NORMAL FORMS
AND CONSERVED QUANTITIES
IN MULTISYMPLECTIC GEOMETRY

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In loving memory of my grandfathers

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1 Introduction and overview

A multisymplectic form of degree \( k+1 \) on a manifold \( M \) is a closed \((k+1)\)-form \( \omega \) that is non-degenerate in the sense that the bundle map \( \omega^\# : TM \to \Lambda^{k+1} T^* M \), \( v \mapsto \iota_v \omega \) is injective. A couple \((M, \omega)\) consisting of a manifold and a multisymplectic form is also called a multisymplectic manifold, and in case \( \omega \) is a of degree \( k + 1 \), sometimes a \( k \)-plectic manifold.

Multisymplectic geometry emerged, on the one hand, as an effort to give a finite-dimensional Hamiltonian description of field theory in analogy to symplectic geometry giving a finite-dimensional picture of classical mechanics (cf. e.g. [Kij73], [Got91], [CnCI91], [FPR03] and [HK04], we also refer to [Hél12] and [Rog12] for some stimulating historical remarks).

On the other hand, the success of Chern-Weil-Kostant theory that identifies integral 2-forms with complex line bundles generated a tremendous amount of work on the differential-topological interpretation of integral differential forms of degrees three and higher, compare, e.g., the theory of gerbes (cf. [Bry93]).

In this doctoral thesis we are interested in the development of the geometry of multisymplectic manifolds, notably the study of the invariants associated to a multisymplectic structure, and the symmetries of this structure. (Since a symmetry of \((M, \omega)\) has to preserve geometric invariants, such as the properties of \( \omega_p \) for \( p \in M \) and foliations of \( M \) associated to \( \omega \), the theory sharply deviates from the symplectic case, well-known to have no local invariants.) Since multisymplectic geometry is a new, “emerging” field of mathematical research, we insisted on including many examples in order to show the broadness of the subject.

The first multisymplectic surprise one encounters is the fact that there are inequivalent linear multisymplectic \((k+1)\)-forms (“linear types”) on a real vector space of a given dimension. For a multisymplectic manifold \((M, \omega)\) this immediately gives an obstruction against “flatness of \( \omega \)”, i.e., the existence of local coordinates such that \( \omega \) has constant coefficients. We illustrate these phenomena in detail by explicit examples and study several situations where the linear type is constant and higher-order obstructions to flatness appear. For instance, the Darboux theorem from symplectic geometry lead to the findings of Geoffrey Martin in [Mar88], later refined in [CldL99] and [FG13]. In addition, some low-dimensional situations (especially cases where multisymplectic forms are also stable) were studied in e.g. [Tur84], [Hit00],
[Bur04], [LPV08] and [Van08]. As we will see, due to the vast amount of inequivalent linear multisymplectic structures whose systematic study goes back, at least, to the fundamental work of Jean Martinet ([Mar70]), the question of “Darboux type theorems” in multisymplectic geometry is far from being settled.

Obviously, such geometric invariants associated to $\omega$ lead to rigidity phenomena for the group of global diffeomorphisms of $M$ preserving the multisymplectic form, having no analogy in the symplectic case.

On the infinitesimal level, the construction of an “observable algebra” generating “Hamiltonian symmetries” suffers - as to be expected from its relation to Lagrangian classical field theories - from the absence of appropriate Lie algebra structures. Often, exact terms (“total divergences” in physics lingo) prevent, e.g., the validity of the Jacobi identity. Early approaches bypassed this problem by dividing out the inconsistencies and considering everything “up to divergence terms” (cf. e.g. [CnCI91]). With the advent of homological methods in mathematical physics (cf. the work of Maxim Kontsevich and Jim Stasheff) this point of view became obsolete and the Poisson algebra of functions got replaced by Lie $\infty$-algebras of observables, such as the algebra of John Baez and Chris Rogers (see [Rog12]). A crucial ingredient of the above is the “Hamilton-DeDonder-Weyl equation”

$$\iota_X \omega = -d\alpha$$

for a couple $(\alpha, X)$, consisting of a $(k-1)$-form $\alpha$ and a vector field $X$ on $M$.

Using this Lie $\infty$-algebra, a new concept of comoment in multisymplectic geometry was defined as a Lie $\infty$-morphism to the observable Lie $\infty$-algebra in [CFRZ16]. This concept unifies several different approaches to comoments on multisymplectic manifolds with an $\omega$-preserving action, notably those given in [Got91, CnCI91] and [MS12]. We generalize the well-known (“Noether”) relation between the Poisson commutation of the components of a comoment and a function generating a Hamiltonian flow to the multisymplectic situation upon introducing a more adapted version of “conserved quantities” (see Sections 2 and 6).

Let us now go through the content of this thesis in some detail. We start by giving a detailed introduction to multisymplectic manifolds and conserved quantities of Hamiltonian vector fields in Section 2. In Subsection 2.1 we elucidate the definition of multisymplectic manifolds by providing a wide
range of examples and constructions thereof. In Subsection 2.2 we briefly recapitulate the construction of a Lie $\infty$-algebra of observables $L_\infty(M, \omega)$, generalizing the Lie algebra structure $\{\cdot, \cdot\}_\omega$ on the real-valued functions $C^\infty(M)$ of a symplectic manifold $(M, \omega)$. Then, in Subsection 2.3 we proceed to a central definition, formalizing the idea that in certain situations “symmetries preserve densities up to total divergences”.

**Definition (2.22).** Let $M$ be a manifold and $X$ a vector field. A form $\alpha \in \Omega^*(M)$ is called a “conserved quantity” for $X$ if the Lie derivative of $\alpha$ along $X$ is an exact form, i.e. $\mathcal{L}_X \alpha = d\beta$ for some $\beta \in \Omega^*(M)$. We call $\alpha$ a “locally conserved quantity” if $\mathcal{L}_X \alpha$ is a closed form.

The space of conserved quantities satisfies interesting algebraic relations, even more so in the presence of a multisymplectic structure $\omega$ preserved by $X$.

**Theorem (2.34).** Let $(M, \omega)$ be a multisymplectic manifold and $X$ a vector field satisfying $\mathcal{L}_X \omega = 0$. Then the conserved quantities for $X$ form a Lie $\infty$-subalgebra of $L_\infty(M, \omega)$.

Finally, in Subsection 2.4, we legitimize the name “conserved” of conserved quantities by treating their impact on the dynamics of a brane $\Sigma$ moving through the ambient manifold $M$. For instance, we prove the following:

**Theorem (2.35).** Let $\Sigma$ be a compact and oriented $d$-dimensional manifold without boundary, $X$ a vector field on $M$ with flow $\phi_t$, and $\sigma_0: \Sigma \to M$ a smooth map. Consider $\sigma_t := \phi_t \circ \sigma_0: \Sigma \to M$. If $\alpha \in \Omega^d(M)$ is a differential form conserved by $X$, then the value of the integral

$$\int_\Sigma (\sigma_t)^* \alpha$$

is independent of the time parameter $t$.

We close this subsection by putting these results into the context of transgression to mapping spaces. The theory of conserved quantities in multisymplectic geometry treated in Subsections 2.3 and 2.4, as well as Subsections 6.2-6.4 and 6.6 was developed in collaboration with Marco Zambon from the KU Leuven. For more details on this theory please consult [RWZ16], where we developed the theory slightly more generally by including the premultisymplectic case.

In Section 3 we turn to the linear structures underlying a multisymplectic form and the question how many equivalence classes of such structures exist once dimension and degree are fixed.
Definition (3.1). Let $V$ be a finite-dimensional real vector space. An alternating tensor $\eta \in \Lambda^{k+1}V^*$ is called non-degenerate (or multisymplectic) if the contraction $V \to \Lambda^k V^*$, $v \mapsto \iota_v \eta$ is an injective map. Alternating tensors $\eta$ and $\eta'$ are called equivalent if there exists a linear automorphism $\phi$ of $V$, such that $\phi^*(\eta) = \eta'$.

The goal of this section is to complete the classification result given below, which was started by Jean Martinet in [Mar70].

Theorem (3.2). The numbers $\Sigma^k_n$ of equivalence classes of non-degenerate $k$-forms over an $n$-dimensional vector space $V$ are

- $\Sigma^n_n = 1$ for all $n$, and $\Sigma^1_n$ as well as $\Sigma^{n-1}_n$ are zero for $n > 1$.
- $\Sigma^2_n$ is 0 for $n$ odd and one for $n$ even.
- $\Sigma^{n-2}_n = \lfloor \frac{n}{2} \rfloor - 1$, when $(n \mod 4) \neq 2$ (for $n \geq 4$) and $\Sigma^{n-2}_n = \frac{n}{2}$, when $(n \mod 4) = 2$ (for $n \geq 4$).
- $\Sigma^3_6 = 3$, $\Sigma^3_7 = 8$, $\Sigma^3_8 = 21$, $\Sigma^4_7 = 15$ and $\Sigma^5_8 = 31$.
- $\Sigma^k_n = \infty$ in all other cases.

After reviewing the necessary linear algebra in Subsection 3.1, we go through the individual cases in Sections 3.2 through 3.8. Our contribution to the proof of this theorem is the determination of the formerly unknown numbers $\Sigma^4_7$ and $\Sigma^5_8$. Supplemented by Appendix A, where we enlist the non-degenerate three-forms in dimensions seven and eight, our proofs also include the construction of the equivalence classes of non-degenerate $k$-forms in dimension $n$, whenever $\Sigma^k_n$ is finite.

Section 4 is inspired by the Darboux theorem of symplectic geometry, which can be expressed by our notion of flatness.

Definition (4.3). A differential form $\alpha$ on a manifold $M$ is called “flat” if around any point $p \in M$ there exist local coordinates such that $\alpha$ has constant coefficients in these coordinates.

In this language the Darboux theorem essentially states that for a form $\omega \in \Omega^2(M)$ symplecticity implies flatness. In Subsection 4.1, we explain why its proof fails in the general multisymplectic setting. Then, in Subsection 4.2, we recall a Darboux-type theorem from [Mar88], which characterizes the obstruction to flatness for a certain type of “standard” multisymplectic manifolds. Surprisingly at first sight, in addition to non-degeneracy and
closedness of the form, the involutivity of a certain characteristic distribution is a necessary (and sufficient) condition for flatness. In the subsequent Subsections 4.3 and 4.4 we characterize the flatness of two other important classes of multisymplectic manifolds.

**Theorem (4.11).** Let \( k \geq 2, m > 2 \) and \( U \subset \mathbb{R}^{km} \) be open and \( \omega \in \Omega^m_{cl}(U) \) be of linear type \( dx^{1,2,\ldots,m} + dx^{m+1,\ldots,2m} + \ldots + dx^{(k-1)m+1,\ldots,km} \). Then there is a decomposition \( \omega = \omega_1 + \ldots + \omega_k \), with \( \omega_1,\ldots,\omega_k \in \Omega^m(U) \) such that \( \text{rank}(\omega_i) = m \). The forms \( \omega_i \) are unique up to permutation. Furthermore, \((U,\omega)\) is flat if and only if \( d\omega_i = 0 \) for all \( i \in \{1,\ldots,k\} \).

**Theorem (4.13).** Let \( m > 2 \) and \( U \subset \mathbb{R}^{2m} \) be open and \( \omega \in \Omega^m_{cl}(U) \) be of linear type \( \text{Re}(dx^1 + idx^2) \wedge \ldots \wedge (dx^{2m-1} + idx^{2m}) \). Then, up to sign, there is a unique almost-complex structure \( J \) such that the following equality holds for all \( p \in U \) and \( v,w \in T_pU \):

\[
\iota_J(w)\iota_J(v)\omega = \iota_w\iota_J(v)\omega
\]

Furthermore, \((U,\omega)\) is flat if and only if \( J \) is integrable.

In Subsection 4.5 we treat special cases of the above theorems by giving a detailed description of the situation of non-degenerate three-forms in dimension 6. We illustrate the subtleties of flatness considerations for multisymplectic forms, by constructing a non-flat multisymplectic manifold with constant linear equivalence class. (Let us note that we wrote a simple SageMath program in this context, see Appendix C.2.)

**Theorem (Examples 4.18 and 4.20\textsuperscript{1}).** The multisymplectic form \( \omega = dx^{135} - dx^{146} - dx^{236} + x^2dx^{245} \in \Omega^3(\mathbb{R}^6) \) has non-constant linear equivalence class in any neighborhood of \( 0 \in \mathbb{R}^6 \). Especially it is non-flat near zero. Its restriction to the open submanifold \( \{x^2 > 0\} \subset \mathbb{R}^6 \) is non-flat despite having constant linear equivalence class.

We conclude Section 4 by treating the Cartan three-forms on real simple Lie groups in Subsection 4.6. Beyond the special case of three dimensions, flatness is impossible for these multisymplectic structures due to the non-vanishing of the curvature of the bi-invariant Riemannian metric.

**Theorem (4.22).** Let \((G,\omega)\) be a real simple Lie group with its canonical bi-invariant non-degenerate three-form. Then \((G,\omega)\) has constant linear equivalence class but is flat if and only if the dimension of \( G \) is three.

\textsuperscript{1}Similar examples have been constructed in e.g. \cite{Van01}
In Section 5 we turn to the diffeomorphisms preserving a multisymplectic structure and measure their degree of transitivity on the underlying connected manifold.

**Definition (5.2).** Let $X$ be a set and $G \times X \to X, (g, p) \mapsto g(p)$ a group action. The action is called “$k$-transitive”, if for any two $k$-tuples $(p_1, \ldots, p_k)$, $(q_1, \ldots, q_k)$ of elements in $X$ satisfying $p_i \neq p_j$ and $q_i \neq q_j$ for $i \neq j$ there exists an element $g \in G$ such that $g(p_i) = q_i$ for $i = 1, \ldots, k$.

In Subsection 5.1 we recall the classical cases of symplectic forms (and their powers) and volume forms, where the diffeomorphisms act $k$-transitively for all $k$. We also explain, why the diffeomorphisms preserving the multisymplectic form $\omega = \Re(e^{dz_1 \wedge \cdots \wedge dz_n})$ on $\mathbb{C}^n$ act $k$-transitively for all $k$.

In Subsections 5.2 and 5.3 we present cases, where the multisymplectic diffeomorphisms act 1-transitively but do not act 2-transitively respectively do not act transitively at all. In case the linear type is constant these phenomena relate to the existence of foliations intrinsically defined by $\omega$. In the non-flat case, even more situations can arise, as illustrated by the following:

**Theorem** (Propositions 5.14 and 5.20). Let $M = \{x^2 > 0\} \subset \mathbb{R}^6$ and $\omega^f = dx^{135} - dx^{146} - dx^{236} + f(x^2) \cdot dx^{245} \in \Omega^3(M)$, where $f : \mathbb{R}^\geq 0 \to \mathbb{R}^\geq 0$ is smooth. Then $(M, \omega^f)$ is multisymplectic and of constant linear type. Furthermore,

(i) Let $f(x^2) = x^2$, then $(M, \omega^f)$ is non-flat and its multisymplectic diffeomorphism group acts transitively but not 2-transitively on $M$.

(ii) Let $f$ satisfy $f|_{[0,1]} = 1$ and $f|_{[2,\infty]}(t) = t$, then there are open subsets of $M$ where $\omega^f$ is flat resp. non-flat and therefore the group of multisymplectic diffeomorphisms can not act transitively on $M$.

Our analysis was prompted by determining all vector fields preserving $\omega^f$ for case (i), which we provide in Appendix B.

Finally, in Section 6, we return to the notion of conserved quantities of multisymplectic manifolds and how they can be generated by Hamiltonian Lie group (or Lie algebra) actions.

**Definition (6.3).** Let $(M, \omega)$ be a multisymplectic manifold and $v : \mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ an infinitesimal right action (i.e. a Lie algebra homomorphism). A “homotopy comoment” for $v$ is a Lie $\infty$-morphism $F = \{f_i\} : \mathfrak{g} \to L_\infty(M, \omega)$, such that $df_1(x) = -i_{v_x} \omega$. The action $v$ is called “Hamiltonian” if it admits a comoment.
After reviewing the notion of comoments in Subsection 6.1, we explain when the components \( \{ f_i \} \) of a comoment give rise to conserved quantities in Subsections 6.2 to 6.4. In Subsection 6.5 we prove the theorem following characterizing Hamiltonian actions.

**Theorem (6.30).** Let \( v : \mathfrak{g} \to \mathfrak{X}(M, \omega) \) be an infinitesimal action. There is a closed element \( g = g(\omega, v) \in \Lambda^* \mathfrak{g}^* \otimes \Omega^*(M) \) canonically associated to \( v \) and \( \omega \). The class of \( g \) vanishes in \( H^*(\mathfrak{g}) \otimes H_{dR}^*(M) \), if and only if \( v \) is Hamiltonian.

This result has already been proven in [Ryv16c], but a new and more conceptual proof has been developed in the course of the dissertation project (cf. also [FLGZ15] for a similar proof developed independently). Finally, using this theorem, we give a cohomological reformulation of Subsections 6.2 to 6.4 in Subsection 6.6:

**Theorem (6.42).** Let \( v : \mathfrak{g} \to \mathfrak{X}(M) \) be a Hamiltonian action with comoment \( \{ f_i \} \) on an \( n \)-plectic manifold \( (M, \omega) \) and \( H \) be an \( (n-1) \)-form fulfilling the Hamilton-DeDonder-Weyl equation \( \iota_{X_H} \omega = -dH \) for some vector field \( X_H \). Assume that the action is locally \( H \)-preserving, i.e. \( \mathcal{L}_{X_H} H \) is closed for all \( x \in \mathfrak{g} \).

(i) The form \( f_k(p) \) is locally conserved by \( X_H \) if \( p \in Z_k(\mathfrak{g}) \) and conserved if \( p \in B_k(\mathfrak{g}) \).

(ii) If the action is strictly \( H \)-preserving (i.e. \( \mathcal{L}_{X_H} H = 0 \) for all \( x \in \mathfrak{g} \)), \( f_k(p) \) is conserved by \( X_H \) for all \( p \in Z_k(\mathfrak{g}) \).
2 Observables of multisymplectic manifolds

In this section we introduce multisymplectic manifolds, their Lie $\infty$-algebras of observables and conserved quantities. We accompany our considerations by several examples to illustrate the aspects of multisymplectic geometry not caught by “symplectic intuition”. We remark that most constructions in Subsections 2.2 to 2.4 can be adapted to the degenerate setting without any difficulty.

2.1 Multisymplectic manifolds

The main body of this subsection consists of examples showing that interesting classes of multisymplectic manifolds abound.

Definition 2.1. A “multisymplectic manifold” is a pair $(M, \omega)$, where $M$ is a manifold, $k \geq 1$ and $\omega \in \Omega^{k+1}_{cl}(M)$ is a closed differential form satisfying the following non-degeneracy condition: The bundle map

$$\iota_v \omega : TM \to \Lambda^k T^* M, \ v \mapsto \iota_v \omega$$

is injective. For fixed degree $k+1$ of the form such manifolds are also called “$k$-plectic”. Such a form is sometimes simply called a “multisymplectic form” or a “multisymplectic structure”.

Example 2.2 (The classical cases).

- A symplectic manifold is, by definition, a 1-plectic manifold.
- An $n$-dimensional manifold equipped with a volume form is an $(n-1)$-plectic manifold.

Example 2.3 (Sums and products). As in the symplectic case, given two $k$-plectic manifolds $(M, \omega)$ and $(\tilde{M}, \tilde{\omega})$, there is a natural $k$-plectic structure $\pi^*_M \omega + \pi^*_\tilde{M} \tilde{\omega}$ on $M \times \tilde{M}$. Additionally $M \times \tilde{M}$ carries the structure given by $\pi^*_M \omega \wedge \pi^*_\tilde{M} \tilde{\omega}$, which is multisymplectic even when $\omega$ and $\tilde{\omega}$ have different degrees.

Example 2.4 (Multicotangent bundles). Let $Q$ be a manifold and $1 \leq n \leq \dim(Q)$. In generalization of the canonical symplectic structure on $T^* Q$ we will now construct an $n$-plectic structure on $M = \Lambda^n T^* Q$. We define $\theta \in \Omega^n(M)$ by

$$\theta_{\alpha}(v_1, ..., v_n) = (T_\alpha \pi)^* \alpha(v_1, ..., v_n) = \alpha(T_\alpha \pi(v_1), ..., T_\alpha \pi(v_n))$$
for \( \alpha \in M = \Lambda^n T^* Q, \) \( v_1, ..., v_n \in T_\alpha M \) and \( T\pi : T(\Lambda^n T^* Q) \to TQ \) the differential of the map \( \pi : \Lambda^n T^* Q \to Q. \) Setting \( \omega = -d\theta \) we turn \( (M, \omega) \) into an \( n \)-plectic manifold. To show non-degeneracy let us regard the situation in local coordinates:

Let \( x = (x^1, ..., x^m) : U \subset Q \to \mathbb{R}^m \) be a coordinate system with \( x^i \in C^\infty(U, \mathbb{R}). \) Then \( dx^1, ..., dx^m \in \Omega^1(U) = \Gamma(U, T^* U) \) form a basis of \( T^*_a U \) at any point \( a \in U. \) Thus we get a coordinate system for \( T^* U, \) where we define \( p_i : T^* U \to \mathbb{R} \) to be \( p_i(\alpha) = \alpha_i, \) by mapping \( \alpha = \sum_{i=1}^m \alpha_i dx^i|_a \in T^*_a Q \) to \( (x^1(\alpha), ..., x^m(\alpha), p_1(\alpha), ..., p_m(\alpha)). \) Accordingly for \( 1 \leq i_1 < ... < i_n \leq m \) the \( (dx^{i_1} \wedge ... \wedge dx^{i_n})|_a =: dx^{|}_a \) form a basis of \( \Lambda^n T^*_a U \) and we get a coordinate system for \( \Lambda^n T^* U \) by

\[
\alpha = \sum_i \alpha_i dx^i|_a \quad \mapsto \quad ((x^i(a)), (p_I(\alpha))_I) = ((x^i(\pi(\alpha))), (p_I(\alpha)))_I,
\]

where \( i \) runs from 1 to \( m, I \) runs through all strictly increasing multi-indices of length \( n \) and \( p_I(\alpha) = \alpha_I. \) Every single \( p_I \) is a map from \( \Lambda^n T^* U \) to \( \mathbb{R}. \)

With respect to these coordinates \( \theta_{(x^i, p_I)} = \sum_I p_I dx^i \) and consequently

\[
\omega_{(x^i, p_I)} = -\sum_I dp_I \wedge dx^i.
\]

To see non-degeneracy let \( v_0 = \sum_i a_i \frac{\partial}{\partial x^i} + \sum_I b_I \frac{\partial}{\partial p_I} \) be a non-zero tangent vector to \( M. \) If there exists an \( I = (i_1, ..., i_n) \) such that \( b_I \neq 0, \) then \( v_j = \frac{\partial}{\partial x^j} \) for \( j \in \{1, ..., n\} \) satisfy \( \omega(v_0, v_1, ..., v_n) = b_I \neq 0. \) If \( b_I \) is zero for all \( I, \) then there is at least one \( i \) such that \( a_i \neq 0. \) Without loss of generality we assume \( i = 1. \) Then for \( v_1 = \frac{\partial}{\partial p_{i_1 + ... + i_n}} \) and \( v_j = \frac{\partial}{\partial x^j} \) for \( j \in \{2, ..., n\} \) we have \( \omega(v_0, v_1, ..., v_n) = -a_1 \neq 0. \) Hence, \( t_{v_0} \omega \neq 0 \) for all nonzero \( v_0, \) i.e. \( \omega \) is non-degenerate.

**Example 2.5** (Subbundles of multicotangent bundles, [CuCI91]). Assume that in the setting above \( TQ \) has a subbundle \( V \) (of dimension \( d \)). Then for \( 0 < i < d \) we define the following submanifolds of \( M = \Lambda^n T^* Q. \)

\[
\Lambda^n T^* Q = \{ \alpha \in \Lambda^n T^* Q \mid t_{v_{i+1}} ... t_{v_i} \alpha = 0 \text{ for all } v_1, ..., v_{i+1} \in V \}.
\]

The multisymplectic form of \( \Lambda^n T^* Q \) restricted to these submanifolds is still multisymplectic for \( n - \text{dim}(Q) + d \leq i \leq n. \)

**Example 2.6** (Multicotangent bundles with magnetic terms, [CiLD99]). Let \( M = \Lambda^k T^* Q \to Q \) and \( \theta \) be as above. For any \( \gamma \in \Omega_{cl}^{n+1}(Q) \) the form \( \omega = -d\theta + \pi^* \gamma \) is multisymplectic. The new component \( \gamma \) is called “magnetic term”.

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Example 2.7 (Complex manifolds with holomorphic volumes). Let \((M, J)\) be a complex manifold of dimension \(m > 1\), interpreted as a \(2m\)-dimensional real manifold with an integrable almost-complex structure \(J\). Let \(\omega = \omega^R + i\omega^I \in \Omega^{m,0}(M) \subset \Omega^m(M) \otimes \mathbb{C}\) be a holomorphic volume form, i.e. \(\omega\) is a \(\mathbb{C}\)-valued smooth \(m\)-form, such that \(\iota_{J(v)}\omega = i\iota_v\omega\) and \(\partial\omega = \bar{\partial}\omega = 0\). Then \(\omega^R\) and \(\omega^I\) are multisymplectic structures on \(M\).

Example 2.8 (Semi-simple Lie groups). Let \(G\) be a real semi-simple Lie group. We construct a 2-plectic form on \(G\) using the following facts:

- The Lie bracket is \(Ad_g\)-equivariant for all \(g \in G\). As \(G\) is semi-simple, we have \([g, g] = g\).
- The (symmetric) Killing-form \(\langle \cdot, \cdot \rangle : g \times g \to \mathbb{R}\) is \(Ad_g\)-invariant for all \(g \in G\) and \(ad_X\) is a skew-adjoint linear map for all \(X \in g\). It is non-degenerate for semi-simple Lie groups.
- The Maurer-Cartan 1-form \(\theta^L \in \Omega^1(G, g)\) defined by \(\theta^L_g(u, v, w) = [\theta^L_g(u), [\theta^L_g(v), \theta^L_g(w)]]\) for all \(g \in G\) and \(u, v, w \in T_gG\). Non-degeneracy follows from \([g, g] = g\) and the non-degeneracy of the Killing form. The left-invariance of \(\theta^L\) implies that \(\omega\), too, is left-invariant. Using the description \(Ad_g = T(L_g) \circ T(R_{g^{-1}})\), the \(Ad_g\)-invariance of the Killing form and the Ad-equivariance of the Lie bracket one can also show that \(\omega\) is right-invariant. Any bi-invariant form on a Lie group is automatically closed, so \(\omega\) is in \(\Omega^3_{cl}(G)\) and non-degenerate and thus defines a 2-plectic structure on \(G\).

Example 2.9 \((G_2\)-structures\). A closed \(G_2\)-structure for a seven-dimensional manifold \(M\) is a closed differential 3-form \(\omega\), such that for all \(p \in M\), there exists a basis \(e^1, ..., e^7\) of \(T^*_pM\) such that

\[\omega_p = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356},\]

where \(e^{ijk}\) denotes \(e^i \wedge e^j \wedge e^k\). Especially, for a closed \(G_2\)-structure \(\omega\), the pair \((M, \omega)\) is a 2-plectic manifold.

Example 2.10 (Exact 2-plectic structure on \(S^6\)). We regard the standard closed \(G_2\)-structure on \(\mathbb{R}^7\), given by

\[
\omega = dx^{123} + dx^{145} - dx^{167} + dx^{246} + dx^{257} + dx^{347} - dx^{356},
\]
and pull it back to $S^6$ by the canonical inclusion $\rho: S^6 \to \mathbb{R}^7$. This form $\rho^*\omega$ is still closed, so for 2-plecticity we only need to verify its non-degeneracy. Since the linear action of $G_2$ on $\mathbb{R}^7$ preserves $\omega$ and restricts to a transitive action on $S^6$ (in fact $\text{Aut}_{\text{Lin}}(\mathbb{R}^7, \omega) = G_2$, cf. eg. [Bry06]), it suffices to show non-degeneracy at one point. We regard the point $p = (0, 0, 0, 0, 0, 0, 1) \in S^6 \subset \mathbb{R}^7$ and see

$$\rho^*\omega = (dx^{123} + dx^{145} + dx^{246} - dx^{356})_{|T_p S^6}.$$  

This form is non-degenerate, as one can see, e.g., by applying Theorem 4.16 from the next section or by direct verification. It follows that $(S^6, \rho^*\omega)$ is a 2-plectic manifold with a homogenous 2-plectic structure. As $H^3_{dR}(S^6) = 0$, $\rho^*\omega$ is exact.

**Remark 2.11.** A more general construction for generating multisymplectic manifolds is described in [MS12]. Their method recovers all homogenous strictly nearly Kähler 6-manifolds (especially $S^6$) as 2-plectic manifolds.

**Example 2.12** (Exact 3-plectic structure on $S^6$). Let $R$ be the radial vector field $\sum x^i \frac{\partial}{\partial x^i}$ on $\mathbb{R}^7$. The differential 2-form $\tau = \rho^*(i_R \omega)$ with $\omega$ as in Example 2.10 is non-degenerate and $G_2$-invariant. However it is not symplectic, in fact $d\tau = 3(\rho^*\omega)$ (as one can see upon using Section 4.1 of [Bry06]). We have $\tau_p = (-dx^{16} + dx^{25} + dx^{34})_{|T_p S^6}$, especially $\tau_p \wedge (\rho^*\omega)_p = 0$. As both $\tau$ and $\rho^*\omega$ are $G_2$-invariant it follows that

$$d(\tau \wedge \tau) = 2d\tau \wedge \tau = 6(\rho^*\omega) \wedge \tau = 0.$$  

Thus $(S^6, \tau \wedge \tau)$ is a 3-plectic manifold, with a $G_2$-homogenous 3-plectic form. As $H^4_{dR}(S^6) = 0$, this form is also exact.

As the above examples indicate, multisymplectic structures on closed manifolds do not, in general, give rise to non-trivial cohomology classes. This is part of a very general phenomenon. In many degrees, multisymplectic structures exist in all cohomology classes (especially in the zero class).

**Theorem 2.13** (Genericity, Theorem 2.2 of [Mar70]). For $n \geq 7$ and $3 \leq k \leq n - 2$ an $n$-dimensional manifold has a $(k-1)$-plectic structure in every class in $H^k_{dR}(M)$. For such degrees the non-degenerate forms are $C^1$-open and dense in the (closed) forms.

### 2.2 The Lie $\infty$-algebra of observables

One of the key features of a symplectic form $\omega$ on a manifold $M$, is the Lie algebra structure $\{\cdot, \cdot\}_\omega$ it induces on $C^\infty(M)$. The bracket of two functions
\( f_1, f_2 \) is defined by \( \{ f_1, f_2 \}_\omega = \omega(X_{f_1}, X_{f_2}) \), where \( X_{f_i} \) is the unique vector fields satisfying \( \iota_{X_{f_i}}\omega = -df \). Trying to generalize the equation defining \( X_{f_i} \) to \( n \)-plectic manifolds with \( n > 1 \), one has to either turn \( X_{f_i} \) into multivector fields or to concentrate on differential forms \( f_i \) of degree \( n-1 \). Following Baez and Rogers, we choose the latter approach here but observe new subtleties: In general, neither do all \( (n-1) \)-forms \( \alpha \) admit a vector field \( X_\alpha \) satisfying the “Hamilton-DeDonder-Weyl equation” \( \iota_{X_\alpha} \omega = -d\alpha \), nor do those admitting such a vector field form a Lie algebra. However, they do form a Lie \( \infty \)-algebra, cf. [Rog12, Ryv16c].

**Definition 2.14.** Let \((M, \omega)\) be an \( n \)-plectic manifold. We define the “Lie \( n \)-algebra of observables” \((L_\infty(M, \omega), \{ l_k \}_{k \in \{1, \ldots, n+1\}})\) as follows. As a graded vector space it is given by

\[
L_\infty(M, \omega) = \bigoplus_{i=0}^{n-2} \Omega^i(M) \oplus \Omega^{n-1}_{Ham}(M, \omega),
\]

where

\[
\Omega^{n-1}_{Ham}(M, \omega) = \{ \alpha \mid d\alpha = -\iota_{X_\alpha} \omega \text{ for some } X_\alpha \in \mathfrak{X}(M) \} \subset \Omega^{n-1}(M).
\]

We turn \( L_\infty(M, \omega) \) into a differential graded vector space with differential \( l_1 = d \) on \( \bigoplus_{i=0}^{n-2} \Omega^i(M) \) and differential \( l_1 = 0 \) on \( \Omega^{n-1}_{Ham}(M, \omega) \). Furthermore, for \( 1 < k \leq n+1 \), we introduce maps

\[
l_k : \Lambda^k \Omega^{n-1}_{Ham}(M, \omega) \to L_\infty(M, \omega),
\]

\[
l_k(\alpha_1, \ldots, \alpha_k) = -(-1)^{k(k+1)/2} \iota_{X_{\alpha_k}} \ldots \iota_{X_{\alpha_1}} \omega,
\]

where \( d\alpha_i = -\iota_{X_{\alpha_i}} \omega \). We extend them to operations \( \Lambda^k L_\infty(M, \omega) \to L_\infty(M, \omega) \) trivially (i.e. by zero). The vector field \( X_\alpha \) is called the “Hamiltonian vector field of \( \alpha \)”.

**Remark 2.15.** The operations \( \{ l_k \}_{k \in \{1, \ldots, n+1\}} \) satisfy the relations

\[
\delta l_k = l_1 l_{k+1},
\]

for \( 1 < k < n+2 \), where \( l_{n+2} \) should be interpreted as the zero map. These relations show that \((L_\infty(M, \omega), \{ l_k \}_{k \in \{1, \ldots, n+1\}})\) is a Lie \( \infty \)-algebra, cf. e.g. [Ryv16c]. Here \( \partial \) denotes the Chevalley-Eilenberg-operator given by

\[
(\partial l_k)(\alpha_1, \ldots, \alpha_{k+1}) = \sum_{i<j} (-1)^{i+j} l_k(l_2(\alpha_i, \alpha_j), \alpha_1, \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \alpha_{k+1}),
\]

where \( \hat{\alpha_i} \) means that \( \alpha_i \) is left out. This operator is defined for any skew-symmetric map with domain a Lie \( \infty \)-algebra (especially a Lie algebra).
Example 2.16 (Symplectic forms). Let \((M, \omega)\) be a \(1\)-plectic (i.e. symplectic) manifold. Then \(L_\infty(M, \omega) = C^\infty(M)\), \(l_1 = 0\) and \(l_2 = \{\cdot, \cdot\}\) is the classical Poisson multiplication of functions on a symplectic manifold.

Example 2.17 (Volumes). We regard \(\mathbb{R}^n\) with \(n \geq 3\) with the standard volume form \(\omega = dx^1 \wedge \cdots \wedge dx^n\) as an \((n-1)\)-plectic manifold and describe its \(l_2\) operation. Let \(\alpha, \tilde{\alpha}\) be \((n-2)\)-forms. They can be written as follows:

\[
\alpha = \sum_{i<j} f_{ij} \frac{\partial}{\partial x^i} \frac{a}{\partial x^j} \omega, \quad \tilde{\alpha} = \sum_{i<j} \tilde{f}_{ij} \frac{\partial}{\partial x^i} \frac{a}{\partial x^j} \omega.
\]

Hence, we have

\[
d\alpha = -\sum_{i<j} \left( \frac{\partial f_{ij}}{\partial x^i} \frac{a}{\partial x^j} \omega - \frac{\partial f_{ij}}{\partial x^j} \frac{a}{\partial x^i} \omega \right), \quad X_\alpha = \sum_{i<j} \left( \frac{\partial f_{ij}}{\partial x^i} \frac{\partial}{\partial x^j} \frac{a}{\partial x^i} \omega - \frac{\partial f_{ij}}{\partial x^j} \frac{\partial}{\partial x^i} \frac{a}{\partial x^i} \omega \right).
\]

Setting \(f_{ji} = -f_{ij}\) this can be rewritten to

\[
X_\alpha = \sum_j \left( \sum_{k \neq j} \frac{\partial f_{kj}}{\partial x^k} \right) \frac{\partial}{\partial x^j} \frac{a}{\partial x^j} \omega.
\]

Consequently, we have

\[
l_2(\alpha, \tilde{\alpha}) = \sum_{i<j} \left( \left( \sum_{k \neq j} \frac{\partial \tilde{f}_{kj}}{\partial x^k} \right) \left( \sum_{l \neq i} \frac{\partial f_{li}}{\partial x^l} \right) - \left( \sum_{l \neq i} \frac{\partial \tilde{f}_{li}}{\partial x^l} \right) \left( \sum_{k \neq j} \frac{\partial f_{kj}}{\partial x^k} \right) \right) \frac{a}{\partial x^i} \frac{a}{\partial x^j} \omega.
\]

As any volume form on an \(n\)-dimensional manifold is locally diffeomorphic to \(\omega\), (the binary operation of) the observable Lie \((n-1)\)-algebra of any \((n-1)\)-plectic \(n\)-dimensional manifold locally has the above form.

Example 2.18 (Sums). Let \((M, \omega)\) and \((\tilde{M}, \tilde{\omega})\) be \((n-1)\)-plectic. There is a (strict) morphism of Lie \(\infty\)-algebras

\[
L_\infty(M, \omega) \oplus L_\infty(\tilde{M}, \tilde{\omega}) \to L_\infty(M \times \tilde{M}, \pi^* M \omega + \pi^* \tilde{\omega}),
\]

given by

\[
(\alpha, \tilde{\alpha}) \mapsto \pi^*_M \alpha + \pi^*_\tilde{M} \tilde{\alpha}.
\]

Example 2.19 (Products). Given an \(n\)-plectic manifold \((M, \omega)\) and an \(m\)-plectic manifold \((\tilde{M}, \tilde{\omega})\) of not necessarily equal degrees, there is a morphism of Lie \(\infty\)-algebras

\[
L_\infty(M, \omega) \oplus L_\infty(\tilde{M}, \tilde{\omega}) \to L_\infty(M \times \tilde{M}, \pi^*_M \omega \wedge \pi^*_\tilde{M} \tilde{\omega}).
\]
constructed in [SZ16]. It is an extension of the bilinear map
\[
\Omega_{\text{Ham}}^{n-1}(M,\omega) \oplus \Omega_{\text{Ham}}^{m-1}(\tilde{M},\tilde{\omega}) \to \Omega_{\text{Ham}}^{n+m}(M \times \tilde{M}, \pi_M^* \omega \wedge \pi_{\tilde{M}}^* \tilde{\omega}),
\]
\[
(\alpha, \tilde{\alpha}) \mapsto \pi_M^* \alpha \wedge \pi_{\tilde{M}}^* \tilde{\alpha} + \pi_M^* \omega \wedge \pi_{\tilde{M}}^* \tilde{\alpha}.
\]
Unlike the previous case, in general this morphism has "higher" components of the type
\[
\Lambda^k \left( L_\infty(M,\omega) \oplus L_\infty(\tilde{M},\tilde{\omega}) \right) \to L_\infty(M \times \tilde{M}, \pi_M^* \omega \wedge \pi_{\tilde{M}}^* \tilde{\omega})
\]
also for \(k > 1\).

**Example 2.20** (Compact simple Lie groups). In the case of connected compact simple Lie groups, we can get a feeling for \(L_\infty(G,\omega)\) by regarding the Lie \(\infty\)-subalgebra of left-invariant differential forms:
\[
L_\infty(G,\omega)^G = C^\infty(M)^G \longrightarrow \Omega_{\text{Ham}}^1(G,\omega)^G.
\]

Identifying \(g^*\) with \(g\) by use of the Killing form, we can interpret the operations \(l_2\) and \(l_3\) as follows.
\[
l_2 : \Lambda^2 g \to g, \quad l_2(X,Y) = [X,Y],
\]
\[
l_3 : \Lambda^3 g \to \mathbb{R}, \quad l_3(X,Y,Z) = -\langle X, [Y,Z] \rangle.
\]
Thus, \(L_\infty(G,\omega)^G \cong (\mathbb{R} \xrightarrow{l_1} g, l_1 = 0, l_2 = [\cdot,\cdot], l_3 = -\langle \cdot, [\cdot,\cdot] \rangle)\).

**Example 2.21** (Abelian Lie \(\infty\)-algebra). In [CST13] a 2-plectic 7-manifold with no non-trivial Hamiltonian vector fields is constructed. Thus, \(\Omega_{\text{Ham}}^{n-1}(M,\omega) = \Omega_{\text{cl}}^{n-1}(M)\), \(l_2 = 0\) and \(l_3 = 0\).

### 2.3 Conserved quantities and their algebraic structure

In this subsection we define and compare various flavours of conserved quantities associated to a vector field. For Hamiltonian vector fields of multisymplectic manifolds we discover interesting relations to the Lie \(\infty\)-algebra of observables defined in the previous subsection.

**Definition 2.22.** Let \(M\) be a manifold and \(X\) a vector field on \(M\). A form \(\alpha \in \Omega^*(M)\) is called a
(i) “locally conserved quantity” if $\mathcal{L}_X \alpha$ is a closed form,

(ii) “(globally) conserved quantity” if $\mathcal{L}_X \alpha$ is an exact form,

(iii) “strictly conserved quantity” if $\mathcal{L}_X \alpha = 0$.

We denote the graded vector spaces of those quantities by $\mathcal{C}_{\text{loc}}(X)$ resp. $\mathcal{C}(X)$ and $\mathcal{C}_{\text{str}}(X)$.

**Remark 2.23.** In the sequel we will observe that condition (iii) is very restrictive, whereas condition (i) is often too weak.

The following inclusions are directly implied by Cartan’s formula for Lie derivatives ($\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega$).

**Lemma 2.24.** Let $M$ be a manifold and $X$ a vector field on $M$. Then we have

(i) $\mathcal{C}_{\text{str}}(X) \subset \mathcal{C}(X) \subset \mathcal{C}_{\text{loc}}(X)$,

(ii) $\Omega_{\text{cl}}^*(M) \subset \mathcal{C}(X)$,

(iii) $d(\mathcal{C}_{\text{loc}}(X)) \subset \mathcal{C}_{\text{str}}(X)$.

We will be especially interested in the case where $(M, \omega)$ is $n$-plectic and $X$ preserves $\omega$. In this case additional results hold.

**Lemma 2.25.** Let $(M, \omega)$ be an $n$-plectic manifold and $X$ a vector field on $M$ such that $\mathcal{L}_X \omega = 0$. Let $\alpha \in \Omega_{\text{Ham}}^{n-1}(M)$ and $X_\alpha$ its Hamiltonian vector field. Then we have

(i) $\alpha$ is locally conserved by $X$ if and only if $[X_\alpha, X] = 0$.

If moreover $X = X_H$ is the Hamiltonian vector field of $H \in \Omega_{\text{Ham}}^{n-1}(M)$, then

(ii) $\alpha$ is locally conserved by $X_H$ if and only if $\mathcal{L}_{X_H} H$ is closed.

(iii) $\alpha$ is globally conserved by $X_H$ if and only if $\mathcal{L}_{X_H} H$ is exact.

(iv) $H \in \mathcal{C}(X_H)$.

**Proof.** Assertion (i) follows from the identity $\mathcal{L}_X \circ \iota_Y = \iota_Y \circ \mathcal{L}_X + \iota_{[X,Y]}$ applied to $\omega$. Assertions (ii) - (iv) follow from Cartan’s formula. \qed

**Remark 2.26.** We observe that the closedness resp. exactness of $\mathcal{L}_{X_\alpha} H$ is equivalent to the closedness resp. exactness of $l_2(\alpha, H)$.
As the following example illustrates, in general, $\mathcal{L}_{X_H} H \neq 0$.

**Example 2.27.** Let $M = \mathbb{R}^3$, $\omega = dx \wedge dy \wedge dz$ and $H = xdy + zdz$. Then $X_H = -\frac{\partial}{\partial z}$, so $\iota_{X_H} H = -z$ and $\mathcal{L}_{X_H} H = -dz$.

**Remark 2.28.** In the symplectic (i.e. the 1-plectic) case with $H \in C^\infty(M) = \Omega^0_{\text{Ham}}(M)$ we have the following statements for $f \in C^\infty(M)$ and $X = X_H$:

(i) $f$ is “globally conserved” if and only if $f$ is “strictly conserved” and this is the case if and only if $\{H, f\} = 0$,

(ii) $f$ is “locally conserved” if and only if $\{H, f\}$ is locally constant.

Locally conserved quantities on symplectic manifolds were studied in some detail in [RW71]. However, as the following example shows, in the symplectic situation local conservedness does not suffice to formulate a “conservation law”.

**Example 2.29.** Let $M = \mathbb{R}^2$ with coordinates $q, p$, $\omega = dp \wedge dq$ and $H = p$. Taking $f = q$, the Hamiltonian vector field is given by $X_H = \frac{\partial}{\partial q}$ and thus $\iota_{X_H} f = 1$ i.e. $f$ is locally but not globally conserved. Then for any integral curve $\gamma(t) = (q_0 + (t - t_0), p_0)$ of $X_H$ we have $f(\gamma(t)) = f(\gamma(t_0)) + (t - t_0)$, i.e., $f$ is not a constant of motion.

The following proposition shows that for multicotangent bundles with exact magnetic terms (cf. Example 2.6) globally conserved quantities on the base manifold induce Hamiltonian forms on the multicotangent bundle.

**Proposition 2.30.** Let $n \geq 1$ and $Q$ be a manifold, $b \in \Omega^n(Q)$ and $X$ a vector field on $Q$, such that $\mathcal{L}_X b = da$ for some $a \in \Omega^{n-1}(M)$ (i.e. $b$ is globally conserved by $X$). Denote the canonical lift of $X$ to $M = \Lambda^n T^* Q$ by $X^h$. Then $X^h$ is a Hamiltonian vector field on $(M, \omega = -d\theta + \pi^*db)$, with the following Hamiltonian $(k-1)$-form:

$$H = -\pi^* a + \iota_{X^h} (-\theta + \pi^*b).$$

**Proof.** Upon observing $\iota_{X^h}(\pi^*b) = \pi^*(\iota_X b)$ and consequently $\mathcal{L}_{X^h}(\pi^*b) = \pi^*(\mathcal{L}_X b)$, we have:

$$dH = -\pi^* \mathcal{L}_X b + d(\iota_{X^h} \theta) + \pi^*(d\iota_X b)$$

$$= -\pi^* da + \iota_{X^h} d\theta - \mathcal{L}_{X^h} \theta - \pi^*(\iota_X db) + \pi^* \mathcal{L}_X b$$

$$= -\pi^* da + \iota_{X^h} d\theta - \pi^*(\iota_X db) + \pi^* da = -\iota_{X^h} (-d\theta + \pi^*db)$$

$$= -\iota_{X^h} \omega,$$

where in the third equality we used $\mathcal{L}_{X^h} \theta = 0$, as in the symplectic case. □
Now, we present elementary methods to construct new conserved quantities from known ones by wedge multiplication.

**Lemma 2.31.** The space $C_{str}(X)$ is a graded subalgebra of $\Omega^*(M)$.

As the following example illustrates, the spaces $C(X)$ and $C_{loc}(X)$, unlike $C_{str}(X)$, are not closed under wedge-multiplication.

**Example 2.32.** Let $M = \mathbb{R}^3$, $\omega = dx \wedge dy \wedge dz$ and $H = -xdy$. We observe that $dH = -dx \wedge dy$ and consequently $L_X H = \partial z$. We set $\alpha = zdx$ and $\beta = zdy$. Then $L_X (\alpha \wedge \beta) = 2zdx \wedge dy$ is not even closed.

However stability under multiplication with elements from the following graded-commutative subalgebra of $\Omega^*(M)$ is assured:

$$A(X) := \{ \beta \in \Omega(M) \mid d\beta = 0 \text{ and } L_X \beta = 0 \} \subset C_{str}(X).$$

**Lemma 2.33.** The spaces $C(X)$ and $C_{loc}(X)$ are graded modules over $A(X)$.

**Proof.** We prove the statement for $C(X)$, the proof for $C_{loc}(X)$ being identical.

Let $\alpha \in C(X)$ (that is, there is a form $\gamma$ with $L_X \alpha = d\gamma$) and $\beta \in A(X)$. Then

$$L_v (\alpha \wedge \beta) = L_X \alpha \wedge \beta + \alpha \wedge L_X \beta = d\gamma \wedge \beta = d(\gamma \wedge \beta).$$

Again, even more can be said if $(M, \omega)$ is $n$-plectic and $X$ preserves $\omega$.

**Theorem 2.34.** Let $(M, \omega)$ be $n$-plectic and $X$ a vector field satisfying $L_X \omega = 0$. The graded vector spaces

$$L_\infty(M, \omega) \cap C_{loc}(X), \ L_\infty(M, \omega) \cap C(X) \text{ and } L_\infty(M, \omega) \cap C_{str}(X)$$

are Lie $\infty$-subalgebras of $L_\infty(M, \omega)$. Moreover $L_X (l_k(\beta_1, ..., \beta_k)) = 0$ for $k \geq 1$ and $\beta_1, ..., \beta_k \in L_\infty(M, \omega) \cap C_{loc}(X)$.

**Proof.** We claim that brackets of locally conserved quantities in $L_\infty(M, \omega)$ are strictly conserved. The only bracket which is nontrivial on components other than $\Omega^{n-1}_{Ham}(M)$ is $l_1 = d$. It follows from part (iii) of Lemma 2.24 that $l_1 = d$ applied to a locally conserved quantity is strictly conserved. Now for $k \geq 2$ consider $\beta_1, ..., \beta_k \in \Omega^{n-1}_{Ham}(M)$, such that $L_X \beta_i$ is closed for all $i$. We want to show that

$$L_X (l_k(\beta_1, ..., \beta_k)) = 0.$$
As \( l_k(\beta_1, \ldots, \beta_k) = \pm t(X_{\beta_1} \wedge \ldots \wedge X_{\beta_k})\omega \), this is equivalent to showing
\[
\mathcal{L}_{X_{\beta_k}} \ldots t_{X_{\beta_1}} \omega = 0.
\]
Using the identity \( \mathcal{L}_X \circ \iota_Y = \iota_Y \circ \mathcal{L}_X + \iota_{[X,Y]} \) we can move \( \mathcal{L}_X \) past the \( t_{X_{\beta_k}} \) since \( t_{[X,X_{\beta}]} \omega = 0 \) by part (i) of Lemma 2.25. We find
\[
\mathcal{L}_{X_{\beta_k}} \ldots t_{X_{\beta_1}} \omega = t_{X_{\beta_k}} \ldots t_{X_{\beta_1}} \mathcal{L}_X \omega = 0,
\]
proving our claim.

2.4 Transgression of conserved quantities

In this subsection we show some geometric consequences of the existence of conserved quantities on a manifold \( M \) by looking at maps from a compact oriented manifold \( \Sigma \) into \( M \). In most of our statements \( M \) does not need any additional geometric structure, but we specialize to the \( n \)-plectic case e.g. in Proposition 2.43. First, we consider conserved quantities whose degree, as differential forms on \( M \), equals \( \dim(\Sigma) \). Afterwards, we extend some of the results to arbitrary degrees. We view \( \Sigma \) as a “membrane” in \( M \), which evolves under the flow of the vector field, and want to find quantities which are unchanged under the evolution. The following theorem, which can be considered to be folklore, can be viewed as a general version of Kelvin’s circulation theorem, as we explain in Remark 2.40 below.

**Theorem 2.35.** Let \( \Sigma \) be a compact, oriented \( d \)-dimensional manifold (possibly with boundary), \( X \) a vector field on \( M \) with flow \( \phi_t \), and \( \sigma_0 : \Sigma \to M \) a smooth map. Consider \( \sigma_t := \phi_t \circ \sigma_0 : \Sigma \to M \). If \( \alpha \in \Omega^d(M) \) is a differential form, then the number
\[
\int_{\Sigma} (\sigma_t)^* \alpha
\]
is independent of the time parameter \( t \) if one of the following conditions holds:

(i) \( \alpha \) is strictly conserved by \( X \),

(ii) \( \alpha \) is globally conserved by \( X \) and \( \Sigma \) has no boundary,

(iii) \( \alpha \) is locally conserved by \( X \) and there exists a compact, oriented manifold with boundary \( N \) such that \( \Sigma = \partial N \) and a map \( \tilde{\sigma}_0 : N \to M \) with \( \tilde{\sigma}_0|_{\partial N} = \sigma_0 \).

**Remark 2.36.** Since \( \Sigma \) is compact, there exists an \( \varepsilon = \varepsilon(\sigma_0) > 0 \) such that \( \phi_t \) is defined at least on \( (-\varepsilon, \varepsilon) \times \sigma_0(\Sigma) \subset \mathbb{R} \times M \). Obviously in (i) and (ii) we can consider \( |t| < \varepsilon = \varepsilon(\sigma_0) \). Mutatis mutandis we consider only \( |t| < \varepsilon(\tilde{\sigma}_0) \) in case (iii). Notice that if \( \dim(M) = d \), or more generally if \( \alpha \) is closed, then \( \alpha \) is globally conserved.
Proof. We only prove that condition (ii) suffices, for the other implications follow analogously. The diffeomorphisms $\phi_t$ satisfy $\frac{d}{dt}(\phi_t^*\alpha) = \phi_t^*(L_X\alpha)$. Pre-composing with the pullback $(\sigma_0)^*$ we obtain
\[
\frac{d}{dt}(\sigma_t^*\alpha) = \sigma_t^*(L_X\alpha).
\] (1)
Hence, by compactness of $\Sigma$,
\[
\frac{d}{dt} \int_{\Sigma} \sigma_t^*\alpha = \int_{\Sigma} \sigma_t^*(L_X\alpha) = \int_{\Sigma} d(\sigma_t^*\gamma) = 0,
\]
where in the first equality we used Equation (1), in the second that $L_X\alpha = d\gamma$ for some form $\gamma$, and in the last one Stokes’ theorem.

Remark 2.37. The sufficiency of Condition (ii) in Theorem 2.35 is not surprising. By assumption $X$ preserves $\alpha$ up to an exact form, and by Stokes’ theorem the contribution given by exact forms vanishes upon integration over $\Sigma$.

The following statement addresses a variation of condition (iii) in Theorem 2.35.

Proposition 2.38. Let $\Sigma$ be a compact, oriented manifold without boundary of dimension $d$, $X$ a vector field on $M$ with flow $\phi_t$. If $\alpha \in \Omega^d(M)$ is locally conserved then for every fixed time $t$ one obtains a well-defined map
\[
F_t: [\Sigma, M] \to \mathbb{R}, \quad [\sigma_0] \mapsto \int_{\Sigma} (\sigma_t)^*\alpha - \int_{\Sigma} (\sigma_0)^*\alpha.
\]
Here $[\Sigma, M]$ denotes the set of smooth homotopy classes of maps from $\Sigma$ to $M$, $\sigma_0: \Sigma \to M$ denotes a smooth map and $\sigma_t := \phi_t \circ \sigma_0: \Sigma \to M$.
Furthermore, the dependence on $t$ is linear: $F_t[\sigma_0] = t \cdot c([\sigma_0])$ where $c([\sigma_0]) := \int_{\Sigma}(\sigma_0)^*(L_v\alpha)$.

Proof. We have
\[
\int_{\Sigma} (\sigma_t)^*\alpha - \int_{\Sigma} (\sigma_0)^*\alpha = \int_0^t \left[ \frac{d}{ds} \int_{\Sigma} (\sigma_s)^*\alpha \right] ds = \int_0^t \left[ \int_{\Sigma} \sigma_s^*(L_v\alpha) \right] ds,
\]
where the last equality is obtained as in the proof of Theorem 2.35. Now recall that $L_v\alpha$ is a closed form on $M$. Hence, by Stokes’ theorem the term in the square bracket depends only on the homotopy class of $\sigma_s$, which agrees with the homotopy class of $\sigma_0$ since $\sigma_s = \phi_s \circ \sigma_0$. We conclude that the above expression equals $t \cdot c([\sigma_0])$. \qed
We present an example for Theorem 2.35 and Proposition 2.38.

**Example 2.39.** Let $M$ be a manifold, $X$ a vector field, and $\alpha \in \Omega^d(M)$. Take a map $\sigma_0: S^d \to M$ defined on the $d$-dimensional sphere, denote by $\sigma_i$ the composition of $\sigma_0$ with the time $t$ flow $\phi_t$ of $X$. The number $\int_{S^d}(\sigma_i)^*\alpha$ is independent of the time parameter $t$ if the following occurs: either i) $\mathcal{L}_X\alpha$ is exact, or ii) $\mathcal{L}_X\alpha$ is closed and $\sigma_0$ is homotopy equivalent to a constant map. This follows from Theorem 2.35 (ii) and (iii).

Further, assuming that $\mathcal{L}_X\alpha$ is closed, one obtains a well-defined group homomorphism

$$
\pi_d(M, x) \to \mathbb{R}, \quad [\sigma_0] \mapsto \int_{S^d}(\sigma_i)^*\alpha - \int_{S^d}(\sigma_0)^*\alpha
$$

defined on the $d$-th homotopy group of $M$ based at some point $x$, whose dependence on $t$ is linear. This follows from Proposition 2.38 and the following argument showing the group homomorphism property. We denote the group multiplication of $\pi_d(M, x)$ by $\ast$. It is given by the following composition, where $p$ denotes a distinguished point on the sphere:

$$
f \ast g : (S^d, p) \to (S^d / S^d - 1, p) \to (S^d \vee S^d, p) = (S^d \vee S^d, p) \to (M, x)
$$

Choosing appropriate representatives of the respective homotopy classes we may assume that $f, g$ and $f \ast g$ are smooth. Then for $\alpha \in \Omega^d(M)$ we calculate:

$$
\int_{S^d}(f \ast g)^*\alpha = \int_{S^d \setminus \{p\} \cup S^d \setminus \{p\}}(f \vee g)^*\alpha = \int_{S^d}f^*\alpha + \int_{S^d}g^*\alpha.
$$

**Remark 2.40 (Kelvin circulation theorem).** A variant of Theorem 2.35 (ii) for a time-dependent vector field $X^t$ and time-dependent differential form $\alpha^t \in \Omega^d(M)$ is the following: if $\mathcal{L}_{X^t}\alpha^t + \frac{d}{dt}\alpha^t$ is exact and $\Sigma$ has no boundary, the number

$$
\int_{\Sigma}(\sigma_i)^*(\alpha^t)
$$

is independent of the time parameter $t$.

We mention this because the Kelvin circulation theorem in fluid mechanics can be understood as a special case of the above. Let $X^t = \sum_i X^t_i \partial x_i$ be a time-dependent vector field on $\mathbb{R}^3$, and use the standard metric on $\mathbb{R}^3$ to obtain from $X^t$ the 1-form $\alpha^t = \sum_i X^t_i dx_i$. One computes $i_{X^t}d\alpha^t = \sum_{i,k} X^t_k \frac{\partial X^t_i}{\partial x_k} dx_i - \frac{1}{2} d \sum_i (X^t_i)^2$, so that $\mathcal{L}_{X^t}\alpha^t + \frac{d}{dt}\alpha^t$ is exact if and only if

$$
\sum_{i,k} X^t_k \frac{\partial X^t_i}{\partial x_k} dx_i + \sum_i \left( \frac{d}{dt} X^t_i \right) dx_i \quad (2)
$$

21
is exact. By the above, it then follows that \( \int_\Sigma (\sigma_t)^* \alpha^t \) is independent of \( t \).

Upon rewriting the exactness of (2) as \((X^t \cdot \nabla)X^t + \frac{\partial X^t}{\partial t} = -\nabla w\), with \( \nabla \) the usual gradient in \( \mathbb{R}^3 \), we recognize the first of the isentropic Euler equations (see e.g. [CM93, p. 15]). It is well-known that this equation implies the classical Kelvin circulation theorem ([CM93, p. 21]), which is exactly the time-independence of \( \int_\Sigma (\sigma_t^*)(\alpha^t) \) in this case.

Next, we will put the above results into the more general context of transition to mapping spaces. Let, from now on until the end of the subsection, \( \Sigma \) be without boundary.

Given a vector field \( X \) on \( M \), there is a naturally associated vector field \( X^\ell \) on \( M^\Sigma = C^\infty(\Sigma, M) \), the space of smooth maps from \( \Sigma \) to \( M \). It is given as follows:

\[
X^\ell|_\sigma = \sigma^* X \in \Gamma(\sigma^* TM) = T_{\sigma} M^\Sigma,
\]

for all \( \sigma \in M^\Sigma \). Notice that, denoting by \( \phi_t \) the flow of \( X \) on \( M \), the flow of \( X^\ell \) maps \( \sigma \in M^\Sigma \) to \( \phi_t \circ \sigma \in M^\Sigma \). Similarly, associated to a differential form on \( M \) there is a differential form on \( M^\Sigma \) of lower degree. It is defined by the transgression map

\[
\ell := \int_\Sigma \circ ev^* : \Omega^\bullet(M) \to \Omega^{\bullet-s}(M^\Sigma),
\]

where \( ev : \Sigma \times M^\Sigma \to M \) is the evaluation map and \( \int_\Sigma \) denotes the integration along the fiber (cf. eg. [Aud04, Cap. VI.4]) of the projection \( \Sigma \times M^\Sigma \to M^\Sigma \).

**Proposition 2.41.** Let \( \Sigma \) be a compact, oriented manifold (without boundary) of dimension \( d \) and let \( v \) be a vector field on \( M \). If \( \alpha \in \Omega^k(M) \) is globally (resp. locally resp. strictly) conserved by \( X \) then \( \alpha^\ell \in \Omega^{k-d}(M^\Sigma) \) is globally (resp. locally resp. strictly) conserved by \( X^\ell \).

**Proof.** The transgression map \( \ell \) commutes with de Rham differentials, and furthermore we have \((\iota_X \alpha)^\ell = \iota_{X^\ell} \alpha^\ell\). Therefore it commutes with Lie derivatives in the following sense: \( \mathcal{L}_{X^\ell} \alpha^\ell = (\mathcal{L}_X \alpha)^\ell \). Assume \( \alpha \) is globally conserved. We have to show that \( \mathcal{L}_{X^\ell} \alpha^\ell \) is an exact form. Since \( \alpha \in \Omega^k(M) \) is a globally conserved quantity, there is a \( \gamma \in \Omega^{k-1}(M) \) with \( \mathcal{L}_X \alpha = d\gamma \). Hence

\[
\mathcal{L}_{X^\ell} \alpha^\ell = (\mathcal{L}_X \alpha)^\ell = (d\gamma)^\ell = d\gamma^\ell.
\]

The other cases follow similarly.

**Remark 2.42.** When \( k = d \), Proposition 2.41 recovers Theorem 2.35 (ii). Indeed, let \( \alpha \in \Omega^d(M) \). By Proposition 2.41 the function \( \alpha^\ell \) on \( M^\Sigma \) is
invariant under the flow of $X^\ell$. The latter maps a point $\sigma \in M^\Sigma$ to $\phi_t \circ \sigma$, where $\phi_t$ denotes the flow of $X$ on $M$. Finally, for all $\sigma \in M^\Sigma$ we have

$$\alpha^{\ell}|_\sigma = (\int_{\Sigma} \circ ev^*(\alpha))|_\sigma = \int_{\Sigma} (ev|_{\Sigma \times \{\sigma\}})^* \alpha = \int_{\Sigma} \sigma^* \alpha,$$

where in the second equality we used that $ev|_{\Sigma \times \{\sigma\}} = \sigma$.

Now we specialize to an $n$-plectic manifold $(M, \omega)$ together with a vector field $X_H$ which is Hamiltonian for some $H \in \Omega^{n-1}_{\text{Ham}}(M)$. Notice that $(X_H)^\ell$ is a Hamiltonian vector field of $H^\ell$ on $(M^\Sigma, \omega^\ell)$, as follows from

$$\iota_{(X_H)^\ell} \omega^\ell = (\iota_{X_H} \omega)^\ell = (-dH)^\ell = -dH^\ell.$$

A special case of Proposition 2.41 reads:

**Proposition 2.43.** Consider an $n$-plectic manifold $(M, \omega)$ together with a vector field $X_H$ which is Hamiltonian for some $H \in \Omega^{n-1}_{\text{Ham}}(M)$. Let $\Sigma$ be a compact, oriented manifold (without boundary) of dimension $d$. If $\alpha \in \Omega^k(M)$ is a globally conserved quantity for $X_H$, then

$$\alpha^{\ell} \in \Omega^{k-d}(M^\Sigma)$$

is a globally conserved quantity for $(X_H)^\ell$, i.e. $\mathcal{L}_{(X_H)^\ell} \alpha^{\ell}$ is an exact form.
3 Linear types of multisymplectic manifolds

In this section we will discuss results concerning multi-linear forms on finite-dimensional real vector spaces.

Definition 3.1. A “$(k-1)$-plectic vector space” (over $\mathbb{R}$) is a pair $(V, \eta)$, where $V$ is a finite-dimensional $\mathbb{R}$-vector space and $\eta \in \Lambda^{k-1} V^*$ is non-degenerate, i.e. $\iota_v \eta : V \to \Lambda^{k-1} V^*$, $v \mapsto \iota_v \omega$ is injective. A “linear multisymplectomorphism” $L$ between $(V, \eta)$ and $(\tilde{V}, \tilde{\eta})$ is a linear isomorphism $L : V \to \tilde{V}$ satisfying $L^* \tilde{\eta} = \eta$. A “$(k-1)$-plectic linear type” is an isomorphism class of such pairs $(V, \eta)$.

Multisymplectomorphic vector spaces have equal dimensions, so we can ask: “How many $(k-1)$-plectic linear types are there in dimension $n$?” An answer is given by the following theorem:

Theorem 3.2. Let $\Sigma^k_n$ denote the number of $(k-1)$-plectic linear types in dimension $n$. Then we have

- $\Sigma^n_n = 1$ for all $n$, and $\Sigma^1_n$ as well as $\Sigma^{n-1}_n$ are zero for $n > 1$.
- $\Sigma^2_n$ is 0 for $n$ odd and one for $n$ even.
- $\Sigma^{n-2}_n = \lfloor \frac{n}{2} \rfloor - 1$, when $(n \mod 4) \neq 2$ (for $n \geq 4$) and $\Sigma^{n-2}_n = \frac{n}{2}$, when $(n \mod 4) = 2$ (for $n \geq 4$).
- $\Sigma^3_6 = 3$, $\Sigma^3_7 = 8$, $\Sigma^3_8 = 21$, $\Sigma^4_7 = 15$ and $\Sigma^5_8 = 31$.
- $\Sigma^k_n = \infty$ in all other cases.

Most cases have been settled in [Mar70]. Three-forms in dimensions six, seven and eight have been handled by [Cap72, Wes81, Djo83] and the remaining cases are solved in [Ryv16a]. In this section, we will go through this theorem case by case. For the sake of completeness we will reiterate most of the proof except for $\Sigma^3_7$ and $\Sigma^3_8$, for which we cite [Djo83]. Most of this section has been known, except for the cases $\Sigma^7_4$ and $\Sigma^5_5$, which we solved in the course of the dissertation project.

For dimensions up to 10 the numbers look as follows, where the rows range from 0-forms (the “$-$” in the table) to $n$-forms:
3.1 Prerequisites

Let $V$ be an $n$-dimensional real vector space and $V^*$ its linear dual. We denote their $k$-th exterior powers by $\Lambda^k V$ and $\Lambda^k V^*$. Let $GL(V)$ denote the group of all invertible linear maps from $V$ to $V$ and $\vartheta : G \times V \to V, \ v \mapsto g v$ its natural left-action. We regard the natural left-action of $GL(V)$ on $\Lambda^k V$ defined by

$$g_*(v_1 \wedge \ldots \wedge v_k) := g v_1 \wedge \ldots \wedge g v_k$$

on monomials $v_1 \wedge \ldots \wedge v_k$ and linearly extended to general elements of $\Lambda^k V$. Its dual right action $\Lambda^k V^* \times GL(V) \to \Lambda^k V^*$ is defined by $(\alpha, g) \mapsto g^* \alpha = \alpha \circ g_*$.  

First, we observe that the classifications of “multivectors” are equivalent for isomorphic vector spaces.
Lemma 3.3. Let $L : V \rightarrow W$ be a vector space isomorphism. Then $L^* : \Lambda^k V \rightarrow \Lambda^k W$ defined on monomials by $L^*(v_1 \wedge ... \wedge v_k) = L(v_1 \wedge ... \wedge Lv_k)$ is a vector space isomorphism and maps $GL(V)$-orbits bijectively to $GL(W)$-orbits.

Proof. The first statement follows from the fact that $L^{-1}$ is an inverse of $L^*$. For the second statement let $\eta \in \Lambda^k V$ and $g \in GL(V)$. Then $(L \circ g \circ L^{-1})^*(\eta) = (L \circ g \circ L^{-1})_* L^*(\eta)$. I.e., $L^*$ maps $GL(V)$-orbits to $GL(W)$-orbits, but $L^{-1}$ does the converse. Hence, the isomorphism $L^*$ preserves orbits. □

As $V$ and $V^*$ are isomorphic, this directly implies that the classification of multivectors over $V$ is equivalent to the classification of multivectors over $V^*$ i.e. alternating forms over $V$.

Corollary 3.4. Let $L : V \rightarrow V^*$ be an isomorphism. Then $L^* : \Lambda^k V \rightarrow \Lambda^k V^*$ induces a 1-to-1 correspondence between the orbits of the natural $GL(V)$-actions on $\Lambda^k V$ and $\Lambda^k V^*$.

Given a volume form $\Omega \in \Lambda^n V^* \setminus \{0\}$, we can construct a linear isomorphism $\iota_\Omega : \Lambda^k V \rightarrow \Lambda^{n-k} V^*$ given by $\eta \mapsto \iota_\eta \Omega$. Here, $\iota$ is the contraction operator $\Lambda^k V \times \Lambda^l V^* \rightarrow \Lambda^{l-k} V^*$, which is defined for $k \leq l$ by

$$(\iota_{v_1 \wedge ... \wedge v_k} \alpha)(w_1, ..., w_{l-k}) := \alpha(v_1, ..., v_k, w_1, ..., w_{l-k})$$

on monomials.

As it turns out, this linear isomorphism does not necessarily induce a bijection between the $\Lambda^k V$- and the $\Lambda^{n-k} V^*$-orbits. This is a consequence of the fact that $\mathbb{R}$ is not algebraically closed, so a multivector $\eta \in \Lambda^k V$ need not be in the same $GL(V)$-orbit as $(-\eta)$, when $k$ is even. (When $k$ is odd we have $(-id)_* \eta = -\eta$.) The following lemma helps specify for which $(n,k)$ this phenomenon may occur.

Lemma 3.5. Let $V$ be $n$-dimensional and $\Omega \in \Lambda^n V^* \setminus \{0\}$ a volume form. Then the linear isomorphism $L = \iota_\Omega$ satisfies:

$L(\text{GL}(V) \cdot \eta) \cup L(\text{GL}(V) \cdot (-\eta)) = \text{GL}(V) \cdot L(\eta) \cup \text{GL}(V) \cdot L(-\eta)$.

Proof. This Lemma is a direct consequence of the equation

$g^* (\iota_\eta \Omega) = \iota_{(g^{-1})^* \eta} g^* \Omega = \text{det}(g) \iota_{(g^{-1})^* \eta} \Omega$

and the fact that arbitrary roots of positive numbers exist in $\mathbb{R}$. □
This leads to the following

**Corollary 3.6.** In the setting above the following holds:

a) Let \( \eta \) be an element of \( \Lambda^k V \). Then the orbits of \( L(\eta) \) and \( -L(\eta) \) lie in the same orbit if and only if there exists a \( g \in GL(V) \) satisfying \( g \cdot \eta = -\det(g) \eta \). This is the case whenever the stabilizer of \( \eta \) contains an element of negative determinant.

b) When \( k \) is odd, \( L^{-1} \) induces an injection from \( GL(V) \)-orbits of \( \Lambda^{n-k}V^* \) to \( GL(V) \)-orbits of \( \Lambda^k V \).

c) When \( n-k \) is odd, \( L \) induces an injection from \( GL(V) \)-orbits of \( \Lambda^k V \) to \( GL(V) \)-orbits of \( \Lambda^{n-k}V^* \).

d) When \( k \) and \( n-k \) are odd \( L \) induces a bijection between the \( GL(V) \)-orbits of \( \Lambda^k V \) and \( \Lambda^{n-k}V^* \).

If \( i : \tilde{V} \to V \) is the inclusion of a vector subspace, then \( \Lambda^k \tilde{V} \) is naturally a vector subspace of \( \Lambda^k V \). When discussing multivectors or alternating forms, we will be primarily interested in orbits not arising from such vector subspaces:

**Definition 3.7.** Let \( \eta \in \Lambda^k V \) be a multivector. The “rank” of \( \eta \) is defined by

\[
\text{rank}(\eta) := \min\{\dim(\tilde{V}) \mid \eta \in \Lambda^k \tilde{V} \subset \Lambda^k V\}.
\]

The element \( \eta \) is of “full rank” if \( \text{rank}(\eta) = \dim(V) \). An element \( \alpha \in \Lambda^k V^* \) is called “non-degenerate” if it has full rank in the above sense, or equivalently, if the map \( \iota \alpha : V \to \Lambda^{k-1}V^* \) is injective. Otherwise the form \( \alpha \) will be called “degenerate”.

If \( \eta \) is an element of \( \Lambda^k \tilde{V} \), then \( g \cdot \eta \) is an element of \( \Lambda^k (g \cdot \tilde{V}) \), i.e. the rank is an invariant of the \( GL(V) \)-orbit. Moreover, the following holds:

**Lemma 3.8.** Let \( \text{rank}(\eta) \leq l \) and \( \tilde{V} \) be an \( l \)-dimensional subspace of \( V \). Then \( \Lambda^k \tilde{V} \cap GL(V) \cdot \eta \) is non-empty. Furthermore if \( \eta, \xi \in \Lambda^k \tilde{V} \subset \Lambda^k V \) lie in the same \( GL(V) \)-orbit, then they also lie in the same \( GL(V) \)-orbit.

**Proof.** The first statement is a consequence of the fact that \( GL(V) \) acts transitively on the space of \( l \)-dimensional hyperplanes in \( V \).

For the second statement, let us first assume \( \text{rank}(\xi) = l \). Let \( g \) be an element of \( GL(V) \) satisfying \( g \cdot \xi = \eta \). From \( \eta = g \cdot \xi \in \Lambda^k (g \cdot \tilde{V}) \) and \( \eta \in \Lambda^k \tilde{V} \), we can derive \( g \cdot \xi \in \Lambda^k (\tilde{V} \cap g \tilde{V}) \). As \( g \cdot \xi \) has rank \( l = \dim(\tilde{V}) \), we know that \( g \tilde{V} = \tilde{V} \). Hence, \( g \) induces an element \( \tilde{g} = g|_{\tilde{V}} \in GL(\tilde{V}) \), which satisfies
We will now reduce the case $\text{rank}(\xi) < l$ to the case $\text{rank}(\xi) = l$. Assume $\text{rank}(\xi) < l$. Then there is a $\text{rank}(\xi)$-dimensional subspace $W \subset \tilde{V}$ such that $\eta \in \Lambda^k W$. By the first statement of the theorem there exists an element $h \in GL(\tilde{V})$ such that $h_\ast \xi \in \Lambda^k W$. Then by the case $\text{rank}(\xi) = l$, we know that there exists an element $\hat{g} \in GL(W)$ satisfying $\hat{g}_\ast (h_\ast \xi) = \eta$. By extending $\hat{g} \in GL(W)$ to an element $g \in GL(\tilde{V})$, we obtain the element $(g \circ h) \in GL(\tilde{V})$ satisfying $(g \circ h)_\ast \xi = \eta$ as required.

This lemma directly yields the following corollary characterizing degenerate forms.

**Corollary 3.9.** Let $\tilde{V} \subset V$ be a vector subspace of codimension 1 and $s: \tilde{V}^* \to V^*$ any right-inverse of $i^*: V^* \to \tilde{V}^*$. Then the degenerate forms in $\Lambda^k V^*$ are given by $s_\ast (\Lambda^k \tilde{V}^*) \cdot GL(V) \subset \Lambda^k V^*$.

The next Lemma describes non-degeneracy in terms of the isomorphism from $\Lambda^k V$ to $\Lambda^{n-k} V^*$.

**Lemma 3.10.** Let $\Omega$ be a volume form and $\eta \in \Lambda^k V \setminus \{0\}$ for $k \geq 1$. Then $\alpha = \iota_v \Omega \in \Lambda^{n-k} V^*$ is non-degenerate if and only if $\eta = v \wedge \tilde{\eta}$ for some $\tilde{\eta} \in \Lambda^{k-1} V$.

**Proof.** If $\eta = v \wedge \tilde{\eta}$, then $\iota_v (\iota_v \eta) = 0$, so $\alpha$ is degenerate. Conversely If $\iota_v \alpha = 0$, then necessarily $v \wedge \eta = 0$, which in its turn implies $\eta = v \wedge \tilde{\eta}$.

Finally in this subsection, we will shortly discuss stability and its immediate consequences.

**Definition 3.11.** A multivector $\eta \in \Lambda^k V$ is called “stable” if its orbit $GL(V) \cdot \eta$ is an open subset of $\Lambda^k V$. The orbit is also called “stable” in this case.

Stability is preserved by vector space isomorphism, especially by the discussed isomorphisms between $\Lambda^k V$ and $\Lambda^k V^*$ resp. $\Lambda^{n-k} V^*$. Also, for any number $l$, the multivectors of maximal rank are open and dense in the space of all multivectors:

**Lemma 3.12.** Let $\omega \in \Lambda^k V^*$ be of maximal rank, i.e., for all $\beta \in \Lambda^k V^*$ the inequality $\text{rank}(\beta) \leq \text{rank}(\omega)$ holds. Then for $l = \text{rank}(\omega)$ the following holds:

The space $\{ \beta \in \Lambda^k V^* | \text{rank}(\beta) = l \}$ is an open and dense subset of $\Lambda^k V^*$.
Proof. We begin with the case, where \( l = n \), i.e. non-degenerate forms exist. We observe that the rank of an element \( \beta \in \Lambda^k V^* \) is equivalent to its rank interpreted as a linear map \( L_\beta = \iota_\beta : V \to \Lambda^{k-1} V^* \). We choose scalar products on \( V \) and \( \Lambda^{k-1} V^* \). Then \( \beta \) is non-degenerate if and only if \( \det(L_\beta \circ L_\beta^*) \neq 0 \). But \( P(\beta) = \det(L_\beta \circ L_\beta^*) \) is a polynomial, so the degenerate forms are the zero locus of a polynomial, which is non-constant because \( P(\omega) \neq 0 = P(0) \). This implies the claim for the non-degenerate case.

When \( l < n \) we can not use the determinant as an argument. The openness statement comes from the fact that the map \( \text{rank} : \text{Hom}(V, \Lambda^{k-1} V^*) \to \mathbb{R} \) is lower semi-continuous i.e. in a sufficiently small neighborhood of a form of rank \( l \) all forms will have rank \( \geq l \). The density statement is obtained as follows: Fix any form \( \beta \). Pick any \( l \)-dimensional vector space \( \tilde{V} \) such that \( \beta \in \Lambda^k \tilde{V}^* \), then apply the statement for the non-degenerate case.

3.2 The case \( k = n \)

Let \( V \) be an \( n \)-dimensional vector space for some \( n \geq 1 \). Then \( \Lambda^n V^* \) is a one-dimensional vector space and \( \Lambda^n V^* \times GL(V) \to \Lambda^n V^* \) is given by \( (\alpha, g) \to \det(g) \cdot \alpha \). Hence, the following lemma holds:

Lemma 3.13. Let \( \{e_1, \ldots, e_n\} \) be a basis of \( V^* \). Then

\[
\Lambda^n V^* = (e_1 \wedge \ldots \wedge e_n) \cdot GL(V) \\
\sqcup 0 \cdot GL(V).
\]

The orbit of \( e_1 \wedge \ldots \wedge e_n \) is stable and non-degenerate.

3.3 The cases \( k = 1 \) and \( k = n - 1 \)

The group \( GL(V) \) acts transitively on \( V \setminus \{0\} \), thus it acts transitively on \( V^* \setminus \{0\} \), so we obtain

Lemma 3.14. Let \( \{e_1, \ldots, e_n\} \) be a basis of \( V^* \). Then

\[
\Lambda^1 V^* = e^1 \cdot GL(V) \\
\sqcup 0 \cdot GL(V).
\]

The orbit of \( e^1 \) is stable. It is non-degenerate if and only if \( n = 1 \).

Proof. We can interpret \( e^1 \) as a map from \( V \) to \( \mathbb{R} \). If \( n > 1 \), it must necessarily have a nonzero kernel element \( v \), which then satisfies \( \iota_v e^1 = 0 \). If \( n = 1 \), then \( e^1 \) is a volume form. \[ \square \]

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Lemma 3.15. Let \( \{e^1, ..., e^n\} \) be a basis of \( V^* \). Then
\[
\Lambda^{n-1}V^* = e^1 \wedge ... \wedge e^{n-1} \cdot GL(V)
\]
\[
\square \quad 0 \cdot GL(V).
\]
The nonzero orbit is stable but degenerate.

Proof. By Corollary 3.6 and the above Lemma, we know that there are at most two orbits. The claims of the lemma then follow directly from the fact that \( 0 \cdot GL(V) = \{0\} \). \( \square \)

3.4 The cases \( k = 2 \) and \( k = n - 2 \)

In the case of 2-forms the statement is the well-known symplectic basis theorem:

Lemma 3.16 (Symplectic basis theorem). Let \( \{e^1, ..., e^n\} \) be a basis of \( V^* \). Then
\[
\Lambda^2V^* = \bigsqcup_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \sum_{i=1}^{k} e^{2i-1} \wedge e^{2i} \right) \cdot GL(V).
\]

Especially there are \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) orbits. If \( n \) is odd, no orbit is non-degenerate otherwise only the orbit of \( \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} e^{2i} \wedge e^{2i+1} \) is non-degenerate. In either case it is the unique stable orbit.

Proof. Let \( \alpha \in \Lambda^2V^* \) be given. Instead of finding an element \( g \in GL(V) \), which transforms \( \alpha \) into one of the orbits representatives, we will construct a basis \( \{v_1, ..., v_n\} \) such that \( \alpha \) already looks like one of the above forms with respect to the dual basis \( \{v^1, ..., v^n\} \). The statements then will follow, because \( GL(V) \) acts transitively on the space of bases of \( V \).

We will first assume \( \alpha \) is non-degenerate and proceed inductively. If \( V \) is zero-dimensional there is nothing to prove. So assume \( \text{dim}(V) > 0 \). Let \( v_n \neq 0 \). As \( \alpha \) is non-degenerate there exists some \( \tilde{v}_{n-1} \) in \( V \) such that \( \alpha(\tilde{v}_{n-1}, v_n) \neq 0 \). Then we set \( v_{n-1} = \frac{1}{\alpha(\tilde{v}_{n-1}, v_n)} \tilde{v}_{n-1} \). Next we regard the kernel \( \tilde{V} \) of the surjective linear map
\[
V \rightarrow \mathbb{R}^2, \quad v \mapsto \left( \frac{\alpha(v, v_{n-1})}{\alpha(v, v_n)} \right)
\]
By construction \( \tilde{V} \) is an \( n - 2 \)-dimensional vector space on which \( \alpha \) is non-degenerate, so we can repeat the above procedure until we arrive at a basis.
\{v_1, ..., v_n\} of V. Then \(\alpha = v^1 \land v^2 + ... + v^{n-1} \land v^n\) by construction. Especially \(n\) is even.

The basis for degenerate \(\alpha\) can be retrieved from the non-degenerate case together with the considerations of Lemma 3.8.

Now it remains to show that the orbit of \(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} e^{2i} \land e^{2i+1}\) is stable, but this follows directly from Lemma 3.12. \(\square\)

Knowing the situation for 2-forms (and thus for bivectors), we can now fix a volume form \(\Omega \in \Lambda^n V^* \setminus \{0\}\) and use the isomorphism induced by it.

**Lemma 3.17 ([Mar70]).** Let \(\text{dim}(V) = n > 4\) and let \(\{e_1, ..., e_n\}\) be a basis of \(V\). Let \(\{e^1, ..., e^n\}\) be its dual basis and \(\Omega = e^1 \land ... \land v^n\). If \(n \not\equiv 2 \mod 4\), then

\[
\Lambda^{n-2} V^* = \bigcup_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left( \iota_{\left(\sum_{i=1}^{k} e_{2i-1} \land e_{2i}\right)} \Omega \right) \cdot GL(V).
\]

If \(n \equiv 2 \mod 4\), then

\[
\Lambda^{n-2} V^* = \bigcup_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left( \iota_{\left(\sum_{i=1}^{k} e_{2i-1} \land e_{2i}\right)} \Omega \right) \cdot GL(V) \;
\bigcup \left( -\iota_{\left(\sum_{i=1}^{n/2} e_{2i-1} \land e_{2i}\right)} \Omega \right) \cdot GL(V).
\]

In both cases all orbits except \(0 \cdot GL(V)\) and \((\iota_{e_1 \land e_2} \Omega) \cdot GL(V)\) are non-degenerate. In the former case \(\iota_{\left(\sum_{i=1}^{n/2} e_{2i-1} \land e_{2i}\right)} \Omega \cdot GL(V)\) is the unique stable orbit, in the latter \(\left( \iota_{\left(\sum_{i=1}^{n/2} e_{2i-1} \land e_{2i}\right)} \Omega \right) \cdot GL(V)\) and \(\left( -\iota_{\left(\sum_{i=1}^{n/2} e_{2i-1} \land e_{2i}\right)} \Omega \right) \cdot GL(V)\) are stable.

**Proof.** We first observe that \(\iota_\eta \Omega\) is equivalent to \(-\iota_\eta \Omega\) for all \(\eta\) of non-full rank. This follows from Corollary 3.6 and the fact that the stabilizer of a bivector of non-full rank always includes an element of negative determinant. To see such a vector pick some subspace \(\tilde{V}\) of \(V\) such that \(V = \tilde{V} \oplus \text{span}(w)\) and \(\eta \in \Lambda^2 \tilde{V}\). Then the element \(g = \text{id}_{\tilde{V}} \oplus -\text{id}_{\text{span}(w)}\) is of negative determinant and stabilizes \(\eta\).

Bivectors of full rank appear only when \(n\) is even. Then \(\eta = \sum_{i=1}^{n/2} e_{2i-1} \land e_{2i} \land e_{2i+1} \land ... \land e_{2n}\).
When $n/2$ is even, we construct a linear isomorphism $g$ satisfying the condition from 3.6 a), as the following:

$$g(e_i) = \begin{cases} 
ei & \text{if } i \text{ even} \\ \nei + 1 & \text{if } i \text{ odd} \end{cases}$$

We observe that $g^*\eta = -\eta$ and $det(g) = 1$ because $n/2$ is even. Hence, when $n/2$ is even $\iota_{\eta}\Omega$ is in the same orbit as $-\iota_{\eta}\Omega$.

Let us now turn to the case $n/2$ is odd. Assume there exists a $g$ satisfying the equation $g^*\eta = -\det(g)\eta$. Then we regard the $n$-fold exterior power of $g^*\eta$:

$$det(g)\eta^{n/2} = (g^*\eta)^{n/2} = (-\det(g)\eta)^{n/2} = -\det(g)^{n/2}\eta^{n/2}$$

But $\det(g) = -\det(g)^{n/2}$ is a contradiction for $n/2$ odd. Thus no such $g$ can exist and the orbits of $\iota_{\eta}\Omega$ and $-\iota_{\eta}\Omega$ are distinct.

As stability is preserved by $\iota_{\eta}: \Lambda^2V \to \Lambda^{n-2}V^*$, what remains to show is the non-degeneracy statement. By Lemma 3.10 $\iota_{\eta}\Omega$ is non-degenerate if and only if $\eta = v \wedge \tilde{\eta}$ for some $v, \eta \in V$. If $v$ or $\tilde{\eta}$ is zero, then $\iota_{\eta}\Omega = 0$, otherwise it lies in the orbit of $\iota_{e_1 \wedge e_2}\Omega$.

\[\Box\]

3.5 The cases $3 \leq k \leq n-3$ for big $n$

If $3 \leq k \leq n-3$, then the dimension of $\Lambda^kV^*$ is usually much greater than the dimension of $GL(V)$, such that there are infinitely many orbits, however none of them are stable. The following lemma describes the situation in this case:

**Lemma 3.18.** Let $dim(V) = n > 8$ and $3 \leq k \leq n-3$ or $n = 8$ and $k = 4$. Then the natural $GL(V)$-action on $\Lambda^kV^*$ has infinitely many orbits, infinitely many non-degenerate orbits and no stable orbits. The aforementioned infinities are uncountable.

**Proof.** As in the given situation $dim(\Lambda^kV^*) = \binom{n}{k} > n^2 = dim(GL(V))$, there are infinitely many orbits and no orbit is open. To show that there are infinitely many non-degenerate forms, we will construct one non-degenerate form and then apply Lemma 3.12.

Given a non-degenerate $k$-form $\alpha$ on an $n-1$-dimensional vector space $\tilde{V}$, we can construct a non-degenerate $k+1$ form $\beta$ on the $n$-dimensional vector space $V = \tilde{v} \oplus span(w)$ by setting $\beta = \alpha \wedge w^*$, where $w^*$ is determined.
by $w^*(w) = 1$ and $w|\tilde{V} = 0$. Thus, for our claim it suffices to build non-degenerate 3-forms on vector spaces of dimension $\geq 5$.

Given a non-degenerate 3-form $\alpha$ on $\tilde{V}$, we can construct a non-degenerate 3-form on $V = \tilde{V} \oplus \mathbb{R}^3$, by $\beta = \alpha + \Omega_{\mathbb{R}^3}$, where $\Omega_{\mathbb{R}^3}$ is the standard volume of $\mathbb{R}^3$. So it suffices to show that non-degenerate 3-forms exist in dimensions 5, 6 and 7.

In dimension 5 and 7, this can be seen as a consequence of the existence of non-degenerate 2-forms in dimensions 4 and 6. In dimension 6 it can be seen as a consequence of the fact that non-degenerate 3-forms exist in dimension 3.

3.6 The case $(n, k) = (6, 3)$

In this subsection we are going to describe the $GL(V)$-orbits of 3-forms in a 6-dimensional vector space $V$, following the argumentation of [Bry06].

By the results so far we can already deduce the situation for the degenerate orbits, which are equivalent to orbits of three-forms in a 5-dimensional vector space by Corollary 3.9. Thus we have:

**Lemma 3.19.** Let $\{e^1, ..., e^6\}$ be a basis of $V^*$. Then the set degenerate forms $(\Lambda^3 V^*)_{\text{deg}}$ fulfills

$$(\Lambda^3 V^*)_{\text{deg}} = (e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5) \cdot GL(V)$$

$\square \quad e^1 \wedge e^2 \wedge e^3 \cdot GL(V)$

$\square \quad 0 \cdot GL(V).$

To classify the non-degenerate forms, we will introduce an endomorphism naturally associated them.

**Definition 3.20.** Let $(V, \Omega)$ be a 6-dimensional vector space equipped with a volume form and $L = \iota_* \Omega : V \rightarrow \Lambda^5 V^*$. Then We define

$$J : \Lambda^3 V^* \rightarrow \text{End}(V), \ \alpha \mapsto J_\alpha$$

by the formula $J_\alpha(v) = L^{-1}(\iota_* \alpha \wedge \alpha)$.

The next lemma describes the behavior of $J_\alpha$, when we alter $\alpha$ by an element of $GL(V)$.
Lemma 3.21. Let \((V, \Omega)\) and \(\alpha\) be as above. Then
\[ J_{g^*\alpha} = \det(g) \cdot g \circ J_\alpha \circ g^{-1} \]

Proof.
\[
L(J_{g^*\alpha}(v)) = \iota_{J_{g^*\alpha}(v)} \Omega = (\iota_{g^*\alpha}) \wedge g^* \alpha \\
= g^*((\iota_{g^*\alpha}) \wedge \alpha) \\
= g^* \iota_{J_\alpha(gv)} \Omega \\
= \iota_{g^{-1}J_\alpha gv} g^* \Omega \\
= \iota_{g^{-1}J_\alpha gv \det(g)} \Omega \\
= \iota_{\det(g)g^{-1}J_\alpha gv} \Omega = L(\det(g) \cdot g^{-1} \circ J_\alpha \circ g(v))
\]
As \(L\) is an isomorphism this proves the claim.

Lemma 3.22. With the setting as above \(\text{trace}(J_\alpha) = 0\) holds for all \(\alpha\). Moreover
\[ J^2_\alpha = \frac{1}{6} \text{trace}(J^2_\alpha) \cdot \text{id}_V. \]

Proof. Let \(\{e_1, \ldots, e_6\}\) be a basis of \(V\) and \(\{e^1, \ldots, e^6\}\) be a basis of \(V^*\), such that \(\Omega = e^1 \wedge \ldots \wedge e^6\). As \(\alpha \mapsto \text{trace}(J_\alpha)\) is a polynomial in \(\Lambda^3 V^*\), so it will be 0 if we find an open set on which it vanishes. If \(\text{trace}(J_\bullet)\) vanishes on \(\alpha\), it will also vanish on \(\alpha \cdot GL(V)\), by the above Lemma. So, it suffices to find a stable form \(\alpha\) on which \(\text{trace}(J_\bullet)\) vanishes. We claim that \(\alpha_{(i)} = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6\) is stable and that \(\text{trace}(J_{\alpha_{(i)}}) = 0\).

First we calculate \(J_{\alpha_{(i)}}\) with respect to \(\{e_1, \ldots, e_6\}\). It takes the form:
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}.
\]
So its trace is zero. We will now show that its orbit is open. For this we will show that its stabilizer is 16-dimensional, which implies that its orbit is \(36 - 16 = 20 = \binom{6}{3}\)-dimensional. We regard the space
\[ C(\alpha_{(i)}) = \{v \in V | \iota_v \alpha_{(i)} \text{ has rank 2} \} = \text{span}(e_1, e_2, e_3) \cup \text{span}(e_4, e_5, e_6). \]
The stabilizer of $\alpha(i)$ preserves this space. Let $g \in \text{Stab}(\alpha(i)) \subset GL(V)$. Being linear, it either preserves or permutes $V_1 = \text{span}(e_1, e_2, e_3)$ and $V_2 = \text{span}(e_4, e_5, e_6)$. Thus the stabilizer of $\alpha(i)$ is a subgroup of

$$(GL(V_1) \times GL(V_2)) \rtimes \mathbb{Z}/2\mathbb{Z},$$

where the $\mathbb{Z}/2\mathbb{Z}$-action is generated by

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

Let us regard the stabilizer subgroup also preserving the subspaces, i.e. it is a subgroup of $GL(V_1) \times GL(V_2)$. It must also preserve the form restricted to the subspaces. But on both of them, $\alpha(i)$ is a volume form. Hence, we get:

$$\text{Stab}(\alpha(i)) = SL(V_1, \alpha(i)|_{V_1}) \times SL(V_2, \alpha(i)|_{V_2}) \rtimes \mathbb{Z}/2\mathbb{Z}$$

This group is 16-dimensional, which finishes the proof of the first claim.

As the map $\alpha \mapsto P(\alpha) = J_\alpha^2 - \frac{1}{6}\text{trace}(J_\alpha^2) \cdot \text{id}_V$ is polynomial, it again suffices to show that $P(\alpha) = 0$ for all $\alpha \in \alpha(i) \cdot GL(V)$. We calculate

$$P(g^*\alpha) = \det^2(g) \cdot g \circ P(\alpha) \circ g^{-1}$$

Hence, $P(\alpha_i) = 0$ finishes the proof of the Lemma.

**Remark 3.23.** In the course of the proof, we have computed the stabilizer of $\alpha(i)$. It is isomorphic to

$$(SL_3(\mathbb{R}) \times SL_3(\mathbb{R})) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Now we will treat the cases $\text{trace}(J_\alpha^2) > 0$, $\text{trace}(J_\alpha^2) < 0$ and $\text{trace}(J_\alpha^2) = 0$ separately.

**Lemma 3.24.** Assume $\Omega$ and $\alpha \in \Lambda^3 V^*$ are given. If $\frac{1}{6}\text{trace}(J_\alpha^2) = \lambda^2$ with $\lambda > 0$, then there exists a basis $\{e^1, ..., e^6\}$ of $V^*$ such that

$$\alpha = \alpha(i) = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6.$$
Proof. As $J^2_α = \lambda^2 \cdot \text{id}_V$ is a scalar multiple of the identity, $J_α$ is diagonalizable with eigenvalues $\pm \lambda$. The identity $\text{trace}(J_α) = 0$ implies that $\text{dim}(E(J_α, \lambda)) = \text{dim}(E(J_α, -\lambda)) = 3$. We will now show that $\alpha = \alpha|_{E(J_α, \lambda)} + \alpha|_{E(J_α, -\lambda)}$, then the statement will follow from Lemma 3.13.

Let us take $v^+ \in E(J_α, \lambda)$ and $v^- \in E(J_α, -\lambda)$. We have to show that $t_{v^-} - t_{v^+} \alpha = 0$. For that, we consider $\lambda t_{v^-} - t_{v^+} = t_{v^-} - t_{J_α(v^+)} = t_{v^-} + t_{J_α(v^-)}$.

Subtracting the equations and dividing by 2 we get

$$(t_{v^-} + t_{J_α(v^-)}) \alpha = (t_{v^-} - t_{v^+} \alpha) \wedge (t_{v^-} + t_{J_α(v^-)}) \alpha = (t_{v^-} - t_{v^+} \alpha) \wedge (t_{v^-} + t_{a^-} \alpha + t_{a^-} \alpha + t_{v^-} \alpha)$$

If $\kappa = t_{v^+} - t_{v^-} \alpha \in V^*$ is nonzero, then this equation implies $\alpha = \kappa \wedge \beta$ for some non-degenerate $\beta \in \Lambda^2(\ker(\kappa))$. As $\ker(\kappa)$ is 5-dimensional no such $\beta$ can exist. Thus $t_{v^-} - t_{v^+} \alpha = 0$ and the Lemma holds.

Lemma 3.25. Assume $\Omega$ and $\alpha \in \Lambda^3 V^*$ are given. If $\frac{1}{3} \text{trace}(J^2_α) = -\lambda^2$ with $\lambda > 0$, then there exists a basis $\{e^1, \ldots, e^6\}$ of $V^*$ such that

$\alpha = \alpha_{(ii)} = e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5$.

Proof. As $J^2_α = -\lambda^2 \cdot \text{id}_V$ is a scalar multiple of the identity, $J_α$ interpreted as an endomorphism of $V \otimes \mathbb{C}$ is diagonalizable with eigenvalues $\pm i\lambda$. As $J_α$ is a real endomorphism, we have $\overline{E(J_α, i\lambda)} = E(J_α, -i\lambda)$. As in Lemma 3.24 we can show that $\alpha^C \in \Lambda^3 V^* \otimes \mathbb{C}$ satisfies

$$\alpha^C = \alpha^C|_{E(J_α, i\lambda)} + \alpha^C|_{E(J_α, -i\lambda)},$$

where $\alpha^C|_{E(J_α, -i\lambda)} = \overline{\alpha^C|_{E(J_α, i\lambda)}}$. Now $\alpha^C|_{E(J_α, i\lambda)}$ is a complex nonzero 3-form on a 3-dimensional complex vector space, so by the complex analogue of 3.13 there exists a basis $\{w^1, w^2, w^3\}$ of $E(J_α, i\lambda)^*$ such that $\alpha^C|_{E(J_α, i\lambda)} = \frac{1}{2} w^1 \wedge w^2 \wedge w^3$. This leads us to $\alpha = \Re(w^1 \wedge w^2 \wedge w^3)$. By picking a basis of $V$ such that $w^1 = e^1 + ie^2$, $w^2 = e^3 + ie^4$ and $w^3 = e^5 + ie^6$, we arrive at the desired result.

Remark 3.26. The proof of the above lemma implicitly tells us how to interpret forms with negative trace($J^2_α$): They are the real part of a volume form in complex 3-space. Also the proof enables us to calculate the stabilizer of $\alpha_{(ii)} = \Re(w^1 \wedge w^2 \wedge w^3) = e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5$. 37
As before, the form $\alpha$ either preserves or interchanges the eigenspaces of $J_\alpha$. If it preserves them, then it especially preserves $\alpha^\mathbb{C}|_{E(J_\alpha, i\lambda)}$. Hence, the stabilizer of $\alpha_{(ii)}$. It is isomorphic to $SL(3, \mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$.

**Lemma 3.27.** Assume $\Omega$ and $\alpha \in \Lambda^3 V^*$ are given. If $\frac{1}{2}\text{trace}(J^2_\alpha) = 0$. Then either $\alpha$ is degenerate, or there exists a basis $\{e^1, ..., e^6\}$ of $V^*$ such that

$$\alpha = \alpha_{(iii)} = e^1 \wedge e^5 \wedge e^6 - e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^4 \wedge e^5.$$

**Proof.** As $J^2_\alpha = 0$, we know that $\text{Image}(J_\alpha) \subset \ker(J_\alpha)$. Thus $\dim(\text{Image}(J_\alpha)) + \dim(\ker(J_\alpha)) = 6$ implies $\dim(\ker(J_\alpha)) \geq 3$. For kernel elements $v_1, v_2$, the identity $v_i \alpha \wedge \alpha = 0$ implies the equations

$$(t_{v_1} t_{v_2} \alpha) \wedge \alpha + t_{v_2} \alpha \wedge t_{v_1} \alpha = 0,$$

$$(t_{v_2} t_{v_1} \alpha) \wedge \alpha + t_{v_1} \alpha \wedge t_{v_2} \alpha = 0.$$

As the left term is skew-symmetric, whereas the right term is symmetric in $v_1$ and $v_2$, these imply

$$(t_{v_1} t_{v_2} \alpha) \wedge \alpha = t_{v_2} \alpha \wedge t_{v_1} \alpha = 0.$$

The equality $(t_{v_1} t_{v_2} \alpha) \wedge \alpha$ implies $(t_{v_1} t_{v_2} \alpha) = 0$, because otherwise we would have $\alpha = (t_{v_1} t_{v_2} \alpha) \wedge \beta$ for some non-degenerate $\beta$ in $\Lambda^2(\ker(t_{v_1} t_{v_2} \alpha))$ and such a $\beta$ cannot exist, as $\ker(t_{v_1} t_{v_2} \alpha)$ is 5-dimensional. Hence, $\alpha$ lies in the space $\Lambda^2(\text{Ann}(\ker(J_\alpha))) \wedge V^*$, where the annihilator of $U \subset V$ is defined as $\text{Ann}(U) = \{\phi \mid \phi|_U = 0\} \subset V^*$. For $\Lambda^2(\text{Ann}(\ker(J_\alpha)))$ to be non-zero, the dimension of $\ker(J_\alpha)$ must be less or equal to 4. If it is equal to 4, the dimension of $\Lambda^2(\text{Ann}(\ker(J_\alpha)))$ is one, with some generator $\gamma \wedge \delta$. But then $\alpha = \gamma \wedge \delta \wedge \eta$ for some $\eta \in V^*$, which contradicts the non-degeneracy of $\alpha$. Thus $\dim(\ker(J_\alpha)) = 3$.

Taking any basis $\{e^4, e^5, e^6\}$ of $\text{Ann}(\ker(J_\alpha))$ we can find $e^1, e^2, e^3 \in V^*$ and a constant $c \in \mathbb{R}$ such that

$$\alpha = e^1 \wedge e^5 \wedge e^6 - e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^4 \wedge e^5 + c e^4 \wedge e^5 \wedge e^6.$$

If $c \neq 0$, we arrive at the desired statement by substituting $e^1$ by $e^1 + ce^4$. \qed

**Remark 3.28.** As before, the proof implicitly tells us what the stabilizer of $\alpha_{(iii)} = e^1 \wedge e^5 \wedge e^6 - e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^4 \wedge e^5$ is. For simplicity we will calculate it in terms of $\text{Stab}(\alpha_{(iii)}) \subset \text{GL}(V^*)$. It is clear that $g \in \text{Stab}(\alpha_{(iii)})$
has to preserve $\text{Ann}(\ker(J_\alpha)) = \text{span}\{e^4, e^5, e^6\}$. On the other hand any transformation of those can be part of a stabilizer element. Hence, there is an exact sequence

$$0 \to \text{Stab}(\alpha_{(iii)}, e^4, e^5, e^6) \to \text{Stab}(\alpha_{(iii)}) \to \text{GL}(\text{span}(e^4, e^5, e^6)) \to 0$$

Now the leftmost group fixes $e^4, e^5$ and $e^6$. However, it includes the transformations

$$e^1 \mapsto e^1 + \lambda_{14}e^4 + \lambda_{15}e^5 + \lambda_{16}e^6$$
$$e^2 \mapsto e^2 + \lambda_{24}e^4 + \lambda_{25}e^5 + \lambda_{26}e^6$$
$$e^3 \mapsto e^3 + \lambda_{34}e^4 + \lambda_{35}e^5 + \lambda_{36}e^6,$$

where $\{\lambda_{ij}\}$ are any real numbers such that $\lambda_{14} + \lambda_{25} - \lambda_{36} = 0$. So the stabilizer of $\alpha_{(iii)}$ is 17-dimensional and isomorphic to

$$\mathbb{R}^8 \rtimes \text{GL}(3, \mathbb{R}).$$

**Remark 3.29.** The form $\alpha_{(iii)}$ can be interpreted as the canonical 3-form on $W \oplus \Lambda^2 W^*$, where $W$ is a 3-dimensional vector space. Especially, it is the linear type of an instance of Example 2.4.

All of the above can be summarized in the following theorem:

**Theorem 3.30 ([Cap72, Bry06]).** Let $\{e^1, ..., e^6\}$ be a basis of $V^*$. Then we have

$$\Lambda^3 V^* = (e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6) \cdot \text{GL}(V)$$
$$\cup (e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5) \cdot \text{GL}(V)$$
$$\cup (e^1 \wedge e^5 \wedge e^6 - e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^4 \wedge e^5) \cdot \text{GL}(V)$$
$$\cup (e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5) \cdot \text{GL}(V)$$
$$\cup e^1 \wedge e^2 \wedge e^3 \cdot \text{GL}(V)$$
$$\cup 0 \cdot \text{GL}(V),$$

where the first two orbits are open, and the first three orbits are non-degenerate.

**Remark 3.31.** We provide the source code to a SageMath function determining the linear type of a real 3-form in dimension 6 in Appendix C.2.
3.7 The cases \((n, k) = (7, 3)\) and \((7, 4)\)

In this subsection we are going to explore the \(GL(V)\)-orbits of 3-forms and 4-forms over a seven-dimensional vector space \(V\).

From the above discussion, we already know that there are 6 degenerate orbits of 3-forms in seven-dimensional space. The following proposition settles the non-degenerate cases.

**Proposition 3.32** ([Wes81, LPV08]). Let \(V\) be a seven-dimensional vector space. Then there are 8 non-degenerate \(GL(V)\)-orbits of \(\Lambda^3 V^*\). Two of these orbits are stable. Representatives of the orbits are listed in Appendix A.1.

Hence, we have the following.

**Theorem 3.33** ([Wes81, LPV08]). Let \(\dim(V) = 7\). Then the natural \(GL(V)\)-action on \(\Lambda^3 V^*\) has 14 orbits, eight of which are non-degenerate. Two of the non-degenerate orbits are stable. Representatives of the degenerate orbits are given by the list from Theorem 3.30 and the non-degenerate ones by Appendix A.1, the stable ones are 5) and 8) in this enumeration.

**Theorem 3.34.** Let \(\dim(V) = 7\). Then the natural \(GL(V)\)-action on \(\Lambda^4 V^*\) has 20 orbits, 16 of which are non-degenerate. Four of the non-degenerate orbits are stable.

**Proof.** We fix a basis \(\{e_1, ..., e_7\}\) of \(V\). We denote its dual basis by \(\{e^1, ..., e^7\}\) and the volume \(\Omega = e^1 \wedge ... \wedge e^8\). Let \(L\) be the isomorphism \(\Lambda^3 V \rightarrow \Lambda^3 V^*\) determined by \(e_i \mapsto e^i, \ i \in \{1, ..., 7\}\).

By Corollary 3.6 and Corollary 3.4, we know that any 4-form \(\beta \in \Lambda^4 V^*\) is in the orbit of \(i_L(\alpha) \Omega\) or \(-i_L(\alpha) \Omega\) where \(\alpha\) is one of the 14 representatives from Theorem 3.33. Thus, to determine the total number of orbits, we just have to find out whether \(i_L(\alpha) \Omega\) and \(-i_L(\alpha) \Omega\) lie in the same orbit. This is the case if and only if the stabilizer of \(L(\alpha)\) (or equivalently the stabilizer of \(\alpha\)) contains an element of negative determinant. This holds because \(g^*(i_L(\alpha) \Omega) = -i_L(\alpha) \Omega\) is equivalent to

\[
-det(g) \left( (g^{-1})^* L(\alpha) \right) = L(\alpha),
\]

I.e., if \(g\) maps \(i_L(\alpha) \Omega\) to \(-i_L(\alpha) \Omega\), then \(h = -\sqrt{-det(g) \circ g^{-1}}\) is an element of the stabilizer of \(L(\alpha)\) with negative determinant. On the other hand any element \(h\) of the stabilizer of \(L(\alpha)\) with negative determinant gives rise to \(g = \sqrt{-det[h]} h\) which satisfies \(g^*(i_L(\alpha) \Omega) = -i_L(\alpha) \Omega\).

We know that the stabilizers of the degenerate 3-forms contain elements of negative determinant and by Appendix A.1 we know that only two of the
stabilizers of non-degenerate 3-forms contain elements of negative determinant. Hence, there are $2 \cdot 6 + 8 = 20$ orbits of 4-forms in seven-dimensional space.

As the stabilizers of the stable 3-forms are connected, each of them has two corresponding 4-forms, so there are 4 stable orbits of 4-forms in seven-dimensional space.

What remains to calculate is the number of non-degenerate 4-forms. By Corollary 3.9 and Lemma 3.17 there are five degenerate orbits, i.e. $20 - 5 = 15$ non-degenerate ones.

3.8 The cases $(n, k) = (8, 3)$ and $(8, 5)$

In this subsection we are going to explore the $GL(V)$-orbits of 3-forms and 5-forms over a eight-dimensional vector space $V$.

From the above discussion, we already know that there are 14 degenerate orbits of 3-forms in eight-dimensional space. The following Proposition settles the non-degenerate cases.

**Proposition 3.35 ([Djo83, LPV08]).** Let $V$ be a eight-dimensional vector space. Then there are 21 non-degenerate $GL(V)$-orbits of $\Lambda^3 V^*$. Three of these orbits are stable. Representatives of the orbits are listed in Appendix A.2.

Hence, for 3-forms in eight-dimensional space we have

**Theorem 3.36 ([Djo83, LPV08]).** Let $\dim(V) = 8$. Then the natural $GL(V)$-action on $\Lambda^3 V^*$ has 35 orbits, 21 of which are non-degenerate. Three of the non-degenerate orbits are stable. Representatives of the degenerate orbits are given by the representatives from Theorem 3.30 and the non-degenerate ones by Appendix A.2, the stable ones are 19),20) and 21) in this enumeration.

**Theorem 3.37.** Let $\dim(V) = 8$. Then the natural $GL(V)$-action on $\Lambda^5 V^*$ has 35 orbits, 31 of which are non-degenerate. Three of the non-degenerate orbits are stable.

**Proof.** By Corollary 3.6 the orbits of 5-forms are in one-to-one correspondence with the orbits of 3-forms, thus by Theorem 3.37, there are 35 orbits. Stability is preserved by this correspondence, hence there are 3 stable orbits. By Corollary 3.9 and Lemma 3.17 there are four degenerate orbits, i.e. $35 - 4 = 31$ non-degenerate ones. 

$\square$
4 Darboux type theorems

A very important tool in symplectic geometry is the Darboux theorem stating that, given a point \( p \) in a symplectic manifold \( (M, \omega) \), there exist local coordinates \( (x^1, ..., x^{2m}) \) near \( p \) such that in these coordinates \( \omega = dx^1 \wedge dx^2 + ... + dx^{2m-1} \wedge dx^{2m} \). The existence of such coordinates relies on the two facts that all \( 2m \)-dimensional symplectic vector spaces are linearly isomorphic and that locally a symplectic manifold \( (M, \omega) \) is diffeomorphic to the linear symplectic manifold \( (T_p M, \omega_p) \) for any \( p \in M \). Neither of these results pertain to a general multisymplectic manifold \( (M, \omega) \) and “flatness” of such a manifold, i.e. the existence of local coordinates such that \( (M, \omega) \) can locally be identified with \( (T_p M, \omega_p) \) for \( p \in M \), turns out to be a rather special situation. We considered the former problem in Section 3 and will focus on the latter in this section. We will begin by recalling the advantageous cases of symplectic and volume forms and will explain why their proofs fail to work in the general situation. After reporting on the case of certain “multicotangent type manifolds” in Subsection 4.2, we give new results in Subsections 4.3 and 4.4. More precisely, we give necessary and sufficient conditions for flatness of multisymplectic manifolds of product type respectively \((m-1)\)-plectic complex \(m\)-manifolds. In Subsection 4.5 we recall the three possible cases for flat 2-plectic manifolds in dimension six before giving an elementary construction of 2-plectic forms on \(\mathbb{R}^6\). Using this construction we show that the linear type can change in a multisymplectic manifold, as well as that flatness may fail even when the linear type is constant throughout the manifold. In the last subsection, 4.6, we show that the canonical 2-plectic structure on a real simple Lie group is not flat unless the dimension of the Lie group is three, though the linear type of these 2-plectic structures is constant. The result is rather natural, but we were not aware of a proof of it in the literature.

4.1 Symplectic and volume forms

The next few subsections are motivated by these two classical theorems:

**Theorem 4.1 ([Wei71, Arn89]).** Let \((M, \omega)\) be a 1-plectic (i.e. symplectic) manifold of dimension \(n = 2m\) and \(p \in M\). Then there exists a chart near \(p\) \(M \supset U \xrightarrow{\phi} \mathbb{R}^{2m}\) such that
\[
\omega = \phi^*(dx^1 \wedge dx^2 + ... + dx^{2m-1} \wedge dx^{2m}).
\]

**Theorem 4.2 ([Mos65]).** Let \(n \geq 2\) and \((M, \omega)\) be an \((n-1)\)-plectic manifold of dimension \(n\) (i.e. a manifold with a volume form), and \(p \in M\). Then there
exists a chart near $p \; M \supset U \to \mathbb{R}^n$ such that
\[ \omega = \phi^*(dx^1 \wedge \ldots \wedge dx^n). \]

Each of these theorems can be decomposed into two statements:

(i) Any symplectic form (resp. volume form) has the linear type of $dx^1 \wedge dx^2 + \ldots + dx^{2m-1} \wedge dx^{2m}$ (resp. $dx^1 \wedge \ldots \wedge dx^n$).

(ii) Around any point $p$ the symplectic resp. $(n-1)$-plectic manifold $(M, \omega)$ is locally isomorphic to $(T_p M, \omega_p)$. (i.e. around $p$ there exists a chart $\phi : M \supset U \to T_p M$ such that $\phi(p) = 0$ and $\phi^* \omega_p = \omega$.)

As we have seen in the last section, there is no hope for (i) to hold $k$-plectically for $k$ other than $1, n-1$. In the sequel we will investigate conditions for (ii) to hold. Let us define this property as follows:

**Definition 4.3.** A multisymplectic manifold $(M, \omega)$ is called “flat near $p$” for $p \in M$, if there exists a chart $\phi : U \to T_p M$ such that $\phi(p) = 0$ and $\phi^* \omega_p = \omega$. It is called “flat” if it is flat near all $p$. Of course, $\omega_p$ is here interpreted as a constant-coefficient differential form on the manifold $T_p M$.

We will discuss the proof of Theorems 4.1 and 4.2 via the “Moser trick” and explain its failure in the general multisymplectic setting. For this proof, we introduce the following

**Definition 4.4.** Let $(M, \omega_0)$ and $(M, \omega_1)$ be multisymplectic forms of degree $k$. We call them isotopic if there exists a (smooth) family of multisymplectic forms $\omega_t$ interpolating between them, such that $[\omega_t] \in H^k_{dR}(M)$ is constant.

By restricting to an open neighborhood around $p \in M$ we can always achieve the latter condition. Especially:

**Example 4.5.** If $\omega_1$ is any multisymplectic form on a contractible $0 \in U \subset \mathbb{R}^n$, then we can define $\omega_0$ as the constant form whose value coincides with $\omega_1$ in some $p \in U$. Then we can define the family $\omega_t = t \omega_1 - (1-t) \omega_0$. As $(\omega_t)_p = (\omega_0)_p$ and because non-degeneracy is an open condition, we can achieve non-degeneracy of $\omega_t$ by shrinking $U$.

Now we describe the construction of the Moser trick for isotopic forms: As $[\omega_t]$ is constant (in de Rham cohomology), the family of classes $[\frac{d\omega_t}{dt}]$ is zero, i.e. $\frac{d\omega_t}{dt}$ is exact for all $t$. We choose a family of potentials $\alpha_t \in \Omega^{k-1}(M)$ of $-\frac{d\omega_t}{dt}$. If there exists a (smooth) family of vector fields $X_t$ satisfying $\iota_{X_t} \omega_t = \alpha_t$, then the flow $\phi^t$ of $X_t$, satisfies:

\[ \frac{d}{dt} (\phi^t)^* \omega_t = (\phi^t)^* \mathcal{L}_{X_t} \omega_t + (\phi^t)^* \frac{d\omega_t}{dt} = (\phi^t)^* \left( d\iota_{X_t} \omega_t + \frac{d\omega_t}{dt} \right) = 0, \]
i.e. it is an isotopy between $\omega_0$ and $\omega_1$. Especially $(M,\omega_1)$ and $(M,\omega_2)$ are multisymplectomorphic. In the case of Example 4.5, this implies that if $X_t$ exists, then $\omega$ is flat. Knowing this, the proof of Theorems 4.1 and 4.2 is a consequence of the following remark, as the existence of $X_t$ is always guaranteed for two-forms and volume forms.

**Remark 4.6.** If $\omega \in \Lambda^2(\mathbb{R}^n)^*$ is non-degenerate then $\iota_{\bullet}\omega : \mathbb{R}^n \to (\mathbb{R}^n)^*$ is a bijection. Also if $\omega \in \Lambda^n(\mathbb{R}^n)^*$ is non-degenerate then $\iota_{\bullet}\omega : \mathbb{R}^n \to \Lambda^{n-1}(\mathbb{R}^n)^*$ is a bijection.

The failure of the above remark to be true for alternating forms is why the Moser trick can not be directly applied to general multisymplectic manifolds.

### 4.2 Multicotangent bundles

In this subsection we recall the situation for multisymplectic manifolds, whose linear types correspond to that of a multicotangent bundle $(\Lambda^n T^*Y, \omega = -d\theta)$ from Example 2.4.

**Definition 4.7.** A real $n$-plectic vector space $(V, \omega)$ is called “standard” if there exists a linear subspace $W \subset V$ such that $\forall u, v \in W$, $\iota_u \wedge v \omega = 0$ and

$$\omega^w : W \to \Lambda^n(V/W)^*, \quad w \mapsto ((v_1 + W, ..., v_n + W) \mapsto (v_1, ..., v_n))$$

is an isomorphism.

**Remark 4.8.** In the above situation $W$ is unique if $n \geq 2$ and then often denoted $W_\omega$.

From [Mar88] the following result can easily be derived:

**Theorem 4.9.** Let $n \geq 2$ and $(M, \omega)$ be a standard $n$-plectic manifold, i.e. $(M, \omega)$ has as constant linear type a fixed standard $n$-plectic vector space. Then $W_\omega = \bigcup_{p \in M} W_{\omega_p} \subset \bigcup_{p \in M} T_p M = TM$ is a smooth distribution. Furthermore, $(M, \omega)$ is flat if and only if $W_\omega$ is integrable.

A variant of this theorem, which takes into account magnetic terms (cf. Example 2.6) has been proven in [CidL99]. Furthermore, we can show that the subbundles of $\Lambda^n T^*Q$ associated to a distribution are flat, when the distribution is involutive.

**Remark 4.10** (cf. also [FG13]). In the setting of Example 2.5, let $V$ be involutive, we can find a chart $x^1, ..., x^n$ of $Q$ such that $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$ span $V$. We can define the fiber coordinates $p_I$ of $\Lambda^n T^*Q$ accordingly where now $I$ runs.
through all multi-indices which contain at most \( i \) of the elements \( \{1, \ldots, d\} \). The conditions on \( i \) guarantee that such multi-indices exist. With respect to these coordinates, \( \omega \) again takes the following constant-coefficient form

\[
\sum dp_i dx^i.
\]

### 4.3 Multisymplectic manifolds of product type

In this subsection we study the local normal form for multisymplectic structures having as (constant) linear type the sum of \( k \) \( m \)-dimensional vector spaces, each supplied with a volume form. It turns out that flatness arises exactly if all elements in a certain intrinsically defined collection of \( m \)-forms are closed.

**Theorem 4.11.** Let \( k \geq 2 \), \( m > 2 \) and \( U \subset \mathbb{R}^{km} \) be open and \( \omega \in \Omega^m_{cl}(U) \) be of linear type \( dx_1, dx_2, \ldots, dx_m + dx_{m+1}, \ldots, dx_{2m} + \ldots + dx_{(k-1)m+1}, \ldots, dx_{km} \). Then there is a decomposition \( \omega = \omega_1 + \ldots + \omega_k \), where \( \omega_1, \ldots, \omega_k \in \Omega^m(U) \) such that \( \text{rank}(\omega_i) = m \). The forms \( \omega_i \) are unique up to permutation.

Furthermore, \( (U, \omega) \) is flat if and only if \( d\omega_i = 0 \) for all \( i \in \{1, \ldots, k\} \).

The condition \( \text{rank}(\omega_i) = m \) guarantees that \( (\omega_i)_p \) is decomposable for all \( p \), i.e. a wedge product of one-forms. For the proof we need the following lemma:

**Lemma 4.12.** Let \( V = \mathbb{R}^{km} \) where \( k \geq 2 \) and \( m > 2 \) and \( \{e_1, \ldots, e_{km}\} \) dual to the standard basis \( \{e_1, \ldots, e_{km}\} \) of \( \mathbb{R}^{km} \). Let \( \alpha \in \Lambda^mV^* \) be given by \( \omega = e_1, e_2, \ldots, e_m + e_{m+1}, \ldots, e_{2m} + \ldots + e_{(k-1)m+1}, \ldots, e_{km} \). Then, up to permutation, the forms \( \omega_i = e^{(i-1)m+1}, \ldots, e^{im} \) are the unique decomposable forms satisfying \( \omega = \sum_{i=1}^n \omega_i \).

**Proof.** Let \( \{ \hat{\omega}_i \} \) be an alternative collection of decomposable forms with the above property, which are no permutation of \( \{\omega_i\} \). We define the subspaces \( \tilde{E}_i = \{v \in V \mid \iota_v \hat{\omega}_j = 0 \ \forall j \neq i\} \). Since we have by construction \( V = \bigoplus \tilde{E}_i \), the projections \( \tilde{\pi}_i : V \to \tilde{E}_i \subset V \) are well-defined. We can reconstruct \( \{ \hat{\omega}_i \} \) from \( \{ \tilde{E}_i \} \) by setting \( \hat{\omega}_i = \tilde{\pi}_i^* (\omega|_{\tilde{E}_i}) \). Hence, as \( \{ \omega_i \} \) is no permutation of \( \{\omega_i\} \), \( \{ \tilde{E}_i \} \) is no permutation of \( \{E_i\} \) (defined correspondingly). I.e., there exists a vector \( v \in E_i \), which does not lie in a single \( \tilde{E}_j \). As \( v_i \in E_i \), \( \iota_v \omega = \iota_v \omega_i \) is decomposable. However, \( \iota_v \omega = \iota_v (\sum \hat{\omega}_i) \) has several nonzero summands, i.e. is not decomposable, which yields a contradiction. Hence, any collection \( \{ \hat{\omega}_i \} \) of decomposable forms summing up to \( \omega \) is a permutation of \( \{\omega_i\} \). □
Proof of the Theorem. By the preceding lemma we know that forms $\omega_i$ exist pointwise. To prove their smoothness, we begin with showing that the distributions $E_i$, defined in Lemma 4.12, are smooth, i.e. subbundles. Assume $U$ to be open and contractible. Then there is a canonical isomorphism $TU = U \times \mathbb{R}^{km}$. We consider $\omega$ as a map $U \to \Lambda^m(\mathbb{R}^{km})$. As $\omega$ is of constant linear type, it maps into

$$\eta \cdot GL(\mathbb{R}^{km}) = (e^{1,2,\ldots,m} + e^{m+1,\ldots,2m} + \ldots + e^{(k-1)m+1,\ldots,km}) \cdot GL(\mathbb{R}^{km}) \subset \Lambda^m(\mathbb{R}^{km})^*.$$ 

By the above lemma, the stabilizer of $\eta$ is isomorphic to $S_k \ltimes SL(\mathbb{R}^d)^k$, where $S_k$ is the permutation group of $k$ elements. We regard the following diagram:

$$U \xrightarrow{\omega} \eta \cdot GL(\mathbb{R}^{km}) \xrightarrow{\pi} GL(\mathbb{R}^{km}) \cong Gr_m(\mathbb{R}^{km}),$$

where $\pi$ is induced by the inclusion of $SL(\mathbb{R}^m)^{k-1}$ into $GL(\mathbb{R}^{(k-1)m})$ and $Gr_m(\mathbb{R}^{km})$ is the Grassmann manifold of all $m$-dimensional vector subspaces of $\mathbb{R}^{km}$. The map $\pi_\sigma$ is a $k!$-fold covering and $U$ is contractible, so the horizontal map admits $k!$ sections. We choose one section for each orbit of $S_{k-1}$, the stabilizer of $\{1\}$ of the $S_k$-action on $\{1,\ldots,k\}$, acting on $GL(\mathbb{R}^{km})/SL(\mathbb{R}^{km})^k$ and denote them as $s_1,\ldots,s_k$. Composed with $\pi$, we get $k$ smooth maps $\pi \circ s_i : U \to Gr_d(\mathbb{R}^{km})$. By the definition of $Gr_m(\mathbb{R}^{km})$ they yield $k$ smooth subbundles $E_i$ of $TU$, which correspond pointwise to the $E_i$ of the above lemma. Thus the elements $\omega_i = \omega|_{E_i}$ are smooth.

Obviously, if $\omega$ is flat, then the $\omega_i$ are closed. Conversely assume all $\omega_i$ are closed. Then the $(k-1)m$-forms $\Omega_i = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_i \wedge \ldots \wedge \omega_k$ are also closed. Consequently the subbundles $E_i = \{v|_E \Omega_i = 0 \}$ are involutive and hence integrable. Also, for any $I \subset \{1,\ldots,k\}$ the sums $\bigoplus_{i \in I} E_i$ are integrable by the same argument. Especially $E'_i = \bigoplus_{j \neq i} E_j$ is integrable. Thus for any $p \in U$ there exist open sets $U_i \subset M$ containing $p$ and submersions $\phi_i : U_i \to \Phi_i(U_i) \subset \text{open } \mathbb{R}^m$, satisfying $\ker(D\phi_i) = E_i'|_{U_i}$. Then automatically $D\phi_i|_{E_i} : E_i|_{U_i} \to T\mathbb{R}^m$ is injective and thus there exists an open neighborhood $V \subset \bigcap U_i$ of $p$ on which $\Phi = (\Phi_1,\ldots,\Phi_k) : V \to (\mathbb{R}^m)^k$ is a diffeomorphism onto its image, i.e. a chart. We know that the pullbacks $(\Phi^{-1})^* \omega_i$ are closed and of the form

$$f_i dx^{\text{(i-1)m+1}} \wedge \ldots \wedge dx^m,$$

so $f_i$ only depends on $x^{\text{(i-1)m+1}},\ldots,x^m$. The theorem then follows from applying the Darboux theorem for volume forms to the $(\Phi^{-1})^* \omega_i$. (For a similar statement proven differently cf. also [ZM13].) 

\[ \square \]
4.4 \((m-1)\)-plectic complex \(m\)-manifolds

We consider here, for \(m > 2\), \((2m)\)-dimensional real manifolds with an \((m-1)\)-plectic structure having as (constant) linear type the real part of a complex volume form, and show that such multisymplectic manifolds are flat if and only if a certain associated almost-complex structure is integrable.

**Theorem 4.13.** Let \(m > 2\) and \(U \subset \mathbb{R}^{2m}\) be open and \(\omega \in \Omega^m_m(U)\) be of linear type \(\text{Re}((dx^1 + idx^2) \wedge \cdots \wedge (dx^{2m-1} + idx^{2m}))\). Then, up to sign, there is a unique almost-complex structure \(J\) such that the following equality holds for all \(p \in U\) and \(v, w \in T_p U\):

\[
\iota_{J(v)} \iota_w \omega = \iota_{\iota_w J(v)} \omega \tag{3}
\]

Furthermore, \((U, \omega)\) is flat if and only if \(J\) is integrable.

For the proof we need the following lemma from [Van08]:

**Lemma 4.14.** Let \(m > 2\) and \(J\) a linear complex structure on the \(2m\)-dimensional real vector space \(V\). Let \(\omega = \omega^R + i \omega^I \in \Lambda^{m,0} V^*\) be non-zero. Then

\[
\mathcal{A}_{\omega^R} = \{ A \in \text{End}_{\mathbb{R}}(V) | \iota_A \iota_v \omega^R = \iota_{\iota_v A} \omega^R \} = \mathbb{R} \cdot \text{id} \oplus \mathbb{R} \cdot J
\]

**Proof.** The “\(\supset\)”:inclusion is clear. For the other inclusion we first observe that the elements of \(\mathcal{A}_{\omega^R}\) commute:

\[
\omega^R(ABv, w, x, \ldots) = \omega^R(v, Av, Bx, \ldots) = \omega^R(BAv, w, x, \ldots).
\]

Especially \(\mathcal{A}_{\omega^R} \subset \text{End}_{\mathbb{C}}(V)\), as every element has to commute with \(J\). Moreover, any element \(A \in \mathcal{A}_{\omega^R}\) has to be diagonal as a complex matrix. To see that, we observe that \(A(v)\) is always \(\mathbb{C}\)-linearly dependent on \(v\). We have

\[
\omega^R(v, Av, x, \ldots) = \omega^R(v, v, Ax, \ldots) = 0,
\]

so \(\iota_v \iota_A(v) \omega^R = 0\) for all \(v\). As \(\omega^R\) is at least a 3-form, this implies \(\iota_v \iota_A(v) \omega = 0\) for all \(v\). Now \(\omega\) is a complex volume, so \(v\) is a complex eigenvector of \(A\). In remains to show that all eigenvalues are equal, but this follows from

\[
\lambda_1 \cdot \omega^R(v_1, v_2, \ldots) = \omega^R(Av_1, v_2, \ldots) = \omega^R(v_1, Av_2, \ldots) = \lambda_2 \cdot \omega^R(v_1, v_2, \ldots),
\]

again using the fact that \(\omega^R\) is the real part of a complex volume form. \(\square\)
Proof of Theorem 4.13. At any point we choose $J_p$ to be the unique almost-complex structure compatible with the standard orientation on $U$ and satisfying (3), existing by the above lemma. The smoothness of $\omega$ assures that the almost-complex structure $J$ varies smoothly. If $(U, \omega)$ is flat, then with respect to some chart $J$ has constant coefficients, i.e. is integrable. On the other hand if $J$ is integrable, then we can extend $\omega$ to an element $\omega^C$ of $\Omega^{m,0}(U)$, by $\iota_v \omega^C = \iota_v \omega - i \cdot \iota_{J(v)} \omega$. By the integrability of $J$, this form is still closed, i.e. a holomorphic volume form (of the complex manifold $(U,J)$). By the holomorphic version of the Darboux-Moser Theorem for volume forms, there exist holomorphic local coordinates $(z_1, ..., z_n)$, such that $\omega^C$ (and hence $\omega = \Re (\omega^C)$) has constant coefficients.

Remark 4.15. For odd $m$, the almost-complex structures defined by Equation (3), could also be described by the equation

$$(\iota_v \omega^R) \wedge \omega^R = \pm \iota_{J(v)} \left( \frac{1}{2} \omega^R \wedge \omega^I \right),$$

as has been done for the $m = 3$ case in, for example, [Bry06, Hit00].

4.5 2-plectic 6-manifolds

In this subsection we construct new 2-plectic structures on $\mathbb{R}^6$ that do not have constant linear type respectively are not flat despite having constant linear type, showing that flatness of multisymplectic manifolds is a subtle issue.

For non-degenerate three-forms in dimension six there are three linearly inequivalent normal forms. We will recall their flatness conditions, as described in [Bry]. The different cases were already presented in [Bur04, PV08, Van01, Mar88, KN69], but also follow from the previous three subsections.

Theorem 4.16. Let $U \subset \mathbb{R}^6$ be open and $\omega \in \Omega^3_{cl}(U)$ (possibly degenerate). Choose any volume form $\Omega \in \Omega^6(U)$. There is a unique $J \in \Gamma(U, \text{End}(TU)) = C^\infty(U, \mathbb{R}^{6 \times 6})$ satisfying $\iota_v \omega \wedge \omega = \iota_{J(v)} \Omega$ for all $v \in TU$. Then we have:

(i) If $\text{trace}(J(p)^2) > 0$, then $\omega_p$ has the linear type of $e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$. If this is the case on an open subset $V \subset U$, then $\omega|_V = \omega_1 + \omega_2$ for decomposable forms $\omega_1, \omega_2 \in \Omega^3(V)$, unique up to order. In such cases $(V, \omega|_V)$ is flat, if and only if $d\omega_1 = d\omega_2 = 0$.

(ii) If $\text{trace}(J(p)^2) < 0$, then $\omega_p$ has the linear type of $e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5$. If this is the case on an open subset
$V \subset U$, then $\tilde{J} = \sqrt{\frac{-6}{\text{trace}(J^2)}} \cdot J$ defines an almost-complex structure. In such cases $(V, \omega|_V)$ is flat, if and only if $\tilde{J}$ is an integrable almost-complex structure.

(iii) If $\text{trace}(J(p)^2) = 0$ and $\omega_p$ is non-degenerate, then $\omega_p$ has the linear type of $e^1 \wedge e^5 \wedge e^6 - e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^4 \wedge e^5$. If this is the case on an open subset $V \subset U$, then $E = \ker(J) \subset TV$ yields a distribution. In such cases $(V, \omega|_V)$ is flat, if and only if $E$ is an integrable distribution.

Proof. The linear statements were proven in Section 3.6. The integrability conditions of the three cases can be reduced to special cases of Theorems 4.11, 4.13 and 4.9. □

We will use the above theorems to construct 2-plectic 6-manifolds not having constant linear type or flatness properties. Similar constructions have been investigated in [Bur04, PV08] and other examples arise in the theory of special holonomy cf. ([Ibo01, Bry87]). We will construct our examples using the following lemma.

Lemma 4.17. Let $M = \mathbb{R}^6$ and

$$
\omega = \omega^f = dx^{135} - dx^{146} - dx^{236} + f(x) \cdot dx^{245} \in \Omega^3(M),
$$

where $f : \mathbb{R}^6 \to \mathbb{R}$ only depends on $x^2, x^4$ and $x^5$. Then $(M, \omega)$ is a multisymplectic manifold. Furthermore, $\omega_x$ is of linear type (i) when $f(x) > 0$, (ii) when $f(x) < 0$ and (iii) when $f(x) = 0$, using the numbering from the above theorem.

Proof. The proof is a simple consequence of the above theorem, by explicit calculation of $J$ and noticing that $dx^{135} - dx^{146} - dx^{236}$ is non-degenerate. With respect to the standard volume on $\mathbb{R}^6 = T_x \mathbb{R}^6$, we obtain that

$$
J(x) = \begin{pmatrix}
0 & -2f(x) & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2f(x) & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 2f(x) & 0
\end{pmatrix}.
$$

Squaring and taking the trace completes the proof. For a SageMath function facilitating the computation, we refer to Appendix C.2. □
Example 4.18 (Non-constant linear type). We set \( f = x^2 \). Then in any neighborhood of \( 0 \in M \), there exist points where \( f \) is positive and points, where \( f \) is negative. Hence, \( (M, \omega^f) \) does not have constant linear type around \( 0 \). Consequently, it cannot satisfy the Darboux property at \( 0 \).

Remark 4.19. An example of non-constant linear type for non-degenerate four-forms in dimension six can be given as follows. Let \( M = \mathbb{R}^6 \) and \( \omega = dx^{1234} + dx^{1256} + x^3 dx^{3456} \). At \( x^3 = 0 \) the linear type changes.

Example 4.20 (Constant linear type but non-flat). We regard the multisymplectic submanifold \( M^{>0} = \{ x \in \mathbb{R}^6 \mid x^2 > 0 \} \subset M \) from Example 4.18. We set

\[
\begin{align*}
\alpha_1 &= (\sqrt{x^2} dx^2 - dx^1) \\
\alpha_2 &= (\sqrt{x^2} dx^4 - dx^3) \\
\alpha_3 &= (\sqrt{x^2} dx^5 + dx^6) \\
\alpha_4 &= (\sqrt{x^2} dx^2 + dx^1) \\
\alpha_5 &= (\sqrt{x^2} dx^4 + dx^3) \\
\alpha_6 &= (\sqrt{x^2} dx^5 - dx^6)
\end{align*}
\]

and

\[
\begin{align*}
\omega_1 &= \frac{1}{(2\sqrt{x^2})} \alpha_1 \wedge \alpha_2 \wedge \alpha_3, \\
\omega_2 &= \frac{1}{(2\sqrt{x^2})} \alpha_4 \wedge \alpha_5 \wedge \alpha_6.
\end{align*}
\]

It follows that

\[
\omega = \omega_1 + \omega_2 \quad \text{and} \quad (\omega_1 \wedge \omega_2)_p \neq 0 \quad \forall p \in M^{>0},
\]

and the linear type is thus constantly type (i) from Theorem 4.16. We observe that \( \omega_1 = \frac{1}{2} \omega + \frac{1}{2} \sqrt{x^2} (dx^{246} - dx^{235} - dx^{145}) + \frac{1}{2\sqrt{x^2}} dx^{136} \) and hence

\[
d\omega_1 = \frac{1}{4\sqrt{x^2}} dx^{1245} + \frac{1}{4\sqrt{x^2}} dx^{1236}.
\]

As \( d\omega_1 \neq 0 \) on any nonempty open subset of \( M^{>0} \), we know that \( (M^{>0}, \omega) \) is nowhere flat.

Remark 4.21. Similar examples can be constructed already for three-forms in \( \mathbb{R}^5 \). In [Tur84] it is shown that \( (dx^{12} + dx^{34}) \wedge (dx^5 + x^2 dx^4) \in \Omega^3(\mathbb{R}^5) \) is nowhere flat, although it is non-degenerate, closed and has constant linear type.
4.6 2-plectic Lie groups

We prove that the canonical 2-plectic structure on a simple Lie group is flat only if the group is three-dimensional.

**Theorem 4.22.** Let \((G, \omega)\) be a real simple Lie group with its canonical three-form, as described in Example 2.8. Then \((G, \omega)\) has constant linear type but is flat if and only if its dimension is three.

**Proof.** Constancy of linear type follows immediately from the bi-invariance of \(\omega\). Without loss of generality, we can assume for the rest of the proof that \(G\) is connected. In the three-dimensional case the flatness is a consequence of the Darboux theorem for volume forms (Theorem 4.2). For all real simple Lie groups of dimension higher than three, we have

\[
\text{Aut}(\mathfrak{g}, \omega_e) = \text{Aut}(\mathfrak{g}, [\cdot, \cdot]) \subset \text{Aut}(\mathfrak{g}, \langle \cdot, \cdot \rangle),
\]

where the leftmost and rightmost terms are linear automorphisms preserving the respective tensor and the middle term are the Lie algebra automorphisms of \(\mathfrak{g}\). The left equality is the statement of Theorem 2.2 of [Lê13] and the right inclusion follows, because the Killing form is intrinsically defined from the Lie bracket.

Let us assume that \(G\) admits a chart \(\phi : U \subset G \to V \subset \mathfrak{g}\) near \(e\), such that \((T_g\phi)^*\omega_e = \omega_g\), where \(\omega_e\) should be interpreted as the constant coefficient extension of \(\omega_e \in \mathfrak{g} = T_e g\). The natural left-invariant pseudo-Riemannian metric on \(G\) is defined by \(h_g = -\langle \cdot, \cdot \rangle\), where \(\theta_L^g : T_g G \to \mathfrak{g}\) is the Maurer-Cartan one-form. By construction we have

\[
(\theta_L^g) \circ (T_g\phi)^{-1} \in \text{Aut}(\mathfrak{g}, \omega_e).
\]

So \((\theta_L^g) \circ (T_g\phi)^{-1}\) preserves \(h_e = -\langle \cdot, \cdot \rangle\), i.e.

\[
(T_g\phi)^*h_e = (T_g\phi)^*((\theta_L^g) \circ (T_g\phi)^{-1})^*h_e = (\theta_L^g)^*h_e = h_g
\]

This means that \(\phi\) is a flat chart for \((G, h)\), where \(h\) is the canonical left-invariant metric on \(G\). Such a chart cannot exist, because real simple Lie groups with canonical left-invariant metric have non-zero curvature (cf. e.g. [O'N83]).
5 The group of multisymplectic diffeomorphisms

In the last section we studied the local structure of multisymplectic manifolds. In this section we will investigate the diffeomorphisms preserving this structure.

Definition 5.1. A “local diffeomorphism” \( \varphi \) of \( M \) is a diffeomorphism between two open subsets \( U, V \) of \( M \). It is called “local multisymplectic diffeomorphism” if it satisfies \( \varphi^*(\omega|_V) = \omega|_U \). The pseudogroup of local multisymplectic diffeomorphisms is called \( \text{Diff}_{loc}(M, \omega) \). Its subgroup of global diffeomorphisms is denoted by \( \text{Diff}(M, \omega) \) and called “group of multisymplectic or multisymplectomorphisms” of \( (M, \omega) \). The elements of the Lie algebra \( \mathfrak{X}(M, \omega) = \{ X \mid \mathcal{L}_X \omega = 0 \} \subset \mathfrak{X}(M) \) are called “multisymplectic or locally Hamiltonian vector fields”.

We will consider the following question:

“Let \( (M, \omega) \) be multisymplectic. How transitive is the action of \( \text{Diff}(M, \omega) \) on \( M \)?”

To make this question precise, we will distinguish several degrees of transitivity:

Definition 5.2. Let \( X \) be a set and \( G \times X \to X, (g, p) \mapsto g(p) \) a group action. The action is called “\( k \)-transitive”, if for any two \( k \)-tuples \( (p_1, ..., p_k), (q_1, ..., q_k) \) of elements in \( X \) satisfying \( p_i \neq p_j \) and \( q_i \neq q_j \) for \( i \neq j \) there exists an element \( g \in G \) such that \( g(p_i) = q_i \) for \( i = 1, ..., k \).

In this section we will answer this question for several classes of examples. First, in Subsection 5.1 we will review the classical results that symplectic and volume-preserving diffeomorphisms act \( k \)-transitively for all \( k \). We then show that the same holds true for the multisymplectomorphism group of \( \mathbb{C}^n \) with the real part of a complex volume form. Subsection 5.2 will treat several situations, where the multisymplectic diffeomorphisms act 1-transitively but not 2-transitively, including some examples from the last section and those discussed in [Mar88]. Finally, we will briefly discuss examples, where the action is not even 1-transitive.

5.1 Very transitive cases

The following theorem shows that the multisymplectomorphisms of symplectic and volume forms act highly transitively:
Theorem 5.3 ([Boo69]). Let $(M, \omega)$ be a connected symplectic manifold or a connected manifold equipped with a volume form. Then $\text{Diff}(M, \omega)$ acts $k$-transitively on $M$ for all $k \in \mathbb{N}$.

Corollary 5.4. Let $(M, \omega)$ be a connected symplectic $2n$-dimensional manifold. Then $(M, \omega^j)$ is a multisymplectic manifold for all $j \in \{1, \ldots, n\}$. Moreover $\text{Diff}(M, \omega^j)$ always acts $k$-transitively for all $k$.

Proof. The non-degeneracy of $\omega^j$ follows from $\iota_X(\omega^j) \wedge \omega^{n-j} = j \cdot (\iota_X\omega) \wedge \omega^{n-1} = \frac{1}{n} \iota_X(\omega^n)$. For $X \neq 0$ the latter is non-zero, as $\omega^n$ is a volume form. The second statement follows immediately as $\text{Diff}(M, \omega) \subset \text{Diff}(M, \omega^j)$. \qed

We note that infinitesimally the converse of the last inclusion in the preceding proof is also true, except for the case $j = n$:

Lemma 5.5. Let $(M, \omega)$ be as in Corollary 5.4 with $n > 1$. Then $\mathfrak{X}(M, \omega^j) = \mathfrak{X}(M, \omega)$ for $1 \leq j < n$ and $\mathfrak{X}(M, \omega^n) \supseteq \mathfrak{X}(M, \omega)$.

Proof. The $j = n > 1$ case is a consequence of Gromov's non-squeezing theorem, cf. e.g. [Gro85]. If $j < n$ we calculate:

$$
L_X(\omega^j) = j \cdot L_X\omega \wedge \omega^{j-1}.
$$

So $L_X\omega = 0$ implies $L_X(\omega^j) = 0$. But on the other hand $\cdot \wedge \omega^j : \Omega^2(M) \to \Omega^{2+2j}(M)$ is injective for $2j \leq 2n-2$ by Lepage’s divisibility Theorem, which is stated below. \qed

Theorem 5.6 (Lepage's divisibility Theorem, [LM87]). Let $\Omega \in \Lambda^2(\mathbb{R}^{2n})^*$ be non-degenerate. Then the following map is a bijection for $0 \leq p < n$:

$$
\Lambda^p(\mathbb{R}^{2n})^* \to \Lambda^{2n-p}(\mathbb{R}^{2n})^*, \quad \eta \mapsto \eta \wedge \Omega^{n-p}.
$$

Another class of multisymplectic manifolds with very transitive multisymplectomorphism groups arises from complex volume forms.

Theorem 5.7. Let $n \geq 2$, $M = \mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\omega = \Re(dz^1 \wedge \ldots \wedge dz^n)$. Then $\text{Diff}(M, \omega)$ acts $k$-transitively on $M$ for all $k$.

We will prove the stronger statement that the group $\text{Aut}^{alg}_1(\mathbb{C}^n) \subset \text{Diff}(M, \omega)$ of polynomial biholomorphisms of determinant one acts $k$-transitively on $\mathbb{C}^n$ for all $k$. The idea of the proof below was explained to us by Frank Kutzschebauch who showed a much more general result on biholomorphism groups in [KRP17].

Proof. Let $p_j = (x^j, y^j, z^j)_{1 \leq j \leq k} \in \mathbb{C} \times \mathbb{C}^{n-2} \times \mathbb{C}$ be pairwise different points.
1. Without loss of generality, we may assume that all $x^i$ are pairwise different. To see this, we will find a map $T$ in $SL(n, \mathbb{C}) \subset Aut_{alg}^1(\mathbb{C}^n)$ such that $T(p_i)$ have different first components for $i \in \{1, ..., k\}$. Consider the $\binom{k}{2}$ hyperplanes $H_{ij} = \{\psi \in (\mathbb{C}^n)^* | \psi(p_i - p_j) = 0\} \subset (\mathbb{C}^n)^*$ for $1 \leq i \leq j \leq k$. As there are only finitely many $H_{ij}$, the space $(\mathbb{C}^n)^* \setminus \bigcup H_{ij}$ is non-empty. Let $\phi_1 \in (\mathbb{C}^n)^* \setminus \bigcup H_{ij}$. Extend $\phi_1$ to a basis $\{\phi_1, ..., \phi_n\}$ of $(\mathbb{C}^*)^n$. Then $\tilde{T}(p) := (\phi_1(p), ..., \phi_n(p))$ is a linear isomorphism, such that $(T(p_j))_{j \in \{1, ..., k\}}$ have different first components. We get the desired map by setting $T = \lambda \tilde{T}$ for an appropriate $\lambda \in \mathbb{C} \setminus \{0\}$.

2. There is an algebraic automorphism with Jacobian determinant 1, moving $(x^i, y^i, z^i)$ (with $x^i$ pairwise different) to $(x^i, 0, j)$. Let $P : \mathbb{C} \to \mathbb{C}^{n-2}$ be a polynomial satisfying $P(x^j) = y^j$ for $j \in \{1, ..., k\}$ and $Q : \mathbb{C} \to \mathbb{C}$ a polynomial satisfying $Q(x^j) = z^j - j$ for $j \in \{1, ..., k\}$. Then the desired algebraic automorphism is given by $(x, y, z) \mapsto (x, y - P(x), z - Q(x))$. Note that no polynomial $P$ is necessary when $n = 2$.

3. There is an algebraic automorphism with Jacobian determinant 1, moving $(x^j, 0, j)$ (with $x^i$ pairwise different) to $(0, 0, j)$. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial satisfying $P(j) = x^j$ for $j \in \{1, ..., k\}$. Then the automorphism $(x, y, z) \mapsto (x - P(z), y, z)$ has the desired property.

4. By composing steps 1., 2. and 3., we can construct an algebraic biholomorphism $\Psi$ of determinant 1, such that $\phi(p_j) = (0, 0, j)$ for $j \in \{1, ..., k\}$. Given an alternative collection of points $\tilde{p}_1, ..., \tilde{p}_k$ we can construct $\tilde{\Psi}$ in the same manner. Then $\tilde{\Psi} \circ \Psi^{-1}$ is an algebraic automorphism of Jacobian determinant 1 such that $\tilde{\Psi} \circ \Psi^{-1}(p_j) = \tilde{p}_j$ for $j \in \{1, ..., k\}$.

Example 5.8. The multisymplectomorphisms of the manifold $M = \mathbb{R}^6$, $\omega = dx^{135} - dx^{146} - dx^{236} - dx^{245}$ act $k$-transitively on $M$ for all $k$. This example is just the real description of the $(n = 3)$-case of the above theorem.

5.2 Slightly transitive cases

In this subsection we will treat several examples, where the action of $\text{Diff}(M, \omega)$ is 1-transitive but not 2-transitive. To identify those cases, we will use the following criterion.

\[ \square \]
Lemma 5.9. Let $M$ be an $n$-dimensional manifold. If a group $G \subset \text{Diff}(M)$ preserves a regular foliation $\mathcal{F}$ of dimension $r \notin \{0, n\}$, its action on $M$ is not 2-transitive.

Proof. Let $\phi$ be a diffeomorphism preserving $\mathcal{F}$. If $p_1, p_2$ are in the same leaf $F$, then $\phi(p_1), \phi(p_2)$ have to be in the same leaf $\phi(F)$. As $r$ is required to be different from $n$ several leaves exist, and as $r$ is nonzero each leaf contains many points. We take leaves $F_1 \neq F_2$ and $p_1 \neq p_2$ in $F_1$ and $q_1 \in F_1$ and $q_2 \in F_2$, then there is no $\phi \in G$, such that $\phi(p_j) = q_j$ for $j = 1, 2$. □

Using this criterion, we will first analyze a few flat examples from the last subsection, and then give an analysis of the non-flat case built in Example 4.20.

Example 5.10. Let $(M, \omega) = (\mathbb{R}^6, dx^{156} - dx^{246} + dx^{345})$, i.e. the flat model with the third linear type from Subsection 4.5. By Theorem 4.16 the integrable distribution $E$ generated by $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\}$ is preserved by $\text{Diff}(M, \omega)$ as it is constructed naturally only using $\omega$. Hence, $\text{Diff}(M, \omega)$ does not act 2-transitively on $M$. However all translations are multisymplectomorphisms. Thus, in this case, $\text{Diff}(M, \omega)$ acts transitively, but not 2-transitively on $M$.

Remark 5.11. This example can be extended to all multisymplectic manifold built as in Example 2.4. By Theorem 4.9 the multisymplectomorphisms of these manifolds preserve the (foliation given by the) fibers of $\pi$, hence they do not act 2-transitively on $M$. In [Mar88, HK04] the multisymplectomorphism groups are explicitly calculated. They are isomorphic to $\text{Diff}(Q) \ltimes \Omega^{\omega}_c(Q)$, i.e. they consist of diffeomorphisms of the base and translations by closed forms on the fibers.

Example 5.12. Let $(M, \omega) = (\mathbb{R}^6, dx^{123} + dx^{456})$. By Theorem 4.16 the forms $\omega_1 = dx^{123}$ and $\omega_2 = dx^{456}$ are either preserved or interchanged by multisymplectic diffeomorphisms. By an argument analogous to Lemma 5.9, we see that starting with two points which are in different leaves with respect to both the foliations generated by $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\}$ respectively $\{\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^5}, \frac{\partial}{\partial x^6}\}$, we can not arrive at a pair of points which share the same $(x^1, x^2, x^3)$-coordinates. Again, we can achieve 1-transitivity by translations. In conclusion, $\text{Diff}(M, \omega)$ acts transitively, but not 2-transitively on $M$.

Remark 5.13. This example also can be generalized to the setting of Theorem 4.11. For $m > 2$ and $k > 1$ the multisymplectic manifold $M = \mathbb{R}^{km}$, $\omega = dx^{1,2,...,m} + dx^{m+1,2m} + ... + dx^{(k-1)m+1,...,km}$ satisfies: $\text{Diff}(M, \omega)$ acts transitively, but not 2-transitively on $M$. 

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Proposition 5.14 (Nonflat). Let \( M > 0 = \{(x^1, x^2, x^3, x^4, x^5, x^6) \in \mathbb{R}^6 \mid x^2 > 0\} \), \( f : \mathbb{R}^3 > 0 \to \mathbb{R}^3 > 0 \), \( f(x^2) = x^2 \) and \( \omega^f = dx^{135} - dx^{146} - dx^{236} + f(x^2) \cdot dx^{245} \). Then \((M > 0, \omega^f)\) is multisymplectic and of constant linear type. Furthermore, it is non-flat and its multisymplectic diffeomorphisms act 1-transitively but not 2-transitively.

Proof. (Notations as in Example 4.20) As discussed in Example 4.20, the 2-plectic manifold \((M > 0, \omega^f)\) has constant linear type and is non-flat. Since the decomposable forms \(\omega_1, \omega_2\) fulfilling \(\omega = \omega^f = \omega_1 + \omega_2\) are unique up to order by Theorem 4.16, a multisymplectic diffeomorphism of \((M > 0, \omega^f)\) preserves or permutes \(\omega_1\) and \(\omega_2\). Hence, it preserves (or reverts the sign of) \(\Omega = \omega_1 \wedge \omega_2 = 2 \sqrt{x^2} dx^{123456}\) and \(d\omega_1\) and thus they preserve the unique bivector field \(\xi\) satisfying the equation \(i_\xi \Omega = d\omega_1\). Moreover, any diffeomorphism preserving \(\omega\) also has to preserve or revert the sign of \(i_\xi i_\xi \Omega \in \Omega^2(M > 0)\).

In our case
\[
\xi = \frac{1}{8x^2} \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^6} + \frac{1}{8(x^2)^2} \frac{\partial}{\partial x^4} \wedge \frac{\partial}{\partial x^5}
\]
and hence
\[
i_\xi i_\xi \Omega = i_\xi d\omega_1 = \frac{1}{16 \sqrt{x^2}} dx^1 \wedge dx^2
\]
So, in our case, \(i_\xi i_\xi \Omega\) is closed, hence its kernel yields a foliation preserved by the multisymplectic diffeomorphisms of \(\omega\). This foliation does not depend on the possible sign ambiguities from above. So, the multisymplectic diffeomorphisms of \(\omega\) do not act 2-transitively on \(M > 0\). However, as it turns out they do act 1-transitively, as we will now see. For 1-transitivity it suffices to check by a direct computation that \(X(M > 0, \omega^f)\) includes the complete vector fields:

\[
\frac{\partial}{\partial x^i}, \ i \neq 2 \text{ and } x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} - \frac{1}{2} x^4 \frac{\partial}{\partial x^4} + \frac{1}{2} x^5 \frac{\partial}{\partial x^5} - x^6 \frac{\partial}{\partial x^6}.
\]

\(\square\)

Remark 5.15. We give a detailed analysis of the infinitesimal symmetries \(X(M > 0, \omega^f)\) in Appendix B. Furthermore, SageMath sourcecode yielding an explicit description of the partial differential equations \(L_X \omega^f = 0\) is provided in Appendix C.1.

In the case of simple Lie groups the proof of Theorem 4.22 implies the following
Proposition 5.16. Let \((G, \omega)\) be a compact real simple Lie group with its canonical three-form, as described in Example 2.8. Then \(\text{Diff}(M, \omega)\) acts 1-transitively on \(G\). However, it acts 2-transitively if and only if \(\dim(G) = 3\).

Proof. For the three-dimensional case, the statement follows from Theorem 5.3. For all other dimensions the statement \(\text{Aut}(g, \omega_e) \subset \text{Aut}(g, \langle \cdot, \cdot \rangle)\) from the proof of Theorem 4.22 implies that \(\text{Diff}(M, \omega) \subset \text{Diff}(M, h)\) (using the left invariance of both tensors). Especially the connected components of the identity satisfy \(\text{Diff}_0(M, \omega) \subset \text{Diff}_0(M, h)\). On the other hand \(\text{Diff}_0(M, h) = (G \times G)/Z(G)\) (cf. eg. [OT76]), acting by \((l_g, r_g^{-1})\), which clearly preserves the bi-invariant form \(\omega\). Thus

\[
\text{Diff}_0(M, \omega) = \text{Diff}_0(M, h).
\]

The statement now follows, because \(\text{Diff}(M, \omega)/\text{Diff}_0(M, \omega)\) is discrete and \(\text{Diff}_0(M, h)\) acts 1-transitively but not 2-transitively. \(\square\)

Remark 5.17. Partial results in this direction have been described in [Sha14]. We also note that \(\text{Diff}(M, \omega) \neq \text{Diff}(M, h)\). For the inversion diffeomorphism \(\phi : G \to G, \phi(g) = g^{-1}\), we have \(T_\phi(X) = -X\), so \(\phi^*h = h\), but \(\phi^*\omega = -\omega\).

5.3 Non-transitive cases

A simple necessary criterion for 1-transitivity of a multisymplectic diffeomorphism group is constant linear type. Two areas, where the multisymplectic form is not of the same linear type can not be multisymplectomorphic.

Example 5.18. In Example 4.18 the spaces \(\{x^2 < 0\}, \{x^2 = 0\}\) and \(\{x^2 > 0\}\) are preserved by (local) multisymplectic diffeomorphisms. Especially, the group \(\text{Diff}(M, \omega)\) does not act transitively on \(M\).

Now we will build a compact version of the above example.

Example 5.19 (Compact). We set \(f = \sin(2\pi x^2)\). Then the can regard the quotient multisymplectic manifold \(M = \mathbb{R}^6/Z^6\) with the form induced by the \(\omega\) above, which we call \(\tilde{\omega}\). Again \((M, \tilde{\omega})\) does not have constant linear type. Consequently it does not satisfy the Darboux property and the group \(\text{Diff}(M, \tilde{\omega})\) does not act transitively on \(M\).

Using the fact that having the Darboux property is preserved by multisymplectic diffeomorphisms, we can build an example of a multisymplectic manifold of constant linear type, on which the local multisymplectic diffeomorphisms do not act transitively.
Proposition 5.20 (Constant linear type). Let $M^> = \{(x^1, x^2, x^3, x^4, x^5, x^6) \in \mathbb{R}^6 \mid x^2 > 0\}$, $f: \mathbb{R}^> \to \mathbb{R}^>$ be a smooth function satisfying $f|_{[0,1]} = 1$ and $f(t) = t$ for $t \geq 2$ and $\omega^f = dx^{135} - dx^{146} - dx^{236} + f(x^2) \cdot dx^{245}$. Then $(M^>, \omega^f)$ is multisymplectic and of constant linear type, but the group of multisymplectic diffeomorphisms does not act transitively on it.

Proof. The form $\omega^f$ is flat on $\{x \in M^> \mid x^2 \in ]0, 1[\}$ and non-flat on $\{x \in M^> \mid x^2 > 2\}$ by Theorem 4.16, Lemma 4.17 and Example 4.20. As a flat subset can not be equivalent to a non-flat one, this means that Diff$(M, \omega)$ does not act transitively on $M^>$. \qed
6 Conserved quantities from homotopy comoments

In this section we return to the topic of conserved quantities. This time, we inspect them with regard to Hamiltonian Lie group or Lie algebra actions. More precisely, \((M, \omega)\) will always denote an \(n\)-plectic manifold\(^2\) and \(G\) a Lie group (resp. \(g\) a Lie algebra) acting on \(M\). We begin Subsection 6.1 by reviewing the notions of multisymplectic Hamiltonian actions and their comoments. We distinguish between three notions of preservation of Hamiltonian forms under a multisymplectic Lie algebra action. In Subsections 6.2, 6.3 and 6.4 we investigate how comoments generate conserved quantities for the three different possible cases of preservation. In Subsection 6.5 we determine the cohomological obstructions to the existence of comoments, which we when use to give a more conceptual approach to the results of Subsections 6.2, 6.3 and 6.4 in Subsection 6.6.

6.1 Actions on multisymplectic manifolds

**Definition 6.1.** Let \((M, \omega)\) be an \(n\)-plectic manifold. A right action \(\vartheta\) of a Lie group \(G\) on \(M\) is called “multisymplectic” if \(\vartheta^*\omega = \omega\) for all \(g \in G\), where \(\vartheta_g = \vartheta(\cdot, g)\). An infinitesimal right action of a Lie algebra \(g\) on \(M\), i.e. a Lie algebra homomorphism \(g \to \mathfrak{X}(M), x \mapsto v_x\), is called “multisymplectic” if \(\mathcal{L}_{v_x}\omega = 0\) for all \(x \in g\). For a connected Lie group \(G\), a right action \(\vartheta\) is multisymplectic if and only if the corresponding infinitesimal right action (given by \(x \mapsto v_x\) where \(v_x(m) = \frac{d}{dt} |_0 \vartheta(m, \exp(tx))\) at all \(m \in M\)) is multisymplectic.

A multisymplectic infinitesimal action is thus a Lie algebra homomorphism from \(g\) to \(\mathfrak{X}(M, \omega) = \{X \in \mathfrak{X}(M)|\mathcal{L}_X\omega = 0\}\). One may ask, whether such an action admits an “Lie \(\infty\)-lift” to \(L_{\infty}(M, \omega)\). For an explicit description of the equations fulfilled by such a lift, the following definition is useful.

**Definition 6.2.** Let \(g\) be a Lie algebra. We define the “Lie algebra homology differential” \(\partial\) by setting \(\partial_k = \partial|_{\Lambda^k g}: \Lambda^k g \to \Lambda^{k-1} g, x_1 \wedge \cdots \wedge x_k \mapsto \sum_{1 \leq i < j \leq k} (-1)^{i+j}[x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_k\), for \(k \geq 1\). We put \(\Lambda^{-1} g = \{0\}\) and \(\partial_0\) to be the zero map.

\(^2\)Most results in this section do not rely on the non-degeneracy of the multisymplectic form, for details please confine [RWZ16].
We recall from [CFRZ16, §5] the higher analogue of momentum map in symplectic geometry, obtained in a natural way replacing Lie algebras morphisms by Lie $\infty$-algebra morphisms:

**Definition 6.3.** A “(homotopy) comoment” for a multisymplectic infinitesimal action $v : g \to \mathfrak{X}(M)$ on $(M,\omega)$ is is a collection of maps $\{f_i\} = \{f_i : \Lambda^i g \to \Omega^{n-i}(M) | 1 \leq i \leq n\}$ such that the generator of the action associated to $x \in g$ is a Hamiltonian vector field for $f_1(x)$ and satisfying the equation

$$-f_{k-1}(\partial(p)) = df_k(p) - (-1)^{(k+1)/2} \iota_{v_p} \omega$$

for all $k = 1, \ldots, n+1$ and $p \in \Lambda^k g$ (setting $f_0$ and $f_{n+1}$ to be zero). Here, we use the short hand notation $v_p := v_{x_1} \wedge \cdots \wedge v_{x_k}$, whenever $p = x_1 \wedge \cdots \wedge x_k$ for $x_i \in g$. The action $v$ is called “Hamiltonian” if it admits a comoment.

**Remark 6.4.** Equation (4) of course is the general definition of an Lie $\infty$-algebra morphism specialized to the case at hand of the Lie algebra $g$ and the Lie $n$-algebra of observables $L_\infty(M,\omega)$.

Now we turn to infinitesimal actions preserving a Hamiltonian $n-1$-form $H$ on an $n$-plectic manifold $(M,\omega)$. As in the case of the conserved quantities, one has to distinguish to which extent the action preserves the Hamiltonian form.

**Definition 6.5.** Let $g \to \mathfrak{X}(M,\omega), x \mapsto v_x$ be an infinitesimal action. It is called

(i) “locally $H$-preserving” if $L_{v_x} H$ is closed for all $x \in g$.

(ii) “globally $H$-preserving” if $L_{v_x} H$ is exact for all $x \in g$.

(iii) “strictly $H$-preserving” if $L_{v_x} H = 0$ for all $x \in g$.

**Remark 6.6.** Usually a differential form would be called “preserved by an infinitesimal action” if condition (iii) is fulfilled.

In the following we will investigate the conserved quantities arising from a comoment separately for these three cases.

### 6.2 Conserved quantities from locally $H$-preserving actions

In this subsection we assume that $(M,\omega)$ is an $n$-plectic manifold, $H \in \Omega^{n-1}_{\text{Ram}}(M)$ and that $\{f_i\} : g \to L_\infty(M,\omega)$ is the comoment of a locally $H$-preserving infinitesimal action $g \to \mathfrak{X}(M,\omega), x \mapsto v_x$. 
By the definition of a comoment, the generator of the infinitesimal action associated to $x$ in $\mathfrak{g}$ is a Hamiltonian vector field of $f_1(x)$. For $p = x_1 \wedge ... \wedge x_k \in \Lambda^k \mathfrak{g}$ we write $v_p := v_{x_1} \wedge ... \wedge v_{x_k}$ and $\iota_{v_p} = \iota_{v_{x_k}} ... \iota_{v_{x_1}}$.

**Lemma 6.7.** Let $\{f_i\} = \{f_i|1 \leq i \leq n\}$ be a comoment for $v : \mathfrak{g} \to \mathfrak{X}(M, \omega)$ and $H \in \Omega^{n-1}_{\text{Ham}}(M)$. Then for the Hamiltonian vector field $X_H$ of $H$ we have

1. $f_1(x) \in C_{\text{loc}}(X_H)$ for all $x \in \mathfrak{g}$,
2. $[X_H, v_x] = 0$ for all $x \in \mathfrak{g}$,
3. $\iota_{v_p} \omega \in C_{\text{str}}(X_H)$ for all $p \in \Lambda^k \mathfrak{g}$.

**Proof.** (i) follows from Lemma 2.25 (ii) and (ii) from Lemma 2.25 (i). Further, (iii) follows upon recalling that $[L_{X_H}, \iota_Y] = \iota_{[X,Y]}$ and part (ii):

$$L_{X_H}(\iota_{v_p} \omega) = L_{X_H} \iota_{v_{x_k}} ... \iota_{v_{x_1}} \omega = -\iota_{v_{x_k}} L_{X_H} ... \iota_{v_{x_1}} \omega = ... = \pm \iota_{v_p} (L_{X_H} \omega) = 0.$$

It turns out that certain subspaces of the image of the higher components of the comoment consist of locally conserved quantities. To specify this we recall the definition of Lie algebra homology.

**Definition 6.8.** Let $\mathfrak{g}$ be a Lie algebra, $k \geq 1$ and $\partial_k$ the $k$-th Lie algebra homology differential. We define

1. the “cycles” $Z_k(\mathfrak{g}) = \ker(\partial_k) \subset \Lambda^k \mathfrak{g}$,
2. the “boundaries” $B_k(\mathfrak{g}) = \text{image}(\partial_{k+1}) \subset \Lambda^k \mathfrak{g}$ and
3. the $k$-th “Lie algebra homology space” $H_k(\mathfrak{g}) = Z_k(\mathfrak{g}) / B_k(\mathfrak{g})$.

**Remark 6.9.** The space $Z_k(\mathfrak{g})$ is denoted by $P_{\mathfrak{g},k}$ and called the “$k$-th Lie kernel” of $\mathfrak{g}$ in [MS13].

**Proposition 6.10.** Let $p \in Z_k(\mathfrak{g})$. Then $f_k(p)$ is locally conserved by the Hamiltonian vector field $X_H$ of $H$.

**Proof.** The case $k = 1$ is part (i) of Lemma 6.7. Assume now $k > 1$. We have to show that $L_{X_H} f_k(p)$ is closed. We have

$$dL_{X_H} f_k(p) = L_{X_H} df_k(p) = (-1)^{(k+1)/2} L_{X_H} \iota_{v_p} \omega = 0,$$

where the first equality holds because the Lie derivative commutes with the exterior derivative, the second one, because of Equation (4), and the last one because of Lemma 6.7 (iii).
Proposition 6.10 states that $\mathcal{L}_{X_H} f_k(p)$ is a closed $(n-k)$-form, hence we obtain:

**Corollary 6.11.** Let $p \in Z_k(\mathfrak{g})$. If $H_{\mathbb{R}}^{n-k}(M)$ is zero, then $f_k(p)$ is globally conserved by the Hamiltonian vector field $X_H$ of $H$.

For boundaries, the statement of Proposition 6.10 can be strengthened:

**Proposition 6.12.** If $p \in B_k(\mathfrak{g}) \subset Z_k(\mathfrak{g})$, then $f_k(p)$ is globally conserved by the Hamiltonian vector field $X_H$ of $H$.

**Proof.** Let $q$ be a potential for $p$, i.e. $\partial_{k+1} q = p$. Then

$$\mathcal{L}_{X_H}(f_k(p)) = \mathcal{L}_{X_H}(f_k(\partial q)) = \mathcal{L}_{X_H}(-df_{k+1}(q) + (-1)^{(k+1)(k+2)/2}\iota v_\omega) = -d\mathcal{L}_{X_H}f_{k+1}(q) + (-1)^{(k+1)(k+2)/2}\mathcal{L}_{X_H}\iota v_\omega,$$

using Equation (4). The statement then follows by Lemma 6.7 (iii).

The following example shows sharpness of the statement of Proposition 6.10, i.e. for $p \in \Lambda^k \mathfrak{g}$ the condition $\partial p = 0$, in general, does not imply that $f_k(p)$ is globally conserved.

**Example 6.13.** Let $M = \mathbb{R}^3$, $\omega = dx \wedge dy \wedge dz$ and $H = -xdy$. We already observed $dH = -dx \wedge dy$ and $X_H = \frac{\partial}{\partial z}$. We consider the two-dimensional abelian Lie algebra $\mathfrak{g} = \langle a, b \rangle _\mathbb{R}$ and the homomorphism $v : \mathfrak{g} \to \mathfrak{X}(M)$ given by $v_a = \frac{\partial}{\partial x}$ and $v_b = \frac{\partial}{\partial y}$. We have that $\mathcal{L}_{v_a} H = -dy$ is exact and $\mathcal{L}_{v_b} H = 0$. We construct a comoment for this action by $f_1(a) = -ydz$, $f_1(b) = xdz$ and $f_2(a \wedge b) = -z$. Then $a \wedge b \in Z_2(\mathfrak{g})$ is a cycle and $-z$ is locally conserved, as predicted by Proposition 6.10, but not globally:

$$\mathcal{L}_{X_H}(-z) = \iota_{\frac{\partial}{\partial z}} d(-z) = -1 \neq 0.$$

### 6.3 Conserved quantities from globally $H$-preserving actions

In this subsection we assume that $(M,\omega)$ is an $n$-plectic manifold, $H \in \Omega_{\text{Ham}}^{n-1}(M)$ and that $\{f_i\} : \mathfrak{g} \to L_{\infty}(M,\omega)$ is the comoment of a globally $H$-preserving infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M,\omega), x \mapsto v_x$.

As Example 6.13 indicates, no significant improvements of the above results are to be expected upon passing from locally to globally $H$-preserving actions. There is only a slight improvement of Lemma 6.7 (i) with essentially the same proof:
Lemma 6.14. Let \( \{ f_i \} = \{ f_i | 1 \leq i \leq n \} \) be a comoment for \( v : \mathfrak{g} \to \mathfrak{X}(M) \). Then \( f_1(x) \in C(X_H) \) for all \( x \in \mathfrak{g} \) and for the Hamiltonian vector field \( X_H \) of \( H \).

Remark 6.15. Notice that Lemma 6.7 and Lemma 6.14 hold for any element in \( \Omega^{n-1}_{\text{Ham}}(M) \) whose Hamiltonian vector field is \( v_x \).

6.4 Conserved quantities from strictly \( H \)-preserving actions

In this subsection we assume that \( (M, \omega) \) is an \( n \)-plectic manifold, \( H \in \Omega_{\text{Ham}}^{n-1}(M) \) and that \( \{ f_i \} : \mathfrak{g} \to L_\infty(M, \omega) \) is the comoment of a strictly \( H \)-preserving infinitesimal action \( \mathfrak{g} \to \mathfrak{X}(M, \omega), x \mapsto v_x \).

To prove a stronger result than Proposition 6.10 in this situation we need the following observation (cf., e.g., [MS13, Lemma 3.4]).

Lemma 6.16. Let \( M \) be a manifold and let \( \Omega \) be a not necessarily closed differential form on \( M \). For all \( m \geq 1 \) and all vector fields \( X_1, \ldots, X_m \) in the Lie algebra \( \mathfrak{X}(M) \) we have:

\[
(-1)^m d_{X_1 \wedge \cdots \wedge X_m} \Omega = \iota_{\partial(X_1 \wedge \cdots \wedge X_m)} \Omega \\
+ \sum_{i=1}^m (-1)^i \iota_{X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_m} \mathcal{L}_{X_i} \Omega \\
+ \iota_{X_1 \wedge \cdots \wedge X_m} d\Omega.
\]

Definition 6.17. Given a differential form \( \Omega \in \Omega^*_{\text{Ham}}(M) \) and a multivector field \( Y \in \Gamma(\Lambda^m T^\ast M) \), the “Lie derivative of \( \Omega \) along \( Y \)” is defined as a graded commutator, by \( \mathcal{L}_Y \Omega := d\iota_Y \Omega - (-1)^m \iota_Y d\Omega \).

Remark 6.18. This definition allows to combine the first and last term in the above formula into a Lie derivative. Hence, the above formula can be written \( \mathcal{L}_{X_1 \wedge \cdots \wedge X_m} \Omega = (-1)^m [\iota_{\partial(X_1 \wedge \cdots \wedge X_m)} + \sum_{i=1}^m (-1)^i \iota_{X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_m} \mathcal{L}_{X_i}] \).

Theorem 6.19. Let \( p \in Z_k(\mathfrak{g}) \). Then \( f_k(p) \) is a globally conserved quantity.

Proof. We have

\[
\iota_{X_H} df_k(p) = (-1)^{k(k+1)/2} \iota_{X_H} \iota_{v_p} \omega = (-1)^k (-1)^{k(k+1)/2} \iota_{v_p} dH = (-1)^{k(k+1)/2} d(\iota_{v_p} H).
\]
where we used Equation (4) in the first equality, and in the last equality Lemma 6.16 applied to the form $H$, as well as the $g$-invariance of $H$ and once more the assumption $p \in Z_k(g)$.

Therefore we conclude from Cartan’s formula that

$$\mathcal{L}_{X_H} f_k(p) = d(\iota_{X_H} f_k(p) - (-1)^{(k+1)/2} \iota_{v_p} H).$$

**Remark 6.20.** In the symplectic case, a homotopy comoment boils down to its first component, $f_1 : g \to \Omega^0(M)$, a classical comoment. Upon observing that $Z_1(g) = g$, the preceding theorem then reduces to the obvious but important fact that if a Hamiltonian function $H$ is $g$-invariant, then for all $x$ in $g$ we have that $\{f_1(x), H\} = \mathcal{L}_{X_H} f_1(x) = 0$.

In particular, by Theorem 6.19, $f_1(x)$ is a globally conserved quantity for all $x \in g$. Even in the case at hand of strictly $H$-preserving actions, $f_1(x)$ is not strictly conserved in general, as the following example shows.

**Example 6.21.** Let $M = \mathbb{R}^3$, $\omega = dx \wedge dy \wedge dz$ and $H = -x dy$. Then $X_H = \frac{\partial}{\partial z}$. Furthermore we consider $\alpha = z dx$ and the $\mathbb{R}$-action given by $g = \mathbb{R} \to \mathfrak{X}(M)$, $1 \mapsto v_\alpha = -\frac{\partial}{\partial y}$. This action clearly admits a comoment determined by $f_1(1) = \alpha$. Then $\mathcal{L}_{v_\alpha} H = 0$ but $\mathcal{L}_{X_H} \alpha = dx \neq 0$.

More is true: even if one assumes that $x \in B_1(g)$ is a boundary, $f_1(x)$ is still not strictly conserved in general, as Remark 6.37 below will show.

Specializing $k$ to $n$ in Theorem 6.19, we obtain scalar functions $f_n(x_1, \ldots, x_n)$ on $M$. Assembling these functions we obtain a map $M \to Z_n(g)^*$, very similar to the multi-momentum maps of Madsen-Swann, cf. [MS13], except that it is not equivariant in general. As in the symplectic case, it satisfies:

**Corollary 6.22.** The vector field $X_H$ is tangent to the level sets of the map $M \xrightarrow{\phi} Z_n(g)^*$ given by $\phi(m)(p) = f_n(p)(m)$.

**Proof.** We have to show that $X_H(m) \in \ker(T_m \phi : T_m M \to T_{\phi(m)} Z_n(g)^*) = Z_n(g)^*)$. We have

$$((T_m \phi)(X_H(m)))(p) = (df_n(p))(X_H(m))$$

$$= (\iota_{X_H} df_n(p))(m) = (\mathcal{L}_{X_H} f_n(p))(m) = 0,$$

where the last equation uses the fact that $\mathcal{L}_{X_H} f_n(p)$ is exact and an exact function is necessarily 0. 

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Remark 6.23. An analogue of this result, where $Z_n(g)$ is substituted by $B_n(g)$ holds in the setting of Proposition 6.12.

We close this subsection by showing how a comoment yields elements of the algebra $A(X_H)$ from Lemma 2.33.

Lemma 6.24. For a strictly $H$-preserving infinitesimal action $x \mapsto v_x$ we have:

(i) Let $p \in Z_k(g)$ for $k \geq 1$. Then $\iota_{v_p}\omega \in A(X_H)$.

(ii) Let $\{f_i\}$ be a comoment for the $g$-action. Let $p \in Z_{k-1}(g)$, for $k \geq 2$. Then $l_k(f_1(x_1), \ldots, f_1(x_{k-1}), H) \in A(X_H)$.

Proof. Let us first observe that if $\alpha \in C(X_H)$, then $d\alpha \in A(X_H)$. In fact $d\alpha$, being exact, is closed. Furthermore $L_{X_H}(d\alpha) = d(L_{X_H}\alpha) = 0$ since $L_{X_H}\alpha$ is zero.

(i) By Theorem 6.19, $f_k(p)$ is a globally conserved quantity. As $\iota_{v_p}\omega = \pm df_k(p)$ due to Equation (4), it is an element of $A(X_H)$ because of the preceding observation.

(ii) By Theorem 2.34, $l_k(f_1(x_1), \ldots, f_1(x_{k-1}), H)$ is conserved. We compute

$$d(\iota_{v_{x_1}} \wedge \cdots \wedge v_{x_{k-1}} H) = (-1)^{k-1}(\iota_{v_{x_1}} \wedge \cdots \wedge v_{x_{k-1}} dH)$$

$$= -(\iota_{v_{x_1}} \wedge \cdots \wedge v_{x_{k-1}} \wedge X_H \omega) = (-1)^{k(k+1)/2} l_k(f_1(x_1), \ldots, f_1(x_{k-1}), H),$$

so $l_k(f_1(x_1), \ldots, f_1(x_{k-1}), H)$ is exact, and in particular closed.

\[\square\]

6.5 Cohomological obstructions to comoments

Before turning to a more concise description of the above, we will give an analysis of comoments from a homological perspective (cf. [Ryv16c, FLGZ15, RW15]). We will begin by reviewing the necessary Lie algebra cohomology groups:

Definition 6.25. Let $g$ be a Lie algebra. For any $k \geq 0$, we define “the $k$-th Lie algebra cohomology group” by $H^k(g) = \ker(\delta : \Lambda^k g^* \to \Lambda^{k+1} g^*) / \text{image}(\delta : \Lambda^{k-1} g^* \to \Lambda^k g^*)$, where $\delta = \partial^*$ is the linear dual of the Lie algebra homology differential (cf. Definition 6.2), also called “Chevalley-Eilenberg differential". 

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To analyse the cohomological obstruction to the existence of a homotopy comoment, we study the double complex $\Lambda^\bullet g^* \otimes \Omega^\bullet(M)$ resulting from tensoring the cochain complexes $(\Lambda^\bullet g^*, \delta)$ with $(\Omega^\bullet(M), d)$. Diagrammatically we consider:

\[
\begin{array}{cccccc}
\delta \otimes id & \delta \otimes id & \delta \otimes id & \delta \otimes id & \delta \otimes id \\
\Delta^3 g^* \otimes \Omega^0(M) & \longrightarrow & \Delta^3 g^* \otimes \Omega^1(M) & \longrightarrow & \Delta^3 g^* \otimes \Omega^2(M) & \longrightarrow & \Delta^3 g^* \otimes \Omega^3(M) \\
\delta \otimes id & \delta \otimes id & \delta \otimes id & \delta \otimes id & \delta \otimes id \\
\Delta^2 g^* \otimes \Omega^0(M) & \longrightarrow & \Delta^2 g^* \otimes \Omega^1(M) & \longrightarrow & \Delta^2 g^* \otimes \Omega^2(M) & \longrightarrow & \Delta^2 g^* \otimes \Omega^3(M) \\
\delta \otimes id & \delta \otimes id & \delta \otimes id & \delta \otimes id & \delta \otimes id \\
\Delta^1 g^* \otimes \Omega^0(M) & \longrightarrow & \Delta^1 g^* \otimes \Omega^1(M) & \longrightarrow & \Delta^1 g^* \otimes \Omega^2(M) & \longrightarrow & \Delta^1 g^* \otimes \Omega^3(M) \\
\delta \otimes id & \delta \otimes id & \delta \otimes id & \delta \otimes id & \delta \otimes id \\
\Delta^0 g^* \otimes \Omega^0(M) & \longrightarrow & \Delta^0 g^* \otimes \Omega^1(M) & \longrightarrow & \Delta^0 g^* \otimes \Omega^2(M) & \longrightarrow & \Delta^0 g^* \otimes \Omega^3(M) \\
\end{array}
\]

We turn $\Lambda^\bullet g^* \otimes \Omega^\bullet(M)$ into a singly graded cochain complex $(C^\bullet, D)$ by defining the total degree of an element as the sum of the individual degrees, i.e. we set $C^k = \bigoplus_{i+j=k} \Lambda^i g^* \otimes \Omega^j(M)$. To assure that $D^2 = 0$ we have to alter the sign of one of the differentials. When $M$ is $n$-plectic, we stick to the following convention:

\[
D|_{\Lambda^\bullet g^* \otimes \Omega(M)} = (\delta \otimes id) + (-1)^n(-1)^{i+j}(id \otimes d).
\]

As usually, we often abusively write $\delta(\alpha_i \otimes \eta_j)$ for $\delta \alpha_i \otimes \eta_j$ and $d(\alpha_i \otimes \eta_j)$ for $\alpha_i \otimes d\eta_j$ in the sequel.

We will now analyse an action $v : g \to \mathfrak{X}(M, \omega)$ in terms of this complex. We define the maps $g_k \in \Lambda^k(g, \Omega^{n+1-k}(M)) = \Lambda^k g^* \otimes \Omega^{n+1-k}(M)$ for $1 \leq k \leq n+1$ by

\[
g_k(x_1, \ldots, x_k) := -(-1)^{k(k+1)/2} v_{x_1} \ldots v_{x_k} \omega.
\]

**Lemma 6.26.** The sum of the above-defined classes $g = \sum_{i=1}^{n+1} g_i$ is a D-co-cycle i.e. $D(g) = 0$.

**Proof.** Obviously $D(g) = 0$ is equivalent to $\delta g_k = dg_{k+1}$ for $0 \leq k \leq n+1$, where $g_0$ and $g_{n+2}$ are interpreted as the zero maps. The latter equalities follow by direct computations, using that $L_v \omega = 0$ for all $x \in g$. \qed

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Remark 6.27. Up to sign conventions, the cocycle $g$ corresponds to the Cartan cocycle $\hat{\omega}$ constructed for different purposes in Chapter 2, §4 of [GS98].

Lemma 6.28. There is a one-to-one correspondence between homotopy comoments and $D$-potentials $p = \sum_{i=1}^{n} p_i \in C^n$ with $p_i \in \Lambda^i g^* \otimes \Omega^{n-i}(M)$ of $g$.

Remark 6.29. We do not have a $p_0$ term as $\delta|_{\Lambda^0 g^*} = 0$ and $dp_0$ would be in $\Lambda^0 g^* \otimes \Omega^{n+1}(M)$. Since $g$ has no component in this bidegree, there is no need for a $p_0$.

Proof of Lemma 6.28. Let $F = \{f_k \mid 1 \leq k \leq n\}$ be a homotopy comoment. Then setting $p_k := -f_k$ we claim that $p = \sum p_k$ is a potential for $g$. As only two components of $D(p)$ have non-vanishing projections on $\Lambda^k(g, \Omega^{n+1-k}(M))$ it suffices to show that $\delta p_k + dp_{k+1} = g_{k+1}$. We have $X_{f_1(x)} = v_x$, which in turn implies $df_1(x) = -v_x \omega$ i.e $dp_1 = g_1$. Further we observe that $g_{k+1}(x_1, \ldots, x_{k+1}) = (f_1 l_{k+1})(x_1, \ldots, x_{k+1})$, which directly implies $\delta p_k + dp_{k+1} = g_{k+1}$ for $k > 0$.

Let now $p = p_1 + \ldots + p_n$ with $p_k \in \Lambda^k g^* \otimes \Omega^{n-k}(M)$ be a potential of $g$. We define $f_k := -p_k$. As $-df_1(x) = dp_1(x) = g_1(x) = v_x \omega$ the map $f_1$ is well-defined and we can express $g_k$ by $f_k^1 l_k$ for $k > 1$. The higher identities directly follow from $\delta p_k + dp_{k+1} = g_{k+1} = f_1 l_{k+1}$.

The following theorem clarifies the existence and unicity question for homotopy comoments.

Theorem 6.30. Let $v : g \to \mathfrak{X}(M, \omega)$ be an infinitesimal $n$-plectic action. The action is Hamiltonian if and only if the above-defined cocycle $g$ vanishes in the cohomology of $C^\bullet$, i.e. $[g] = 0$ as an element of $H^{n+1}(C^\bullet)$. If a homotopy comoment exists it is unique up to $D$-cocycles of total degree $n$ with vanishing $\Lambda^0 g^* \otimes \Omega^n(M)$-component.

Proof. The statement of the theorem follows directly from Lemma 6.28.
In order to formalize this, we introduce the quotient maps 
\[ N \rightarrow \delta g \text{ and Chevalley-Eilenberg} \] have to be divided out at the same time, as 
Using this description of Theorem 6.30, one can derive the following:

By the Künneth theorem (cf., eg., [Wei94], Thm 3.6.3) the total cohomology group \( H^{n+1}(C^\ast) \) is isomorphic to the direct sum \( \bigoplus_{i+j=n+1} H^i(g) \otimes H^j_{dR}(M) \). Thus \( [g] \) can be decomposed into classes \( h_i \in H^i(g) \otimes H^j_{dR}(M) \) beyond \( h_0 = 0 \). These classes correspond to the \( g_i \), but in a slightly subtle way, as the latter are in general neither \( \delta \)-closed nor \( d \)-closed. In order to interpret the \( g_i \) as cocycles representing the \( h_i \) both equivalence relations (de Rham and Chevalley-Eilenberg) have to be divided out at the same time, as \( dg_i \in Im(\delta) \) and \( \delta g_i \in Im(d) \).

In order to formalize this, we introduce the quotient maps \( q_\Lambda : (\Lambda^k g^\ast) \rightarrow (\Lambda^k g^\ast)_{\text{ex}} \) and \( q_\Omega : \Omega^l(M) \rightarrow \Omega^l(M)_{\text{ex}} \), where for a cochain complex \( N \) we denote by \( N_{\text{ex}}^k \subset N^k \) the subspace of exact elements. As \( \delta \) resp. \( d \) are zero on \( (\Lambda^k g^\ast)_{\text{ex}} \) resp. \( \Omega^l_{\text{ex}}(M) \) they induce maps \( \bar{\delta} \) resp. \( \bar{d} \) rendering the following sequences exact:

\[
0 \longrightarrow H^k(g) = \frac{(\Lambda^k g^\ast)_{\text{ex}}}{(\Lambda^k g^\ast)_{\text{ex}}^\delta} \subset \frac{(\Lambda^k g^\ast)}{(\Lambda^k g^\ast)_{\text{ex}}} \stackrel{\bar{\delta}}{\longrightarrow} \frac{(\Lambda^{k+1} g^\ast)_{\text{ex}}}{(\Lambda^{k+1} g^\ast)_{\text{ex}}^\delta} \longrightarrow 0
\]

resp.

\[
0 \longrightarrow H^l_{dR}(M) = \frac{\Omega^l(M)_{\text{ex}}}{\Omega^l(M)_{\text{ex}}^\delta} \subset \frac{\Omega^l(M)}{\Omega^l(M)_{\text{ex}}} \stackrel{\bar{d}}{\longrightarrow} \frac{\Omega^{l+1}(M)_{\text{ex}}}{\Omega^{l+1}(M)_{\text{ex}}^\delta} \longrightarrow 0
\]

where \( \subset \) denotes here the canonical inclusion maps. By the exactness of the preceding two diagrams an element \( a \in \frac{(\Lambda^k g^\ast)}{(\Lambda^k g^\ast)_{\text{ex}}} \otimes \frac{\Omega^l(M)}{\Omega^l(M)_{\text{ex}}} \) satisfying \( (\bar{\delta} \otimes id)a = 0 \) and \( (id \otimes \bar{d})a = 0 \) already fulfills \( a \in H^k(g) \otimes H^l_{dR}(M) \).

For \( k \geq 2 \), \( (q_\Lambda \otimes q_\Omega)(g_k) \) satisfies these equations and consequently we can regard \( (q_\Lambda \otimes q_\Omega)(g_k) = h_k \) as an element of \( H^k(g) \otimes H^{n+1-k}_{dR}(M) \).

Using this description of Theorem 6.30, one can derive the following:

\[
0 \longrightarrow \bigoplus_{i=0}^{n-2} \Omega^i(M) \oplus \Omega_{\text{cl}}^{n-1}(M) \longrightarrow L_{\infty}(M, \omega) \xrightarrow{\delta g} X(M, \omega) \xrightarrow{\gamma} H^n_d(M) \]

where \( \Omega_{\text{cl}}^{n-1}(M) \) denotes the space of closed \((n-1)\)-forms on \( M \), the only non-trivial bracket of the leftmost space is the unary bracket \( d \) and the rightmost term is to be interpreted as an abelian Lie algebra. The map \( v \) can be lifted to a linear map \( j \) if and only if \( \langle [g_1]_{dR} = \gamma v = 0 \).
Theorem 6.31 ([CFRZ16]). If \( \omega \) admits a \( v \)-invariant potential \( \eta \), then it has a comoment given by the formulas

\[
f_k(x_1, \ldots, x_k) = (-1)^k \left( -1 \right)^{\frac{k(k+1)}{2}} \iota_{v_{x_k}} \cdots \iota_{v_{x_1}} \eta, \quad k \in \{1, \ldots, n\}.
\]

Example 6.32 (Symplectic). Let \((M, \omega)\) be symplectic and \(v : g \to X(M)\) a Lie algebra homomorphism. Then the above definition collapses to the classical notion of (equivariant) comoment. I.e. a multisymplectic comoment is a Lie algebra homomorphism \( f = f_1 : g \to C^\infty(M) \) satisfying \( X_{f(x)} = v_x \) for all \( x \in g \). A necessary condition for the existence such of a comoment is the \( v \)-invariance of \( \omega \). In such cases the sufficient condition is given by the classes \( g_1 = (x \mapsto \iota_{v_{x}} \omega) \in H^1(g) \otimes \Omega^1_{dR}(M) \) and \( g_2 = ((x_1, x_2) \mapsto \iota_{v_{x_2}} \iota_{v_{x_1}} \omega) \in H^2(g) \otimes \Omega^0_{dR}(M) \). If \( g_1 \) vanishes, then a linear (not necessarily equivariant) comoment exists and \( g_2 \) is the obstruction against equivariance (compare [Wei77]).

Example 6.33 (Sums and products). Let \((M, \omega)\) and \((\tilde{M}, \omega)\) be multisymplectic manifolds and \(v : g \to X(M)\) and \(\tilde{v} : \tilde{g} \to X(M)\) be Lie algebra homomorphisms. Then there is an induced Lie algebra homomorphism \((v, \tilde{v}) : g \oplus \tilde{g} \to X(M \times \tilde{M})\).

Example 6.34 (Multicotangent bundles). Let \( G \) be a Lie group and \( \vartheta^Q : Q \times G \to Q \) a right action. For each \( g \) the map \( \vartheta^Q_g : Q \to Q \) is a diffeomorphism. Then \( T\vartheta^Q_g : TQ \to TQ \) is a fiberwise linear diffeomorphism, which makes the following diagram commute:

\[
\begin{array}{ccc}
TQ & \xrightarrow{T\vartheta^Q_g} & TQ \\
\downarrow & & \downarrow \\
Q & \xrightarrow{\vartheta^Q_g} & Q
\end{array}
\]

With the map \( T\vartheta^Q_g \) at hand we construct a diffeomorphism \( \Lambda^n T^*(\vartheta^Q_g) : \Lambda^n T^*Q \to \Lambda^n T^*\tilde{Q} \). Let \( \alpha \) be an element of \( \Lambda^n T^*Q \) with \( \pi(\alpha) = p \in Q \) and \( v_1, \ldots, v_n \in T_{\vartheta^Q_g p} Q \).

\[
(\Lambda^n T^*(\vartheta^Q_g))(\alpha)(v_1, \ldots, v_n) = \alpha((T_p \vartheta^Q_g)^{-1} v_1, \ldots, (T_p \vartheta^Q_g)^{-1} v_n),
\]

where \((T_p \vartheta^Q_g)^{-1} : T_{\vartheta^Q_g p} Q \to T_p Q \) is the inverse of the linear map \( T_p \vartheta^Q_g : T_p Q \to T_{\vartheta^Q_g p} Q \). Then \( \vartheta^M_g := \Lambda^n T^*(\vartheta^Q_g) \) defines a right action which makes the following diagram commute:

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Thus we have a right action $\vartheta^M$ of $G$ on $M = \Lambda^n T^* Q$ for $1 \leq n \leq \dim(Q)$. To see that the action is $n$-plectic and even Hamiltonian with respect to the canonical $n$-plectic structure $\omega$ it suffices to show that the $n$-form $\theta$ is $G$-invariant. Regard $\alpha \in \Lambda^n T^* Q$ and $v_1, \ldots, v_n \in T_a(\Lambda^n T^* Q)$,

$$\left( (\vartheta^M_g)^* \theta \right)_a(v_1, \ldots, v_n) = \theta_{\vartheta^M_g}(T(\vartheta^M_g)v_1, \ldots, (T\vartheta^M_g)v_n)$$

$$= \vartheta^M_g(\alpha)((T\pi)(T\vartheta^M_g)v_1, \ldots, (T\pi)(T\vartheta^M_g)v_n)$$

$$= \vartheta^M_g(\alpha)((T(\vartheta^Q_g \circ \pi))v_1, \ldots, (T(\vartheta^Q_g \circ \pi))v_n)$$

$$= \alpha((T\vartheta^Q_g)^{-1}(T(\vartheta^Q_g \circ \pi))v_1, \ldots, (T\vartheta^Q_g)^{-1}(T(\vartheta^Q_g \circ \pi))v_n)$$

$$= \alpha((T\pi)v_1, \ldots, (T\pi)v_n)$$

$$= \theta_a(v_1, \ldots, v_n).$$

Thus $(\vartheta^M_g)^* \theta = \theta$ and thus $\omega$ is $G$-invariant (especially $\mathfrak{g}$-invariant) with an invariant potential. Theorem 6.31 now implies that the action is Hamiltonian with homotopy comoment defined via the $G$-invariant potential $\eta = -\theta$ of $\omega$.

**Example 6.35** (Subbundles of multicotangent bundles, cf. [CnCI91]). Let $V$ be an involutive subbundle of $TQ$ and $\vartheta^Q : Q \times G \to Q$ a right-action preserving $V$, i.e. $T\vartheta^Q_g(V) = V$ for all $g \in G$. Then, for $1 \leq i \leq \operatorname{rank}(V) \leq n \leq \dim(Q)$, $\vartheta^M$ preserves the subbundle $\Lambda^i T^* Q$, and thus defines a multisymplectic action on it. This action inherits the comoment from $M = \Lambda^n T^* Q$.

**Example 6.36** (Multicotangent bundles with exact magnetic term). We return to the setting of Proposition 2.30. Let $M = \Lambda^n T^* Q$ be equipped with the multisymplectic form $\omega = d(-\theta + \pi^* b)$. Let $Q \times G \to Q$ be a right action such that $b$ is $G$-invariant. By Theorem 6.31, the action admits a comoment with first component $f_1 : \mathfrak{g} \to \Omega^{n-1}(M)$ given by $f_1(x) = \iota_{\varphi_x}(\omega - \theta + \pi^* b)$.

**Remark 6.37.** We can use the above, to provide a counterexample announced in Subsection 6.4. Let the vector field $X$ be $G$-invariant and $X^h$ the horizontal lift of $X$. Assume that $\mathcal{L}_X b = da$, where $a$ is also $G$-invariant. Then $H = -\pi^* a + \iota_{X^h}(-\theta + \pi^* b)$ is invariant under the induced $G$-action on $M$ =
ΛnT*Q. Assuming furthermore db = 0, we can choose a := \iota_X b. Specialize to n = 2 and Q = G with the action by Q \times G \to Q, (q, g) \mapsto g^{-1} \cdot q. Thus we have:

- for x ∈ g = T_e G and g ∈ G is v_x(g) = -(r_g)_*(x), where r_g(h) = h \cdot g for h ∈ G. In particular, the generators v_x of the action are right-invariant vector fields on G.

- X is a left-invariant vector field, i.e. there exists \tilde{X} ∈ g such that for g ∈ G, X(g) = (l_g)_*(\tilde{X}) where l_g(h) = g \cdot h for h ∈ G,

- b is a closed left-invariant 2-form, i.e. it exists a \tilde{b} ∈ Λ^2g^* which is closed under the Chevalley-Eilenberg differential (the dual of the Lie algebra homology differential) and b(g) = (l_g^{-1})^*(\tilde{b}) = ((l_g^{-1})_*)^*(\tilde{b}) for all g ∈ G.

For any x ∈ g we compute

\[ L_X f_1(x) = \iota_{v_x} \pi^* (L_X b) = -\pi^* (d(\iota_{v_x} a)), \]

so f_1(x) being a strictly conserved quantity is equivalent to \iota_{v_x} a being a constant function on Q. Evaluating the function \iota_{v_x} a = b(X, v_x) at g ∈ Q = G one obtains

\[ -\tilde{b}(\tilde{X}, Ad_{g^{-1}}(x)). \tag{6} \]

It is clear that the function (6) is not constant in general. For instance, take G = SL_2(\mathbb{R}). A basis for g := sl_2(\mathbb{R}) is

\[ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

and [h, e] = 2e, [h, f] = -2f, [e, f] = h. So notably all elements of g are boundaries. The form b := e^* \wedge f^* ∈ Λ^2g^* is closed (actually exact) with respect to the Chevalley-Eilenberg differential. Taking X := f and x := h ∈ g = B_1(g) one computes that the function (6) attains the value 2\beta \delta at g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ∈ G, hence it is not a constant function on G. We conclude that for this choice of x ∈ B_1(g), the form f_1(x) is not strictly conserved.

Example 6.38 (Simple real Lie groups). Recall from Theorem 21.1. in [CE48] that for a semi-simple Lie algebra g we have H^1(g) = 0 = H^2(g) = 0 and 0 ≠ [\omega_0]_{CE} = [[\cdot, \cdot], \cdot]_{CE} ∈ H^3(g), where ⟨·, ·⟩ again denotes the Killing form of g.
Assume the real connected simple Lie group $G$ acts on itself from the right by $(g, x) \mapsto x \cdot g$. The corresponding infinitesimal action $v$ extends a $x \in g$ to a left-invariant vector field $v(x) = x^l$. This action preserves the multisymplectic structure $\omega$ on $G$. Since $\omega$ is bi-invariant we obtain:

$$\omega(v(x_1), v(x_2), v(x_3)) = \omega_e(x_1, x_2, x_3).$$

Thus $g_3 = [\omega_e]_{CE} \in H^3(g) = H^3(g) \otimes H^0_{dR}(G)$ does not vanish and therefore $v$ cannot admit a comoment.

On the other hand, the conjugation right-action $c : G \times G \to G$, $c_g(x) = g^{-1}xg$ does admit a comoment, cf. [CFRZ16].

### 6.6 The homological perspective on conserved quantities

In this subsection we rephrase the “generation of conserved quantities” via a comoment in a homological fashion. Let $\mathfrak{g}$ be a Lie algebra acting on an $n$-plectic manifold $(M, \omega)$, let $H$ be a Hamiltonian $(n-1)$-form, and $X_H$ be a Hamiltonian vector field of $H$. Assume that the action is locally $H$-preserving, i.e. $\mathcal{L}_v H$ is closed for all $x \in \mathfrak{g}$. The map $\mathfrak{g} \to H^{n-1}_{dR}(M), x \mapsto [\mathcal{L}_v H]$ measures how far the action is from being globally $H$-preserving. This map is 0 on $[\mathfrak{g}, \mathfrak{g}]$ and can thus be defined on $H_1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Furthermore it can be extended to a map on the whole Lie algebra homology.

**Proposition 6.39.** For every $k = 1, \ldots, \dim(\mathfrak{g})$ the map

$$A : H_k(\mathfrak{g}) \to H^{n-k}_{dR}(M), \ [p] \mapsto [\mathcal{L}_v p H]$$

is well-defined.

**Proof.** Let $p \in Z_k(\mathfrak{g})$. We first check that $\mathcal{L}_v p H$ is closed: Putting $v_p = \sum_i v_1^i \wedge \ldots \wedge v_k^i$ one has

$$d\mathcal{L}_v p H = (-1)^{k+1} \mathcal{L}_v p dH$$

$$= - \left( t_{v_p} dH + \sum_i \sum_{l=1}^k (-1)^l v_1^i \wedge \ldots \wedge \hat{v}_l^i \wedge \ldots \wedge v_m^i \mathcal{L} v_l^i dH \right) = 0,$$

where the first equality follows from Definition 6.17, the second from Remark 6.18 and the last one from $\partial p = 0$ and the closeness of $\mathcal{L}_v p H$. 

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Let \( q \in \Lambda^{k+1} \mathfrak{g} \). Similarly to above, we write \( v_q = \sum_l v_l^1 \wedge \ldots \wedge v_{k+1}^l \). We check that \( \mathcal{L}_{v_q} H \) is exact. By the definition of Lie derivative, this follows since

\[
\iota_{v_q} \partial q H = (-1)^{k+1} \mathcal{L}_{v_q} dH = -d\mathcal{L}_{v_q} H
\]

is exact. Again, here in the first equality we used Remark 6.18 and in the second that \( \mathcal{L}_{v_q} H \) is closed since the action is locally \( H \)-preserving.

**Remark 6.40.**

(i) If the action is globally \( H \)-preserving, the map \( \mathfrak{g} \to H^{n-1}_d M \) is zero, but the higher components of \( A \) do not necessarily vanish. This is exhibited by Example 6.13: \( t_{v_q} \wedge v_q dH = -t_\omega \wedge v_q (dx \wedge dy) = -1 \) is closed but not exact.

(ii) If the action is strictly \( H \)-preserving, then the map \( A \) is identically zero. Indeed, for every \( p \in Z_k(\mathfrak{g}) \) we have \( \mathcal{L}_{v_q} H = 0 \), as can be seen applying Lemma 6.16 to \( H \).

When a comoment exists, we can be more explicit:

**Lemma 6.41.** If \( \{ f_i \} \) is a comoment for the \( \mathfrak{g} \)-action, then the map \( A \) can be written as follows: for all \( p \in Z_k(\mathfrak{g}) \),

\[
A([p]) = (-1)^{k(k+1)/2} [\mathcal{L}_{X_H} f_k(p)].
\]

**Proof.** Let \( p \in Z_k(\mathfrak{g}) \). We have \( A([p]) = [\mathcal{L}_{v_p} H] = (-1)^{k} [t_{v_p} t_{X_H} \omega] \) using the definition of Lie derivative for multivector fields (see Remark 6.18). We can express this in terms of the comoment using

\[
(-1)^k t_{v_p} t_{X_H} \omega = t_{X_H} t_{v_p} \omega = (-1)^{k(k+1)/2} t_{X_H} df_k(p) = (-1)^{k(k+1)/2} (-d t_{X_H} f_k(p) + \mathcal{L}_{X_H} f_k(p)).
\]

Passing to the cohomology class finishes the proof.

**Theorem 6.42.** Let \( \nu : \mathfrak{g} \to \mathfrak{X}(M) \) be a Hamiltonian action with comoment \( \{ f_i \} \) on an \( n \)-plectic manifold \( (M, \omega) \) and \( H \) be a Hamiltonian \((n-1)\)-form. Assume that the action is locally \( H \)-preserving, i.e. \( \mathcal{L}_{v_q} H \) is closed for all \( x \in \mathfrak{g} \).
(i) The form $f_k(p)$ is locally conserved if $p \in Z_k(g)$ and globally conserved if $p \in B_k(g)$.

(ii) There is a canonical injective map $J : \frac{\mathcal{C}_{\text{loc}}(X_H)}{\mathcal{C}(X_H)} \hookrightarrow H_{dR}(M), [\alpha] \mapsto [\mathcal{L}_{X_H}\alpha]$. The map $A$ factors as

$$
H_k(g) \rightarrow \frac{\mathcal{C}^{n-k}(X_H)}{\mathcal{C}^{n-k}(X_H)} \xrightarrow{J} H_{dR}^{n-k}(M)
$$

for every $k$, where the first map is induced by $f_k$ multiplied by $(-1)^{k(k+1)/2}$. In particular, the map $A$ takes values in the subspace $J(\frac{\mathcal{C}_{\text{loc}}(X_H)}{\mathcal{C}(X_H)})$ of $H_{dR}(M)$.

(iii) If the action is strictly $H$-preserving, by Remark 6.40 (ii), $f_k(p)$ is globally conserved for all $p \in Z_k(g)$.

As a concluding remark, we note that part (i) implies Propositions 6.10 and 6.12 and part (iii) recovers Theorem 6.19.
Appendices

A Non-degenerate forms in low dimensions

In this appendix we put together known facts about three-forms in dimensions seven and eight scattered in the literature, which we need in Section 3.

A.1 Non-degenerate 3-forms in dimension 7

In the sequel we will give a list of representatives for the non-degenerate \( GL(V) \)-orbits of 3-forms in a seven-dimensional vector space \( V \), based on [Wes81, LPV08, BV03, Sal11]. We will also note, whether their stabilizers contain an element of negative determinant. Let \( \{e^1, ..., e^7\} \) be a basis of \( V^* \).

1) \( e^{127} + e^{134} + e^{256} \)
   The stabilizer of this form contains an element of negative determinant, given by
   \[
   \begin{pmatrix}
   -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 1 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 1 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 1 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & -1 \\
   \end{pmatrix}.
   \]

2) \( e^{125} + e^{127} + e^{147} - e^{237} + e^{346} + e^{347} \)
   The four connected components of the stabilizer of this form are represented by the identity, the following two matrices and their product (see [Sal11, Theorem 2.2])
   \[
   \begin{pmatrix}
   -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 1 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & -1 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & -1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & 1 \\
   0 & 0 & 0 & 0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 0 & 1 & 0 & 0 \\
   \end{pmatrix},
   \begin{pmatrix}
   -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 1 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & -1 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & 1 \\
   0 & 0 & 0 & 0 & 1 & 0 & 0 \\
   0 & 0 & 0 & 1 & 0 & 0 & 0 \\
   \end{pmatrix},
   \]
   which both have positive determinant. Hence, the stabilizer only contains elements of positive determinant.
3) $e^{123} + e^{145} + e^{167}$
   It is basically a symplectic form wedged with one extra direction. The stabilizer of this form contains an element of negative determinant, given by
   $$
   \begin{pmatrix}
   1 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 1 & 0 & 0 & 0 & 0 \\
   0 & 1 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 1 & 0 & 0 \\
   0 & 0 & 0 & 1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 0 & 0 & 1 & 0 \\
   \end{pmatrix}.
   $$

4) $e^{127} - e^{136} + e^{145} + e^{246}$
   The two connected components of the stabilizer of this form are generated by the identity and the matrix (see [Sal11, Theorem 2.4])
   $$
   \begin{pmatrix}
   -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 1 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & -1 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & -1 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & -1 \\
   \end{pmatrix},
   $$
   which has positive determinant. Hence, the stabilizer only contains elements of positive determinant.

5) $e^{123} - e^{145} + e^{167} + e^{246} + e^{257} + e^{347} - e^{356}$
   The stabilizer of this form is isomorphic to $\tilde{G}_2$, which is connected. Hence, the stabilizer only contains elements of positive determinant. Also, this form is stable.

6) $e^{127} - e^{136} + e^{145} + e^{235} + e^{246}$
   The two connected components of the stabilizer of this form are generated by the identity and the matrix (see [Sal11, Theorem 2.5])
   $$
   \begin{pmatrix}
   0 & 1 & 0 & 0 & 0 & 0 & 0 \\
   1 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & -1 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 0 & 1 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & -1 \\
   \end{pmatrix},
   $$
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which has positive determinant. Hence, the stabilizer only contains elements of positive determinant.

7) \(e^{125} + e^{136} + e^{147} + e^{237} - e^{246} + e^{345}\)
   The stabilizer of this form is connected by [BV03, Proposition 10]. Hence, the stabilizer only contains elements of positive determinant.

8) \(e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}\)
   The stabilizer of this form is isomorphic to \(G_2\), which is connected. Hence, the stabilizer only contains elements of positive determinant. Also, this form is stable.

A.2 Non-degenerate 3-forms in dimension 8

In this section we will enumerate representatives of the non-degenerate \(GL(V)\)-orbits in \(\Lambda^3 V^*\), for eight-dimensional \(V\), as they were calculated in [Djo83]. The last three orbits are stable, as proven in [LPV08].

1) \(e^{127} + e^{138} + e^{146} + e^{235}\)
2) \(e^{128} + e^{137} + e^{146} + e^{236} + e^{245}\)
3) \(e^{135} + e^{246} + e^{147} + e^{238}\)
4) \(-e^{135} + e^{146} + e^{236} + e^{245} + e^{127} + e^{348}\)
5) \(e^{138} + e^{147} + e^{156} + e^{235} + e^{246}\)
6) \(e^{128} + e^{137} + e^{146} + e^{247} + e^{256} + e^{345}\)
7) \(e^{156} + e^{178} + e^{234}\)
8) \(e^{158} + e^{167} + e^{234} + e^{256}\)
9) \(e^{148} + e^{157} + e^{236} + e^{245} + e^{347}\)
10) \(e^{134} + e^{244} + e^{156} + e^{278}\)
11) \(e^{135} - e^{245} + e^{146} + e^{236} + e^{678}\)
12) \(e^{137} + e^{237} + e^{256} + e^{148} + e^{345}\)
13) \(e^{135} + e^{245} + e^{146} - e^{236} + e^{678} + e^{127}\)
14) \(e^{138} + e^{147} + e^{245} + e^{267} + e^{356}\)
15) \(-e^{135} + e^{146} + e^{236} + e^{245} + e^{137} + e^{247} + e^{568}\)
16) \(-e^{135} + e^{146} + e^{236} + e^{245} + e^{127} + e^{347} + e^{568}\)
17) \(e^{128} + e^{147} + e^{236} + e^{257} + e^{358} + e^{456}\)
18) \(-e^{135} + e^{146} + e^{236} + e^{245} + e^{137} + e^{247} + e^{128} - e^{568}\)
19) \(e^{124} + e^{134} + e^{256} + e^{378} + e^{157} + e^{468}\)
20) \(e^{135} + e^{245} + e^{146} - e^{236} + e^{127} + e^{348} + e^{678}\)
21) \(e^{135} - e^{146} + e^{236} + e^{245} + e^{347} + e^{568} + e^{127} + e^{128}\)
B The infinitesimal symmetries of \((M^{>0}, \omega^f)\)

In this appendix we will determine all infinitesimal symmetries (i.e. multi-
symplectic vector fields) of the multisymplectic manifold \(M^{>0}, \omega = \omega^f\) with
\(f(x) = f(x^1, \ldots, x^6) = x^2\) from Example 4.20 and Proposition 5.14. As a con-
sequence of our calculation we get the 1-transitivity of the multisymplectic
diffeomorphisms of \((M^{>0}, \omega)\).

The differential equations

We will try to find the Lie algebra of symmetries of \(\omega\). For that we calculate
d\(i_X \omega = 0\) for \(X = X^i \partial_i\), where we use the Einstein summation convention
for the indices. We give a list of the coefficients of d\(i_X \omega\), which all have to
equal zero. The function given after "dx\(^{ijk}\)" is exactly the coefficient of dx\(^{ijk}\)
in d\(i_X \omega\).

\[
\begin{align*}
\text{dx}^{135} &: \partial_1 X^1 + \partial_3 X^3 + \partial_5 X^5 \\
\text{dx}^{146} &: - \partial_1 X^1 - \partial_4 X^4 - \partial_6 X^6 \\
\text{dx}^{236} &: - \partial_2 X^2 - \partial_5 X^5 - \partial_6 X^6 \\
\text{dx}^{245} &: x^2 (\partial_2 X^2 + \partial_4 X^4 + \partial_3 X^5) + X^2 \\
\text{dx}^{136} &: \partial_6 X^5 - \partial_3 X^4 - \partial_1 X^2 \\
\text{dx}^{145} &: \partial_4 X^3 - \partial_5 X^6 + x^2 \partial_1 X^2 \\
\text{dx}^{235} &: \partial_2 X^1 - \partial_5 X^6 + x^2 \partial_3 X^4 \\
\text{dx}^{246} &: - \partial_2 X^1 - \partial_4 X^3 + x^2 \partial_6 X^5 \\
\text{dx}^{125} &: \partial_3 X^3 - x^2 \partial_1 X^4 \\
\text{dx}^{126} &: \partial_1 X^3 - \partial_2 X^4 \\
\text{dx}^{256} &: \partial_5 X^3 + x^2 \partial_6 X^4 \\
\text{dx}^{156} &: \partial_6 X^3 + \partial_5 X^4 \\
\text{dx}^{123} &: \partial_2 X^5 + \partial_1 X^6 \\
\text{dx}^{124} &: x^2 \partial_1 X^5 + \partial_2 X^6 \\
\text{dx}^{134} &: \partial_4 X^5 + \partial_3 X^6 \\
\text{dx}^{234} &: x^2 \partial_3 X^5 + \partial_4 X^6
\end{align*}
\]
\begin{align*}
\text{dx}^{345} &= x^2 \partial_3 X^2 - \partial_1 X^1 \\
\text{dx}^{236} &= - \partial_2 X^1 + \partial_4 X^2 \\
\text{dx}^{356} &= \partial_6 X^1 + \partial_5 X^2 \\
\text{dx}^{456} &= x^2 \partial_6 X^2 + \partial_5 X^1
\end{align*}

The above equations can be obtained either by direct calculation or by using the SageMath source code provided in Appendix C.1.

The invariant foliation

From the calculations in Proposition 5.14, we know that any multisymplectic diffeomorphism preserves \( \iota_\xi \iota_\xi \Omega \) or reverts its sign. As the infinitesimal symmetries are tangent to the identity component of the multisymplectic diffeomorphisms, we know that any vector field \( X = X^i \partial_i \) preserving \( \omega \) also preserves \( \iota_\xi \iota_\xi \Omega = \frac{1}{16 \sqrt{x^2}} \text{dx}^1 \wedge \text{dx}^2 \). This gives us the auxiliary conditions:

\[
\begin{align*}
X^1 &= X^1(x^1, x^2) \\
X^2 &= X^2(x^1, x^2) \\
\partial_1 X^1 + \partial_2 X^2 - \frac{5}{2} \frac{1}{x^2} X^2 &= 0 \quad (*)
\end{align*}
\]

This renders block (e) from above obsolete as all partial derivatives involved are zero.

Calculating \( X^1 \) and \( X^2 \)

Dividing the fourth equation from (a) by \( x^2 \) we get:

\[
\begin{align*}
\partial_1 X^1 + \partial_3 X^3 + \partial_5 X^5 \\
\partial_1 X^1 + \partial_4 X^4 + \partial_6 X^6 \\
\partial_2 X^2 + \partial_3 X^3 + \partial_6 X^6 \\
\partial_2 X^2 + \partial_4 X^4 + \partial_5 X^5 + X^2 / x^2
\end{align*}
\]

Now we can add the first and second equation and subtract the third and fourth to get:

\[
\partial_1 X^1 - \partial_2 X^2 - \frac{1}{2} \frac{1}{x^2} X^2 = 0 \quad (**)\]
The equations (*) and (**) together imply:

\[ X^1 = \frac{3}{2} C_1(x^1) + C_2(x^2) \]
\[ X^2 = C'_1(x^1) \cdot x^2 \]

With the new information we can rewrite systems (a) and (b), where before rewriting (b) we need to multiply the first equation by \(x^2\):

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\partial_3 X^3 \\
\partial_4 X^4 \\
\partial_5 X^5 \\
\partial_6 X^6
\end{pmatrix}
= - \begin{pmatrix}
3/2 \\
3/2 \\
1 \\
2
\end{pmatrix}
C'_1(x^1) \quad (a')
\]
\[
\begin{pmatrix}
-1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x^2 \partial_3 X^4 \\
\partial_4 X^3 \\
\partial_5 X^6 \\
x^2 \partial_6 X^5
\end{pmatrix}
= \begin{pmatrix}
(x^2)^2 C''_1(x^1) \\
(x^2)^2 C''_1(x^1) \\
C'_2(x^2) \\
C'_2(x^2)
\end{pmatrix} \quad (b')
\]

This matrix (b’) is not invertable. If we sum up the rows 1 and 2 and subtract the rows 3 and 4, we get 0 on the LHS and the following equation on the RHS:

\[ 2(x^2)^2 C''_1(x^1) = 2C'_2(x^2) \]

or equivalently \(C''_1(x^1) = \frac{1}{(x^2)^2} C'_2(x^2)\). Hence, neither side depends on a variable i.e. we set \(\lambda := C''_1(x^1) = \frac{1}{(x^2)^2} C'_2(x^2)\). Then we know:

\[ C_1(x^1) = \frac{\lambda}{2} (x^1)^2 + bx^1 + c \]
\[ C_2(x^2) = \frac{\lambda}{3} (x^2)^3 + d \]

**Regrouping the system**

We observe that systems (a’) and (b’) can be simplified to:
Each of the systems has rank three, i.e. one system of equations could be left out. We subtract row 4 from row one and row 2 from row 4 in the first system and leave out the third row to get:

\[
\begin{align*}
\partial_3 X^3 - \partial_4 X^4 &= \frac{1}{2}(\lambda x^1 + b) \\
\partial_5 X^5 - \partial_6 X^6 &= -\frac{1}{2}(\lambda x^1 + b) \\
\partial_4 X^4 + \partial_5 X^5 &= -2(\lambda x^1 + b)
\end{align*}
\]  

(a’’)

Now we subtract the first from the fourth line and the first from the third line from the second matrix equation and leave out the first equation to get

\[
\begin{align*}
x^2 \partial_3 X^4 - \partial_4 X^3 &= 0 \\
\partial_5 X^6 - x^2 \partial_6 X^5 &= 0 \\
-\partial_4 X^3 + \partial_5 X^6 &= \lambda(x^2)^2
\end{align*}
\]  

(b’’)

Next we regroup the systems to the equations containing \(X^3, X^4\) (A), the equations containing \(X^5, X^6\) (B) and the two equations relating them (C).

126 : \(\partial_1 X^3 = \partial_2 X^4\)  
125 : \(\partial_2 X^3 = x^2 \partial_1 X^4\)  
a34 : \(\partial_3 X^3 = \partial_4 X^4 + \frac{1}{2}(\lambda x^1 + b)\)  
b34 : \(x^2 \partial_3 X^4 = \partial_4 X^3\)  
256 : \(\partial_5 X^3 = -x^2 \partial_6 X^4\)  
156 : \(\partial_6 X^3 = -\partial_5 X^4\)
\[
\frac{\partial}{\partial x^2} X^5 + \frac{\partial}{\partial x^2} X^6 = (B)
\]

\[
\frac{\partial}{\partial x^2} X^5 + \frac{\partial}{\partial x^2} X^6 = (B)
\]

\[
\frac{\partial}{\partial x^2} X^5 = \frac{\partial}{\partial x^2} X^6 - \frac{1}{2}(\lambda x^1 + b)
\]

\[
x^2 \frac{\partial}{\partial x^2} X^5 = \frac{\partial}{\partial x^2} X^6
\]

\[
\frac{\partial}{\partial x^5} + \frac{\partial}{\partial x^6} = -2(\lambda x^1 + b)
\]

\[
-\frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^5} = \lambda (x^2)^2
\]

\[
\frac{\partial}{\partial x^4} + \frac{\partial}{\partial x^5} = -2(\lambda x^1 + b)
\]

\[
-\frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^5} = \lambda (x^2)^2
\]

\[
\textbf{Simplifying the systems}
\]

We apply \( \frac{\partial}{\partial x} \) to equation (b34). This yields:

\[
\frac{\partial}{\partial x} X^4 + x^2 \frac{\partial}{\partial x} X^4 = \frac{\partial}{\partial x} X^3
\]

Next we regroup and use equation (125):

\[
\frac{\partial}{\partial x} X^4 = \frac{\partial}{\partial x} X^3 - x^2 \frac{\partial}{\partial x} X^4
\]

\[
\frac{\partial}{\partial x} X^4 = x^2 \frac{\partial}{\partial x} X^3 - x^2 \frac{\partial}{\partial x} X^4
\]

Now we use equations (a34) and (126) on the right hand side:

\[
\frac{\partial}{\partial x} X^4 = x^2 \frac{\partial}{\partial x} (\frac{\partial}{\partial x} X^3 - \frac{1}{2}(\lambda x^1 + b)) - x^2 \frac{\partial}{\partial x} \frac{\partial}{\partial x} X^3
\]

I.e:

\[
\frac{\partial}{\partial x} X^4 = -\frac{1}{2} \lambda x^2
\]

This implies, using equations (b34), (b56) and the third equation from (C):

\[
\begin{pmatrix}
\frac{\partial}{\partial x^4} X^4 \\
\frac{\partial}{\partial x^3} X^3 \\
\frac{\partial}{\partial x^5} X^5 \\
\frac{\partial}{\partial x^6} X^6
\end{pmatrix} =
\begin{pmatrix}
-\frac{1}{2} \lambda x^2 \\
-\frac{1}{2} \lambda (x^2)^2 \\
\frac{1}{2} \lambda (x^2)^2 \\
\frac{1}{2} \lambda x^2
\end{pmatrix}
\]

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We note that using the same technique on equation (b56) yields the same result.

We apply $\partial_2$ to equation (234):

$$\partial_3 X^5 + x^2 \partial_2 \partial_3 X^5 + \partial_2 \partial_4 X^6 = 0$$

Then we apply equation (124):

$$\partial_3 X^5 + x^2 \partial_2 \partial_3 X^5 - x^2 \partial_1 \partial_4 X^5 = 0$$

Then we apply equations (123) and (134):

$$\partial_3 X^5 - x^2 \partial_1 \partial_3 X^6 + x^2 \partial_1 \partial_4 X^6 = 0$$

I.e. $\partial_3 X^5 = 0$ and hence $\partial_4 X^6 = 0$.

We can do the same thing with the third block to arrive at: $\partial_6 X^4 = 0$ and hence $\partial_3 X^5 = 0$. (The order of the equation use is (256), (125), (126), (156)).

Hence, we can simplify the equations (b34) and (256) in System (A), the equations (b56) and (234) in system (B) and leave out the second equation in system (C).

**Solving System (A)**

The updated system (A) looks as follows:

126 : $\partial_1 X^3 = \partial_2 X^4$ (A')
125 : $\partial_2 X^3 = x^2 \partial_1 X^4$

a34 : $\partial_3 X^3 = \partial_4 X^4 + \frac{1}{2}(\lambda x^1 + b)$

b34 : $\partial_3 X^4 = -\frac{1}{2} \lambda x^2$

$\partial_4 X^3 = -\frac{1}{2} \lambda (x^2)^2$

256 : $\partial_5 X^3 = 0$
$\partial_6 X^4 = 0$

156 : $\partial_6 X^3 = -\partial_5 X^4$
Using equations (b34) and (256) we get:

\[ X^3 = -\frac{1}{2} \lambda (x^2)^2 x^4 + f(x^1, x^2, x^3, x^6) \]
\[ X^4 = -\frac{1}{2} \lambda x^2 x^3 + g(x^1, x^2, x^4, x^5) \]

The remaining equations turn into:

\[ \partial_1 f = -\frac{1}{2} \lambda x^3 + \partial_2 g \]
\[ \partial_2 f - \lambda x^2 x^4 = x^2 \partial_1 g \]
\[ \partial_3 f = \partial_4 g + \frac{1}{2}(\lambda x^1 + b) \]
\[ \partial_6 f = -\partial_5 g \]

Rewriting a little bit we get:

\[ \partial_1 f + \frac{1}{2} \lambda x^3 = \partial_2 g \]
\[ \partial_2 f = \lambda x^2 x^4 + x^2 \partial_1 g \]
\[ \partial_3 f = \partial_4 g + \frac{1}{2}(\lambda x^1 + b) \]
\[ \partial_6 f = -\partial_5 g \]

As the left hand sides only depend on \((x^1, x^2, x^3, x^6)\) and the right hand sides only depend on \((x^1, x^2, x^4, x^5)\) actually both sides only depend on \(x^1\) and \(x^2\). This implies\(^3\):

\[ f = \alpha(x^1, x^2) x^3 + \beta(x^1, x^2) x^6 + \gamma(x^1, x^2) \]
\[ g = (\alpha(x^1, x^2) - \frac{1}{2}(\lambda x^1 + b)) x^4 - \beta(x^1, x^2) x^5 + \delta(x^1, x^2) \]

Now we can use the first equation from the system \((A")\) to get:

\[ \partial_1 \alpha = -\frac{1}{2} \lambda \]
\[ \partial_1 \beta = 0 \]
\[ \partial_2 \alpha = 0 \]
\[ \partial_2 \beta = 0 \]
\[ \partial_1 \gamma = \partial_2 \delta \]

\(^3\)using the latter two equations from system \((A")\)
The second equation gives the same first four lines and the additional equation:

\[ \partial_2 \gamma = x^2 \partial_1 \delta \]

So \( \beta = \beta_0 \) is constant and \( \alpha = -\frac{1}{2} \lambda x^1 + \alpha_0 \). All in all we get:

\[
X^3 = -\frac{1}{2} \lambda (x^2)^2 x^4 + (-\frac{1}{2} \lambda x^1 + \alpha_0) x^3 + \beta_0 x^6 + \gamma(x^1, x^2)
\]

\[
X^4 = -\frac{1}{2} \lambda x^2 x^3 + (-\lambda x^1 + \alpha_0 - \frac{1}{2} b) x^4 - \beta_0 x^5 + \delta(x^1, x^2),
\]

where \( \gamma, \delta \) satisfy \( \partial_1 \gamma = \partial_2 \delta, \partial_2 \gamma = x^2 \partial_1 \delta \).

Solving System (B)

The same procedure can be done with system (B):

\[
124 : \quad x^2 \partial_1 X^5 + \partial_2 X^6 \quad \text{(B')} \\
123 : \quad \partial_2 X^5 + \partial_1 X^6 \\
234 : \quad \partial_3 X^5 = 0 \\
\quad \partial_4 X^6 = 0 \\
134 : \quad \partial_4 X^5 + \partial_3 X^6 \\
a56 : \quad \partial_5 X^5 = \partial_6 X^6 - \frac{1}{2} (\lambda x^1 + b) \\
b56 : \quad \partial_6 X^5 = \frac{1}{2} \lambda x^2 \\
\quad \partial_5 X^6 = \frac{1}{2} \lambda (x^2)^2
\]

\[
X^5 = \frac{1}{2} \lambda x^2 x^6 + f(x^1, x^2, x^4, x^5) \\
X^6 = \frac{1}{2} \lambda (x^2)^2 x^5 + g(x^1, x^2, x^3, x^6)
\]
We note that these are new functions not related a priori to the functions from the last section.

\[
\partial_1 f = \frac{1}{x^2} (\lambda x^2 x^5 + \partial_2 g) \quad (B'')
\]

\[
\partial_2 f + \frac{1}{2} \lambda x^6 = - \partial_1 g
\]

\[
\partial_4 f = - \partial_3 g
\]

\[
\partial_5 f = \partial_6 g - \frac{1}{2} (\lambda x^1 + b)
\]

This implies

\[
f = \epsilon (x^1, x^2)x^4 + \zeta (x^1, x^2)x^5 + \eta (x^1, x^2)
\]

\[
g = - \epsilon (x^1, x^2)x^3 + (\zeta (x^1, x^2) + \frac{1}{2} (\lambda x^1 + b))x^6 + \theta (x^1, x^2)
\]

Again we get \( \epsilon = \epsilon_0, \zeta = -\lambda x^1 + \zeta_0 \) and \( \partial_1 \eta = -\frac{1}{2} \partial_2 \theta, \partial_2 \eta = -\partial_1 \theta \). So all in all we have:

\[
X^5 = \frac{1}{2} \lambda x^2 x^6 + \epsilon_0 x^4 + (-\lambda x^1 + \zeta_0) x^5 + \eta (x^1, x^2)
\]

\[
X^6 = \frac{1}{2} \lambda (x^2)^2 x^5 - \epsilon_0 x^3 + (-\frac{1}{2} \lambda x^1 + \zeta_0 + \frac{1}{2} b) x^6 + \theta (x^1, x^2),
\]

where \( \partial_1 \eta = -\frac{1}{2x^2} \partial_2 \theta, \partial_2 \eta = -\partial_1 \theta \).

**Solving the remaining equation from (C)**

The remaining equation from (C) reads:

\[
-\lambda x^1 + \alpha_0 - \frac{1}{2} b + (-\lambda x^1 + \zeta_0) = -2(\lambda x^1 + b)
\]

I.e. \( \zeta_0 = \frac{1}{2} b - \alpha_0 \).

**Putting everything together**

A general multisymplectic vector field takes the form:
\[ X^1 = \frac{3}{2} \lambda (x^1)^2 + bx^1 + \frac{\lambda}{3} (x^2)^3 + c \]
\[ X^2 = (\lambda x^1 + b) \cdot x^2 \]
\[ X^3 = -\frac{1}{2} \lambda x^2 x^4 + (-\frac{1}{2} \lambda x^1 + \alpha_0) x^3 + \beta_0 x^6 + \gamma(x^1, x^2) \]
\[ X^4 = -\frac{1}{2} \lambda x^2 x^3 + (-\lambda x^1 + \alpha_0 - \frac{1}{2} b) x^4 - \beta_0 x^5 + \delta(x^1, x^2) \]
\[ X^5 = \frac{1}{2} \lambda x^2 x^6 + \epsilon_0 x^4 + (-\lambda x^1 + \frac{1}{2} b - \alpha_0) x^5 + \eta(x^1, x^2) \]
\[ X^6 = \frac{1}{2} \lambda (x^2)^2 x^5 - \epsilon_0 x^3 + (-\frac{1}{2} \lambda x^1 + b - \alpha_0) x^6 + \theta(x^1, x^2), \]

where \( \gamma, \delta \) and \( \eta, \theta \) satisfy

\[
\partial_2 \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 & x^2 \\ 1 & 0 \end{pmatrix} \partial_1 \begin{pmatrix} \gamma \\ \delta \end{pmatrix},
\]

\[
\partial_2 \begin{pmatrix} \theta \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & -x^2 \\ -1 & 0 \end{pmatrix} \partial_1 \begin{pmatrix} \theta \\ \eta \end{pmatrix}.
\]
C Computations in SageMath

In the following two subsections we provide the SageMath sourcecode (and output), which facilitated the calculations in Appendix B and the search for suitable examples of three-forms in $\mathbb{R}^6$ in 4.5. We have used the [Sag17] programming language.

C.1 Determining the equations for Appendix B

The following code calculates the differential form $\mathcal{L}_X \omega$, where $f(x^1, \ldots, x^6) = x^2$ for a general vector field $\sum_{i=1}^{6} y_i \partial_{x_i} = \sum y_i \partial_i$.

```
M = Manifold(6, 'M', r'M')
cart_ch.<x1,x2,x3,x4,x5,x6> = M.chart('x1/uni2423x2:(0,+oo)/uni2423x3/uni2423x4/uni2423x5/uni2423x6')
XM = M.vector_field_module()
Om3 = M.diff_form_module(3)
w=Om3([],'w')
w[0,2,4]=1
w[0,3,5]=-1
w[1,2,5]=-1
w[1,3,4]=-x2
y1 = M.scalar_field(function('y1')(x1,x2,x3,x4,x5,x6), name='y1')
y2 = M.scalar_field(function('y2')(x1,x2,x3,x4,x5,x6), name='y2')
y3 = M.scalar_field(function('y3')(x1,x2,x3,x4,x5,x6), name='y3')
y4 = M.scalar_field(function('y4')(x1,x2,x3,x4,x5,x6), name='y4')
y5 = M.scalar_field(function('y5')(x1,x2,x3,x4,x5,x6), name='y5')
y6 = M.scalar_field(function('y6')(x1,x2,x3,x4,x5,x6), name='y6')
u = M.vector_field('u')
u[0] = y1
u[1] = y2
u[2] = y3
u[3] = y4
u[4] = y5
u[5] = y6
r=w.lie_der(u)
r.display()
```

We get the following result:
\[
-d(y5)/dx2 - d(y6)/dx1) dx1/dx2 + (x2*d(y5)/dx1 + d(y6)/dx2)
\]
\[
+ dx1/dx2/dx4 + (-x2*d(y4)/dx1 + d(y3)/dx2) dx1/dx2/dx5
\]
\[
+ (d(y3)/dx1 - d(y4)/dx2) dx1/dx2/dx6 + (d(y5)/dx4 + d(y6))/dx3
\]
\[
dx3/dx3/dx4 + (d(y1)/dx1 + d(y3)/dx3 + d(y5)/dx5) dx1
\]
\[
+ dx3/dx5 + (-d(y2)/dx1 - d(y4)/dx3 + d(y5)/dx6) dx1/dx3
\]
\[
+ dx6 + (x2*d(y2)/dx1 + d(y3)/dx4 - d(y6)/dx5) dx1/dx4/dx5
\]
\[
+ (-d(y1)/dx1 - d(y4)/dx4 - d(y6)/dx6) dx1/dx4/dx6 + (-d(y3)/dx6 - d(y4)/dx5) dx1/dx5/dx6 + (-x2*d(y5)/dx3 - d(y6))
\]
\[
dx4/dx2/dx3/dx1608.074244 + (x2*d(y4)/dx3 + d(y1)/dx2 - d(y6)/dx5) dx2/dx3/dx5 + (-d(y2)/dx2 - d(y3)/dx3 - d(y6))
\]
\[
dx6/dx2/dx3/dx6 + (x2*d(y2)/dx2 + d(y4)/dx4 + d(y5)/dx5)
\]
\[
+ y2(x1, x2, x3, x4, x5, x6)) dx2/dx4/dx5 + (x2*d(y5)/dx6 - d(y1)/dx2 - d(y3)/dx4) dx2/dx4/dx6 + (-x2*d(y4)/dx6 - d(y3))
\]
\[
dx5/dx2/dx5/dx6 + (x2*d(y2)/dx3 - d(y1)/dx4) dx3/dx4/dx5/dx6 + (-d(y1)/dx3 + d(y2)/dx4) dx3/dx4/dx5/dx6 + (d(y1)/dx6 + d(y2)/dx5/dx6 + (x2*d(y2)/dx6 + d(y1)/dx5) dx4/dx5/dx6
\]
\[
dx5/dx6
\]

C.2 Code for analyzing three-forms in $\mathbb{R}^6$

The following SageMath code defines a function, which analyses a three-form (with one symbolic parameter $t$), and yields as output:

- an expression in $t$, which is zero if and only if the form is degenerate,
- the characteristic endomorphism $J$ (with respect to the standard volume of $\mathbb{R}^6$),
- the trace of $J^2$,
- the eigenspaces of $J$.

```python
def analyse(componentdictionary):
    basering=SR;
    V = FiniteRankFreeModule(basering, 7,name='V') #7-dimensional,
    L3V = V.dual_exterior_power(3)
    e = V.basis('e')
    differentialform = L3V([], name='differentialform')
```

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for d in componentdictionary:
    differentialform[d]=componentdictionary[d]

AAt=matrix(basering,6,36)  #where A is a matrix representing the
↔ contraction
for i in range(6):
    for j in range(6):
        for k in range(6):
            AAt[i,5*j+k]=differentialform.contract(e[i+1]).components
↔ ()[j+1,k+1]
print("The_form_is_non-degenerate_if_and_only_if_the_following_
↔ expression_is_nonzero:")
print(det(AAt*AAt.transpose()))
print(""")

J=matrix(basering,6,6)
for i in range(6):
    fiveform=differentialform.contract(e[i+1]).wedge(differentialform)
    for j in range(6):
        index=range(7);index.remove(j+1);index.remove(0)
        J[j,i]=(-1)**j*fiveform[index[0],index[1],index[2],index[3],index
↔ [4]]

print("The_characteristic_endomorphism_J_is_given_by:")
print(J)
print(""")

print("The_trace_of_its_square_is_given_by:")
print((J*J).trace())
print(""")

print("The_eigenspace_of_J_are:")
print((J).eigenvectors_left())
print(""")

Once the above function is defined the form can be entered by a list of
components, each of which has the form (i, j, k) : f(t) and stands for the
summand f(t)e^i ∧ e^j ∧ e^k, and then analysed. For the form

α_t = e^1 ∧ e^3 ∧ e^5 − e^1 ∧ e^4 ∧ e^6 − e^2 ∧ e^3 ∧ e^6 + t · e^2 ∧ e^4 ∧ e^5
the function call then reads as follows:

```
t=SR.var('t')
alphat={}(1,3,5):1,(1,4,6):-1,(2,3,6):-1,(2,4,5):t}
analyse(alphat)
```

The program then returns the following:

The form is non–degenerate if and only if the following expression is nonzero:

\[288*t^6 + 576*t^4 + 360*t^2 + 72\]

The characteristic endomorphism J is given by:

\[
\begin{bmatrix}
0 & -2*t & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2*t & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 2*t & 0 \\
\end{bmatrix}
\]

The trace of its square is given by:

\[24*t\]

The eigenspaces of J are:

\[
\begin{bmatrix}
(-2*sqrt(t)), [(1, sqrt(t), 0, 0, 0, 0), (0, 0, 1, sqrt(t), 0, 0), (0, 0, 0, 0, 1, -1/sqrt(t))], 3), (2*sqrt(t), [(1, -sqrt(t), 0, 0, 0, 0), (0, 0, 1, -sqrt(t), 0, 0), (0, 0, 0, 0, 1/sqrt(t))], 3)
\end{bmatrix}
\]
References


