^ν(X_ν)_R\) defines an even vector field on \(C^{0|k} \times Aut(M)\), and \(\exp(Y)\) is a finite sum since \(Y\) is nilpotent. Furthermore, using Lemma 1.2.6 we get

\[
\exp(Y)(f) = (f \exp(Y)(g)) (\exp(Y)(h)) \text{ and thus } \exp(Y) = \text{ the pullback of an automorphism } \text{ from } Aut(M) \rightarrow C^{0|k} \times Aut(M) \text{ with the identity as the underlying map. This implies that } χ^* \text{ is indeed the pullback of a morphism.}
\]

We still need to verify that the morphism \( χ \) satisfies \( φ_χ = φ \). For this it is enough to check \( pr\_M \circ φ_χ = pr\_M \circ φ \) since we already know that \( pr\_C^{0|k} \circ φ_χ = pr\_C^{0|k} \circ φ \). We have \( pr\_M \circ φ_χ = Ψ \circ (χ \times id_M) \). Using the relation \( ((X_ν)_R \otimes id^*) \circ Ψ^* = Ψ^* \circ X_ν \), we get

\[
\left( \left( \sum τ^ν(X_ν)_R \right)^n \otimes id^* \right) \circ (id^* \otimes Ψ^*) = (id^* \otimes Ψ^*) \circ \left( \sum τ^νX_ν \right)^n
\]

for any \( n \geq 0 \), and thus

\[
(exp(Y) \otimes id^*) \circ (id^* \otimes Ψ^*) = \left( \exp \left( \sum τ^ν(X_ν)_R \right) \otimes id^* \right) \circ (id^* \otimes Ψ^*) = (id^* \otimes Ψ^*) \circ \exp \left( \sum τ^νX_ν \right).
\]

This implies

\[
\left( pr\_M \circ φ_χ \right)^* = (χ^* \otimes id^*) \circ Ψ^* = (id^* \otimes ev\_φ_0 \otimes id^*) \circ (exp(Y) \otimes id^*) \circ (id^* \otimes Ψ^*) \circ pr\_M^*
\]

\[
= (id^* \otimes ev\_φ_0 \otimes id^*) \circ (id^* \otimes Ψ^*) \circ \exp \left( \sum τ^νX_ν \right) \circ pr\_M^*
\]

\[
= (id^* \otimes φ_0^*) \circ \exp \left( \sum τ^νX_ν \right) \circ pr\_M^* = φ^* \circ pr\_M^*,
\]

where we used the special form of \( φ^* \) and the fact that \( ev\_φ_0 \otimes id^* \circ Ψ^* = φ_0^* \) by definition of the action Ψ of Aut(M) on M; cf. Corollary 5.1.19.

As a direct consequence of this proposition and Lemma 5.4.2, we get:

**Corollary 5.4.5.** The functors Aut(M) and Hom(−, Aut(M)) are isomorphic. This isomorphism is realized by the natural transformation introduced in Lemma 5.4.2.
5.5 The case of a superdomain with bounded underlying domain

In the classical case, the automorphism group of a bounded domain $U \subset \mathbb{C}^n$ is a (real) Lie group (see Theorem 13 in “Sur les groupes de transformations analytiques” in [Car79]). If $U \subset \mathbb{C}^{m/n}$ is a superdomain whose underlying set $U$ is a bounded domain in $\mathbb{C}^n$, it is in general not possible to endow its set of automorphisms with the structure of a Lie group such that the action on $U$ is smooth, as will be illustrated in an example. In particular, there is no Lie supergroup satisfying the universal property as the automorphism group of a compact complex supermanifold $\mathcal{M}$ does as formulated in Theorem 5.3.2.

Example 5.5.1. Let $U$ be a superdomain of dimension $(1|2)$ whose underlying set is an arbitrary bounded domain $U \subset \mathbb{C}$. Let $z, \theta_1, \theta_2$ denote coordinates for $\mathcal{M}$. For any holomorphic function $f$ on $U$, define the even vector field $X_f = f(z)\theta_1\theta_2 \frac{\partial}{\partial z}$. The reduced vector field $\bar{X}_f = 0$ is completely integrable and thus the flow of $X_f$ can be defined on $\mathbb{C} \times U$ (cf. [GW13] Lemma 5.2). The flow is given by $(\varphi_t)^*(z) = z + t \cdot f(z)\theta_1\theta_2$ and $(\varphi_t)^*(\theta_j) = \theta_j$. For any holomorphic functions $f$ and $g$ we have $[X_f, X_g] = 0$, and thus their flows commute by Corollary 2.3.8. Therefore, $\{X_f | f \in \mathcal{O}(U)\} \cong \mathcal{O}(U)$ is an uncountably infinite-dimensional abelian Lie algebra. If the set of automorphisms of $U$ carried the structure of a Lie group such that its action on $U$ was smooth, its Lie algebra would necessarily contain $\{X_f | f \in \mathcal{O}(U)\} \cong \mathcal{O}(U)$ as a Lie subalgebra, which is not possible.

5.6 Examples

In this section, we determine the automorphism group $\text{Aut}(\mathcal{M})$ for some complex supermanifolds $\mathcal{M}$ with underlying manifold $M = \mathbb{P}_1 \mathbb{C}$.

Let $L_1$ denote the hyperplane bundle on $M = \mathbb{P}_1 \mathbb{C}$ with sheaf of sections $\mathcal{O}(1)$, and $L_k = (L_1)^\otimes k$ the line bundle of degree $k$, $k \in \mathbb{Z}$, on $\mathbb{P}_1 \mathbb{C}$, and sheaf of sections $\mathcal{O}(k)$. Each holomorphic vector bundle on $\mathbb{P}_1 \mathbb{C}$ is isomorphic to a direct sum of line bundles $L_{k_1} \oplus \ldots \oplus L_{k_n}$ (see [Gro57]). Therefore, if $\mathcal{M}$ is a split supermanifold with $M = \mathbb{P}_1 \mathbb{C}$ and $\text{dim} \mathcal{M} = (1|n)$, there exist $k_1, \ldots, k_n \in \mathbb{Z}$ such that the structure sheaf $\mathcal{O}_\mathcal{M}$ of $\mathcal{M}$ is isomorphic to

$$\bigwedge (\mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_n)).$$

Let $U_j = \{[z_0 : z_1] \in \mathbb{P}_1 \mathbb{C} | z_j \neq 0\}$, $j = 1, 2$, and $U_j = (U_j, \mathcal{O}_\mathcal{M}|U_j)$. Moreover, define $U_0^* = U_0 \setminus \{[1 : 0]\}$ and $U_1^* = U_1 \setminus \{[0 : 1]\}$, and let $U_j^* = (U_j^*, \mathcal{O}_\mathcal{M}|U_j^*)$. We can now choose local coordinates $z, \theta_1, \ldots, \theta_n$ for $\mathcal{M}$ on $U_0$, and local coordinates $w, \eta_1, \ldots, \eta_n$ on $U_1$ so that the transition map $\chi : U_0^* \to U_1^*$, which determines the supermanifold structure of $\mathcal{M}$, is given by

$$\chi^*(w) = \frac{1}{z} \quad \text{and} \quad \chi^*(\eta_j) = z^{k_j} \theta_j.$$

Example 5.6.1. Let $\mathcal{M} = (\mathbb{P}_1 \mathbb{C}, \mathcal{O}_\mathcal{M})$ be a complex supermanifold with $\text{dim} \mathcal{M} = (1|1)$. Since the odd dimension is 1, the supermanifold $\mathcal{M}$ has to be split. Let $-k \in \mathbb{Z}$ be the degree of the associated line bundle. Choose local coordinates $z, \theta$ for $\mathcal{M}$ on $U_0$.
and $w, \eta$ on $U_1$ as above so that the transition map $\chi : U_0^* \to U_1^*$ is given by $\chi^*(w) = \frac{1}{z}$ and $\chi^*(\eta) = \frac{1}{z} \theta$.

We first want to determine the Lie superalgebra $\text{Vec}(\mathcal{M})$ of vector fields on $\mathcal{M}$. A calculation in local coordinates verifying the compatibility condition with the transition map $\chi$ yields that the restriction to $U_0$ of any vector field on $\mathcal{M}$ is of the form

$$
\left( (\alpha_0 + \alpha_1 z + \alpha_2 z^2) \frac{\partial}{\partial z} + (\beta + k\alpha_2 z) \frac{\partial}{\partial \theta} + \left( p(z) \frac{\partial}{\partial \theta} + q(z) \theta \frac{\partial}{\partial z} \right) \right),
$$

where $\alpha_0, \alpha_1, \alpha_2, \beta \in \mathbb{C}$, $p$ is a polynomial of degree at most $k$, and $q$ is a polynomial of degree at most $2 - k$. If $k < 0$ (resp. $2 - k < 0$), the polynomial $p$ (resp. $q$) is 0.

The Lie algebra $\text{Vec}_0(\mathcal{M})$ of even vector fields is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$, where an isomorphism $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C} \to \text{Vec}_0(\mathcal{M})$ is given by

$$
\left( \begin{array}{cc} a & b \\ c & -a \end{array} \right), d \mapsto (-b - 2az + cz^2) \frac{\partial}{\partial z} + ((d - ka) + kcz) \theta \frac{\partial}{\partial \theta}.
$$

Note that since the odd dimension of $\mathcal{M}$ is one, each automorphism $\varphi : \mathcal{M} \to \mathcal{M}$ gives rise to an automorphism of the line bundle $L_{-k}$ and vice versa. Hence, the automorphism group $\text{Aut}(L_{-k})$ of the line bundle $L_{-k}$ and $\text{Aut}_0(\mathcal{M})$ coincide.

A calculation yields that the group $\text{Aut}_0(\mathcal{M})$ of automorphisms $\mathcal{M} \to \mathcal{M}$ can be identified with $\text{PSL}_2(\mathbb{C}) \times \mathbb{C}^*$ if $k$ is even and with $\text{SL}_2(\mathbb{C}) \times \mathbb{C}^*$ if $k$ is odd. Consider the element $\left( \left( \begin{array}{cc} a & b \\ c & -a \end{array} \right), s \right)$, where $s \in \mathbb{C}^*$ and $\left( \begin{array}{cc} a & b \\ c & -a \end{array} \right)$ is either an element of $\text{SL}_2(\mathbb{C})$ or the representative of the corresponding class in $\text{PSL}_2(\mathbb{C})$. The action of the corresponding element $\varphi \in \text{Aut}_0(\mathcal{M})$ on $\mathcal{M}$ is then given by

$$
\varphi^*(z) = \frac{c + dz}{a + bz} \quad \text{and} \quad \varphi^*(\theta) = \frac{1}{(a + bz)^k} s \theta
$$

as a morphism over appropriate subsets of $U_0$ and by

$$
\varphi^*(w) = \frac{aw + b}{cw + d} \quad \text{and} \quad \varphi^*(\eta) = \frac{1}{(cw + d)^k} s \eta
$$

over appropriate subsets of $U_1$.

The Lie supergroup structure on $\text{Aut}(\mathcal{M})$ is now uniquely determined by $\text{Aut}_0(\mathcal{M})$, $\text{Vec}(\mathcal{M})$, and the adjoint action of $\text{Aut}_0(\mathcal{M})$ on $\text{Vec}(\mathcal{M})$. Since $\text{Aut}_0(\mathcal{M})$ is connected it is enough to calculate the adjoint action of $\text{Vec}_0(\mathcal{M}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$ on $\text{Vec}_1(\mathcal{M})$.

Let $P_l$ denote the space of polynomials of degree at most $l$, and set $P_l = \{0\}$ for $l < 0$. The space of odd vector fields $\text{Vec}_1(\mathcal{M})$ is isomorphic to $P_k \oplus P_{-k}$ via $(p(z) \frac{\partial}{\partial z} + q(z) \theta \frac{\partial}{\partial \theta}) \mapsto (p(z), q(z))$.

The element $H = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \in \mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C} \cong \text{Vec}_0(\mathcal{M})$ corresponds to $-2z \frac{\partial}{\partial z} - k \theta \frac{\partial}{\partial \theta}$. The adjoint action of this vector field on the first factor $P_k$ of $\text{Vec}_1(\mathcal{M})$ is given by $-2z \frac{\partial}{\partial z} + k \cdot \text{Id}$, and on the second factor $P_{-k}$ by $-2z \frac{\partial}{\partial z} + (2 - k) \cdot \text{Id}$. Calculating the weights of the $\mathfrak{sl}_2(\mathbb{C})$-representation on $P_k$ and $P_{-k}$, we get that $P_k$ is the unique irreducible $(k + 1)$-dimensional representation and $P_{-k}$ the unique irreducible $(3 - k)$-dimensional representation. Moreover, a calculation yields that $d \in \mathbb{C}$ corresponding to $d \cdot \theta \frac{\partial}{\partial \theta} \in \text{Vec}_0(\mathcal{M})$ acts on $P_k$ by multiplication with $-d$ and on $P_{-k}$ by multiplication with $d$. 

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If $k < 0$ or $k > 2$, we have

$$\text{[Vec}_1(\mathcal{M}), \text{Vec}_1(\mathcal{M})] = 0.$$  

In the case $k = 0$, we have $P_k \cong \mathbb{C}$. Since $\left[ \frac{\partial}{\partial \theta}, q(z) \frac{\partial}{\partial z} \right] = q(z) \frac{\partial}{\partial z}$ for any $q \in P_2$, we get

$$\text{[Vec}_1(\mathcal{M}), \text{Vec}_1(\mathcal{M})] = \left\{ a(z) \frac{\partial}{\partial z} \bigg| a \in P_2 \right\} \cong \mathfrak{sl}_2(\mathbb{C}),$$

and the map $P_0 \times P_2 \rightarrow \text{Vec}_0(\mathcal{M})$, $(X, Y) \mapsto [X, Y]$, corresponds to $\mathbb{C} \times P_2 \rightarrow \text{Vec}_0(\mathcal{M})$, $(p, q(z)) \mapsto p \cdot q(z) \frac{\partial}{\partial z}$.

Similarly, if $k = 2$, we have $P_{2-k} \cong \mathbb{C}$, and

$$\text{[Vec}_1(\mathcal{M}), \text{Vec}_1(\mathcal{M})] = \left\{ (\alpha_0 + \alpha_1 z + \alpha_2 z^2) \frac{\partial}{\partial z} + (\alpha_1 + 2\alpha_2 z) \theta \frac{\partial}{\partial \theta} \bigg| \alpha_0, \alpha_1, \alpha_2 \in \mathbb{C} \right\} \cong \mathfrak{sl}_2(\mathbb{C})$$

since $[p(z) \frac{\partial}{\partial \theta}, \theta \frac{\partial}{\partial z}] = p(z) \frac{\partial}{\partial z} + p'(z) \theta \frac{\partial}{\partial \theta}$. The map $P_2 \times P_0 \rightarrow \text{Vec}_0(\mathcal{M})$, $(X, Y) \mapsto [X, Y]$, corresponds to $\mathbb{C} \times P_0 \rightarrow \text{Vec}_0(\mathcal{M})$, $(p(z), q) \mapsto q \cdot p(z) \frac{\partial}{\partial z} + q \cdot p'(z) \theta \frac{\partial}{\partial \theta}$.

If $k = 1$, then $P_k \oplus P_{2-k} \cong \mathbb{C}^2 \oplus \mathbb{C}^2$. We have

$$\left[ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right] = \frac{\partial}{\partial \theta}, \quad \left[ z \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right] = z \frac{\partial}{\partial z} + \theta \frac{\partial}{\partial \theta},$$

and consequently $[\text{Vec}_1(\mathcal{M}), \text{Vec}_1(\mathcal{M})] = \text{Vec}_0(\mathcal{M})$.

Note that $\text{Aut}(\mathcal{M})$ carries the structure of a split Lie supergroup if and only if $k < 0$ or $k > 2$. For the definition of a split Lie supergroup see e.g. Proposition 4 in [Vis11].

**Example 5.6.2.** Let $\mathcal{M} = (P_1 \mathbb{C}, O_{\mathcal{M}})$ be a split complex supermanifold of dimension $\dim \mathcal{M} = (1|2)$ associated with $O(-k_1) \oplus O(-k_2)$, $k_1, k_2 \in \mathbb{Z}$. We determine the group $\text{Aut}_0(\mathcal{M})$ of automorphisms $\mathcal{M} \rightarrow \mathcal{M}$.

We choose coordinates $z, \theta_1, \theta_2$ for $U_0$ and $w, \eta_1, \eta_2$ for $U_1$ as described above so that the transition map $\chi$ is given by $\chi^*(w) = z^{-1}$ and $\chi^*(\eta_j) = z^{-k_j} \theta_j$.

The action of $\text{PSL}_2(\mathbb{C})$ on $P_1 \mathbb{C}$ by Möbius transformations lifts to an action of $\text{SL}_2(\mathbb{C})$ on $\mathcal{M}$ by letting $A = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{C})$ act by the automorphism $\varphi_A : \mathcal{M} \rightarrow \mathcal{M}$ with pullback

$$\varphi_A^*(z) = \frac{c + dz}{a + bz} \quad \text{and} \quad \varphi_A^*(\theta_j) = (a + bz)^{-k_j} \theta_j$$

as a morphism over appropriate subsets of $U_0$, and

$$\varphi_A^*(w) = \frac{aw + b}{cw + d} \quad \text{and} \quad \varphi_A^*(\eta_j) = (cw + d)^{-k_j} \eta_j$$

over appropriate subsets of $U_1$. Using the transition map $\chi$ one might also calculate the representation of $\varphi$ in coordinates as a morphism over subsets $U_0 \rightarrow U_1$ and $U_1 \rightarrow U_0$.

If $k_1$ and $k_2$ are both even, we have $\varphi_A = \text{Id}_{\mathcal{M}}$ for $A = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ and thus we get an action of $\text{PSL}_2(\mathbb{C})$ on $\mathcal{M}$.

Consider the homomorphism of Lie groups $\Psi : \text{Aut}_0(\mathcal{M}) \rightarrow \text{Aut}(P_1 \mathbb{C})$ assigning to each automorphism $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ the underlying biholomorphic map $\tilde{\varphi} : P_1 \mathbb{C} \rightarrow P_1 \mathbb{C}$.
This homomorphism is surjective since \(\text{Aut}(\mathbb{P}_1 \mathbb{C}) \cong \text{PSL}_2(\mathbb{C})\) and since the \(\text{PSL}_2(\mathbb{C})\)-action on \(\mathbb{P}_1 \mathbb{C}\) lifts to an action of \(\text{SL}_2(\mathbb{C})\) on the supermanifold \(\mathcal{M}\). The kernel \(\ker \Psi\) consists of those automorphisms \(\varphi: \mathcal{M} \to \mathcal{M}\) whose underlying map \(\tilde{\varphi}\) is the identity \(\mathbb{P}_1 \mathbb{C} \to \mathbb{P}_1 \mathbb{C}\). This kernel is a normal subgroup, \(\text{SL}_2(\mathbb{C})\) acts on \(\ker \Psi\), and we obtain:

\[
\ker \Psi \times \text{SL}_2(\mathbb{C}) \quad \text{if } k_1 \text{ and } k_2 \text{ are both even}
\]

\[
\ker \Psi \times \text{PSL}_2(\mathbb{C}) \quad \text{otherwise}
\]

Thus, it remains to determine \(\ker \Psi\).

Let \(\varphi: \mathcal{M} \to \mathcal{M}\) be an automorphism with \(\tilde{\varphi} = \text{Id}\). Let \(f\) and \(b_{jk}\), \(j, k = 1, 2\), be holomorphic functions on \(U_0 \cong \mathbb{C}\) such that the pullback of \(\varphi\) over \(U_0\) is given by

\[
\varphi^* = z + f(z)\theta_1\theta_2 \quad \text{and} \quad \varphi^*(\theta) = B(z)\theta,
\]

where \(B(z) = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix}\) and \(\varphi^*(\theta) = B(z)\theta\) is an abbreviation for \(\varphi^*(\theta_j) = b_{j1}(z)\theta_1 + b_{j2}(z)\theta_2\) for \(j = 1, 2\). Similarly, let \(g\) and \(c_{jk}\) be holomorphic functions on \(U_1 \cong \mathbb{C}\) such that the pullback of \(\varphi\) over \(U_1\) is given by

\[
\varphi^*(w) = w + g(w)\eta_1\eta_2 \quad \text{and} \quad \varphi^*(\eta) = C(z)\eta,
\]

where \(C(z) = \begin{pmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{pmatrix}\). The compatibility condition with the transition map \(\chi\) now gives the relation

\[
f(z) = -z^{2-(k_1+k_2)}g(z) \quad \text{for } z \in \mathbb{C}^*.
\]

Therefore, \(f\) and \(g\) are both polynomials of degree at most \(2 - (k_1 + k_2)\), and they are zero in the case \(k_1 + k_2 > 2\). For the matrices \(B\) and \(C\) we get the relation

\[
B(z) = \begin{pmatrix} z^{k_1} & 0 \\ 0 & z^{k_2} \end{pmatrix} C \begin{pmatrix} 1/z \end{pmatrix} \begin{pmatrix} z^{-k_1} & 0 \\ 0 & z^{-k_2} \end{pmatrix} \quad \text{for } z \in \mathbb{C}^*.
\]

If \(k_1 = k_2\), this implies \(B(z) = C \begin{pmatrix} 1/z \end{pmatrix}\) for all \(z \in \mathbb{C}^*\). Thus, \(B(z) = B\) and \(C(w) = C\) are constant matrices, and \(B = C \in \text{GL}_2(\mathbb{C})\) since \(\varphi\) was assumed to be invertible. Consequently, we have

\[
\ker \Psi \cong P_{2-(k_1+k_2)} \rtimes \text{GL}_2(\mathbb{C})
\]

in the case \(k_1 = k_2\), where \(P_l\) denotes again the space of polynomials of degree at most \(l\) if \(l \geq 0\) and \(P_l = \{0\}\) otherwise. The group structure on the semidirect product is given by

\[
(f_1(z), B_1) \cdot (f_2(z), B_2) = (\det B_1 f_1(z) + f_2(z), B_1 B_2).
\]

Let now \(k_1 \neq k_2\). After possibly changing coordinates we may assume \(k_1 > k_2\). We have

\[
B(z) = \begin{pmatrix} z^{k_1} & 0 \\ 0 & z^{k_2} \end{pmatrix} C \begin{pmatrix} 1/z \end{pmatrix} \begin{pmatrix} z^{-k_1} & 0 \\ 0 & z^{-k_2} \end{pmatrix} = \begin{pmatrix} c_{11} \left( \frac{1}{z} \right) & c_{12} \left( \frac{1}{z} \right) \\ c_{21} \left( \frac{1}{z} \right) & c_{22} \left( \frac{1}{z} \right) \end{pmatrix}
\]

for all \(z \in \mathbb{C}^*\). This implies that \(b_{11} = c_{11}\) and \(b_{22} = c_{22}\) are constants. Since we assume \(k_1 > k_2\), we also get \(b_{21} = c_{21} = 0\) and \(b_{12}\) and \(c_{12}\) are polynomials of degree at most \(k_1 - k_2\). Therefore,

\[
\ker \Psi \cong P_{2-(k_1+k_2)} \rtimes \left\{ \left( \begin{array}{c} \lambda \\ 0 \end{array} \right) \begin{pmatrix} p(z) \\ \mu \end{pmatrix} \middle| \lambda, \mu \in \mathbb{C}^*, p \in P_{k_1-k_2} \right\},
\]
and the group structure is again given by \((f_1(z), B_1) \cdot (f_2(z), B_2) = (\det B_1 f_1(z) + f_2(z), B_1 B_2)\) for \(f_1, f_2 \in P_2 \cdot (k_1 + k_2)\), \(B_1, B_2 \in \{ \binom{\lambda \mu(z)}{0} \mid \lambda, \mu \in \mathbb{C}, p \in P_{k_1 - k_2} \}\).

The semidirect product \(\ker \Psi \rtimes \text{SL}_2(\mathbb{C})\) (or \(\ker \Psi \rtimes \text{PSL}_2(\mathbb{C})\)) is a direct product if and only if \(k_1 = k_2\) and \(k_1 + k_2 \geq 2\).

### Example 5.6.3.
Let \(\mathcal{M} = (\mathbb{P}_1 \mathbb{C}, \mathcal{O}_\mathcal{M})\) be the complex supermanifold of dimension \(\dim \mathcal{M} = (1|2)\) given by the transition map \(\chi : \mathcal{U}_1^* \to \mathcal{U}_2^*\) with pullback

\[
\chi^*(w) = \frac{1}{z} + \frac{1}{z^2} \theta_1 \theta_2 \quad \text{and} \quad \chi^*(\eta_j) = \frac{1}{z^2} \theta_j.
\]

The supermanifold \(\mathcal{M}\) is not split and the associated split supermanifold corresponds to \(\mathcal{O}(-2) \oplus \mathcal{O}(-2)\); see e.g. [BO96].

As in the previous example, the action of \(\text{PSL}_2(\mathbb{C})\) on \(\mathbb{P}_1 \mathbb{C}\) by Möbius transformations lifts to an action of \(\text{SL}_2(\mathbb{C})\) on \(\mathcal{M}\). Let \(A\) denote the class of \(\binom{a}{c} \in \text{SL}_2(\mathbb{C})\) in \(\text{PSL}_2(\mathbb{C})\). Then \(A\) acts by the morphism \(\varphi_A : \mathcal{M} \to \mathcal{M}\) whose pullback as a morphism over appropriate subsets of \(\mathcal{U}_0\) is given by

\[
\varphi_A^*(z) = \frac{c + dz}{a + bz} - \frac{b}{(a + bz)^2} \theta_1 \theta_2 \quad \text{and} \quad \varphi_A^*(\theta_j) = \frac{1}{(a + bz)^2} \theta_j.
\]

Let \(\Psi : \text{Aut}_{\mathcal{O}}(\mathcal{M}) \to \text{Aut}(\mathbb{P}_1 \mathbb{C}) \cong \text{PSL}_2(\mathbb{C})\) denote again the Lie group homomorphism which assigns to an automorphism of \(\mathcal{M}\) the underlying automorphism of \(\mathbb{P}_1 \mathbb{C}\). The assignment \(A \mapsto \varphi_A \in \text{Aut}_{\mathcal{O}}(\mathcal{M})\) defines a section \(\text{PSL}_2(\mathbb{C}) \to \text{Aut}_{\mathcal{O}}(\mathcal{M})\) of \(\Psi\), and we have

\[
\text{Aut}_{\mathcal{O}}(\mathcal{M}) \cong \ker \Psi \rtimes \text{PSL}_2(\mathbb{C}).
\]

The section \(\text{PSL}_2(\mathbb{C}) \to \text{Aut}_{\mathcal{O}}(\mathcal{M})\) induces on the level of Lie algebras the morphism \(\sigma : \mathfrak{sl}_2(\mathbb{C}) \to \text{Vec}_\mathcal{O}(\mathcal{M})\), which maps an element \(\binom{a}{c} \in \mathfrak{sl}_2(\mathbb{C})\) to the vector field on \(\mathcal{M}\) whose restriction to \(\mathcal{U}_0\) is

\[
(c - 2az - bz^2 - b\theta_1 \theta_2) \frac{\partial}{\partial z} - 2(a + bz) \left( \theta_1 \frac{\partial}{\partial \theta_1} + \theta_2 \frac{\partial}{\partial \theta_2} \right).
\]

We now calculate the kernel \(\ker \Psi\). Let \(\varphi \in \ker \Psi\). Its underlying map \(\bar{\varphi}\) is the identity and we thus have

\[
\varphi^*(z) = z + a_0(z) \theta_1 \theta_2 \quad \text{and} \quad \varphi^*(\theta) = A_0(z) \theta
\]
on \(\mathcal{U}_0\) and

\[
\varphi^*(w) = w + a_1(w) \eta_1 \eta_2 \quad \text{and} \quad \varphi^*(\eta) = A_1(w) \eta
\]
on \(\mathcal{U}_1\) for holomorphic functions \(a_0\) and \(a_1\) and invertible matrices \(A_0\) and \(A_1\) whose entries are holomorphic functions. The notation \(\varphi^*(\theta) = A_0(z) \theta\) (and similarly \(\varphi^*(\eta) = A_1(w) \eta\)) is again an abbreviation for \(\varphi^*(\theta_j) = (A_0(z))_{j1} \theta_1 + (A_0(z))_{j2} \theta_2\), where \(A_0(z) = ((A_0(z))_{jk})_{1 \leq j,k \leq 2}\). A calculation with the transition map \(\chi\) then yields the relations

\[
A_1(w) = A_0 \left( \frac{1}{w} \right) \quad \text{and} \quad a_1(w) = \frac{1}{w} \left( \left( \det A_0 \left( \frac{1}{w} \right) - 1 \right) - \frac{1}{w} a_0 \left( \frac{1}{w} \right) \right)
\]
for any \( w \in \mathbb{C}^* \). Since \( a_0, a_1, A_0, \) and \( A_1 \) are holomorphic on \( \mathbb{C} \), we get that \( A_0 = A_1 \) are constant matrices, \( \det A_0 = 1 \), and \( a_0 = a_1 = 0 \). Therefore, \( \ker \Psi \cong SL_2(\mathbb{C}) \), and its Lie algebra is

\[
\text{Lie}(\ker \Psi) = \left\{ \begin{pmatrix} a_{11} \theta_1 + a_{12} \theta_2 & \frac{\partial}{\partial \theta_1} + (a_{21} \theta_1 + a_{22} \theta_2) \frac{\partial}{\partial \theta_2} \end{pmatrix} \left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) \right. \right\}.
\]

Since \( \text{Lie}(\ker \Psi) \) and \( \sigma(\text{Lie}(PSL_2(\mathbb{C}))) \) commute, the semidirect product \( \ker \Psi \rtimes PSL_2(\mathbb{C}) \) is direct and we have

\[
\text{Aut}_0(\mathcal{M}) \cong SL_2(\mathbb{C}) \times PSL_2(\mathbb{C}).
\]

Remark in particular that this group is different from the automorphism group of the corresponding split supermanifold \( \mathcal{N} \) which is associated with \( O(-2) \oplus O(-2) \) and for which we calculated that \( \text{Aut}_0(\mathcal{N}) \cong GL_2(\mathbb{C}) \times PSL_2(\mathbb{C}) \).
Bibliography


