Large scale asymptotics for random convex hulls

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Julian Grote

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Meinen Eltern in Dankbarkeit gewidmet.

"*It depends on how many beers one has already drunken when choosing the points.*"
- my mother, on Sylvester's four point problem.

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Chapter 1

Introduction

This chapter begins with a general introduction into the theory of stochastic geometry and random polytopes, presenting some milestones in their historical development. In particular, we focus on Buffon's needle problem from 1777 and Sylvester's four point problem from 1864, which enable the reader to gain some insights into two mathematical problems that are regarded as the starting points in the theory of stochastic geometry and random polytopes, respectively. Then, we use the class of Gaussian polytopes to state some applications of random polytopes to other fields of mathematics, to explain results concerning random polytopes that have been obtained in the last decades and to arrive at a point that can be considered as a kick-off for Chapter 3. Moreover, we provide the reader with a historical background, dealing with the theory of the approximation of smooth convex bodies by random polytopes. Finally, this leads to a starting point for Chapter 5.
Next, we switch to the actual content of this thesis and describe its guideline. We introduce three different random polytope models, combined with the associated issues that are analyzed and solved in the upcoming chapters. Furthermore, we state exemplary results to give an heuristic overview on the main topics of the thesis, and specify the research papers that this thesis is based on.

1.1 General introduction

Stochastic geometry is a branch of mathematics at the borderline between convex geometry and probability theory. Its origin is in the year 1777, when Buffon [21] found an answer to the following question, known as Buffon's needle problem in literature:

> Let us imagine two people stand in a room whose floor consists of parallel ordered boards, having all the same width. Now, one person throws up a needle of given length, betting that it falls on one of the splices between the boards. The other person bets on the opposite event. We assume, additionally, that the needle is not longer than the boards are wide. Under which assumption on the length of the needle in relation to the width of the boards can this game be considered to be fair?

This problem can be regarded as an easier version of the game 'franc Carreau', very popular in France at that time. Here, two people stand in a room and throw a coin onto its chessboard patterned floor. The question, how high is the probability that the coin will be completely contained in one of the squares, initiated also the interest in Buffon's needle problem.

In order to provide the reader with an overview on where both geometry and probability come into play, we have decided to enumerate the main ideas of Buffon's proof. Let $2a$ and $2b$ denote the width of the boards and the length of the needle, respectively, satisfying $2b \leq 2a$ by assumption. Now, consider a strip of width $2a - 2b$ in the middle of one of the boards (see Figure 1.1). If the center of the needle, symbolized by M,

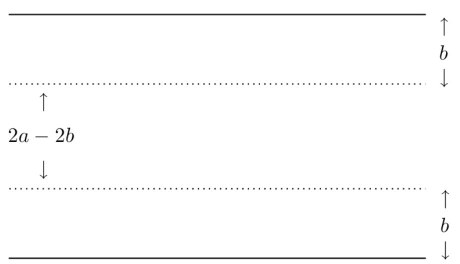

FIGURE 1.1: Buffon's needle problem I.

falls into this strip, the needle cannot touch one of the splices. On the other hand, if M falls into one of the two remaining stripes of width b, the needle can either touch a splice or not. Without loss of generality, let us assume that M falls into the upper one, and denote by x the shortest distance from M to the corresponding upper splice.

If the needle falls down in such a way that it just touches the splice with one of its ends, this direction and the perpendicular dropped of M onto the splice form the angle α (see Figure 1.2). Now, the probability that the needle hits the upper splice is $\frac{2\alpha}{\pi}$,

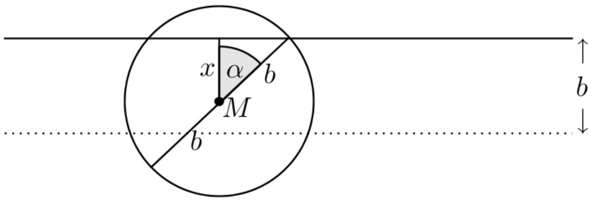

FIGURE 1.2: Buffon's needle problem II.

and in view of $\cos\alpha = \frac{x}{b}$, it follows that $\alpha = \arccos\frac{x}{b}$. Therefore, the probability that the needle hits the splice, given that it is at distance x to M, is

$$\frac{2}{\pi}\arccos\frac{x}{b}.$$

Since x can take values between 0 and b, we average over all of those. Thus, the probability that the needle hits a splice, given that it lies in the upper strip, is

$$\frac{1}{b}\frac{2}{\pi}\int_0^b \arccos\frac{x}{b}\,\mathrm{d}x = \frac{2}{\pi}.$$

If M is located in the lower strip, we obtain the same probability. Because the probability that M falls into one of the two stripes is $\frac{2b}{2a}$, finally, the probability that the needle hits a splice is given by

$$\frac{2b}{2a}\frac{2}{\pi} = \frac{2b}{a\pi}. \tag{1.1}$$

As a consequence, to achieve a fair game between the two players, the length of the needle, $2b$, and the width of the boards, $2a$, need to satisfy

$$1 - \frac{2b}{a\pi} = \frac{2b}{a\pi} \Leftrightarrow 2b = \frac{a\pi}{2}.$$

Hence, their relation has to fulfill

$$\frac{2b}{2a} = \frac{\pi}{4},$$

which is the answer to Buffon's needle problem.

The first proof in the setting of a 'long needle', that is, $2b \geq 2a$ in our above used spelling, is given in an article by Wolf [134], who published results obtained by Merian in 1850. In this situation, the probability that the needle hits a splice is given by

$$\frac{2\beta}{\pi} + \frac{2b}{a\pi}(1 - \sin\beta), \tag{1.2}$$

where β is the angle arising when both endpoints of the needle are located on neighboring splices (see Figure 1.3). In particular, in the case that $2a = 2b$, it follows that

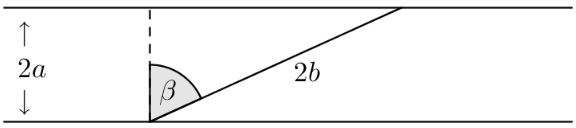

FIGURE 1.3: Buffon's needle problem III.

$\beta = 0$ and, thus, the two probabilities in (1.1) and (1.2) do match.

In the last two centuries, random polytopes, or random convex hulls, have become one of the outstanding models of study in stochastic geometry. Indeed, they have seen numerous applications to other branches of mathematics, such as asymptotic geometric analysis, coding theory, compressed sensing, computational geometry, optimization and multivariate statistics. Also consider the surveys about random polytopes by Bárány [7], Hug [72] and Reitzner [110] for further details and references. We explicitly state some of these applications in the setting of Gaussian polytopes below.
A random polytope emerges as the result of a random experiment, obtained in our setting by choosing $n \in \mathbb{N}$ random points in $\mathbb{R}^d$, $d \geq 2$, according to some probability measure. The random convex hull of this point set, that is, the smallest closed convex set containing all the points, defines the random polytope.
The most natural way of choosing the points might be the one where they are independent and uniformly distributed. In view of this purpose, we restrict ourself to some bounded set K in $\mathbb{R}^d$. Since the induced random polytope is convex and should also be contained in this set, we assume K to be convex itself.
For the sake of precision, we always assume K to be a convex body in $\mathbb{R}^d$, i.e., a convex, compact and non-empty subset of $\mathbb{R}^d$, and $X_1, \dots, X_n$ independent and uniformly distributed points in K. Then, the random convex hull of this point set is denoted by

$$K_n := \operatorname{conv}(X_1, \dots, X_n).$$

Figure 1.4 illustrates some convex body in the planar case, from which we choose uniformly distributed random points and, finally, the corresponding convex hull K_n.

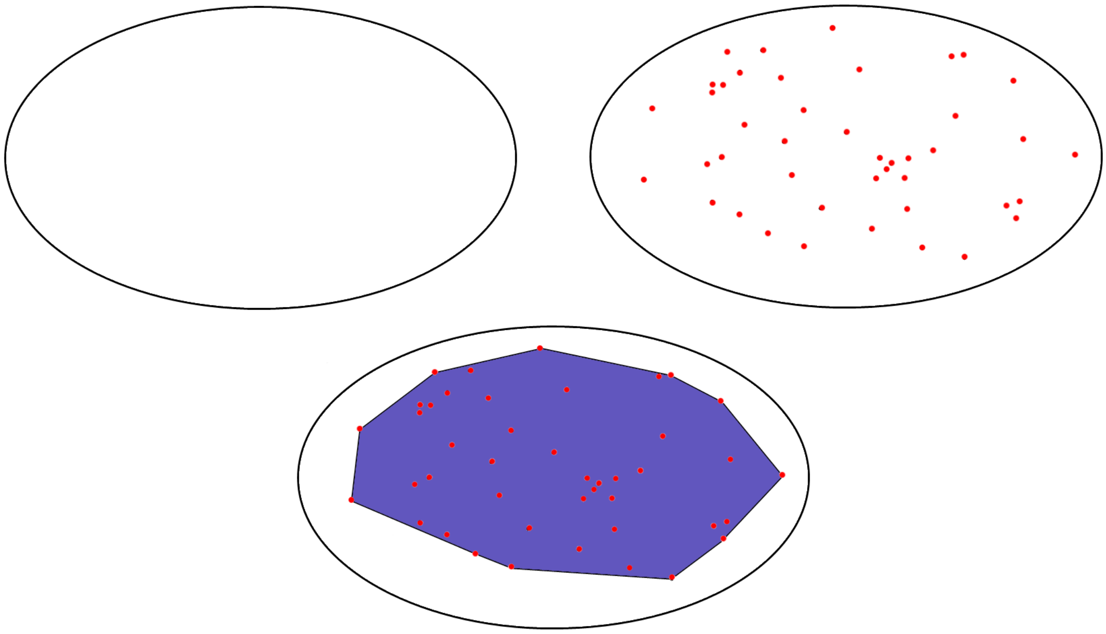

FIGURE 1.4: Construction of K_n.

The origin of random polytopes is traditionally related to the Sylvester's four point problem. In 1864, Sylvester [124] posed a problem, very innocent at the first glance, that in more recent terminology reads as follows:

Show, that the probability that the convex hull of four points taken at random in an indefinite plane is a triangle, is $\frac{1}{4}$.

Some years later, in 1885, Crofton [30, Page 785] underlined the meaning of Sylvester's four point problem by writing:

"*Historically, it would seem that the first question given on local probability, since Buffon, was the remarkable four point problem of Prof. Sylvester.*"

Many people started working on this issue and presented different 'solutions'. While Sylvester [124] himself gave a proof, Ingleby [74] published answers obtained by De-Morgan and Wilson, arguing that the probability has to be $\frac{1}{2}$, respectively $\frac{1}{3}$. If $\boldsymbol{o}$ is the origin of $\mathbb{R}^d$, we denote by $\mathbb{B}^d(\boldsymbol{o}, r)$ the d-dimensional closed ball of radius $r > 0$, centered at the origin. Woolhouse [136] initially chose the four points at random in $\mathbb{B}^2(\boldsymbol{o}, r)$. Then, he computed the probability inside this circle. Finally, he took the limit, as $r \to \infty$. In other words, he treated $\mathbb{R}^2$ as a circle of infinite radius.

Using this idea, he obtained a probability of

$$\frac{35}{12\pi^2}.$$

It was almost immediately understood that these inconsistent results were due to the instruction that the points should be taken at random in an indefinite plane, allowing for different interpretations of the underlying probability measure. To overcome this culprit, Sylvester modified his question. Given a convex body K in the plane, he asked for the probability that the convex hull of four points chosen from K independently and uniformly distributed forms a triangle (see Figure 1.5). We denote this probability by $\mathbb{P}(K)$. In the above introduced notation, this is the probability that K_4 forms

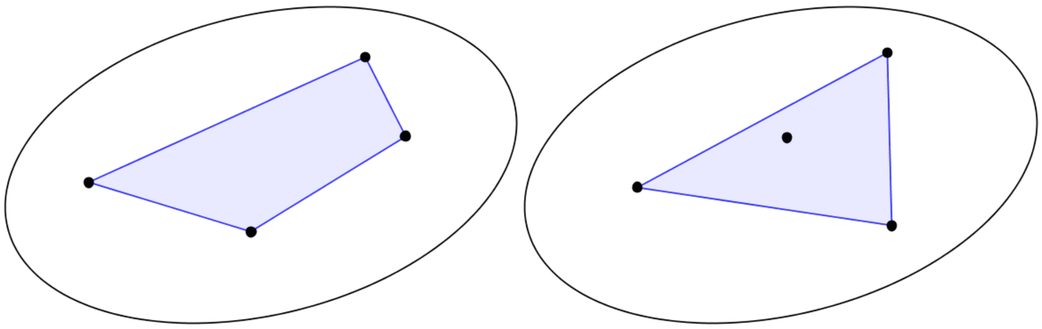

FIGURE 1.5: Sylvester's four point problem.

a triangle. Additionally, Sylvester asked for classes of convex bodies that minimize, respectively maximize $\mathbb{P}(K)$.

Since the points are chosen independently and uniformly distributed, it follows that

$$\begin{aligned}\mathbb{P}(K) &= 4\,\mathbb{P}(\{\text{One point lies inside the triangle formed by the other three points.}\})\\ &= 4\,\frac{\mathbb{E}[\mathrm{vol}_2(K_3)]}{\mathrm{vol}_2(K)}.\end{aligned}$$

Here, $\mathrm{vol}_2(\cdot)$ denotes the 2-dimensional volume, namely, the area, while, generally, $\mathrm{vol}_j(\cdot)$, $j \in \mathbb{N}$, is the j-dimensional volume of the underlying set. Moreover, the expected area of the random polytope K_3 is given by

$$\mathbb{E}[\mathrm{vol}_2(K_3)] = \frac{1}{\mathrm{vol}_2(K)^3} \int_K \int_K \int_K \mathrm{vol}_2(\mathrm{conv}(x_1, x_2, x_3))\, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3,$$

where $\mathrm{d}x_i$, $i \in \{1, 2, 3\}$, is the Lebesgue measure.

Woolhouse [136] computed this integral in the case of K being a circle of arbitrary radius. As mentioned above, he obtained that

$$\mathbb{P}(K) = \frac{35}{12\pi^2}.$$

On the other hand, if K is an arbitrary triangle, Sylvester [125] proved that

$$\mathbb{P}(K) = \frac{1}{3}.$$

In 1867, Woolhouse [135] computed the probability when the underlying convex body is given by a square, a parallelogram and a regular hexagon, respectively. In particular, he noted that for all discussed sets K, it holds that

$$\frac{35}{12\pi^2} \leq \mathbb{P}(K) \leq \frac{1}{3}.$$

Therefore, it seemed natural to conjecture that the minimum and maximum of $\mathbb{P}(K)$ are obtained when K is a circle and a triangle, respectively. In 1885, Crofton [30] proved that the minimum is indeed attained when K is a circle. It has taken 32 more years, until 1917, before a unified proof of the complete conjecture was established by Blaschke [16].

Sylvester's four point problem in the plane can easily be generalized to an arbitrary underlying space dimension. For $d \geq 2$, let K be a convex body in $\mathbb{R}^d$ and K_{d+2} be the convex hull of $d+2$ independent and uniformly distributed points in K. Moreover, we denote by $\mathbb{P}_d(K)$ the probability that one of the $d+2$ chosen points lies in the convex hull of the $d+1$ others. Thus, the probability discussed above can be rewritten as

$$\mathbb{P}(K) = \mathbb{P}_2(K).$$

At the end of his proof in the planar case, Blaschke claimed that his results would easily carry over to higher dimensions. More precisely, he claimed that $\mathbb{P}_d(K)$ would be minimized and maximized when K is a ball and a regular simplex in $\mathbb{R}^d$, respectively. Unfortunately, only in 1973, Groemer [54] verified the minimization property of the ball. However, the conjecture that $\mathbb{P}_d(K)$ is maximized when K is a regular simplex is still unsolved and, therefore, still a current topic of research. This is due to the fact that a positive solution to this problem would immediately imply the famous hyperplane conjecture, one of the major open problems in the asymptotic theory of Banach spaces (see Milman and Pajor [102]). Let us give a small background about this conjecture.

We denote by $GL(\mathbb{R}^d, \mathbb{R}^d)$ the family of linear isomorphisms $T : \mathbb{R}^d \to \mathbb{R}^d$, and let K be a convex body in $\mathbb{R}^d$. Without loss of generality, we assume it to be centered, i.e.,

$$\int_K \langle x, u \rangle \, \mathrm{d}x = 0,$$

for all u on the unit sphere $\mathbb{S}^{d-1}$, where $\langle \cdot, \cdot \rangle$ indicates the standard scalar product on $\mathbb{R}^d$. Then, there exists a $T \in GL(\mathbb{R}^d, \mathbb{R}^d)$ such that $K := T(K)$ is isotropic, i.e., has unit volume and there is an absolute constant $L_K \in (0, \infty)$ satisfying

$$\int_K \langle x, u \rangle^2 \, \mathrm{d}x = L_K^2,$$

for all $u \in \mathbb{S}^{d-1}$ (see [102]). L_K is called the isotropic constant of K. In 1986, Bourgain [20] conjectured that a uniform upper bound on the isotropic constant should hold, simultaneously for any convex body and any space dimension. More precisely, he conjectured that there is an absolute constant $C \in (0, \infty)$ such that

$$L_K \leq C,$$

for any $d \geq 2$ and any convex body K in $\mathbb{R}^d$. Unfortunately, the best known bound in the literature is

$$L_K \leq C \, d^{\frac{1}{4}},$$

where $C \in (0, \infty)$ is an absolute constant (see Klartag [82]). The isotropic constant conjecture is equivalent to the aforementioned hyperplane conjecture, stating that there exists another absolute constant $C \in (0, \infty)$ such that

$$\max\{\mathrm{vol}_{d-1}(K \cap u^\perp) : u \in \mathbb{S}^{d-1}\} \geq C,$$

for any $d \geq 2$ and any centered convex body K in $\mathbb{R}^d$ of unit volume. Here, $u^\perp$ is the hyperplane containing the origin and orthogonal to $u \in \mathbb{S}^{d-1}$. Indeed, for any $d \geq 2$, any $u \in \mathbb{S}^{d-1}$ and any isotropic convex body K in $\mathbb{R}^d$, it holds that

$$C_1 \frac{1}{L_K} \leq \mathrm{vol}_{d-1}(K \cap u^\perp) \leq C_2 \frac{1}{L_K},$$

where $C_1, C_2 \in (0, \infty)$ are absolute constants (see Hensley [65]).

Thus, Sylvester's four point problem from 1864 was not only the starting point in the theory of random polytopes, but even nowadays it still affects and influences the ongoing research in this rapidly developing area of stochastic geometry.

Another way of constructing random polytopes that has attracted particular interest in the last decades concerns the so-called Gaussian polytopes. They arise as convex hulls of a collection of independent random points in $\mathbb{R}^d$, distributed according to the standard Gaussian law. More formally, let $\|\cdot\|$ stand for the Euclidean norm. Then,

$$\phi_d(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \exp\left(-\frac{\|x\|^2}{2}\right), \qquad x \in \mathbb{R}^d, \tag{1.3}$$

is the density of a standard Gaussian random variable in $\mathbb{R}^d$. Now, let $X_1, \ldots, X_n$, $n \in \mathbb{N}$, be independent random points in $\mathbb{R}^d$, $d \geq 2$, distributed according to the Gaussian law. Finally, the random convex hull of this point set, denoted again by K_n, defines the Gaussian polytope.

The main differences towards the already discussed model are that there is here no reference body in which the points are contained with probability 1 and that the induced Gaussian polytope grows unboundedly in all directions, as the number of points increases.

Gaussian polytopes show relevant connections to other fields of mathematics. Firstly, they are highly relevant in asymptotic convex geometry or the local theory of Banach spaces. Indeed, since the breakthrough paper of Gluskin [49], Gaussian polytopes have been used as extremizers in geometric or analytic problems. For example, consider the two random convex hulls

$$K_d^i := \mathrm{conv}([-1,1]^d \cup \{\pm X_1^{(i)}, \ldots, \pm X_d^{(i)}\}),$$

$i \in \{1,2\}$, formed by the union of the d-dimensional unit cube $[-1,1]^d$ with two independent and symmetrized Gaussian polytopes in $\mathbb{R}^d$, arising from d independent Gaussian points $X_1^{(i)}, \ldots, X_d^{(i)}$, $i \in \{1,2\}$. In particular, K_d^i, $i \in \{1,2\}$, is an origin symmetric convex body in $\mathbb{R}^d$, that is, if $x \in K_d^i$, it holds that also $-x \in K_d^i$. Note that there is a one to one correspondence between the class of such origin symmetric convex bodies and the class of d-dimensional Banach spaces (see [49] for further explanations). Now, with probability tending to 1 exponentially fast, as the space dimension d tends to infinity, the two random convex hulls K_d^1 and K_d^2 – or, equivalently, the random d-dimensional normed Banach spaces that have these polytopes as their respective unit balls – have Banach-Mazur distance bounded from below by a constant multiple of d. This distance is defined as follows. Let X and Y be two d-dimensional normed spaces, and let $GL(X,Y)$ be the collection of all linear isomorphisms $T : X \to Y$. If $||T||_{Op}$ denotes the operator norm of T, the Banach-Mazur distance between X and Y is given by

$$\inf\{||T||_{Op}\,||T^{-1}||_{Op} : T \in GL(X,Y)\}.$$

The value d also provides an upper bound for this quantity by the classical John's theorem (see [77]). We refer to the work of Latała, Mankiewicz, Oleszkiewicz and Tomczak-Jaegermann [88] and to the survey by Mankiewicz and Tomczak-Jaegermann [96] for a generalization of this result to certain sub-Gaussian polytopes. Further extremality results in this context are due to, for instance, Gluskin and Litvak [50] and Szarek [126].

Secondly, Gaussian polytopes are prototypical examples of random convex sets that satisfy the (probabilistic version of the) celebrated hyperplane conjecture, already mentioned above. Initially, they where considered as a potential counterexample. More detailed, it was shown by Klartag and Kozma [83] that the isotropic constant of the convex hull of $n \geq d+1$ independent Gaussian random points in $\mathbb{R}^d$ is bounded by an absolute constant with probability at least $1 - e^{-Cd}$, where $C \in (0,\infty)$ is another absolute constant. In other words, Gaussian polytopes satisfy the hyperplane conjecture asymptotically almost surely, as $d \to \infty$. For other random polytope models satisfying this form of the hyperplane conjecture, we refer to the works of Alonso-Gutiérrez [5], Dafnis, Guédon and Giannopoulos [31], Hörrmann, Hug, Reitzner and Thäle [67] and Hörrmann, Prochno and Thäle [68].

Thirdly, Gaussian polytopes are of interest in some branches of coding theory because of the following interpretation, derived by Baryshnikov and Vitale [11]. Fix $n \geq d+1$, and let Δ_n be a regular simplex in $\mathbb{R}^n$. Now, take a random rotation ϱ in $\mathbb{R}^n$, and let $\mathrm{pr}_d^n : \mathbb{R}^n \to \mathbb{R}^d$ be the projection onto the first d coordinates (see Figure 1.6). Then, the randomly rotated and projected simplex $\mathrm{pr}_d^n(\varrho(\Delta_n))$ has the same distribution as a Gaussian polytope that arises as the convex hull of $n+1$ standard Gaussian random points in $\mathbb{R}^d$, up to an affine transformation. In the context of coding theory, it is of interest whether the projection of k vertices of Δ_n is always a $(k-1)$-dimensional face of $\mathrm{pr}_d^n(\varrho(\Delta_n))$. As $n \to \infty$, this holds as long as k and d are both proportional to n. For more details in this direction, we refer to the works of Candes and Tao [24], Donoho and Tanner [35, 36, 37] and Vershik and Sporyshev [132].

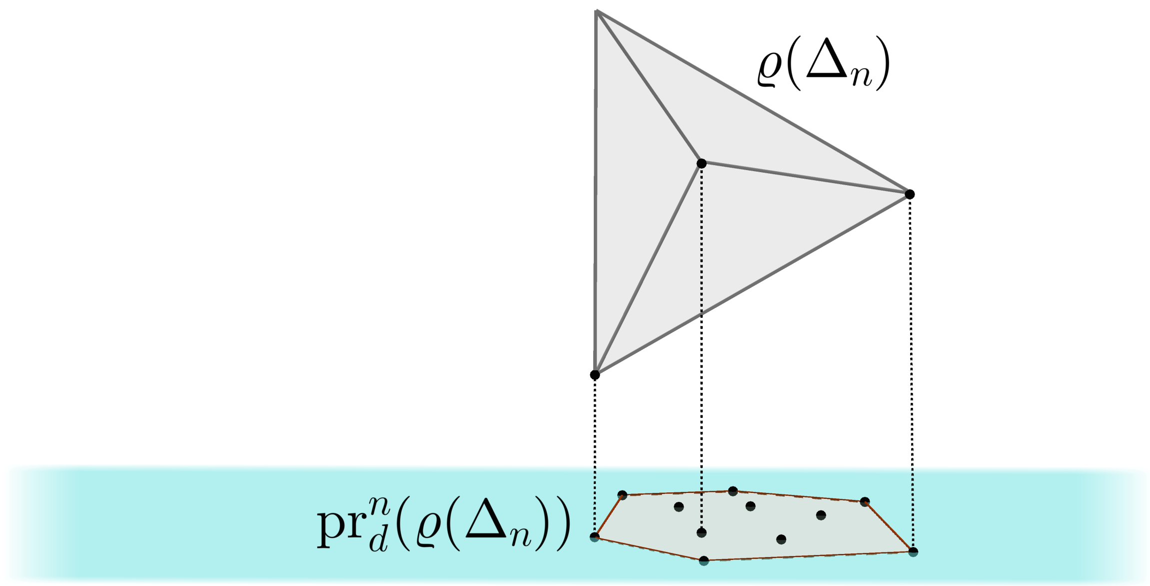

FIGURE 1.6: Projection of the randomly rotated simplex onto its first d coordinates.

Moreover, spatial data can be assumed to follow a Gaussian law. For more information on this point, we refer to the survey article by Cascos [26]. Hence, Gaussian polytopes show a clear relevance also in the area of multivariate statistics. For example, the vertices of a Gaussian polytope can be viewed as the multivariate extremes of the underlying sample.

In the following, the reader is provided with a historical background, dealing with expectation and variance asymptotics, as well as central limit theorems, for different characteristics of Gaussian polytopes. Let us denote the i-th intrinsic volume and the number of j-dimensional faces of K_n by $V_i(K_n)$, $i \in \{1, \dots, d\}$, and $f_j(K_n)$, $j \in \{0, \dots, d-1\}$, respectively (see Section 2.2 for a detailed definition). In particular, $V_d(K_n)$ represents the volume, while $f_0(K_n)$ indicates the number of vertices of K_n.
One of the first issues taken into account concerned their expected values, as the number of points tends to infinity. This line of research starts with the classical work of Rényi and Sulanke [111] in 1963. Their paper can generally be seen as the first milestone in the analysis of expected values of characteristics of random polytopes, as the number of underlying points tends to infinity. Specifically, Rényi and Sulanke analyzed the expected vertex number of a huge class of random polytope models in the plane, including the Gaussian model as a special case.

This research was continued by the papers by Affentranger [2] and Affentranger and Schneider [3], concerning the face numbers of Gaussian polytopes in higher dimensions. Particularly, for all $j \in \{0, \dots, d-1\}$, it holds that

$$\mathbb{E}[f_j(K_n)] \sim c_1 \, (\log n)^{\frac{d-1}{2}} \, ,$$

as $n \to \infty$, where $c_1 \in (0, \infty)$ is an explicitly known constant only depending on d and j. Here, for two functions $f(n)$ and $g(n)$, the notion $f(n) \sim g(n)$ indicates that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$

Now, some simulations are presented to provide an heuristic explanation for the behavior of the expectation asymptotic for the volume of K_n. While Figure 1.7 shows the simulation of different numbers of Gaussian points in the plane, Figure 1.8 contains their respective convex hulls. As already described above, K_n grows unboundedly in all directions, as the number of points increases. On the other hand, the more points thrown in, the more the random polytope K_n looks like a circle. Figure 1.9 indicates that $\mathbb{B}^d(\boldsymbol{o}, \sqrt{2 \log n})$ might be an appropriate reference body to compare the random polytope K_n with. Indeed, it was proved by Geffroy [48] that the Hausdorff distance between K_n and this ball converges to 0 almost surely, as $n \to \infty$. Here, the Hausdorff-distance of two convex bodies K and L in $\mathbb{R}^d$ is given by

$$\inf\{\varepsilon > 0 : K \subseteq (L \oplus \mathbb{B}^d(\boldsymbol{o}, \varepsilon)) \quad \text{and} \quad L \subseteq (K \oplus \mathbb{B}^d(\boldsymbol{o}, \varepsilon))\},$$

where the Minkowski sum of K and L is defined as

$$K \oplus L := \{a + b : a \in K, b \in L\}.$$

Therefore, the expected value of the volume of K_n behaves like $\kappa_d (2 \log n)^{\frac{d}{2}}$, as $n \to \infty$, where κ_d denotes the volume of the d-dimensional unit ball. This was proved by Affentranger [2], showing that even more generally for all intrinsic volumes, i.e., all $i \in \{1, \dots, d\}$, it holds that

$$\mathbb{E}[V_i(K_n)] \sim \binom{d}{i} \frac{\kappa_d}{\kappa_{d-i}} \, (2 \log n)^{\frac{i}{2}},$$

as $n \to \infty$.

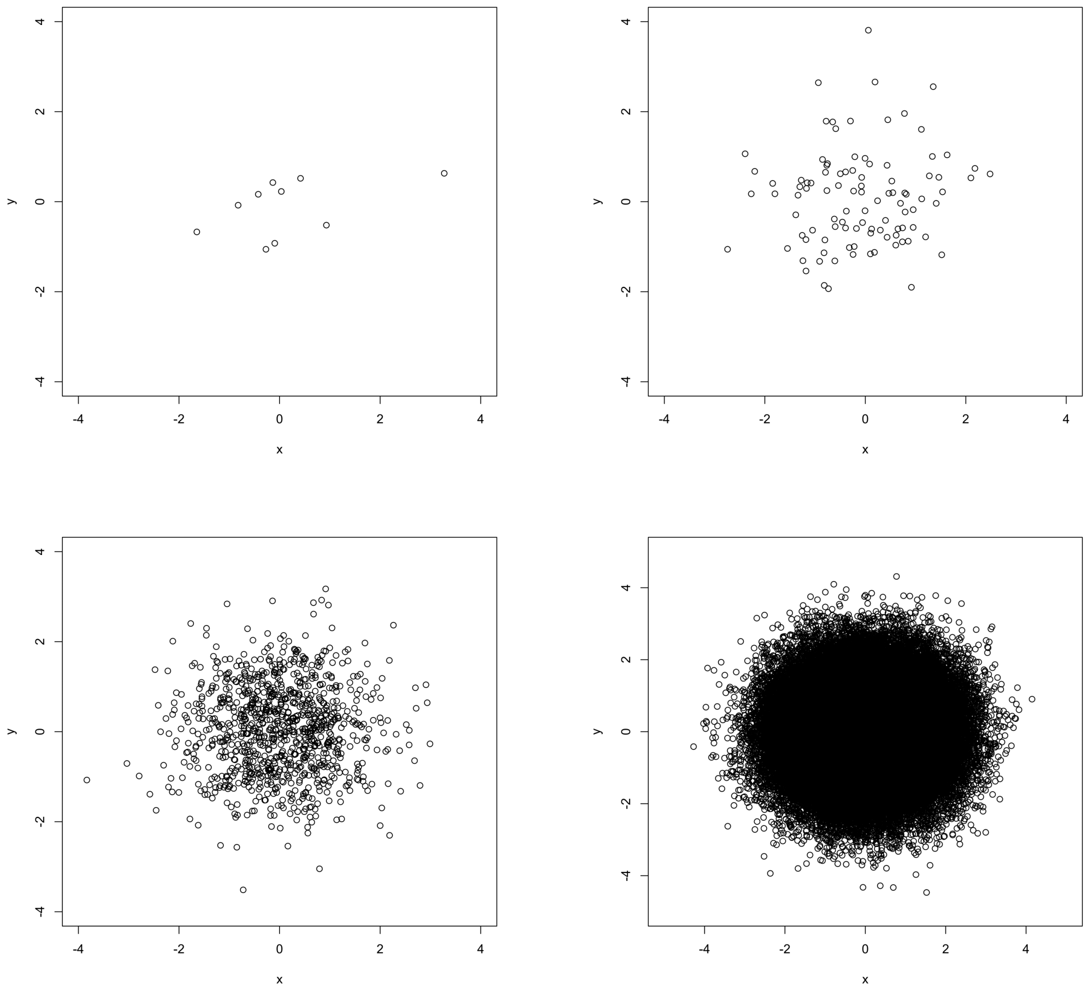

FIGURE 1.7: Simulation of $n = 10, 100, 1.000, 100.000$ Gaussian points.

Hug and Reitzner [73] derived variance upper bounds and used them to establish laws of large numbers. For all $i \in \{1, \dots, d\}$ and $j \in \{0, \dots, d-1\}$, they proved that

$$\mathrm{var}[V_i(K_n)] \le c_1 (\log n)^{\frac{i-3}{2}} \qquad \text{and} \qquad \mathrm{var}[f_j(K_n)] \le c_2 (\log n)^{\frac{d-1}{2}},$$

for sufficiently large n, where $c_1, c_2 \in (0, \infty)$ are constants only depending on d, i and j. Matching lower bounds were obtained by Bárány and Vu [9] for all face numbers and the volume.

Moreover, Hueter [70, 71] computed the precise variance asymptotics for the number of vertices and the volume of K_n, while Calka and Yukich [23] generalized the result to hold for all intrinsic volumes and face numbers.

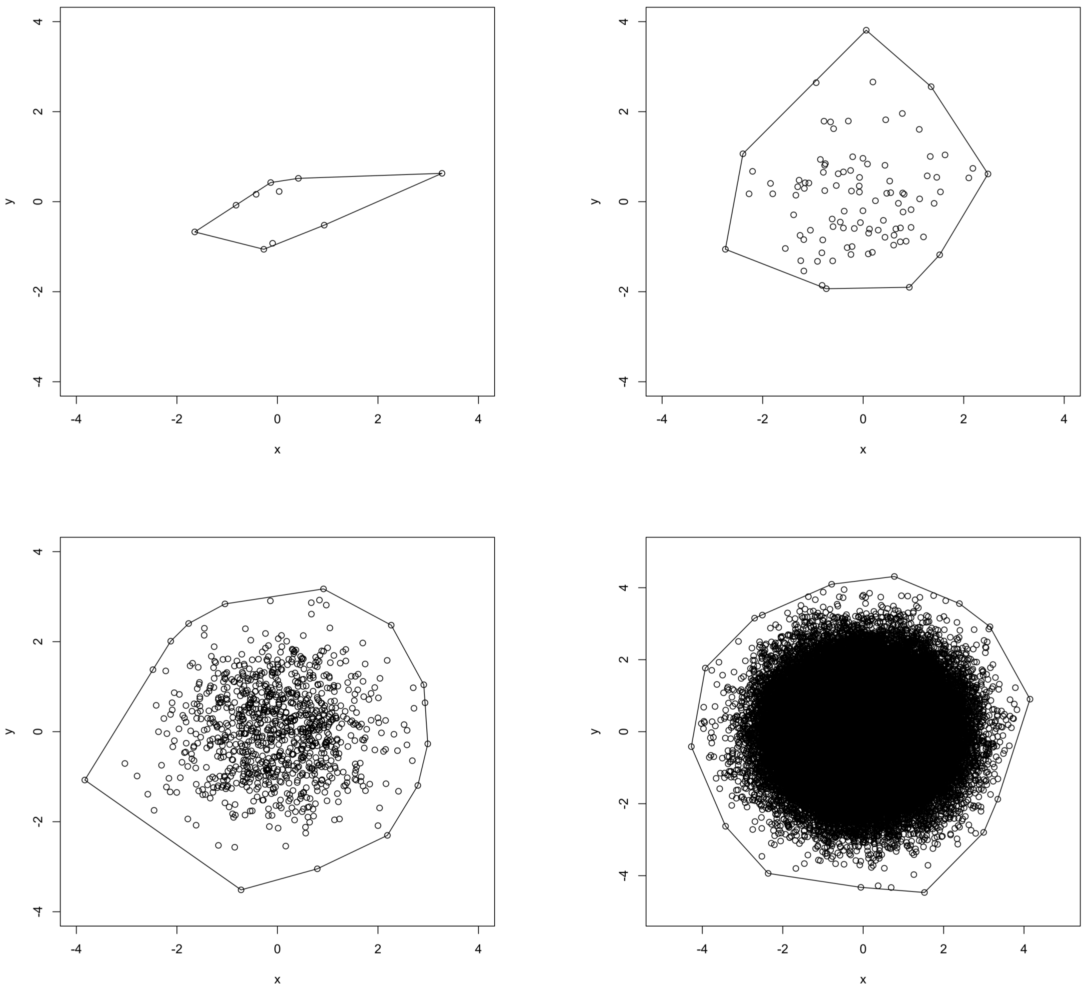

FIGURE 1.8: Simulation of $K_{10}, K_{100}, K_{1.000}$ and $K_{100.000}$.

For all $i \in \{1, \ldots, d\}$ and $j \in \{0, \ldots, d-1\}$, they showed that

$$\operatorname{var}[V_i(K_n)] \sim c_1 \,(2 \log n)^{i-\frac{d+3}{2}} \qquad \text{and} \qquad \operatorname{var}[f_j(K_n)] \sim c_2 \,(2 \log n)^{\frac{d-1}{2}},$$

as $n \to \infty$, where $c_1 \in [0, \infty)$ and $c_2 \in (0, \infty)$ are constants only depending on d, i and j. In particular, the upper bound for the intrinsic volumes derived in [73] does not have the right order of magnitude. However, except for the case that $i = d$, Calka and Yukich were not able to exclude the possibility that $c_1 = 0$. Recently, Bárány and Thäle [8] closed the missing gap and proved that, in fact, $c_1 \in (0, \infty)$ for all other intrinsic volumes, too. Further, it was shown in [23] that the scaling limit of the boundary of K_n, as $n \to \infty$, converges to a 'festoon' of parabolic surfaces.

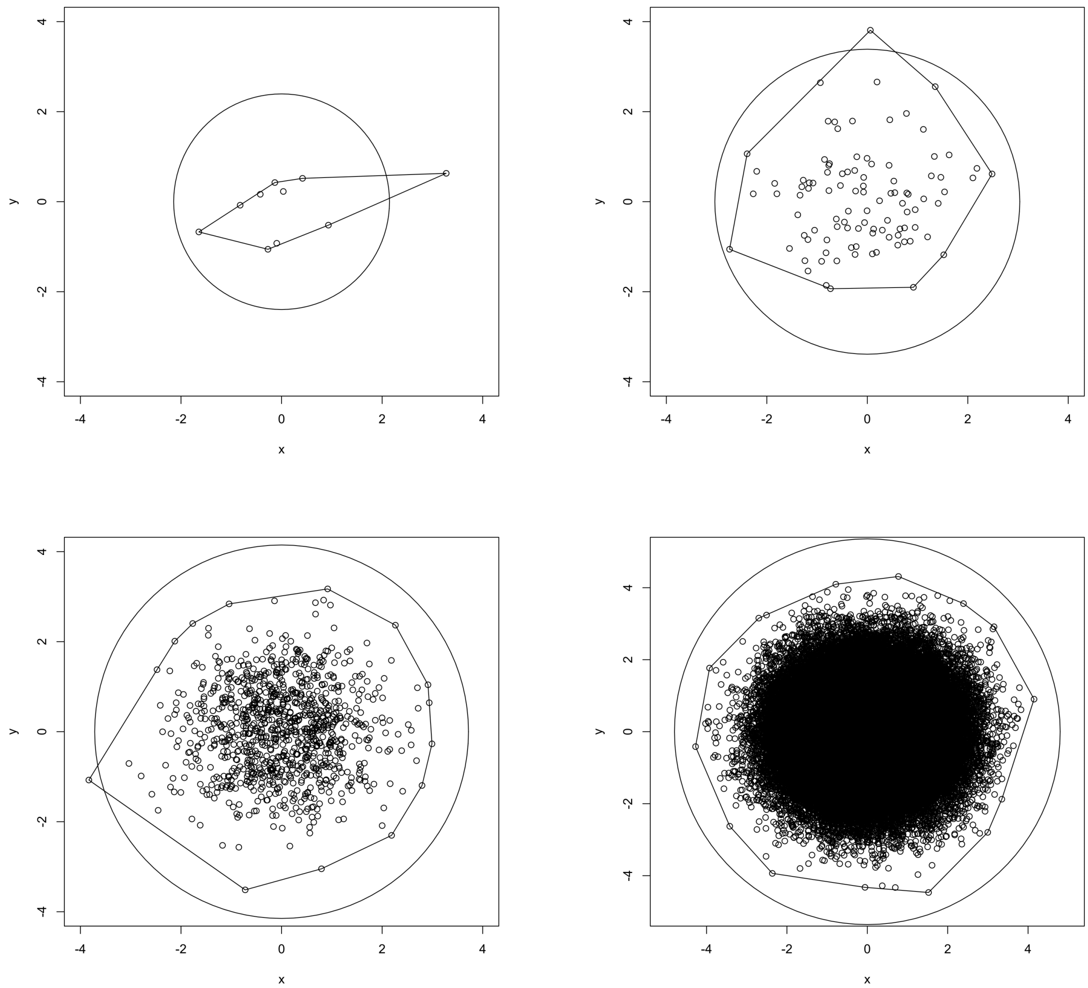

FIGURE 1.9: Comparison of K_n with the respective ball with radius $\sqrt{2\log n}$.

The central limit problem for Gaussian polytopes has first been treated again by Hueter [70, 71] for the number of vertices and the volume of K_n, and been generalized in the breakthrough paper by Bárány and Vu [9] to hold for all other face numbers, too. Finally, Bárány and Thäle [8] added the result for the lower-dimensional intrinsic volumes. More in detail, for all $i \in \{1, \dots, d\}$ and $j \in \{0, \dots, d-1\}$, it holds that

$$\frac{V_i(K_n) - \mathbb{E}[V_i(K_n)]}{\sqrt{\mathrm{var}[V_i(K_n)]}} \xrightarrow{D} \mathcal{N}(0,1) \qquad \text{and} \qquad \frac{f_j(K_n) - \mathbb{E}[f_j(K_n)]}{\sqrt{\mathrm{var}[f_j(K_n)]}} \xrightarrow{D} \mathcal{N}(0,1),$$

as $n \to \infty$. Here, $\xrightarrow{D}$ denotes convergence in distribution and $\mathcal{N}(0,1)$ stands for a standard normally distributed random variable.

Under a further randomization of the model, that is, the family of random points is induced by a Poisson distributed number of points instead of a deterministic one, these central limit theorems yield an excellent starting point for Chapter 3 (see Section 1.2).

A third valuable example of constructing random polytopes is the following. Indeed, it is the underlying model in Chapter 5. Let for now K be a convex body in $\mathbb{R}^d$, $d \geq 2$, having twice continuously differentiable boundary ∂K with strictly positive Gaussian curvature $\kappa_K(x)$, $x \in \partial K$. Moreover, let $f : \partial K \to \mathbb{R}_+$ be a continuous and strictly positive function, satisfying

$$\int\limits_{\partial K} f(x)\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) = 1,$$

where $\mathcal{H}^{d-1}_{\partial K}$ is the Hausdorff measure on ∂K. Additionally, $X_1, \dots, X_n$ are chosen independently on ∂K, distributed according to the probability measure induced by

$$f(x)\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x). \tag{1.4}$$

Finally, K_n indicates the random convex hull of this point set. In this situation, Schütt and Werner [121] proved that

$$\frac{\mathrm{vol}_d(K) - \mathbb{E}[\mathrm{vol}_d(K_n)]}{n^{-\frac{2}{d-1}}} \sim \frac{(d-1)^{\frac{d+1}{d-1}}\, \Gamma\left(d+1+\frac{2}{d-1}\right)}{2\,(d+1)!\, \omega_{d-1}^{\frac{2}{d-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x), \tag{1.5}$$

as $n \to \infty$, where ω_d is the surface area of $\mathbb{S}^{d-1}$. In particular, the latter result will be crucial in Chapter 5, which is concerned with the theory of the approximation of smooth convex bodies by (random) polytopes. We close this section by providing a historical background on this topic.
Let P be a polytope in $\mathbb{R}^d$, defined as the closed convex hull of a finite point set, and let K be a convex body in $\mathbb{R}^d$ as above. The general question,

How well can such a convex body K be approximated by a polytope P?,

has attracted a lot of interest in the last decades and much research has been devoted to its solution. This is due to the fact that it is fundamental in convex geometry and has applications in stochastic geometry, complexity, geometric algorithms and many more. The surveys and books by Gruber [59, 60, 62] and the references cited therein are excellent sources in this context.

As formulated above, the question is quite vague and needs to be phrased more precisely. First of all, we aim to clarify what we mean by approximated and how we want to measure the degree of approximation. The most prominent ways to do this might be the symmetric difference metric and the surface deviation, reflecting the volume deviation, respectively the surface deviation, of the approximating and approximated objects. Here, we focus on the symmetric difference metric, defined for convex bodies K and L in $\mathbb{R}^d$ as

$$\mathrm{vol}_d(K \Delta L) := \mathrm{vol}_d\left(K \cup L\right) - \mathrm{vol}_d\left(K \cap L\right).$$

The blue area in Figure 1.10 gives an example of this set in the planar case. Moreover,

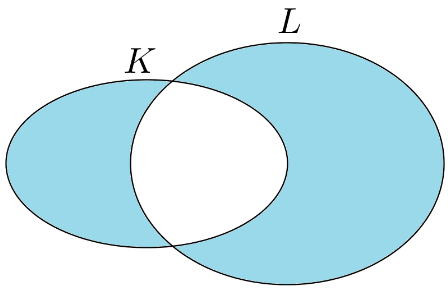

FIGURE 1.10: The symmetric difference metric.

various assumptions on the approximating polytopes can be added. One can restrict only to those polytopes that are contained in or do contain K, respectively. One can focus on polytopes with a fixed number of vertices or, more generally, a fixed number of lower-dimensional faces. Here, we deal with the situation where the number of vertices is fixed and the polytope can be placed arbitrarily in space.

The question about the approximation of K in the symmetric difference metric by a polytope P, having a fixed number of vertices and satisfying $P \subseteq K$, was answered in dimension $d = 2$ by McClure and Vitale [99] and in arbitrary dimension $d \geq 2$ by Gruber [61]. There exists a constant $\mathrm{del}_d \in (0, \infty)$ only depending on d such that

$$\frac{\inf\{\mathrm{vol}_d(K \Delta P) : P \subseteq K,\ f_0(P) \leq n\}}{n^{-\frac{2}{d-1}}} \sim \frac{1}{2}\,\mathrm{del}_d\,\mathrm{as}(K)^{\frac{d+1}{d-1}},$$

as $n \to \infty$, where $f_0(P)$ denotes the number of vertices of P, and the affine surface area of K is given by

$$\mathrm{as}(K) := \int\limits_{\partial K} \kappa_K(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x).$$

The integral involving the Gaussian curvature appears in questions of best approximation of convex bodies by polytopes quite naturally. Indeed, more vertices of the approximating polytope should be placed where the boundary of K is strongly curved and fewer points where the boundary is flat.
Regarding to the constant del_d, Gordon, Reisner and Schütt [52, 53] proved that there exist absolute constants $C_1, C_2 \in (0,\infty)$ such that

$$C_1\, d \leq \mathrm{del}_d \leq C_2\, d. \tag{1.6}$$

Later, Mankiewicz and Schütt [94, 95] enhanced the bounds, providing that

$$\frac{d-1}{d+1}\,\kappa_{d-1}^{-\frac{2}{d-1}} \leq \mathrm{del}_d \leq \left(1+\frac{C\,\log d}{d}\right)\frac{d-1}{d+1}\,\kappa_{d-1}^{-\frac{2}{d-1}},$$

where $C \in (0,\infty)$ is an absolute constant. In particular, the latter result implies that

$$\frac{\mathrm{del}_d}{d} \sim \frac{1}{2\pi e},$$

as $d \to \infty$. Removing the assumption that the polytope P has to be contained in K and, hence, considering all polytopes having at most n vertices, Ludwig [90] showed that there exists another constant $\mathrm{ldel}_d \in (0,\infty)$ only depending on d such that

$$\frac{\inf\{\mathrm{vol}_d(K\Delta P) : f_0(P) \leq n\}}{n^{-\frac{2}{d-1}}} \sim \frac{1}{2}\,\mathrm{ldel}_d\,\mathrm{as}(K)^{\frac{d+1}{d-1}}, \tag{1.7}$$

as $n \to \infty$. The dependences on the number of points n and the convex body K are the same as in the previously mentioned result. Besides, as a corollary from (1.6), one gets that there is an absolute constant $C \in (0,\infty)$ such that

$$\mathrm{ldel}_d \leq C\, d.$$

On the other hand, since we removed the restriction that $P \subseteq K$, this upper bound can be improved. Indeed, in the case of K being the d-dimensional unit ball, it was proved by Ludwig, Schütt and Werner [91] that there exists a polytope P in $\mathbb{R}^d$, having n vertices, such that for all sufficiently large n, it holds that

$$\mathrm{vol}_d(K\Delta P) \leq C\, n^{-\frac{2}{d-1}}\,\kappa_d, \tag{1.8}$$

where $C \in (0,\infty)$ is an absolute constant.

The approximating polytope is obtained via a *random* construction. In particular, if one drops the restriction that P has to be contained in K, one gains by a factor of dimension, that is,

$$\mathrm{ldel}_d \leq C,$$

where $C \in (0, \infty)$ is an absolute constant. The corresponding result for arbitrary K follows from (1.8), together with (1.7). Indeed, applying the affine isoperimetric inequality [118, Equation (6.2.4)], that is,

$$\left(\frac{\mathrm{as}(K)}{\omega_d} \right)^{\frac{d+1}{d-1}} \leq \frac{\mathrm{vol}_d\,(K)}{\kappa_d},$$

yields that

$$\mathrm{vol}_d(K \Delta P) \leq C\, n^{-\frac{2}{d-1}}\ \mathrm{vol}_d(K).$$

However, we aim to produce a direct proof for the approximation of K without using the results of [90, 91]. This is the content of Chapter 5, where we construct a well approximating polytope, obtained via a random construction. It is the convex hull of randomly chosen points with respect to a probability measure with density f, given in (1.4). In fact, it is only via our new approach that different densities can be considered, not merely the uniform distribution as in [91] (see Section 1.2).

1.2 Guideline

This thesis deals with three completely different models of random polytopes and associated issues. The purpose of this guideline is to introduce these models and to provide an overview on the problems afforded, techniques used and answers suggested.

Chapter 2: The second chapter is devoted to some general preliminaries and background material. In Section 2.1, we start with the presentation of notation, used throughout the whole text.
Then, Section 2.2 provides a short review of important concepts from convex geometry. In particular, we define characteristics of convex bodies, namely, the intrinsic volumes, together with their basic properties. Moreover, we introduce a prominent sub-class of convex bodies, namely, convex polytopes. For polytopes, we present not only their

metric parameters like the intrinsic volumes, but also their combinatorial structure, represented by its f-vector.
Thereafter, Section 2.3 is concerned with some 'special functions' and their analytic and asymptotic properties. This includes the Gamma function with associated digamma and polygamma functions, the Beta function, as well as the Barnes G-function.
Next, in Section 2.4, we introduce the reader to the theory of cumulants. We state their main properties, stressing their importance. This is due to the fact that once one can bound the cumulants of a sequence of random variables 'efficiently', one directly achieves not only a central limit theorem for this sequence, but also some related results, listed in this section.
Among them is a so-called moderate deviation principle, explained in Section 2.5, by giving an overview on the even more general theory of large deviations.
Afterwards, in Section 2.6, we define the concept of Poisson point processes, focusing on properties that turn out to be crucial later, namely, the Mecke equation and a mapping theorem.
The final Section 2.7 is concerned with an introduction into the theory of mod-ϕ convergence. Once an appropriate version of mod-ϕ convergence has been established, one gets a collection of companion theorems for free, stated in this section.

Chapter 3: This chapter ties on the results presented above for the class of Gaussian polytopes, after a further randomization of the model. Specifically, we assume that the considered generalized Gamma polytopes are defined as the random convex hulls of a Poisson point process, whose intensity measure is given by a multiple of a huge class of isotropic measures on $\mathbb{R}^d$, $d \geq 2$, including the Gaussian one as a special case.
Such a random polytope is constructed in three steps. First, let N be a Poisson distributed random variable of intensity $\lambda > 0$, i.e.,

$$\mathbb{P}(N = k) = \frac{\lambda^k}{k!} e^{-\lambda},$$

for all $k \in \mathbb{N}_0$. Secondly, choose the random number of N points in $\mathbb{R}^d$, independently and distributed according to the density

$$\phi_{\alpha,\beta}(x) := c^d_{\alpha,\beta} \, ||x||^\alpha \exp\left(-\frac{\|x\|^\beta}{\beta}\right), \qquad x \in \mathbb{R}^d, \tag{1.9}$$

where $\alpha > -1$ and $\beta \geq 1$. We denote this point set by $\mathcal{P}_\lambda$. In a third step, the convex hull of $\mathcal{P}_\lambda$, indicated by K_λ, defines the underlying random polytope.

The family of stated densities can be summarized under the class of the generalized Gamma distribution, giving rise to the description generalized Gamma polytopes. As special cases, it includes the Gaussian distribution ($\alpha = 0, \beta = 2$), the generalized normal distribution ($\alpha = 0, \beta \geq 1$), the Gamma distribution ($\alpha \geq 0, \beta = 1$) and the Weibull distribution ($\alpha > 0, \beta = \alpha + 1$).
In the Gaussian setting, the only difference to the previously discussed Gaussian polytopes concerns the number of chosen points, now no longer deterministic but random, and determined by a Poisson distributed random variable. Furthermore, since we are interested in the large scale asymptotics, as the number of points tends to infinity, we now consider the setting where the parameter in the Poisson distribution tends to infinity, i.e., $\lambda \to \infty$.
Since Bárány and Vu [9] in the Gaussian framework, the line of research under this further randomization has recently been taken up in the remarkable work of Calka and Yukich [23], who computed the precise variance asymptotics for the intrinsic volumes and face numbers of Gaussian polytopes. Moreover, in [23], the scaling limit of the boundary of the K_λ considered is obtained by means of a scaling transformation, developed in previous works of Calka, Schreiber and Yukich [22] and Schreiber and Yukich [120] on random Poisson polytopes in the unit ball.
The general purpose of Chapter 3 is to introduce a new probabilistic viewpoint on our class of generalized Gamma polytopes and to gain new insights into their large scale asymptotic geometry. It is based on sharp bounds for cumulants and the large deviation theory of Saulis and Statulevičius [117]. By means of these techniques, we derive a number of new and powerful results that were not within the reach of other methods available before. The geometric characteristics we consider are, on the one hand, related to the metric and, on the other hand, related to the combinatorial structure of the underlying random polytopes, given by the intrinsic volumes and the face numbers, respectively.

Some very technical, but crucial, preliminaries are described in Section 3.1. In the Gaussian case, i.e., $\alpha = 0$ and $\beta = 2$, it is known from the work of Geffroy [48] that the Hausdorff distance between K_{λ_k} and $\mathbb{B}^d(\boldsymbol{o}, (2 \log \lambda_k)^{\frac{1}{2}})$ converges to 0 almost surely, as $k \to \infty$, along all 'suitable' subsequences λ_k tending to infinity. The goal of Section 3.1.1 is to determine this critical ball in our generalized setting, which turns out to be

$$\mathbb{B}^d(\boldsymbol{o}, (\beta \log \lambda_k)^{\frac{1}{\beta}}),$$

not depending on the parameter α.

Establishing these balls is necessary to define the scaling transformation T_λ in Section 3.1.2, a modified version of the one used in [23, Equation (1.5)]. Via this transformation, the Poisson point process $\mathcal{P}_\lambda$ in $\mathbb{R}^d$ is mapping to another Poisson point process

$$\mathcal{P}^{(\lambda)} := T_\lambda(\mathcal{P}_\lambda)$$

in some bounded region of the product space $\mathbb{R}^{d-1} \times \mathbb{R}$. Its limit, as $\lambda \to \infty$, is given by a third Poisson point process $\mathcal{P}$ in $\mathbb{R}^{d-1} \times \mathbb{R}$, having density

$$(v, h) \mapsto e^h, \qquad (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R},$$

with respect to the Lebesgue measure on $\mathbb{R}^{d-1} \times \mathbb{R}$, independent of the parameter α and β in the underlying density function $\phi_{\alpha,\beta}$.
From this scaling transformation, in Section 3.1.3, we introduce germ-grain processes $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ and $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$. As $\lambda \to \infty$, these processes connect the characteristics of K_λ to limit paraboloid germ-grain processes $\Psi(\mathcal{P})$ and $\Phi(\mathcal{P})$ associated with $\mathcal{P}$ (see also Section 3.4.3). To construct, for instance, $\Phi(\mathcal{P})$, let

$$\Pi^{\downarrow} := \left\{ (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \leq -\frac{\|v\|^2}{2} \right\},$$

and, for $w \in \mathbb{R}^{d-1} \times \mathbb{R}$, put

$$\Pi^{\downarrow}(w) := w \oplus \Pi^{\downarrow}.$$

Then, we denote by $\Phi(\mathcal{P})$ the maximal union of downward parabolic grains $\Pi^{\downarrow}(w)$, whose interior contains no points of the Poisson point process $\mathcal{P}$. Specifically, $\Phi(\mathcal{P})$ arises by filling up the space from below with downward parabolas $\Pi^{\downarrow}$, not containing points from $\mathcal{P}$. Therefore, its boundary $\partial\Phi(\mathcal{P})$ is a union of inverted parabolic surfaces.
Afterwards, in Section 3.1.4, we introduce the key geometric functionals taken into account in this chapter (see [23, Page 9], slightly modified from their Gaussian setting). First, we define the intrinsic volume and face number functionals that correspond to $\mathcal{P}_\lambda$, then, the ones to the rescaled process $\mathcal{P}^{(\lambda)}$. For example, the volume functional is given by the score function

$$\xi^{(\lambda)}_{V_d}(w, \mathcal{P}^{(\lambda)}),$$

$w \in \mathcal{P}^{(\lambda)}$. Thirdly, we present those correlated to the limit Poisson point process $\mathcal{P}$.

The final Section 3.1.5 is concerned with the measure-valued versions of our functionals defined in Section 3.1.4. We also introduce a cluster measure representation of their cumulant measures that is crucial for the proof of the cumulant bound in Section 3.3.

Recall that the case $\alpha = 0$ and $\beta = 2$ is the classical Gaussian setup. In Section 3.2, we generalize the results stated and proved in [23] concerning the functionals of interest and the germ-grain processes to arbitrary parameter $\alpha > -1$ and $\beta \geq 1$ in the underlying density of the Poisson point process $\mathcal{P}_\lambda$. First, we show in Section 3.2.1 that the rescaled functional $\xi^{(\lambda)}_{V_d}(w, \mathcal{P}^{(\lambda)})$, defined on points $w := (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}$, 'localizes' exponentially fast in its spatial coordinate v, as well as height coordinate h. Loosely speaking, the value of this functional is just influenced by changes in the whole point configuration $\mathcal{P}^{(\lambda)}$ in some 'small' neighborhood of the point w itself. This leads to the theory of localization, formally introduced at the beginning of this section.
Furthermore, [23, Lemma 4.4] shows that, in the Gaussian setting, the geometric functional $\xi^{(\lambda)}_{V_d}(w, \mathcal{P}^{(\lambda)})$ has finite moments of all orders. However, we provide some bound in Section 3.2.2 to control the precise growth of these moments by using the localization results from the previous section. For the volume functional, for all $p \in \mathbb{N}$, $w = (v, h) \in \mathcal{P}^{(\lambda)}$ and sufficiently large λ, it holds that

$$\mathbb{E}\big|\xi^{(\lambda)}_{V_d}(w, \mathcal{P}^{(\lambda)})\big|^p \leq c_1\, c_2^p\, (p!)^{2d}\, (1 + |h|)^{p(d-1)+d} \exp\left(-\frac{e^{h\vee 0}}{c_3}\right), \tag{1.10}$$

where $c_1, c_2, c_3 \in (0, \infty)$ are constants only depending on d, α and β, and $h \vee 0$ indicates the maximum between both values. Such precise moment bounds can be seen as the main output of Section 3.2.

The proofs of most results we achieve in Chapter 3 rely on a cumulant estimate, content of Section 3.3. The related proof uses heavily the moment bounds obtained in the foregoing section. Moreover, it is based on the cluster measure representation of the cumulants of the measure-valued versions of the key geometric functionals, presented in Section 3.1.4. Omitting technical details at this point, as an example, we state the bound of the k-th cumulant achieved for the volume of K_λ. For all $k \in \{3, 4, \ldots\}$ and sufficiently large λ, it holds that

$$|c^k[V_d(K_\lambda)]| \leq c_1\, c_2^k\, (\log \lambda)^{\frac{\beta(d-kd-k-1)+2kd}{2\beta}}\, (k!)^{3d+5},$$

where $c_1, c_2 \in (0, \infty)$ are constants only depending on d, α and β.

Thereafter, in Section 3.4, we present our main findings of Chapter 3. In Section 3.4.1, we discuss the intrinsic volumes and face numbers of K_λ. First of all, we generalize the expectation and variance asymptotics, as well as central limit theorems, from the Gaussian setting treated in [8, 9, 23] to our generalized one. For example, as the volume of K_λ is considered, it holds that

$$\mathbb{E}[V_d(K_\lambda)] \sim \kappa_d\,(\beta \log \lambda)^{\frac{d}{\beta}}, \qquad \operatorname{var}[V_d(K_\lambda)] \sim c_1\,(\beta \log \lambda)^{\frac{4d-\beta(d+3)}{2\beta}},$$

and

$$\frac{V_d(K_\lambda) - \mathbb{E}[V_d(K_\lambda)]}{\sqrt{\operatorname{var}[V_d(K_\lambda)]}} \xrightarrow{D} \mathcal{N}(0,1),$$

as $\lambda \to \infty$, where $c_1 \in (0,\infty)$ is a constant only depending on d, α and β. Secondly, we state further probabilistic results for this class of random polytopes that were, to the best of our knowledge, previously unknown, even in the Gaussian case. In particular, this includes

- concentration inequalities,
- bounds for the growth of moments of all orders,
- Marcinkiewicz-Zygmund-type strong laws of large numbers,
- bounds on the relative error in the central limit theorems, and
- moderate deviation principles,

for all key geometric characteristics discussed above. Let us, for instance, state the following concentration inequality. For sufficiently large λ and all $y \geq 0$, it holds that

$$\begin{aligned}\mathbb{P}\big(|V_d(K_\lambda) - \mathbb{E}[V_d(K_\lambda)]| \geq y\,\sqrt{\operatorname{var}[V_d(K_\lambda)]}\,\big)\\ \leq 2\exp\left(-\frac{1}{4}\min\left\{\frac{y^2}{2^{3d+5}}, c_1\,(\log\lambda)^{\frac{d-1}{4(3d+5)}}\,y^{\frac{1}{3d+5}}\right\}\right),\end{aligned}$$

where $c_1 \in (0,\infty)$ is a constant only depending on d, α and β.

Subsequently, in Section 3.4.2, we present the corresponding results for the measure-valued counterparts of all intrinsic volumes and face numbers of K_λ. We emphasize that the latter have the advantage to capture the spatial profile of the considered functionals, not only their total masses.

As aforementioned, the germ-grain processes $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ and $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ connect the characteristics of K_λ via the scaling transformation T_λ to limit paraboloid germ-grain processes $\Psi(\mathcal{P})$ and $\Phi(\mathcal{P})$, as $\lambda \to \infty$. In Section 3.4.3, we discuss more in details the topic. In particular, the scaling limit $T_\lambda(\partial K_\lambda)$ coincides with $\partial\Phi(\mathcal{P})$, as $\lambda \to \infty$, for all parameter α and β in the underlying distribution of the Poisson point process $\mathcal{P}_\lambda$. Thus, this 'festoon' of parabolic surfaces turns out to be a unique scaling limit for the rescaled boundary of our class of generalized Gamma polytopes.

We remark that most of the theorems we have stated in Section 3.4.1 and Section 3.4.2 are the analogues of the results derived by Grote and Thäle [56], where random polytopes arising as convex hulls of a homogeneous Poisson point process in the d-dimensional unit ball are considered. Moreover, the principal technique we use, based on sharp bounds for cumulants in conjunction with the large deviation theory from [117], parallels that in [56]. However, besides of these conceptual similarities, the further details and arguments differ considerably and require much more technical effort, as well as a number of new ideas, if compared to [56]. Indeed, in contrast to random polytopes in the unit ball, our random polytopes in $\mathbb{R}^d$ grow unboundedly in all directions. In particular, for any fixed $\lambda > 0$, there is no centered ball with radius only depending on λ (or any other deterministic set that depends on the parameter λ only) in which K_λ is included with probability 1. This implies that the scaling transformation T_λ maps K_λ into a set in the product space $\mathbb{R}^{d-1} \times \mathbb{R}$, while the scaling transformation for random polytopes in the unit ball has $\mathbb{R}^{d-1} \times [0, \infty)$ as its target space (see [22, 56]). In our situation, the upper half-space $\mathbb{R}^{d-1} \times [0, \infty)$ corresponds to the image of a proper centered ball that contains the random polytope with high probability, while the lower half-space $\mathbb{R}^{d-1} \times (-\infty, 0)$ corresponds to the image of its complement. The probability that the latter contains points from the Poisson point process $\mathcal{P}^{(\lambda)}$ is small, but if there are such points, they have a significant influence on the geometry of the underlying random polytope K_λ.

While the geometric functionals satisfy a weak spatial localization property in the upper half-space, such a behavior holds no more in the global setup. An example is provided by the moment bound stated in (1.10), where the exponential term and, thus, the complete right hand side, is just 'small' if the involved point is located in the upper half-space. This remarkable but unavoidable phenomenon is explained in the Gaussian case in [23] and generalized to all underlying densities of the form (1.9) in Section 3.2.1. It causes considerable technical difficulties that were not present in the previous work [56] and makes the analysis of probabilistic properties of generalized Gamma polytopes a much more demanding task.

The final Section 3.5 of Chapter 3 contains the proofs of all the results presented in Section 3.4. More in details, Section 3.5.1 includes the proof of the expectation and variance asymptotics for the measure-valued versions of the intrinsic volumes and face numbers of K_λ.
Then, in Section 3.5.2, we apply the cumulant bound obtained in Section 3.3 to prove all other theorems stated in Section 3.4.2, while Section 3.5.3 covers the proofs of the results in Section 3.4.1, concerning the intrinsic volumes and face numbers of K_λ themselves.
Finally, in Section 3.5.4, we establish the results regarding to the scaling limit properties of the germ-grain processes $\Phi(\mathcal{P})$ and $\Psi(\mathcal{P})$, respectively.

Chapter 3 is partly based on the paper

- Grote, J., and Thäle, C. [57]: Gaussian polytopes: a cumulant-based approach. *Journal of Complexity* (2018+),

where the authors derived the cumulant bound presented in Section 3.3 in the Gaussian case, i.e., $\alpha = 0$ and $\beta = 2$, leading to the corresponding results stated in Section 3.4.1 and Section 3.4.2, respectively. Compared with that, the content of Chapter 3 goes far beyond. Indeed, it contains a far-reaching generalization of the results stated by Grote and Thäle [57], as well as the variance asymptotics and scaling limits obtained by Calka and Yukich [23], to arbitrary parameter $\alpha > -1$ and $\beta \geq 1$ in the distribution of the underlying Poisson point process $\mathcal{P}_\lambda$. The main steps in the process of this generalization are the following.
First, in Section 3.1, we modify the scaling transformation T_λ from [23, Equation (1.5)] to achieve the uniqueness of the limit process $\mathcal{P}$, giving rise to the uniqueness of the scaling limit of the rescaled boundary $T_\lambda(\partial K_\lambda)$ (see Section 3.4.3). A second demanding task is to prove the localization results and moment bounds in our generalized setting, content of Section 3.2, following roughly [23, Lemma 4.1, 4.2 and 4.3] and [57, Section 5], respectively. Additionally, the proof of the cumulant bound in Section 3.3 generalizes [57, Section 6]. Moreover, the expectation and variance asymptotics for the intrinsic volumes and face numbers of K_λ, as well as there measure-valued counterparts, are obtained as in [23, Section 5.2 and 5.3], while the scaling limit properties are realized as in [23, Lemma 3.1 and Section 5.1]. Finally, all other main results in Section 3.4, based on the cumulant bound, are accomplished as in [57, Section 4].
The general case, dealing with our huge class of generalized Gamma polytopes, will be prepared for publication soon.

Chapter 4: In the last decades, random polytopes have mostly been modeled as follows. First, *fix* a space dimension $d \geq 2$ and a probability measure μ on $\mathbb{R}^d$. Then, let $X_1, \ldots, X_n$, $n \in \mathbb{N}$, be independent random points in $\mathbb{R}^d$, distributed according to μ. A random polytope K_n now arises by taking the convex hull of the point set $X_1, \ldots, X_n$. In particular, if the probability measure μ is the uniform distribution on some convex body K or the Gaussian measure on $\mathbb{R}^d$, we obtain two of the situations described in detail in Section 1.1. As aforementioned, the asymptotic behavior of the expectation and the variance of characteristics like the volume or the number of vertices of K_n has been studied intensively, keeping the dimension fixed and letting n grow to infinity. Moreover, it has been investigated whether these quantities satisfy a 'typical' or 'atypical' behavior, that is, for instance, they fulfill a central limit theorem, large and moderate deviation principles, and concentration inequalities.

However, up to a few exceptions, it has not been investigated what happens if the space dimension d is *not* fixed, but tends to infinity itself. As far as we know, the only exceptions are the papers by Ruben [114], Mathai [98], Anderson [6] and Maehara [93]. In the first two, it is shown that, for any *fixed* $r \in \mathbb{N}$, the r-volume of the convex hull of $r + 1 \leq d + 1$ independent and uniform random points, partly in the interior of the d-dimensional unit ball and partly on its boundary, is asymptotically normally distributed, as $d \to \infty$. Besides, the third one establishes analogous results when the fixed number of points are distributed according to the Beta distribution in the d-dimensional unit ball, while the fourth paper generalizes the setup to the situation of an arbitrary underlying d-fold product distribution on $\mathbb{R}^d$.

Nevertheless, the regime in which r and d tend to infinity *simultaneously* is not treated in these papers. The purpose of Chapter 4 is to close this gap and to prove a collection of probabilistic limit theorems for the r-volume of the convex hull of $r + 1 \leq d + 1$ random points, distributed according to some classes of probability distributions that allow for explicit computations. We focus especially on different regimes of growths of the parameter r, relative to d. More precisely, we distinguish between the following three regimes. The first case concerns r growing as $o(d)$ with the dimension d, that is,

$$\lim_{d \to \infty} \frac{r}{d} = 0.$$

This includes, for example, the case where r is fixed or behaves like d^α, for some $\alpha \in (0, 1)$. Secondly, the underlying situation might be the one where r is asymptotically equivalent to αd, again for some $\alpha \in (0, 1)$. Lastly, we analyze the setting where $d - r = o(d)$, as $d \to \infty$.

In particular, for $r = d$, we choose $d + 1$ random points. Thus, their convex hull is nothing but a full-dimensional simplex in $\mathbb{R}^d$ (see Figure 1.11).

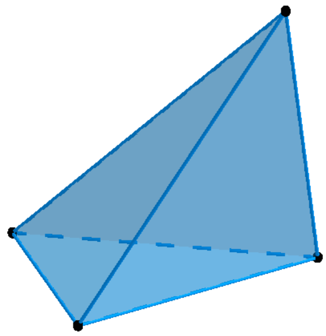

FIGURE 1.11: Full-dimensional simplex in $\mathbb{R}^3$.

This chapter of the thesis is organized as follows. At first, in Section 4.1.1, we introduce the different random point models considered. Besides the spherical and Beta-type, this class includes once more the Gaussian model. For the sake of readability, we focus on the Gaussian model for the rest of this guideline, stressing that similar results hold also for the other two models.
Let $r(d) \leq d$ be an integer, let $X_1, \dots, X_{r+1}$ be Gaussian random points in $\mathbb{R}^d$, and denote by $\mathcal{V}_{d,r}$ the r-dimensional volume of the random simplex with vertices $X_1, \dots, X_{r+1}$. In Section 4.1.2, we state a formula for the moments of $\mathcal{V}_{d,r}$, going back to Miles [101]. More in detail, for all $k \geq 0$, it holds that

$$\mathbb{E}[(r!\mathcal{V}_{d,r})^{2k}] = (r+1)^k \prod_{j=1}^{r} \left[2^k \frac{\Gamma\left(\frac{d-r+j}{2} + k\right)}{\Gamma\left(\frac{d-r+j}{2}\right)} \right]. \tag{1.11}$$

By using these moments, we derive the distribution of $\mathcal{V}_{d,r}$ in Section 4.1.3.

Thereafter, Section 4.2 is devoted to the precise analysis of the cumulants of the logarithmic volume of the Gaussian simplex, given by the random variable

$$\mathcal{L}_{d,r} := \log(r!\mathcal{V}_{d,r}).$$

In Section 4.2.1, we start by analyzing its expectation and variance asymptotics, where the first significant difference between the regimes for the parameter r arises.

Specifically, as $d \to \infty$, it holds that

$$\mathbb{E}[\mathcal{L}_{d,r}] \sim \frac{r}{2} \log d \qquad \text{and} \qquad \operatorname{var}[\mathcal{L}_{d,r}] \sim \begin{cases} \frac{1}{2}\frac{r}{d} & : r = o(d) \\ \frac{1}{2} \log \frac{1}{1-\alpha} & : r \sim \alpha d \\ \frac{1}{2} \log \frac{d}{d-r+1} & : d - r = o(d). \end{cases}$$

Further, we derive a cumulant bound. For all $k \in \{3, 4, \dots\}$ and sufficiently large d, it holds that

$$\left| c^k \left[\frac{\mathcal{L}_{d,r} - \mathbb{E}[\mathcal{L}_{d,r}]}{\operatorname{var}[\mathcal{L}_{d,r}]} \right] \right| \le \begin{cases} \frac{c_1^k \, k!}{(\sqrt{rd})^{k-2}} & : r = o(d) \text{ or } r \sim \alpha d \\ \frac{c_2^k \, k!}{\left(\sqrt{\log \frac{d}{d-r+1}} \right)^k} & : d - r = o(d), \end{cases}$$

where $c_1, c_2 \in (0, \infty)$ are constants not depending on d and k. In particular, in contrast to the cumulant estimates in Chapter 3, these bounds are 'optimal' in the sense that the exponent at $k!$ is 1 and, therefore, as small as possible. Indeed, this can be seen from Theorem 2.4.3.
In Section 4.2.2, we apply these bounds and obtain 'optimal' Berry-Esseen bounds, moderate deviation principles and concentration inequalities for the log-volume $\mathcal{L}_{d,r}$. Then, in Section 4.2.3, we transfer the limit theorem from the log-volume to the volume itself and obtain a phase transition in the limiting behavior. If $r = o(d)$ or $r \sim \alpha d$, $\alpha \in (0, 1)$, the volume of the Gaussian simplex, $\mathcal{V}_{d,r}$, converges to a normally or log-normally distributed random variable, respectively.

Beyond, Section 4.3 establishes results concerning mod-ϕ convergence for the logarithmic volume of the Gaussian simplices. Postponing a detailed introduction into this topic and the used terminology to Section 2.7, let us for now just state the exemplary result from Section 4.3.1 in the case that $r \in \mathbb{N}$ is fixed. As $d \to \infty$, the sequence

$$d \left(\mathcal{L}_{d,r} - \frac{r}{2} \log d - \frac{1}{2} \log(r+1) \right)$$

converges mod-ϕ with parameter rd and limiting function $(t+1)^{-\frac{r(r+1)}{4}}$. More formally,

$$\frac{\mathbb{E}\left[e^{td\left(\mathcal{L}_{d,r} - \frac{r}{2}\log d - \frac{1}{2}\log(r+1)\right)} \right]}{e^{rd\left(\frac{1}{2}((t+1)\log(t+1) - t)\right)}} \sim (t+1)^{-\frac{r(r+1)}{4}},$$

as $d \to \infty$, uniformly as long as t stays in any compact subset of $\mathbb{C} \setminus (-\infty, -1)$.

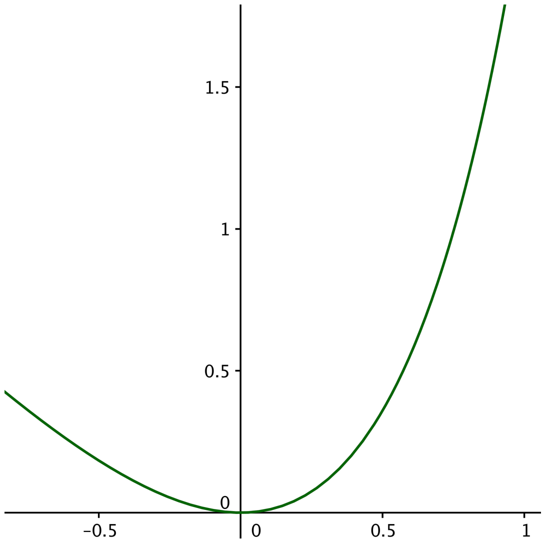

FIGURE 1.12: The rate function $I(x)$.

Section 4.3 is also the starting point to prove the results presented in Section 4.4, concerning large deviation principles. For example, if $r \in \mathbb{N}$ is fixed, the sequence

$$\frac{1}{r}\left(\mathcal{L}_{d,r} - \frac{r}{2}\log d - \frac{1}{2}\log(r+1)\right)$$

satisfies a large deviation principle with speed rd and rate function

$$I(x) = \frac{1}{2}\left(e^{2x} - 1\right) - x, \qquad x \in \mathbb{R},$$

(see Figure 1.12).

Chapter 4 is based on the paper

- GROTE, J., KABLUCHKO, Z., AND THÄLE, C. [55]: Limit theorems for random simplices in high dimensions. arXiv: 1708.00471.

Chapter 5: Let K be a convex body in $\mathbb{R}^d$, $d \geq 2$, with twice continuously differentiable boundary ∂K and strictly positive Gaussian curvature $\kappa_K(x)$, $x \in \partial K$. As aforementioned, it has been derived from [91] that there exists a polytope P in $\mathbb{R}^d$, having n vertices, such that for sufficiently large n, it holds that

$$\mathrm{vol}_d(K \Delta P) \leq C\, n^{-\frac{2}{d-1}}\, \mathrm{vol}_d(K), \tag{1.12}$$

where $C \in (0, \infty)$ is an absolute constant.

In Chapter 5, we generalize the latter result in the following way. Let $f : \partial K \to \mathbb{R}_+$ be a continuous and strictly positive function, satisfying

$$\int_{\partial K} f(x)\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) = 1.$$

Then, there exists a polytope P_f in $\mathbb{R}^d$, having n vertices, such that for sufficiently large n, it holds that

$$\mathrm{vol}_d(K \Delta P_f) \leq C\, n^{-\frac{2}{d-1}} \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x), \tag{1.13}$$

where $C \in (0, \infty)$ is an absolute constant. In particular, in the case of f being the uniform distribution on the boundary of K, we recover the result stated in (1.12). Afterwards, we discuss the influence of different densities f on the right hand side of (1.13). On the one hand, the optimal density is given by the normalized affine surface area measure, distributing the points according to the Gaussian curvature. With this optimal density, the dependence on K in our result is optimal. On the other hand, our result always gives the optimal dependence on the number of vertices, not depending on the underlying density f.

Next, Section 5.2 is devoted to some preliminaries. Besides a Blaschke-Petkantschin type formula for functions with respect to points chosen on the boundary of K due to Zähle [137], this includes the result by Schütt and Werner [121], stated in (1.5).

The proof of the main result is the content of Section 5.3. As in [91], we obtain the approximating polytope in a probabilistic way. To be more precise, we consider a convex body that is slightly bigger than the body K. Then, we choose n points randomly on the boundary of the bigger body and take the convex hull of these points. Since our density functions live on the boundary of K, we choose the random points on ∂K and approximate a slightly smaller body, say $(1-\gamma)K$, where γ depends only on the dimension d and the number of points n. This is established by means of the aforementioned result from [121]. Secondly, we bound the expected volume difference

$$\mathbb{E}[\mathrm{vol}_d((1-\gamma)K \Delta P_n)]$$

between $(1-\gamma)K$ and a random polytope $P_n := \mathrm{conv}(X_1, \ldots, X_n)$ by the right hand side of (1.13).

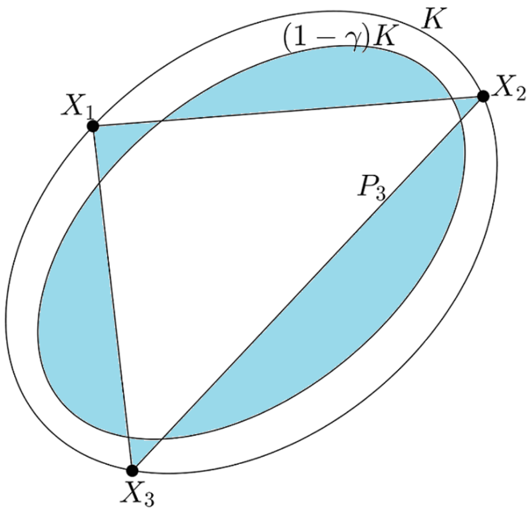

FIGURE 1.13: The set $(1-\gamma)K\Delta P_n$.

The blue area in Figure 1.13 illustrates the set $(1-\gamma)K\Delta P_n$ in the planar setting. Here, the vertices of P_n are chosen randomly from the boundary of K, according to the probability measure

$$f(x)\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x).$$

Consequently, there exists a polytope P_f satisfying (1.13). Finding this bound relies on ideas from [91], in conjunction with approximation results for the boundary of convex bodies due to Reitzner [108], which we develop and apply in detail in a quite technical proof in this section.

Chapter 5 is based on the paper

- GROTE, J., AND WERNER, E. [58]: Approximation of smooth convex bodies by random polytopes. *Electronic Journal of Probability 23* (2018), no. 9, 1–21.

Chapter 2

Preliminaries

In this chapter, we provide the reader with background material. First, we introduce standard notation used throughout this thesis and, then, recall some foundations from convex geometry. Thereafter, we present some special functions together with their basic properties, the theory of cumulants, as well as large deviations and the concept of Poisson point processes. The chapter closes by introducing the notion of mod-ϕ convergence.
Most of the results stated in this chapter are given without a proof. We refer the reader to [118, 119] as a general reference for convex geometry and to [87] for Poisson point processes. For cumulants, we cite [81], while we mention [1] for special functions. An overview on the theory of large deviations and mod-ϕ convergence is provided in [33] and [44], respectively.

2.1 Notation

We work in the Euclidean space $\mathbb{R}^d$ of dimension $d \in \mathbb{N} := \{1, 2, 3, \ldots\}$ with origin $\boldsymbol{o}$. For $x, y \in \mathbb{R}^d$, we denote by $\langle x, y \rangle$ the standard scalar product with associated Euclidean norm $\|x\|$. Moreover, let $\mathbb{B}^d(x, r)$ be the closed ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$. If we parametrize points in $\mathbb{R}^d$ by $(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}$, we write $C_{d-1}(v, r)$ for the infinite vertical cylinder $\mathbb{B}^{d-1}(v, r) \times \mathbb{R}$ around v with base radius $r > 0$. Furthermore, we let

$$\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\| \le 1\} \qquad \text{and} \qquad \mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$$

denote the unit ball and the unit sphere, respectively, whereas the north pole on the unit sphere is given by $u_0 := (0, \ldots, 0, 1)$.
We define $\mathrm{vol}_j(\cdot)$, $j \in \mathbb{N}$, to be the j-dimensional volume of the argument set, and $\kappa_j := \mathrm{vol}_j(\mathbb{B}^j)$ to be the j-volume of $\mathbb{B}^j$. From [119, Page 13], we know that it fulfills

$$\kappa_j = \frac{\pi^{\frac{j}{2}}}{\Gamma\left(1 + \frac{j}{2}\right)},$$

where Γ denotes the Gamma function. Additionally, the surface area of the unit sphere $\mathbb{S}^{j-1}$, $j \in \mathbb{N}$, is given by

$$\omega_j = j\kappa_j = \frac{2\pi^{\frac{j}{2}}}{\Gamma\left(\frac{j}{2}\right)},$$

(see [119, Page 13]).
Besides, for $u \in \mathbb{S}^{d-1}$ and $h \ge 0$, let

$$H(u, h) := \{x \in \mathbb{R}^d : \langle x, u \rangle = h\}$$

be the hyperplane orthogonal to u at distance h from the origin, and let $H^+(u, h)$ be the corresponding half-space containing the origin, that is,

$$H^+(u, h) := \{x \in \mathbb{R}^d : \langle x, u \rangle \le h\}.$$

Similarly, we define $H^-(u, h)$ to be the half-space not containing the origin, i.e.,

$$H^-(u, h) := \{x \in \mathbb{R}^d : \langle x, u \rangle \ge h\}.$$

Now, let K and L be two non-empty subsets of $\mathbb{R}^d$. We denote their union by $K \cup L$, their intersection by $K \cap L$ and their difference by $K \setminus L$, while

$$K \oplus L := \{x + y : x \in K, y \in L\}$$

denotes their Minkowski sum. Moreover, we write K^c for the complement and ∂K for the boundary of K, respectively, and for a Borel set $B \subseteq \mathbb{R}^d$, we write $\mathrm{int}(B)$ and $\mathrm{cl}(B)$ for the interior and the closure of B, respectively.
While $a \wedge b$ denotes the minimum of $a, b \in \mathbb{R}$, $a \vee b$ describes their maximum. For $a \geq 0$, let $\lfloor a \rfloor$ indicate the largest $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ that is smaller than or equal to a. Similarly, $\lceil a \rceil$ is the smallest $n \in \mathbb{N}$ that is bigger than or equal to a.
Besides, let $\| \cdot \|_\infty$ denote the sup-norm of the argument function, and let $\mathbf{1}(\cdot)$ express the indicator function of the underlying event. Further, $| \cdot |$ stands for the cardinality of the argument set or the absolute value of some real number, respectively, depending on the context.
By $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{C}(\mathbb{R}^d)$, we indicate the spaces of bounded measurable and of bounded continuous real-valued functions on $\mathbb{R}^d$, respectively. For a Borel set $B \subseteq \mathbb{R}^d$, we write $\mathcal{C}(\mathbb{R}^d, B)$ for the collection of functions $f \in \mathcal{B}(\mathbb{R}^d)$ whose set of continuity points includes B. Beyond, we write $\mathcal{M}(\mathbb{R}^d)$ for the space of s-finite measures on $\mathbb{R}^d$. Here, a measure on $\mathbb{R}^d$ is called s-finite, if it can be represented as a countable sum of finite measures. For a function $f \in \mathcal{B}(\mathbb{R}^d)$ and a measure $\nu \in \mathcal{M}(\mathbb{R}^d)$, we use the symbol $\langle f, \nu \rangle$ to abbreviate the integral of f with respect to ν, that is,

$$\langle f, \nu \rangle := \int_{\mathbb{R}^d} f \, \mathrm{d}\nu.$$

Finally, if $z \in \mathbb{C}$ is a complex number, i.e., $z = a + b\mathfrak{i}$, where $a, b \in \mathbb{R}$ and $\mathfrak{i}$ is the imaginary unit, we denote by $\mathrm{Re}(z)$ the real part of z, and for $z \in \mathbb{C} \setminus \{0\}$,

$$\arg z := \begin{cases} \arctan \frac{b}{a} & : a > 0, b \in \mathbb{R} \\ \arctan \frac{b}{a} + \pi & : a < 0, b \geq 0 \\ \arctan \frac{b}{a} - \pi & : a < 0, b < 0 \\ \frac{\pi}{2} & : a = 0, b > 0 \\ -\frac{\pi}{2} & : a = 0, b < 0 \end{cases}$$

defines the argument of z.

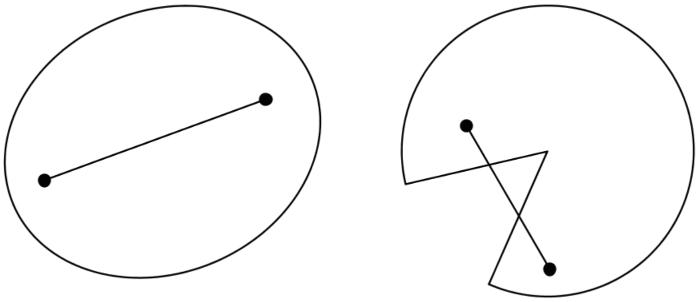

FIGURE 2.1: A convex and a non-convex set.

2.2 Convex geometry

A subset K of $\mathbb{R}^d$ is convex, if for any pair of points $x, y \in K$, every point on the straight line segment between them is also within K (see Figure 2.1). A compact and convex subset of $\mathbb{R}^d$, having non-empty interior, is called a convex body.
Let K be some convex body in $\mathbb{R}^d$. We define its centroid by

$$\frac{1}{\operatorname{vol}_d(K)} \int_K x \, \mathrm{d}x,$$

where $\mathrm{d}x$ stands for the Lebesgue measure. Moreover, the support function and the radial function of K in direction $u \in \mathbb{S}^{d-1}$ are given by

$$h_K(u) := \max\{\langle x, u \rangle : x \in K\},$$

and

$$r_K(u) := \max\{r : ru \in K\},$$

respectively, while for $x \in \partial K$, we denote the corresponding outer unit normal by $N_K(x)$ and the Gaussian curvature by $\kappa_K(x)$.
Let the symmetric difference metric of two convex bodies K and L in $\mathbb{R}^d$ be defined as

$$\operatorname{vol}_d(K \Delta L) := \operatorname{vol}_d(K \cup L) - \operatorname{vol}_d(K \cap L).$$

Now, we introduce the intrinsic volumes of a convex body, which are some of its most important characteristics. As an example, consider the 2-dimensional case and let K be the square $[0,1]^2$.

Then, the area of $K \oplus \mathbb{B}^2(\boldsymbol{o}, r)$, $r > 0$, is given by

$$\mathrm{vol}_2(K \oplus \mathbb{B}^2(\boldsymbol{o}, r)) = \mathrm{vol}_2(K) + 4r + \pi r^2,$$

(see Figure 2.2). In particular, it can be expressed as a polynomial in r of degree 2.

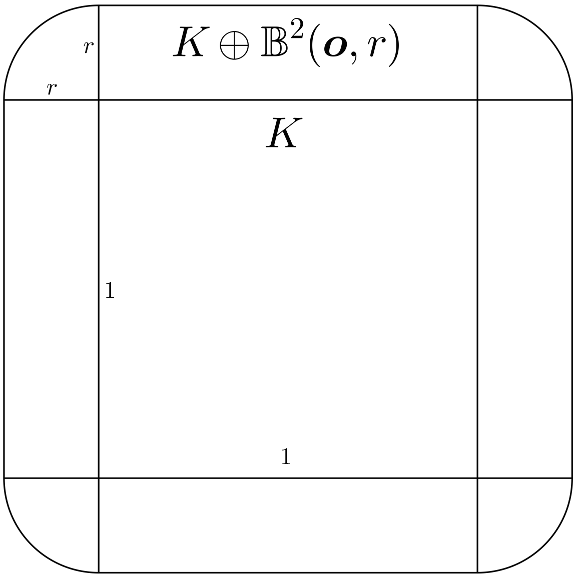

FIGURE 2.2: The Minkowski sum of the square $[0, 1]^2$ and $\mathbb{B}^2(\boldsymbol{o}, r)$.

This phenomenon holds true far more generally. Indeed, the classical Steiner formula [118, Equation (4.2.27)] states that for an arbitrary convex body K in $\mathbb{R}^d$, there exist coefficients $V_i(K)$, $i \in \{0, \ldots, d\}$, such that

$$\mathrm{vol}_d(K \oplus \mathbb{B}^d(\boldsymbol{o}, r)) = \sum_{i=0}^{d} r^{d-i}\, \kappa_{d-i}\, V_i(K).$$

In the latter result, the term $V_i(K)$ is called the i-th intrinsic volume of K. In particular, for all $r > 0$, it satisfies

$$V_i(\mathbb{B}^d(\boldsymbol{o}, r)) = \binom{d}{i} \frac{\kappa_d}{\kappa_{d-i}}\, r^i, \tag{2.1}$$

(see [119, Page 601]). Some special cases are extremely classical measurements. Specially, $V_d(K)$ is the volume of K. Secondly, $V_{d-1}(K)$ is half of its surface area and $V_1(K)$ a multiple of its mean width. We only focus on the case that $i \geq 1$, since for $i = 0$ we get that $V_0(K) = 1$, for all convex bodies K in $\mathbb{R}^d$ (see [119, Page 601]).

In the following lines, K and L are two convex bodies in $\mathbb{R}^d$ and $i \in \{1, \dots, d\}$. It is well-known (see [119, Page 600]), that the intrinsic volumes are valuations, i.e., whenever $K \cup L$ is also a convex body, it holds that

$$V_i(K \cup L) = V_i(L) + V_i(K) - V_i(K \cap L).$$

Furthermore, they are motion-invariant, i.e.,

$$V_i(K) = V_i(gK),$$

for any $g \in G_d$, where G_d is the group of rigid motions on $\mathbb{R}^d$. Additionally, they are non-negative, that is,

$$V_i(K) \geq 0,$$

and monotone, that is,

$$V_i(K) \leq V_i(L),$$

when $K \subseteq L$. Moreover, they are continuous with respect to the Hausdorff distance. The latter property indicates that if a sequence $(K_n)_{n\in\mathbb{N}}$ of convex bodies converges in the Hausdorff distance to a convex body K, then, $V_i(K_n)$ converges to $V_i(K)$, as $n \to \infty$. Here, the Hausdorff distance of two convex bodies K and L in $\mathbb{R}^d$ is given by

$$\inf\{\varepsilon > 0 : K \subseteq (L \oplus \mathbb{B}^d(\boldsymbol{o}, \varepsilon)) \quad \text{and} \quad L \subseteq (K \oplus \mathbb{B}^d(\boldsymbol{o}, \varepsilon))\}.$$

The intrinsic volumes are of outstanding importance in convex geometry since they form a basis of the vector space of all continuous motion invariant valuations on the space of convex bodies. Indeed, Hadwiger's characterization theorem [118, Theorem 4.2.6] states that any motion-invariant, continuous real-valued valuation ϕ on the space of convex bodies in $\mathbb{R}^d$ can be rewritten as a linear combination of intrinsic volumes. Specifically, for all convex bodies K in $\mathbb{R}^d$, it holds that

$$\phi(K) = \sum_{i=0}^{d} a_i V_i(K),$$

where $a_0, \dots, a_d \in \mathbb{R}$.

Kubota's formula [119, Equation (6.11)] yields a second way of introducing the intrinsic volumes. Let $G(d,i)$, $i \in \{1,\dots,d\}$, be the set of all i-dimensional linear subspaces of $\mathbb{R}^d$, or linear i-dimensional Grassmannian of $\mathbb{R}^d$, and denote by ν_i the normalized $SO(d)$-invariant Haar measure on $G(d,i)$, where $SO(d)$ is the group of rotations on $\mathbb{R}^d$. Further, if $L \in G(d,i)$, we indicate by $K|L$ the orthogonal projection of K onto L. Then, for all $i \in \{1,\dots,d\}$, it holds that

$$V_i(K) = \binom{d}{i} \frac{\kappa_d}{\kappa_i \kappa_{d-i}} \int\limits_{G(d,i)} \mathrm{vol}_i(K|L)\, \nu_i(\mathrm{d}L).$$

Following [22, 23], we define, for $x \in \mathbb{R}^d \setminus \{\boldsymbol{o}\}$, the projection avoidance functional as

$$\theta_i(x,K) := \int\limits_{G(\mathrm{lin}[x],i)} (1 - \mathbf{1}(x \notin K|L))\, \mathrm{d}\nu_i^{\mathrm{lin}[x]}(L). \tag{2.2}$$

Here, $\mathrm{lin}[x]$ is the 1-dimensional linear space spanned by x, $G(\mathrm{lin}[x],i)$ the set of i-dimensional linear subspaces of $\mathbb{R}^d$ that do contain $\mathrm{lin}[x]$ and $\nu_i^{\mathrm{lin}[x]}$ the normalized rotational invariant Haar measure on $G(\mathrm{lin}[x],i)$.

Lemma 2.2.1 *Let K be a convex body in $\mathbb{R}^d$ and $i \in \{1,\dots,d\}$. Then, it holds that*

$$V_i(K) = \binom{d-1}{i-1} \frac{1}{\kappa_{d-i}} \int\limits_{\mathbb{R}^d} \frac{1}{\|x\|^{d-i}}\, \theta_i(x,K)\, \mathrm{d}x.$$

Proof. Starting with Kubota's formula yields that

$$V_i(K) = \binom{d}{i} \frac{\kappa_d}{\kappa_i \kappa_{d-i}} \int\limits_{G(d,i)} \int\limits_{L} \big(1 - \mathbf{1}(x \notin K|L)\big)\, \mathrm{d}x\, \nu_i(\mathrm{d}L).$$

To the inner integral over L, we apply the Blaschke-Petkantschin formula [119, Theorem 7.2.1], which leads to

$$\begin{aligned} &\binom{d}{i} \frac{\kappa_d}{\kappa_i \kappa_{d-i}} \int\limits_{G(d,i)} \int\limits_{L} \big(1 - \mathbf{1}(x \notin K|L)\big)\, \mathrm{d}x\, \nu_i(\mathrm{d}L) \\ &= \frac{i\kappa_i}{2} \binom{d}{i} \frac{\kappa_d}{\kappa_i \kappa_{d-i}} \int\limits_{G(d,i)} \int\limits_{G(L,1)} \int\limits_{M} \big(1 - \mathbf{1}(x \notin K|L)\big)\, \|x\|^{i-1}\, \mathrm{d}x\, \nu_1^L(\mathrm{d}M)\, \nu_i(\mathrm{d}L). \end{aligned}$$

Applying [119, Theorem 7.1.2] and Fubini's theorem yields that

$$\begin{aligned}
&\frac{i\kappa_i}{2}\binom{d}{i}\frac{\kappa_d}{\kappa_i\kappa_{d-i}}\int\limits_{G(d,1)}\int\limits_{G(M,i)}\int\limits_{M}\big(1-\mathbf{1}(x\notin K|L)\big)\,\|x\|^{i-1}\,\mathrm{d}x\,\nu_i^M(\mathrm{d}L)\,\nu_1(\mathrm{d}M)\\
&\quad=\frac{i\kappa_i}{2}\binom{d}{i}\frac{\kappa_d}{\kappa_i\kappa_{d-i}}\int\limits_{G(d,1)}\int\limits_{M}\int\limits_{G(M,i)}\big(1-\mathbf{1}(x\notin K|L)\big)\,\|x\|^{i-1}\,\nu_i^M(\mathrm{d}L)\,\mathrm{d}x\,\nu_1(\mathrm{d}M)\\
&\quad=\frac{i\kappa_i}{2}\binom{d}{i}\frac{\kappa_d}{\kappa_i\kappa_{d-i}}\int\limits_{G(d,1)}\int\limits_{M}\int\limits_{G(\mathrm{lin}[x],i)}\big(1-\mathbf{1}(x\notin K|L)\big)\,\|x\|^{i-1}\,\nu_i^{\mathrm{lin}[x]}(\mathrm{d}L)\,\mathrm{d}x\,\nu_1(\mathrm{d}M),
\end{aligned}$$

where we used that $M=\mathrm{lin}[x]$. Next, to this expression we apply the Blaschke-Petkantschin formula, now backwards, to verify that

$$\begin{aligned}
&\frac{i\kappa_i}{2}\binom{d}{i}\frac{\kappa_d}{\kappa_i\kappa_{d-i}}\int\limits_{G(d,1)}\int\limits_{M}\int\limits_{G(\mathrm{lin}[x],i)}\big(1-\mathbf{1}(x\notin K|L)\big)\,\|x\|^{i-1}\,\nu_i^{\mathrm{lin}[x]}(\mathrm{d}L)\,\mathrm{d}x\,\nu_1(\mathrm{d}M)\\
&\quad=\frac{i\kappa_i}{d\kappa_d}\binom{d}{i}\frac{\kappa_d}{\kappa_i\kappa_{d-i}}\int\limits_{\mathbb{R}^d}\int\limits_{G(\mathrm{lin}[x],i)}\big(1-\mathbf{1}(x\notin K|L)\big)\,\frac{\|x\|^{i-1}}{\|x\|^{d-1}}\,\nu_i^{\mathrm{lin}[x]}(\mathrm{d}L)\,\mathrm{d}x\\
&\quad=\frac{\binom{d-1}{i-1}}{\kappa_{d-i}}\int\limits_{\mathbb{R}^d}\int\limits_{G(\mathrm{lin}[x],i)}\big(1-\mathbf{1}(x\notin K|L)\big)\,\|x\|^{-(d-i)}\,\nu_i^{\mathrm{lin}[x]}(\mathrm{d}L)\,\mathrm{d}x.
\end{aligned}$$

Taking into account the definition of $\theta_i(x,K)$ completes the proof. □

A convex polytope in $\mathbb{R}^d$ is defined as the convex hull of a finite point set. Here, the convex hull is the smallest closed convex set containing all these points. More precisely, if $\mathcal{X}$ is a finite point set in $\mathbb{R}^d$, its convex hull can be expressed as

$$\mathrm{conv}(\mathcal{X}):=\Big\{\sum_{i=1}^m\alpha_ix_i: m\in\mathbb{N},x_1,\dots,x_m\in\mathcal{X},\alpha_1,\dots,\alpha_m\ge0,\sum_{i=1}^m\alpha_i=1\Big\}.$$

Let P be a convex polytope in $\mathbb{R}^d$. For $j\in\{0,1,\dots,d-1\}$, we write $\mathcal{F}_j(P)$ for the collection of all j-dimensional faces of P, and put $f_j(P):=|\mathcal{F}_j(P)|$. In particular, $\mathcal{F}_0(P)$ is the set of vertices and $f_0(P)$ the vertex number of P, while the elements of $\mathcal{F}_{d-1}(P)$ are called the facets and $f_{d-1}(P)$ is the facet number of P. The vector

$$(f_0(P),f_1(P),\dots,f_{d-1}(P))$$

is called the f-vector of P and describes its combinatorial structure.

An extreme point of a convex body K in $\mathbb{R}^d$ is a point which does not lie in any open line segment joining two points of K. We write $\mathrm{ext}(K)$ for the set of extreme points of K. The extreme points of a finite point set $\mathcal{X}$ characterize the extreme points of its convex hull, i.e., $\mathrm{ext}(\mathcal{X}) := \mathrm{ext}(\mathrm{conv}(\mathcal{X}))$. If $x \in \mathrm{ext}(\mathcal{X})$, we denote by $\mathcal{F}_j(x, \mathcal{X})$, $j \in \{0, \dots, d-1\}$, the collection of all j-dimensional faces of $\mathrm{conv}(\mathcal{X})$ containing x. Similarly as above, define as $|\mathcal{F}_j(x, \mathcal{X})|$ the cardinality of $\mathcal{F}_j(x, \mathcal{X})$, whereas

$$\mathrm{cone}(x, \mathcal{X}) := \{ry : r \geq 0, y \in \mathcal{F}_{d-1}(x, \mathcal{X})\}$$

is the cone corresponding to the facets $\mathcal{F}_{d-1}(x, \mathcal{X})$.

Now, let K have twice continuously differentiable boundary with strictly positive Gaussian curvature everywhere. Then,

$$\mathrm{as}(K) := \int_{\partial K} \kappa_K(x)^{\frac{1}{d+1}} \, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x)$$

is the affine surface area of K. Here, $\mathcal{H}^{d-1}_{\partial K}$ denotes the $(d-1)$-dimensional Hausdorff measure on ∂K, normalized such that

$$\int_{\mathbb{S}^{d-1}} \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) = \omega_d, \tag{2.3}$$

and satisfying the relation

$$\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) = \kappa_K(x) \, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x), \qquad x \in \partial K, \tag{2.4}$$

(see [118, Equation (2.5.30)]). The affine surface area is an important affine invariant from convex and differential geometry with applications in approximation theory, the theory of valuations, as well as affine curvature flows (see, for example, [19, 63, 64, 121, 130, 131]). Moreover, it has recently been extended to spherical and hyperbolic space by Besau and Werner [14, 15].

Its related affine isoperimetric inequality [118, Equation (6.2.4)] says that

$$\left(\frac{\mathrm{as}(K)}{\omega_d}\right)^{\frac{d+1}{d-1}} \leq \frac{\mathrm{vol}_d\,(K)}{\kappa_d},$$

with equality if and only if K is an ellipsoid.

Further, for $p \in [-\infty, \infty]$, $p \neq -d$, let

$$\mathrm{as}_p(K) := \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{d+p}}}{\langle x, N_K(x)\rangle^{\frac{d(p-1)}{d+p}}} \, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x)$$

be the p-affine surface area of K. The p-affine surface area, an extension of the classical affine surface area, was introduced by Lutwak [92] for $p > 1$ and has been extended to all other $p \neq -d$ by Schütt and Werner [122]. In particular, it is central to the rapidly developing L_p Brunn Minkowski theory (see, for example, [18, 69, 100, 123, 133]).

2.3 Analysis

For $n \in \mathbb{N}$, define

$$n! := \prod_{j=1}^{n} j.$$

The following link between the factorial and the exponential function will be used several times in the subsequent analysis. For all $n \in \mathbb{N}$ and $x > 0$, it holds that

$$\exp(-x) \leq \frac{n!}{x^n}, \tag{2.5}$$

(see [1, Equation (4.2.35)]).

Lemma 2.3.1 *(a) For all $a, b \in \mathbb{N}$, it holds that*

$$(ab)! \leq (a^a)^b \, (b!)^a. \tag{2.6}$$

(b) For all $d, j, k \in \mathbb{N}$, it holds that

$$(d(k+j))! \leq (2dj)^{dj} \, 2^{dkj} \, (dk)!. \tag{2.7}$$

(c) For all $a_1, \ldots, a_n \in \mathbb{N}$, $n \in \mathbb{N}$, and $b := a_1 + \ldots + a_n$, it holds that

$$a_1! \, a_2! \cdots a_n! \leq b!. \tag{2.8}$$

Proof. We prove the first assertion by induction. For $b = 1$, the inequality follows trivially. Now, we assume that the result holds for $b \in \mathbb{N}$. With the induction requirement used in the third step, we achieve

$$\begin{aligned}(a(b+1))! &= (ab+a)(ab+(a-1))\cdots(ab+1)\,(ab)! \\ &\le (ab+a)^a\,(ab)! \le a^a\,(b+1)^a\,(a^a)^b\,(b!)^a = (a^a)^{b+1}\,((b+1)!)^a.\end{aligned}$$

Let us prove (b). By using $k^d \le 2^{dk}$, $d, k \in \mathbb{N}$, we get

$$\begin{aligned}(d(k+j))! &= (dk+dj)(dk+dj-1)\cdots(dk+1)\,(dk)! \\ &\le (dk+dj)^{dj}\,(dk)! \le (2dkj)^{dj}\,(dk)! = (2dj)^{dj}\,k^{dj}\,(dk)! \le (2dj)^{dj}\,2^{dkj}\,(dk)!.\end{aligned}$$

Let us now prove (c). It holds that

$$\begin{aligned}b! &= (a_1+\cdots+a_n)! \\ &= \underbrace{1\cdots a_1}_{=a_1!}\,\underbrace{(a_1+1)\cdots(a_1+a_2)}_{\ge a_2!}\cdots\underbrace{(a_1+\ldots+a_{n-1}+1)\cdots(a_1+\ldots+a_{n-1}+a_n)}_{\ge a_n!} \\ &\ge a_1!\,a_2!\,\cdots\,a_n!.\end{aligned}$$

This completes the proof. □

The Gamma function is given by

$$\Gamma(x) := \int_0^\infty t^{x-1}\,e^{-t}\,\mathrm{d}t, \qquad x > 0.$$

In particular, for all $x > 0$, it fulfills

$$\Gamma(x+1) = x\,\Gamma(x), \tag{2.9}$$

(see [1, Equation (6.1.15)]), and

$$\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right) = \frac{\Gamma(2x)}{\sqrt{\pi}\,2^{2x-1}}, \tag{2.10}$$

(see [1, Equation (6.1.18)]).

Moreover, the Beta function is defined as

$$B(x,y) := \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\mathrm{d}t, \qquad x,y>0.$$

Specifically, for all $x, y > 0$, it satisfies

$$B(x,y) = \int_0^\infty t^{x-1}\,(1+t)^{-x-y}\,\mathrm{d}t, \tag{2.11}$$

(see [1, Equation (6.2.1)]), and

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \tag{2.12}$$

(see [1, Equation (6.2.2)]).
Further, for two functions $g_1, g_2 : \mathbb{C} \to \mathbb{C}$, the relation $g_1 \sim g_2$ indicates that

$$\lim_{z\to\infty} \frac{g_1(z)}{g_2(z)} = 1,$$

while the relations $g_1 = o(g_2)$ and $g_1 = O(g_2)$ indicate that

$$\lim_{z\to\infty} \frac{g_1(z)}{g_2(z)} = 0 \qquad \text{and} \qquad \limsup_{z\to\infty} \left|\frac{g_1(z)}{g_2(z)}\right| < \infty,$$

respectively, as long as $|\arg z| < \pi$.
Building on this notation, for $n > 0$ and $m, m_1, m_2 \in \mathbb{R}$, it holds that

$$\Gamma(nz+m) \sim (2\pi)^{\frac{1}{2}}\, e^{-nz}\, (nz)^{nz+m-\frac{1}{2}}, \tag{2.13}$$

(see [1, Equation (6.1.39)]), and

$$\frac{\Gamma(nz+m_1)}{\Gamma(nz+m_2)} = (nz)^{m_1-m_2} \left(1 + \frac{(m_1-m_2)(m_1+m_2-1)}{2nz} + O\left(\frac{1}{z^2}\right)\right), \tag{2.14}$$

(see [1, Equation (6.1.47)]), as $z \to \infty$, as long as $|\arg z| < \pi$.

Now, let $z = a + b\mathrm{i}$ with $a > 1$ and $b \in \mathbb{R}$. Then, the Riemann ζ-function of z is given by

$$\zeta(z) := \sum_{j=1}^{\infty} \frac{1}{j^z}.$$

As $d \to \infty$, it fulfills the asymptotic

$$\zeta(z) \sim \sum_{j=1}^{d} \frac{1}{j^z} + \frac{1}{(z-1)\, d^{z-1}}, \tag{2.15}$$

(see [1, Equation (23.2.9)]). Moreover, let, for $d \geq 1$,

$$H_d := \sum_{k=1}^{d} \frac{1}{k}$$

denote the d-th harmonic number. In particular, it satisfies

$$H_d = \log d + \gamma + \frac{1}{2d} + O\left(\frac{1}{d^2}\right), \tag{2.16}$$

(see [29, Page 79]), as $d \to \infty$, with the Euler-Mascheroni constant γ.
Additionally, for a function $f : \mathbb{C} \to \mathbb{C}$, let

$$\frac{\mathrm{d}^j}{\mathrm{d}z^j} f(z)$$

indicate its j-th derivative, $j \in \mathbb{N}$. Then, we define the digamma function

$$\psi(z) := \psi^{(0)}(z) := \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z),$$

and the polygamma functions

$$\psi^{(k)}(z) := \frac{\mathrm{d}^k}{\mathrm{d}z^k} \psi(z) = \frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}} \log \Gamma(z), \qquad k \in \mathbb{N}.$$

Each polygamma function has a series representation. Indeed, from [1, Equation (6.4.10)], we know that it fulfills

$$\psi^{(k)}(z) = (-1)^{k+1}\, k! \sum_{j=0}^{\infty} \frac{1}{(z+j)^{k+1}}. \tag{2.17}$$

Lemma 2.3.2 *Let $k \in \mathbb{N}$. Then, as $z \to \infty$, as long as $|\arg z| < \pi$, the digamma and polygamma functions fulfill*

$$\psi(z) = \log z - \frac{1}{2z} + o\left(\frac{1}{z}\right), \tag{2.18}$$

and

$$\psi^{(k)}(z) = (-1)^{k-1}\,\frac{(k-1)!}{z^k} + O\left(\frac{1}{z^{k+1}}\right). \tag{2.19}$$

Besides, for all $z > 0$, it holds that

$$|\psi^{(k)}(z)| \le \frac{(k-1)!}{z^k} + \frac{k!}{z^{k+1}}. \tag{2.20}$$

Proof. The asymptotic relations (2.18) and (2.19) can be found in [1, Equation (6.3.18) and (6.4.11)]. To prove the inequality, note that, in view of (2.17),

$$|\psi^{(k)}(z)| = \sum_{j=0}^{\infty} \frac{k!}{(z+j)^{k+1}} \le \frac{k!}{z^{k+1}} + k! \int_z^{\infty} \frac{1}{x^{k+1}}\,\mathrm{d}x = \frac{k!}{z^{k+1}} + \frac{(k-1)!}{z^k},$$

where we estimated the sum by the integral since $x \mapsto x^{-(k+1)}$, $x > 0$, is decreasing. □

Lemma 2.3.3 *As $d \to \infty$, it holds that*

$$\frac{1}{2}\sum_{j=1}^{d} \psi\left(\frac{j}{2}\right) \sim \frac{d}{2}\log d, \tag{2.21}$$

and

$$\frac{1}{4}\sum_{j=1}^{d} \psi^{(1)}\left(\frac{j}{2}\right) = \frac{1}{2}\log d + c_1 + o(1), \tag{2.22}$$

where $c_1 = \frac{1}{2}(\gamma + 1 + \frac{\pi^2}{8})$. Furthermore, for all $k \ge 3$, it holds that

$$\frac{1}{2^k}\Big|\sum_{j=1}^{d} \psi^{(k-1)}\left(\frac{j}{2}\right)\Big| \le 2\,(k-1)!. \tag{2.23}$$

Proof. The asymptotic relations (2.21) and (2.22) can be found in [38, Page 17]. To prove inequality (2.23), use (2.20) and $(k-2)! \leq \frac{1}{2}(k-1)!$ to get that

$$\frac{1}{2^k}\Big|\sum_{j=1}^{d}\psi^{(k-1)}\left(\frac{j}{2}\right)\Big| \leq \frac{1}{2^k}\sum_{j=1}^{\infty}\left(\frac{(k-2)!}{\left(\frac{j}{2}\right)^{k-1}}+\frac{(k-1)!}{\left(\frac{j}{2}\right)^{k}}\right)$$
$$\leq (k-1)!\sum_{j=1}^{\infty}\left(\frac{1}{4j^{k-1}}+\frac{1}{j^k}\right)$$
$$\leq (k-1)!\left(\frac{1}{4}\zeta(2)+\zeta(3)\right).$$

Bounding the ζ-functions using [1, Equation (23.2.17) and (23.2.24)] yields that

$$\zeta(2)+\zeta(3) \leq \frac{\pi^2}{6}+\frac{7\pi^3}{180} \approx 1,6.$$

This completes the proof. □

Moreover, for $z \in \mathbb{C}$, the Barnes G-function is defined by

$$G(z) := (2\pi)^{\frac{z}{2}} e^{-\frac{1}{2}(z+(1+\gamma)z^2)} \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^k e^{\frac{z^2}{2k}-z}.$$

In particular, it satisfies the functional equation

$$G(z+1) = \Gamma(z)\,G(z), \tag{2.24}$$

(see [10, Page 265]), and fulfills a Stirling-type formula of the form

$$\log G(z+1) = \frac{1}{2}z^2\log z - \frac{3}{4}z^2 + \frac{z}{2}\log(2\pi) - \frac{1}{12}\log z + \frac{\mathrm{d}}{\mathrm{d}z}\zeta(z)\Big|_{z=-1} + O\left(\frac{1}{z}\right), \tag{2.25}$$

(see [10, Page 285]), as $z \to \infty$, as long as $|\arg z| < \pi$.

Lemma 2.3.4 *For all $d \in \mathbb{N}$ and $z \geq 0$, it holds that*

$$\prod_{k=1}^{d}\Gamma\left(\frac{k}{2}+z\right) = \frac{G\left(\frac{d+1}{2}+z\right)G\left(\frac{d+2}{2}+z\right)}{G\left(\frac{1}{2}+z\right)G\left(\frac{2}{2}+z\right)}. \tag{2.26}$$

Proof. We prove the statement by induction. For $d = 1$, by equation (2.24), we obtain

$$\frac{G(1+z)\,G\left(\frac{3}{2}+z\right)}{G\left(\frac{1}{2}+z\right)G(1+z)} = \frac{G\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}+z\right)}{G\left(\frac{1}{2}+z\right)} = \Gamma\left(\frac{1}{2}+z\right) = \prod_{k=1}^{1}\Gamma\left(\frac{k}{2}+z\right).$$

Let us assume that the assertion holds for $d \in \mathbb{N}$ and consider the case $d+1$. By applying the induction hypothesis and (2.24), it follows that

$$\prod_{k=1}^{d+1}\Gamma\left(\frac{k}{2}+z\right) = \frac{G\left(\frac{d+1}{2}+z\right)G\left(\frac{d+2}{2}+z\right)}{G\left(\frac{1}{2}+z\right)G\left(\frac{2}{2}+z\right)}\frac{G\left(\frac{d+3}{2}+z\right)}{G\left(\frac{d+1}{2}+z\right)} = \frac{G\left(\frac{d+2}{2}+z\right)G\left(\frac{d+3}{2}+z\right)}{G\left(\frac{1}{2}+z\right)G\left(\frac{2}{2}+z\right)},$$

as claimed. □

Lemma 2.3.5 *Let $z \to \infty$ be such that $|\arg z| < \pi$ and $a = a(z) \in \mathbb{C}$ be such that $a = o(z)$. Then, it holds that*

$$\begin{aligned}\log G(z+a+1) &- \log G(z+1)\\ &= a\left(z\log z - z + \log\sqrt{2\pi}\right) + \frac{1}{2}a^2\log z + O\left(\frac{|a|^3+1}{z}\right).\end{aligned}$$

Proof. As $z \to \infty$, as long as $|\arg z| < \pi$, applying (2.25) yields

$$\log G(z+a+1) - \log G(z+1) = \frac{1}{2}A + B + C + D + O\left(\frac{1}{z}\right),$$

where

$$\begin{aligned}
A &= (z+a)^2\,\log(z+a) - z^2\,\log z\\
&= (z^2+a^2+2za)\left(\log z + \frac{a}{z} - \frac{a^2}{2z^2} + O\left(\frac{a^3}{z^3}\right)\right) - z^2\,\log z\\
&= za - \frac{1}{2}a^2 + a^2\log z + 2za\log z + 2a^2 + O\left(\frac{a^3}{z}\right),\\
B &= -\frac{3}{4}\left((z+a)^2 - z^2\right) = -\frac{3}{4}a^2 - \frac{3}{2}za,\\
C &= \frac{1}{2}a\log(2\pi), \quad \text{and}\\
D &= -\frac{1}{12}(\log(z+a) - \log z) = O\left(\frac{a}{z}\right).
\end{aligned}$$

Combining these terms completes the proof. □

2.4 Cumulants

Let X be a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution function $F(t) := \mathbb{P}(X \leq t)$ and tail distribution $\bar{F}(t) := 1 - F(t)$, $t \in \mathbb{R}$. If $\mathbb{E}|X|^k < \infty$, we denote by $\mathbb{E}[X^k]$ the k-th moment of X. The moments stand in a direct connection to the so-called cumulants, introduced by the Danish mathematician and astronomer Thiele [127] in 1889 under the name of semi-invariants. In 1931, Fisher and Wishart [47] were the first to call them cumulants.

Let us write $c^k[X]$, $k \in \mathbb{N}$, for the k-th cumulant of a random variable X with $\mathbb{E}|X|^k < \infty$, that is,

$$c^k[X] := (-\mathrm{i})^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \log \mathbb{E}[\exp(\mathrm{i}tX)]\Big|_{t=0}.$$

In particular, it holds that

$$c^1[X] = \mathbb{E}[X] \qquad \text{and} \qquad c^2[X] = \mathrm{var}[X].$$

In general, the k-th cumulant of X can be expressed as a combination of its moments up to order k. Indeed, it holds that

$$c^k[X] = \sum_{L_1,\dots,L_p \preceq [\![k]\!]} (-1)^p \, (p-1)! \, \mathbb{E}[X^{|L_1|}] \cdots \mathbb{E}[X^{|L_p|}],$$

(see [117, Equation (1.34)]), where the sum ranges over all unordered partitions of the set $[\![k]\!] := \{1, \dots, k\}$. This is indicated by the symbol $L_1, \dots, L_p \preceq [\![k]\!]$ in what follows. The next lemma lists some basic properties of cumulants. Further details can be found in [104, Page 33].

Lemma 2.4.1 *Let X and Y be independent random variables with $c^k[X], c^k[Y] < \infty$, for some $k \in \mathbb{N}$. Then, for all $b \in \mathbb{R}$, it holds that*

(a) $c^1[X+b] = c^1[X] + b$ *and* $c^k[X+b] = c^k[X]$, *for* $k \geq 2$,

(b) $c^k[bX] = b^k c^k[X]$,

(c) $c^k[X+Y] = c^k[X] + c^k[Y]$.

We provide the reader with a short example. Let $X \overset{D}{\sim} \mathcal{N}(\mu, \sigma^2)$, where $\mathcal{N}(\mu, \sigma^2)$ denotes a normally distributed random variable with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. While $\overset{D}{\sim}$ indicates equality in distribution, by $\overset{D}{\longrightarrow}$ we mean convergence in distribution.

Then, it holds that

$$\log \mathbb{E}[\exp(\mathfrak{i}tX)] = \mu\mathfrak{i}t - \frac{1}{2}\sigma^2\mathfrak{i}t^2,$$

and, thus,

$$c^1[X] = \mu, \quad c^2[X] = \sigma^2 \quad \text{and} \quad c^k[X] = 0, \quad \text{for all} \quad k \geq 3. \tag{2.27}$$

Thiele [128, Page 25] underlined the meaning of this property of the Gaussian distribution by writing:

"*This remarkable proposition has originally led me to prefer the semi-invariants to every other system of symmetrical functions.*"

It is noteworthy that the property (2.27) with $\mu = 0$ and $\sigma = 1$ characterizes uniquely the standard normal distribution, as the following result due to Marcinkiewicz [97] shows.

Theorem 2.4.2 *Let X be a random variable. Then, it holds that*

$$X \overset{\mathcal{D}}{\sim} \mathcal{N}(0,1) \quad \Leftrightarrow \quad c^1[X] = 0, \quad c^2[X] = 1 \quad \textit{and} \quad c^k[X] = 0, \quad \textit{for all} \quad k \geq 3.$$

In view of this universality of the Gaussian distribution, cumulants have become a key concept in probability theory. Indeed, the latter theorem suggests a method of proving a central limit theorem for a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables, having the properties $\mathbb{E}[X_n] = 0$ and $\mathrm{var}[X_n] = 1$, for all $n \in \mathbb{N}$. Showing that the cumulants of order three and higher vanish, as $n \to \infty$, implies that the underlying sequence of random variables automatically fulfills a central limit theorem.
If one can not only show that the cumulants of order three and higher vanish, but also bound them 'efficiently', one simultaneously achieves a list of companion theorems, stated in the following result. We denote by

$$\Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} \mathrm{d}x, \qquad y \in \mathbb{R},$$

the distribution function of a standard normally distributed random variable.

Theorem 2.4.3 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with $\mathbb{E}[X_n] = 0$ and $\mathrm{var}[X_n] = 1$, for all $n \in \mathbb{N}$. Suppose that, for all $k \in \{3, 4, \dots\}$ and sufficiently large n, it holds that*

$$|c^k[X_n]| \leq \frac{(k!)^{1+\gamma}}{(\Delta_n)^{k-2}}, \tag{2.28}$$

with a constant $\gamma \in [0, \infty)$ not depending on n, and a constant $\Delta_n \in (0, \infty)$ that may depends on n. Then, the following assertions are true:

(i) *For all $y \geq 0$ and sufficiently large n, it holds that*

$$\mathbb{P}(|X_n| \geq y) \leq 2 \exp\left(-\frac{1}{4} \min\left\{ \frac{y^2}{2^{1+\gamma}}, (y\,\Delta_\lambda)^{\frac{1}{1+\gamma}} \right\} \right).$$

(ii) *For all $0 \leq y \leq c_1 \, \Delta_n^{\frac{1}{1+2\gamma}}$ and sufficiently large n, it holds that*

$$\left| \log \frac{\mathbb{P}(X_n \geq y)}{1 - \Phi(y)} \right| \leq c_2 \, (1 + y^3) \, \Delta_n^{-\frac{1}{(1+2\gamma)}},$$

and

$$\left| \log \frac{\mathbb{P}(X_n \leq -y)}{\Phi(-y)} \right| \leq c_2 \, (1 + y^3) \, \Delta_n^{-\frac{1}{(1+2\gamma)}},$$

where $c_1, c_2 \in (0, \infty)$ are constants only depending on γ.

(iii) *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that*

$$\lim_{n\to\infty} a_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} a_n \, \Delta_n^{-\frac{1}{1+2\gamma}} = 0.$$

Then, $(a_n^{-1} X_n)_{n\in\mathbb{N}}$ satisfies a moderate deviation principle on $\mathbb{R}$ with speed a_n^2 and rate function $\frac{x^2}{2}$.

(iv) *For sufficiently large n, we get the Berry-Esseen bound*

$$\sup_{y\in\mathbb{R}} |\mathbb{P}(X_n \leq y) - \Phi(y)| \leq c \, \Delta_n^{-\frac{1}{1+2\gamma}},$$

with a constant $c \in (0, \infty)$ only depending on γ.

Proof. Part (i) is a reformulation of [117, Lemma 2.4] in a form taken from [43, Lemma 3.9] with $H = 2^{1+\gamma}$ there. Moreover, the statement in (ii) corresponds to [117, Lemma 2.3], modified in the form of [43, Corollary 3.2]. Next, the moderate deviation principle for the family $(a_n^{-1}X_n)_{n\in\mathbb{N}}$ follows from [39, Theorem 1.1]. Finally, the Berry-Esseen bound is implied by [117, Corollary 2.1]. □

Due to Janson [76], Theorem 2.4.2 can be weakened as follows. It is enough to show that all cumulants of order higher than some level $j \geq 3$ are 0 to assure that it is Gaussian. To the best of our knowledge, there is no result in literature providing that the respective assumption on (2.28) also implies the results presented in Theorem 2.4.3.

Theorem 2.4.4 *Let X be a random variable and $j \geq 3$. Then, it holds that*

$$X \overset{D}{\sim} \mathcal{N}(0,1) \quad \Leftrightarrow \quad c^1[X] = 0, \quad c^2[X] = 1 \quad \textit{and} \quad c^k[X] = 0, \quad \textit{for all} \quad k \geq j.$$

Part three of Theorem 2.4.3 makes a statement about a so-called moderate deviation principle, formally introduced in the next section.

2.5 Large and moderate deviations

The theory of large and moderate deviations is concerned with the study of the probability of 'rare events'. In order to provide an intuitive idea of the concept of rare, we start with an example. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $\mathbb{E}[X_1] = 0$ and $\mathrm{var}[X_1] = 1$, and put $S_n := X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Then, by the strong law of large numbers, it holds that

$$\frac{1}{n} S_n \longrightarrow 0 \qquad \text{almost surely,}$$

as $n \to \infty$. Thus, for all $x > 0$, it follows that

$$\mathbb{P}(|S_n| \geq xn) \longrightarrow 0, \tag{2.29}$$

as $n \to \infty$, while from the central limit theorem for the sequence S_n, for all $x \in \mathbb{R}$,

$$\mathbb{P}(S_n \geq x\sqrt{n}) \longrightarrow 1 - \Phi(x) > 0.$$

Summarizing, it follows that

$$\frac{1}{n^p} S_n \longrightarrow 0 \tag{2.30}$$

is almost surely true for $p \geq 1$ and almost surely false for $p \leq \frac{1}{2}$, as $n \to \infty$. The question of what happens in between, that is, for $p \in (\frac{1}{2}, 1)$, is answered by the so-called Marcinkiewicz-Zygmund type strong law of large numbers. It states that for $p \in (\frac{1}{2}, 1)$, the condition $\mathbb{E}|X_1|^p < \infty$ is equivalent to the almost sure convergence in (2.30) (see [27, Page 122]).
The central limit theorem makes a precise statement about deviations of order $\sqrt{n}$, i.e., for the 'typical' behavior of S_n, while the probability of 'rare events' of order n tends to 0 (see (2.29)). The question about the speed of this convergence finds an explanation in Cramér's theorem. Before stating it, take another look at the Gaussian setting, that is, $X_1 \overset{D}{\sim} \mathcal{N}(0,1)$. Then, for all $x > 0$, it holds that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} S_n \geq x\right) = -\frac{x^2}{2}, \tag{2.31}$$

(see [33, Page 2]). Now, one could guess that the central limit theorem implies (2.31) for all sums of independent and identically distributed random variables. However, this turns out to be wrong. For example, let us consider the case of X_1 being a Bernoulli random variable and taking the values -1 and 1 with probability $\frac{1}{2}$. Then, for all $x > 0$, it holds that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} S_n \geq x\right) = -I(x),$$

where

$$I(x) = \begin{cases} \frac{1+x}{2} \log(1+x) + \frac{1-x}{2} \log(1-x) & : x \in (0,1) \\ \infty & : \text{otherwise}, \end{cases}$$

(see [33, Page 35]). This is due to the fact that the probability of such rare events, or 'large deviations', is much more sensitive by the tail behavior of the involved random variables, while the central limit theorem only requires the existence of certain moments.

Let X be a random variable. We denote its moment generating function by

$$\varphi_X(t) := \mathbb{E}[e^{tX}], \qquad t \in \mathbb{R},$$

its cumulant generating function by

$$\Delta_X(t) := \log \mathbb{E}[e^{tX}], \qquad t \in \mathbb{R},$$

and the Legendre-Fenchel transformation of $\Delta_X(t)$ by

$$\Delta_X^*(x) := \sup_{t\in\mathbb{R}} \{tx - \Delta_X(t)\}, \qquad x \in \mathbb{R}.$$

How to establish the so-called rate function $I(x)$ in general, is the content of the next theorem, which can be found in [33, Theorem 2.2.3].

Theorem 2.5.1 (Cramér's theorem) *Let $X_1, X_2, \dots$ be independent and identically distributed random variables with $\mathbb{E}[X_1] = 0$, and let $\Delta_{X_1}(t) < \infty$, for all $t \in (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$. Then, for all $x > 0$, it holds that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} S_n \geq x\right) = -I(x),$$

where $I(x) := \Delta_{X_1}^*(x)$.

Going back to the example $X_1 \overset{D}{\sim} \mathcal{N}(0,1)$, we get that

$$\Delta_{X_1}(t) = \frac{t^2}{2} \qquad \text{and, thus,} \qquad I(x) = \Delta_{X_1}^*(x) = \sup_{t\in\mathbb{R}} \left\{tx - \frac{t^2}{2}\right\} = \frac{x^2}{2}.$$

Cramér's theorem yields information about the rate function $I(x)$ in the setting of a sequence of independent and identically distributed random variables. Moreover, it identifies the exponential rate of the probability that $\frac{1}{n}S_n$ lies in an interval of the form $[x, \infty)$. In particular, it implies that for all $x > 0$ and $A := [x, \infty)$, it holds that

$$\lim_{n\to\infty} \log \mathbb{P}\left(\frac{1}{n} S_n \in A\right) = -\inf_{a\in A} I(a).$$

On the one hand, this illustrates a key principle in large deviation theory:

Any large deviation is done in the least unlikely of all the unlikely ways.

On the other hand, one would like to generalize such a statement to other $A \subseteq \mathbb{R}$, resulting in large and moderate deviation principles. Let us recall from [66, Chapter III.1] what this formally means.

A family $(\nu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{R}$ fulfills a large deviation principle with speed a_n and rate function $I : \mathbb{R} \to [0, \infty]$, if I is lower semi-continuous, has compact level sets, and if it holds that

$$- \inf_{x\in \mathrm{int}(B)} I(x) \leq \liminf_{n\to\infty} \frac{1}{a_n} \log \nu_n(B) \leq \limsup_{n\to\infty} \frac{1}{a_n} \log \nu_n(B) \leq - \inf_{x\in \mathrm{cl}(B)} I(x),$$

for every Borel set $B \subseteq \mathbb{R}$. Here, a function $f : \mathbb{R} \to [-\infty, \infty]$ is lower-semicontinuous, if it has closed sub-level sets, i.e., for all $c \in \mathbb{R}$, it holds that the set

$$f^{-1}([-\infty, c]) = \{x \in \mathbb{R} : f(x) \leq c\}$$

is closed. A sequence $(X_n)_{n\in\mathbb{N}}$ of real-valued random variables satisfies a large deviation principle with speed a_n and rate function $I : \mathbb{R} \to [0, \infty]$, if the family of their distributions does.

Moreover, if the involved random elements $(X_n)_{n\in\mathbb{N}}$ satisfy a strong law of large numbers and a central limit theorem, and if the rescaling a_n lies 'between' that of a law of large numbers and that of a central limit theorem, one usually speaks about a moderate deviation principle, instead of a large deviation principle, with speed a_n and rate function I.

While, in Section 2.4, we presented a way to prove moderate deviation principles for a general sequence of random variables via cumulant bounds, the approach to prove large deviation principles applied throughout this thesis relies on the following theorem (see [33, Theorem 2.3.6]).

Theorem 2.5.2 (Gärtner-Ellis theorem) *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, and let $\Delta_{X_n}(t)$, $t \in \mathbb{R}$, be its cumulant generating function. Suppose that for each $t \in \mathbb{R}$ and any sequence a_n tending to infinity, as $n \to \infty$, there exists the logarithmic moment generating function, defined as the extended real-valued limit*

$$\Delta(t) := \lim_{n\to\infty} \frac{1}{a_n} \Delta_{X_n}(a_n t), \qquad t \in \mathbb{R}.$$

Moreover, assume that the origin belongs to the interior of the set

$$\mathcal{D}_\Delta := \{t \in \mathbb{R} : \Delta(t) < \infty\},$$

$\Delta(t)$ *is lower-semicontinuous, differentiable on* $\mathrm{int}(\mathcal{D}_\Delta)$*, and, either* $\mathcal{D}_\Delta = \mathbb{R}$*, or*

$$\lim_{t \to \partial \mathcal{D}_\Delta} \left| \frac{\mathrm{d}}{\mathrm{d}t} \Delta(t) \right| = \infty.$$

Then, the sequence $(X_n)_{n\in\mathbb{N}}$ *satisfies a large deviation principle with speed* a_n *and rate function*

$$\Delta^*(x) := \sup_{t\in\mathbb{R}} \{tx - \Delta(t)\}.$$

By using a continuous function, one can shift a large deviation principle due to the following theorem (see [33, Theorem 4.2.1]).

Theorem 2.5.3 (Contraction principle) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function, and let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of random variables that fulfills a large deviation principle on* $\mathbb{R}$ *with speed* a_n *and rate function* $I(x)$*. Then, the sequence* $(f(X_n))_{n\in\mathbb{N}}$ *also fulfills a large deviation principle on* $\mathbb{R}$ *with rate function*

$$I^*(y) := \inf\{I(x) : x \in \mathbb{R}, f(x) = y\}, \qquad y \in \mathbb{R},$$

and the same speed a_n*.*

2.6 Poisson point processes

A point process on $\mathbb{R}^d$ can be viewed as a random collection of at most countably many points. More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space, and let $\mathscr{B} := \mathscr{B}(\mathbb{R}^d)$ be the Borel σ-field on $\mathbb{R}^d$. Now, let $N := N(\mathbb{R}^d)$ be the space of all s-finite measures ν on $\mathbb{R}^d$, having the property

$$\nu(B) \in \mathbb{N}_0 \cup \{\infty\}, \qquad B \in \mathscr{B}.$$

Moreover, let $\mathcal{N} := \mathcal{N}(\mathbb{R}^d)$ be the σ-field generated by the sets

$$\{\nu \in N : \nu(B) = k\},$$

where $k \in \mathbb{N}_0$ and $B \in \mathscr{B}$. Thus, $\mathcal{N}$ is the smallest σ-field on N such that the map $\nu \to \nu(B)$ becomes measurable.

Based on this formalism, a point process η on $\mathbb{R}^d$ is a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(N, \mathcal{N})$, i.e., a measurable mapping $\eta : (\Omega, \mathcal{F}) \to (N, \mathcal{N})$. For a point process η on $\mathbb{R}^d$, $\eta(B)$ describes the number of points of η contained in some $B \in \mathscr{B}$, that is, we denote by $\eta(B)$ the mapping

$$\omega \to \eta(\omega, B), \qquad \omega \in \Omega.$$

By the definition of the σ-field $\mathcal{N}$, the function η takes values in $\mathbb{N}_0 \cup \{\infty\}$. Furthermore, the intensity measure of a point process η on $\mathbb{R}^d$ is, for all $B \in \mathscr{B}$, defined by

$$\nu(B) := \mathbb{E}[\eta(B)].$$

The most prominent example in the class of point processes might be the Poisson point process. It describes a point process, whose number of points in a prescribed set has a Poisson distribution. Moreover, the number of points in disjoint sets are stochastically independent. More in detail, let ν be a s-finite measure on $\mathbb{R}^d$ without atoms. A Poisson point process η on $\mathbb{R}^d$ with intensity measure ν is a point process with the following two additional properties:

- The number $\eta(B)$ of points falling into some $B \in \mathscr{B}$ is Poisson distributed with mean $\nu(B)$, i.e., it holds that

  $$\mathbb{P}(\eta(B) = k) = \frac{\nu(B)^k}{k!} e^{-\nu(B)},$$

  for $k \in \mathbb{N}_0$.
- Let $n \in \mathbb{N}$, and let $B_1, \dots, B_n \in \mathscr{B}$ be pairwise disjoint. Then, the random variables $\eta(B_1), \dots, \eta(B_n)$ are stochastically independent.

Poisson point processes play an outstanding role in probability theory. Indeed, they have applications in, for example, the theory of Lévy processes [13, 86], Brownian excursion theory [113] and extreme value theory [112]. Besides, Poisson point processes

are fundamental to stochastic geometry. In particular, they are often used to construct more complex random structures such as the Boolean model, the Gilbert graph and the Voronoi, Delaunay and hyperplane tessellations (see, for example, [103, 105, 119] and the references cited therein).
Additionally, Poisson point processes have been used several times as the underlying point sets in the theory of random polytopes (see, for example, [9, 22, 23, 56, 109]). In order to illustrate why it can be especially advantageous to consider Poisson point processes here, consider the Gaussian polytope setting described in the general introduction. Here, the Gaussian polytope arises as the convex hull of $n \in \mathbb{N}$ independent random points in $\mathbb{R}^d$, distributed according to the standard Gaussian law.
In the corresponding 'Poissonized' Gaussian model, the number of points is no longer deterministic but random, and determined by a Poisson distributed random variable N with mean $n > 0$. In other words, the underlying point set is induced by a Poisson point process whose intensity measure is a multiple $n > 0$ of the standard Gaussian law in $\mathbb{R}^d$.
The effect of this additional randomization is a further independence property, namely, the number of points in two disjoint regions are independent random variables. Under this randomization, Bárány and Vu [9] were able to apply Stein's method for weakly dependent random variables. In particular, they deduced a central limit theorem for the volume of the Gaussian polytopes in the Poissonized model. Then, a so-called coupling can be constructed to push the central limit theorem for the Poissonized model back to the deterministic setup.

Now, we state two important properties of Poisson point processes. We start with the Mecke equation [87, Theorem 4.1].

Theorem 2.6.1 (Mecke equation) *Let η be a Poisson point process on $\mathbb{R}^d$ with intensity measure ν, and let ξ be a non-negative measurable function acting on pairs (x, η), $x \in \mathbb{R}^d$. Then, it holds that*

$$\mathbb{E}\left[\sum_{x\in\eta} \xi(x,\eta)\right] = \int_{\mathbb{R}^d} \mathbb{E}[\xi(x,\eta\cup\{x\})]\,\mathrm{d}\nu.$$

Moreover, let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping. For a measure ν on $\mathbb{R}^d$, we denote the push-forward of ν under T to be the measure $T(\nu)$, defined by

$$T(\nu)(C) := \nu(T^{-1}(C)), \qquad C \in \mathscr{B}.$$

If η is a point process on $\mathbb{R}^d$, then, for any $\omega \in \Omega$, $T(\eta(\omega))$ is a measure on $\mathscr{B}$, given by

$$T(\eta(\omega))(C) := \eta(\omega, T^{-1}(C)), \qquad C \in \mathscr{B}.$$

Finally, we state the Mapping theorem [87, Theorem 5.1] of (Poisson) point processes.

Theorem 2.6.2 (Mapping theorem) *Let η be a Poisson point process on $\mathbb{R}^d$ with intensity measure ν, and let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function. Then, $T(\eta)$ is also a Poisson point process on $\mathbb{R}^d$ whose intensity measure is given by $T(\nu)$.*

2.7 Mod-ϕ convergence

Consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ with existing moment generating functions $\varphi_{X_n}(t)$ on some strip

$$S(a, b) := \{t \in \mathbb{C} : a < \mathrm{Re}(t) < b\},$$

where $a < 0 < b$ are extended real numbers. We assume that there exists an infinitely divisible distribution ϕ with moment generating function

$$\int_{-\infty}^{\infty} e^{tx}\, \phi(\mathrm{d}x) = e^{\eta(t)},$$

well-defined on $S(a, b)$, and an analytic function $\psi(t)$ that does not vanish on the real part of $S(a, b)$, such that locally uniformly in $t \in S(a, b)$, it holds that

$$\lim_{n\to\infty} \mathbb{E}[e^{tX_n}]\, e^{-w_n\eta(t)} = \psi(t),$$

where $(w_n)_{n\in\mathbb{N}}$ is some sequence of real numbers converging to infinity. Then, $(X_n)_{n\in\mathbb{N}}$ is said to converge mod-ϕ on $S(a, b)$ with parameter w_n and limiting function ψ.
The idea behind the concept of mod-ϕ convergence is to look for a renormalization of the moment generating function of random variables instead of looking at one of the random variables themselves, as it is done in the central limit theorem. After this renormalization, the sequence of moment generating functions converges to some non-trivial limit.

Intuitively, a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges mod-ϕ, if

- it has approximately the same distribution as the w_n-th convolution power of ϕ and the 'difference' between these distributions is measured by ψ, or
- it can be seen as a large renormalization of ϕ plus residue, asymptotically encoded by ψ.

Mod-ϕ convergence is a powerful notion, introduced and studied in the context of models from number theory, random matrix theory and probability theory in [32, 44, 45, 75, 84, 85], to mention only a few references. In the case of ϕ being the standard Gaussian distribution, i.e., $\eta(t) = \frac{t^2}{2}$, $t \in \mathbb{R}$, one usually speaks of mod-Gaussian convergence. Besides this, the most basic case is probably mod-Poisson convergence, but there are also examples of mod-Cauchy and even mod-uniform convergence in the aforementioned list of references.
Let us continue with a short example. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of centered, independent and identically distributed random variables with distribution ϕ. Put $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Then, it holds that

$$\mathbb{E}[e^{tS_n}] = e^{n\eta(t)},$$

and, therefore, $(S_n)_{n\in\mathbb{N}}$ converges mod-ϕ with parameter $\omega_n = n$ and limiting function $\psi \equiv 1$.
While it is quite immediate to see that mod-ϕ convergence for a sequence $(X_n)_{n\in\mathbb{N}}$ implies a central limit theorem (see, for example, [42, Page 12]), there is in fact much more information encoded in mod-ϕ convergence. In particular, extended central limit theorems and large deviation results can be derived. The following theorem can be found in [44, Theorem 4.2.1 and Theorem 4.3.1].

Theorem 2.7.1 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables that converges mod-ϕ on some strip $S(a,b)$ with parameter w_n and limiting function ψ. We further assume that ϕ is a non-lattice infinitely divisible distribution, which is absolutely continuous with respect to the Lebesgue measure. Then, the following assertions are true:*

(i) *For any sequence x_n satisfying $x_n = o\big(w_n^{\frac{1}{12}}\big)$, as $n \to \infty$, it holds that*

$$\mathbb{P}\left(\frac{X_n - w_n \frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\Big|_{t=0}}{\sqrt{w_n \frac{\mathrm{d}^2}{\mathrm{d}t^2}\eta(t)\Big|_{t=0}}} \le x_n\right) = \Phi(x_n)(1 + o(1)).$$

In the special case of mod-Gaussian convergence, the 'normality zone' is even $o(w_n^{\frac{1}{2}})$*. In this regime, it holds that*

$$\mathbb{P}\left(\frac{X_n}{\sqrt{w_n}} \le x_n\right) = \Phi(x_n)(1+o(1)),$$

as $n \to \infty$*, since we have that* $\eta(t) = \frac{t^2}{2}$ *and, therefore, for all* $a \in \mathbb{R}$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\Big|_{t=a} = a \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\eta(t)\Big|_{t=a} = 1.$$

(ii) *For* $x \in \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\Big|_{t=0}, \frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\Big|_{t=b}\right)$*, it holds that*

$$\mathbb{P}(X_n \ge w_n x) = \frac{\exp(-w_n F(x))}{h\sqrt{2\pi w_n \frac{\mathrm{d}^2}{\mathrm{d}t^2}\eta(t)\Big|_{t=h}}}\,\psi(h)\,(1+o(1)),$$

and, for $x \in \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\Big|_{t=a}, \frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\Big|_{t=0}\right)$*,*

$$\mathbb{P}(X_n \ge w_n x) = \frac{\exp(-w_n F(x))}{|h|\sqrt{2\pi w_n \frac{\mathrm{d}^2}{\mathrm{d}t^2}\eta(t)\Big|_{t=h}}}\,\psi(h)\,(1+o(1)),$$

as $n \to \infty$*. Here,* h *is defined by the implicit equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\Big|_{t=h} = x,$$

and

$$F(x) := \sup_{t\in\mathbb{R}}\{tx - \eta(t)\}$$

is the Legendre-Fenchel transformation of η*.*
In the special case of mod-Gaussian convergence, we have

$$h = x \qquad \text{and} \qquad F(x) = \frac{x^2}{2}.$$

Thus, in this case, for $x \in (0, b)$,

$$\mathbb{P}(X_n \geq w_n x) = \frac{\exp\left(-w_n \frac{x^2}{2}\right)}{x\sqrt{2\pi w_n}}\, \psi(x)\,(1 + o(1)),$$

and, for $x \in (a, 0)$,

$$\mathbb{P}(X_n \geq w_n x) = \frac{\exp\left(-w_n \frac{x^2}{2}\right)}{|x|\sqrt{2\pi w_n}}\, \psi(x)\,(1 + o(1)),$$

as $n \to \infty$.

A stable distribution with scale parameter $c > 0$, stability parameter $\alpha \in (0, 2]$ and skewness parameter $\beta \in [-1, 1]$ is defined as the infinitely divisible distribution $\phi_{c,\alpha,\beta}$, whose Fourier transform

$$\int_{-\infty}^{\infty} e^{\mathrm{i}tx}\, \phi_{c,\alpha,\beta}(\mathrm{d}x) = e^{\eta_{c,\alpha,\beta}(\mathrm{i}t)}$$

satisfies

$$\eta_{c,\alpha,\beta}(\mathrm{i}t) = |ct|^{\alpha}\,(1 - \mathrm{i}\,\mathrm{sign}(t)\beta h(\alpha, t)),$$

where $\mathrm{sign}(t)$ denotes the sign of t and

$$h(\alpha, t) := \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & : \alpha \neq 1 \\ -\frac{2}{\pi}\log|t| & : \alpha = 1. \end{cases}$$

In particular, if $c = \frac{1}{\sqrt{2}}$, $\alpha = 2$ and $\beta = 0$, one gets the Gaussian case, that is,

$$\eta_{c,\alpha,\beta}(\mathrm{i}t) = \frac{t^2}{2}, \qquad t \in \mathbb{R}.$$

Other prominent examples in this class are the Cauchy distribution ($c = 1, \alpha = 1, \beta = 0$), as well as the Lévy distribution ($c = 1, \alpha = \frac{1}{2}, \beta = 1$).

Now, if a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in the mod-$\phi_{c,\alpha,\beta}$ sense with parameter w_n, then,

$$Y_n := \begin{cases} \frac{X_n}{w_n^{1/\alpha}} & : \alpha \neq 1 \\ \frac{X_n}{w_n} - \frac{2c\beta}{\pi\alpha} \log w_n & : \alpha = 1 \end{cases} \tag{2.32}$$

converges in distribution to $\phi_{c,\alpha,\beta}$ (see [45, Proposition 3]).

To continue, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables, let $\phi_{c,\alpha,\beta}$ be a stable distribution, and let $(w_n)_{n\in\mathbb{N}}$ be some sequence of positive real numbers tending to infinity. Consider the two following assertions:

(A) Fix $v \geq 1$ and $w, \gamma \geq 0$. There exists a zone of convergence $[-Kw_n^\gamma, Kw_n^\gamma]$, $K > 0$, such that for all $t \in \mathbb{R}$ in this zone,

$$|\psi_n(\mathrm{i}t) - 1| \leq K_1\, |t|^v \, \exp(K_2|t|^w),$$

where $K_1, K_2 \in (0, \infty)$ are constants independent of n and

$$\psi_n(z) := \mathbb{E}[e^{zX_n}]\, e^{-w_n \eta_{c,\alpha,\beta}(z)}, \qquad z \in \mathbb{C}.$$

(B) It holds that

$$\alpha \leq w, \qquad \gamma \leq \frac{1}{w-\alpha} \qquad \text{and} \qquad 0 < K \leq \left(\frac{c^\alpha}{2K_2}\right)^{\frac{1}{w-\alpha}}.$$

If the above conditions are satisfied, one says that $(X_n)_{n\in\mathbb{N}}$ has a zone of control $[-Kw_n^\gamma, Kw_n^\gamma]$ and index of control (v, w). These conditions give rise to the following Berry-Esseen bound, which can be found in [45, Theorem 11].

Theorem 2.7.2 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables that converges mod-$\phi_{c,\alpha,\beta}$. Moreover, assume that conditions (A) and (B) are satisfied, together with the inequality $\gamma \leq \frac{v-1}{\alpha}$. If Y denotes a random variable with distribution $\phi_{c,\alpha,\beta}$, then,*

$$\sup_{x\in\mathbb{R}} |\mathbb{P}(Y_n \leq x) - \mathbb{P}(Y \leq x)| \leq \frac{3}{2\pi\alpha c}\left(\frac{2^{\frac{v}{\alpha}}\,\Gamma\left(\frac{v}{\alpha}\right)\, K_1}{c^{v-1}} + \frac{7\,\Gamma\left(\frac{1}{\alpha}\right)}{K}\right)\frac{1}{w_n^{\gamma+\frac{1}{\alpha}}},$$

where Y_n is the random variable defined in (2.32).

The aim of the last theorem in this section is to add one more item to this list of mod-ϕ convergences, by proving a convergence modulo a tilted 1-stable totally skewed distribution. To the best of the author's knowledge, this type of mod-ϕ convergence has not been treated in the literature before.

Let X_n be a random variable having a Gamma distribution with parameter $(n, 1)$, $n \in \mathbb{N}$, that is, the probability density of X_n is given by

$$\frac{1}{\Gamma(n)}\, x^{n-1}\, e^{-x}, \qquad x > 0.$$

Then, the distribution of $\log X_n$ is called the exp-Gamma distribution and the probability density of $\log X_n$ is given by

$$\frac{1}{\Gamma(n)}\, e^{-e^x}\, e^{xn}, \qquad x \in \mathbb{R}.$$

It follows that

$$\begin{aligned}
\mathbb{E}\left[\log X_n\right] &= \frac{1}{\Gamma(n)} \int\limits_{-\infty}^{\infty} x\, e^{xn}\, e^{-e^x}\, \mathrm{d}x = \frac{1}{\Gamma(n)} \int\limits_{0}^{\infty} (\log y)\, y^{n-1}\, e^{-y}\, \mathrm{d}y \\
&= \frac{1}{\Gamma(n)} \int\limits_{0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}n}(y^{n-1}\, e^{-y})\, \mathrm{d}y = \frac{1}{\Gamma(n)} \frac{\mathrm{d}}{\mathrm{d}n} \int\limits_{0}^{\infty} y^{n-1}\, e^{-y}\, \mathrm{d}y \\
&= \psi(n),
\end{aligned}$$

where $\psi(n)$ is the digamma function (see Section 2.3).

Theorem 2.7.3 *The sequence of random variables $n(\log X_n - \psi(n))$ converges in the mod-ϕ sense with $\eta(t) = (t+1)\log(t+1) - t$ and parameter $w_n = n$, namely,*

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[e^{tn(\log X_n - \psi(n))}\right]}{e^{n((t+1)\log(t+1)-t)}} = \frac{e^{\frac{t}{2}}}{\sqrt{t+1}},$$

uniformly as long as t stays in any compact subset of $\mathbb{C}\backslash(-\infty, -1)$.

Proof. By the properties of the Gamma distribution (see [79, Page 168]), we have that

$$\mathbb{E}\left[e^{tn(\log X_n - \psi(n))}\right] = e^{-tn\psi(n)}\, \mathbb{E}\left[X_n^{tn}\right] = e^{-tn\psi(n)}\, \frac{\Gamma(tn+n)}{\Gamma(n)}.$$

Using (2.13) and (2.18), it follows that

$$e^{-tn\psi(n)}\,\frac{\Gamma(tn+n)}{\Gamma(n)} \sim e^{-tn(\log n-\frac{1}{2n})}\,\frac{\sqrt{\frac{2\pi}{tn+n}}\,\left(\frac{tn+n}{e}\right)^{n+tn}}{\sqrt{\frac{2\pi}{n}}\,\left(\frac{n}{e}\right)^{n}}$$
$$= \frac{e^{\frac{t}{2}}}{\sqrt{t+1}}\,e^{n((t+1)\log(t+1)-t)},$$

as $n \to \infty$. This concludes the proof. □

Now, let Z_1 be a random variable with stable distribution $\phi_{c,\alpha,\beta}$, where $c = \frac{\pi}{2}$, $\alpha = 1$ and $\beta = -1$. Then, it follows from [116, Proposition 1.2.12] that the cumulant generating function of this random variable is given by

$$\log \mathbb{E}\left[e^{tZ_1}\right] = t\log t, \qquad \mathrm{Re}(t) \geq 0.$$

Furthermore, $\mathbb{E}[e^{Z_1}] = 1$, and consider an exponential tilt of Z_1, denoted by Z_2. Observe that

$$\mathbb{E}[Z_2] = \mathbb{E}[e^{Z_1}Z_1] = \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\mathbb{E}[e^{tZ_1}]}_{=t^t}\Big|_{t=1} = t^t(\log t+1)\Big|_{t=1} = 1,$$

and consider the centered version $Z := Z_2 - 1$. Then, the cumulant generating function of Z is given by

$$\log \mathbb{E}\left[e^{tZ}\right] = (t+1)\log(t+1) - t, \qquad \mathrm{Re}(t) \geq -1,$$

(see [116, Proposition 1.2.12]). As an exponential tilt of an infinitely divisible distribution, Z is itself infinitely divisible. Thus, in Theorem 2.7.3, we have a mod-ϕ convergence modulo a tilted totally skewed 1-stable distribution.

Chapter 3

Generalized Gamma polytopes

Calka and Yukich [23] obtained precise expectation and variance asymptotics for the intrinsic volumes and face numbers of the random convex hull of a Poisson point process in $\mathbb{R}^d$, whose intensity measure is a multiple of the standard Gaussian measure. The existing gap, that the limiting variance of all lower-dimensional intrinsic volumes is strictly positive, was closed by Bárány and Thäle [8]. Additionally, Bárány and Thäle [8] and Bárány and Vu [9] proved central limit theorems for all intrinsic volumes and face numbers.

In this chapter, we generalize these results to the situation where the underlying intensity measure of the Poisson point process is a multiple of a huge class of isotropic measures on $\mathbb{R}^d$, including the Gaussian one as a special case.

The second purpose is to introduce a new viewpoint on the resulting generalized Gamma polytopes, based on cumulant bounds and the general large deviation theory of Saulis and Statulevičius [117]. This leads to new and powerful concentration inequalities, moment bounds, Marcinkiewicz-Zygmund-type strong laws of large numbers and moderate deviation principles for the intrinsic volumes and face numbers. To the best of our knowledge, none of these results have counterparts in the existing literature, not even in the Gaussian case. Corresponding results are also derived for the empirical measures induced by these key geometric functionals, thereby taking care of their spatial profiles.

Thirdly, we show that the scaling limit of the boundary of the generalized Gamma polytopes arises as a unique festoon of inverted parabolic surfaces, not depending on the underlying Poisson point process, generalizing once more a result from [23].

3.1 Preliminaries

Fix a space dimension $d \geq 2$, the parameter $\alpha > -1$ and $\beta \geq 1$, and let $\gamma_{d,\alpha,\beta}$ be the measure of an isotropic random variable in $\mathbb{R}^d$ with density

$$\phi_{\alpha,\beta}(x) := c_{\alpha,\beta}^d \,||x||^\alpha \exp\left(-\frac{\|x\|^\beta}{\beta}\right) := \left(\frac{\beta^{\frac{\beta-\alpha-1}{\beta}}}{2\,\Gamma\left(\frac{\alpha+1}{\beta}\right)}\right)^d ||x||^\alpha \exp\left(-\frac{\|x\|^\beta}{\beta}\right), \quad (3.1)$$

$x \in \mathbb{R}^d$, with respect to the Lebesgue measure on $\mathbb{R}^d$. We denote by $\mathcal{P}_\lambda$ a Poisson point process in $\mathbb{R}^d$ with intensity measure $\lambda\gamma_{d,\alpha,\beta}$, where $\lambda > 0$. Recall, if N is a Poisson distributed random variable with mean λ, $\mathcal{P}_\lambda$ is a point set consisting of N points in $\mathbb{R}^d$, independently chosen according to the law $\gamma_{d,\alpha,\beta}$. In the next step, the generalized Gamma polytope K_λ arises as the random convex hull of the point set $\mathcal{P}_\lambda$. Actually, both K_λ and $\mathcal{P}_\lambda$ depend on the parameter α and β, but we suppress this dependence to simplify the notation.

3.1.1 Critical radius

In the Gaussian case, i.e., $\alpha = 0$ and $\beta = 2$, it follows from the work of Geffroy [48] that the Hausdorff distance between K_{λ_k} and $\mathbb{B}^d(\boldsymbol{o}, \sqrt{2\log\lambda_k})$ converges to 0 almost surely, as $k \to \infty$, along 'suitable' subsequences λ_k tending to infinity. The goal of this section is to determine this critical ball in our generalized setting, following from the next theorem (see also [41, Theorem 4.1] for a slightly different statement).

Theorem 3.1.1 *Let $\alpha > -1$, $\beta \geq 1$ and $X_1, X_2, \ldots$ be independent random variables in $\mathbb{R}$, distributed according to the density*

$$f_{\alpha,\beta}(x) := c_{\alpha,\beta}\,|x|^\alpha \exp\left(-\frac{|x|^\beta}{\beta}\right) := \frac{\beta^{\frac{\beta-\alpha-1}{\beta}}}{2\,\Gamma\left(\frac{\alpha+1}{\beta}\right)}\,|x|^\alpha \exp\left(-\frac{|x|^\beta}{\beta}\right), \qquad x \in \mathbb{R}.$$

Put $M_n := \max\{X_1, \ldots, X_n\}$, $n \in \mathbb{N}$. Then, for all $x \in \mathbb{R}$, it holds that

$$\lim_{n\to\infty} \mathbb{P}\left((\beta\log n)^{\frac{\beta-1}{\beta}}\left[M_n - \left((\beta\log n)^{\frac{1}{\beta}} - \frac{(\beta-\alpha-1)\,\log\left(c_\beta^{-\frac{\beta}{\beta-\alpha-1}}\,\beta\log n\right)}{\beta\,(\beta\log n)^{\frac{\beta-1}{\beta}}}\right)\right] \leq x\right)$$
$$= \exp(-e^{-x}).$$

Remark 3.1.2 Loosely speaking, the latter theorem yields that for all $\alpha > -1$, $\beta \geq 1$ and sufficiently large n, the maximum M_n takes values that are 'close' to $(\beta \log n)^{\frac{1}{\beta}}$, independent of the second parameter α. Moreover, the difference between M_n and $(\beta \log n)^{\frac{1}{\beta}}$ is random and of the magnitude

$$\frac{1}{(\beta \log n)^{\frac{\beta-1}{\beta}}}.$$

In our Poissonized model, this indicates that $(\beta \log \lambda)^{\frac{1}{\beta}}$ should be chosen as the critical radius, i.e., K_λ can be expected to grow like $\mathbb{B}^d(\boldsymbol{o}, (\beta \log \lambda)^{\frac{1}{\beta}})$, for all $\beta \geq 1$ and $\alpha > -1$, as $\lambda \to \infty$.
By using the method described in [48], it seems likely to show that also the Hausdorff distance between K_{λ_k} and $\mathbb{B}^d(\boldsymbol{o}, (\beta \log \lambda_k)^{\frac{1}{\beta}})$ converges to 0 almost surely, as $k \to \infty$, along 'suitable' subsequences λ_k. We leave this issue to further research.

In order to prove Theorem 3.1.1, we recall some basic facts from extreme value theory. Consider a sequence of independent and identically distributed random variables $(X_n)_{n\in\mathbb{N}}$, and denote by M_n the maximum of $X_1, \dots, X_n$, $n \in \mathbb{N}$. Moreover, let F be the distribution function of X_1.
If there exist $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, and a non-degenerated distribution function G such that

$$\frac{M_n - b_n}{a_n} \xrightarrow{D} G,$$

as $n \to \infty$, one says that F lies in the maximum domain of attraction of G. Such a G is called an extreme value distribution.
Fisher and Tippett [46] proved that every extreme value distribution belongs to the families of Frechet-, Weibull- or Gumbel-type distributions. For our purpose, it turns out to be enough to focus on the latter one.
A random variable is said to be Gumbel distributed, if its distribution function is given by

$$F(t) = \exp(-e^{-t}), \qquad t \in \mathbb{R}.$$

The following theorem, first proved by Gnedenko [51], yields a complete description of all distribution functions lying in the maximum domain of attraction of the Gumbel distribution.

Theorem 3.1.3 *A distribution function F lies in the maximum domain of attraction of the Gumbel distribution, if and only if there exists a positive and measurable function $g(t)$, fulfilling*

$$\lim_{t\to\infty} \frac{\bar{F}(t + xg(t))}{\bar{F}(t)} = e^{-x}, \tag{3.2}$$

for all $x \in \mathbb{R}$. If (3.2) holds, then, it follows that

$$\frac{M_n - b_n}{a_n} \xrightarrow{D} G,$$

as $n \to \infty$, where G is Gumbel distributed, and $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are given by

$$\lim_{n\to\infty} n\,\bar{F}(b_n) = 1 \qquad \textit{and} \qquad a_n = g(b_n), \tag{3.3}$$

respectively.

Lemma 3.1.4 *Let $\bar{F}$ be the tail distribution of a random variable with density $f_{\alpha,\beta}$. Then, it holds that*

$$\lim_{t\to\infty} \bar{F}(t) = \frac{f_{\alpha,\beta}(t)}{t^{\beta-1}}. \tag{3.4}$$

Proof. By using the rule of L'Hospital in the case that '$\frac{0}{0}$', we achieve that

$$\frac{\bar{F}(t)}{\frac{f_{\alpha,\beta}(t)}{t^{\beta-1}}} \sim \frac{\int_t^\infty s^\alpha\, e^{-\frac{s^\beta}{\beta}}\,\mathrm{d}s}{\frac{1}{t^{\beta-1}}\,t^\alpha\, e^{-\frac{t^\beta}{\beta}}} \sim \frac{-t^\alpha\, e^{-\frac{t^\beta}{\beta}}}{(\alpha-\beta+1)\,t^{\alpha-\beta}\, e^{-\frac{t^\beta}{\beta}} - t^\alpha\, e^{-\frac{t^\beta}{\beta}}} \sim \frac{1}{\frac{\beta-\alpha-1}{t^\beta}+1} \sim 1,$$

as $t \to \infty$. This completes the proof. □

Lemma 3.1.5 *The distribution function F of a random variable with density $f_{\alpha,\beta}$ lies in the maximum domain of attraction of the Gumbel distribution.*

Proof. In view of Theorem 3.1.3, it is enough to show that equation (3.2) holds for some suitable function $g(t)$. Choose

$$g(t) := \frac{1}{t^{\beta-1}}, \qquad t > 0,$$

which is positive and measurable and, therefore, fits into the setting of Theorem 3.1.3.

Using Lemma 3.1.4, it follows that

$$\begin{aligned}
\frac{\bar{F}(t+xg(t))}{\bar{F}(t)} &\sim \frac{\frac{1}{\left(t+\frac{x}{t^{\beta-1}}\right)^{\beta-1}}\left(t+\frac{x}{t^{\beta-1}}\right)^{\alpha} e^{-\frac{\left(t+\frac{x}{t^{\beta-1}}\right)^{\beta}}{\beta}}}{\frac{1}{t^{\beta-1}}\, t^{\alpha}\, e^{-\frac{t^{\beta}}{\beta}}} \\
&= \left(\frac{t}{t+\frac{x}{t^{\beta-1}}}\right)^{\beta-1} \left(\frac{t+\frac{x}{t^{\beta-1}}}{t}\right)^{\alpha} \exp\left(-\frac{1}{\beta}\left(\left(t+\frac{x}{t^{\beta-1}}\right)^{\beta}-t^{\beta}\right)\right) \\
&= \left(1+\frac{x}{t^{\beta}}\right)^{1-\beta} \left(1+\frac{x}{t^{\beta}}\right)^{\alpha} \exp\left(-\frac{1}{\beta}\left(\left(t+\frac{x}{t^{\beta-1}}\right)^{\beta}-t^{\beta}\right)\right) \\
&= \left(1+\frac{x}{t^{\beta}}\right)^{\alpha+1-\beta} \exp\left(-\frac{1}{\beta}\left(\left(t+\frac{x}{t^{\beta-1}}\right)^{\beta}-t^{\beta}\right)\right),
\end{aligned}$$

as $t \to \infty$. Now, using the Taylor-Lagrange expansion up to second order yields that there is an absolute constant $C \in (-\infty, \infty)$ such that

$$\left(t+\frac{x}{t^{\beta-1}}\right)^{\beta} = t^{\beta} + \beta x + \frac{x^2}{2t^{2(\beta-1)}}\beta(\beta-1)(t+C)^{\beta-2}.$$

Since $(t+C)^{\beta-2} \sim t^{\beta-2}$, as $t \to \infty$, we obtain

$$\frac{(t+C)^{\beta-2}}{t^{2(\beta-1)}} \sim \frac{t^{\beta-2}}{t^{2(\beta-1)}} \sim t^{-\beta},$$

as $t \to \infty$. Thus,

$$\begin{aligned}
\exp\left(-\frac{1}{\beta}\left(\left(t+\frac{x}{t^{\beta-1}}\right)^{\beta}-t^{\beta}\right)\right) &= \exp\left(-\frac{1}{\beta}\left(\beta x + \frac{x^2}{2t^{2(\beta-1)}}\beta(\beta-1)(t+C)^{\beta-2}\right)\right) \\
&= e^{-x} \exp\left(-\frac{x^2}{2t^{2(\beta-1)}}(\beta-1)(t+C)^{\beta-2}\right) \\
&\sim e^{-x},
\end{aligned}$$

as $t \to \infty$. Combined with

$$\lim_{t\to\infty}\left(1+\frac{x}{t^{\beta}}\right)^{\alpha+1-\beta} = 1,$$

this yields that

$$\lim_{t\to\infty}\frac{\bar{F}(t+xg(t))}{\bar{F}(t)} = e^{-x},$$

for all $x \in \mathbb{R}$, finishing the proof. □

Proof of Theorem 3.1.1. It remains to find sequences $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that

$$\frac{M_n - b_n}{a_n} \xrightarrow{D} G,$$

as $n \to \infty$, where G is Gumbel distributed. The first condition of (3.3) and Lemma 3.1.4 imply that

$$c_{\alpha,\beta}^{-1}\, b_n^{\beta-\alpha-1}\, e^{\frac{b_n^\beta}{\beta}} \sim n,$$

as $n \to \infty$. As a first approach of the sequence b_n, we choose w_n by $e^{\frac{w_n^\beta}{\beta}} = n$, i.e.,

$$w_n = (\beta \log n)^{\frac{1}{\beta}}.$$

Of course, this is not the right choice for b_n since it holds that

$$c_{\alpha,\beta}^{-1}\, w_n^{\beta-\alpha-1}\, e^{\frac{w_n^\beta}{\beta}} = c_{\alpha,\beta}^{-1}\, (\beta \log n)^{\frac{\beta-\alpha-1}{\beta}}\; n \nsim n,$$

as $n \to \infty$. Thus, we need to modify w_n and, therefore, start with the estimate

$$b_n = (\beta \log n)^{\frac{1}{\beta}} + \delta_n,$$

where $(\delta_n)_{n\in\mathbb{N}}$ is an unknown sequence. By using again the Taylor-Lagrange expansion up to second order, we obtain that there exists an absolute constant $C \in (-\infty, \infty)$ such that

$$\begin{aligned}
&\exp\left(\frac{1}{\beta}\left((\beta \log n)^{\frac{1}{\beta}} + \delta_n\right)^\beta\right)\\
&\quad = \exp\left(\log n + (\beta \log n)^{\frac{\beta-1}{\beta}}\delta_n + \frac{\delta_n^2(\beta-1)}{2}\left((\beta \log n)^{\frac{1}{\beta}} + C\right)^{\beta-2}\right)\\
&\quad = n\,\exp\left((\beta \log n)^{\frac{\beta-1}{\beta}}\delta_n\right)\exp\left(\frac{\delta_n^2(\beta-1)}{2}\left((\beta \log n)^{\frac{1}{\beta}} + C\right)^{\beta-2}\right).
\end{aligned}$$

Similarly, the Taylor-Lagrange expansion up to first order yields that there is an absolute constant $C' \in (-\infty, \infty)$, satisfying

$$\left((\beta \log n)^{\frac{1}{\beta}} + \delta_n\right)^{\beta-1-\alpha} = (\beta \log n)^{\frac{\beta-1-\alpha}{\beta}} + (\beta-1-\alpha)\delta_n\left((\alpha \log n)^{\frac{1}{\beta}} + C'\right)^{\beta-2-\alpha}.$$

This leads to

$$\begin{aligned}
&c_{\alpha,\beta}^{-1}\, b_n^{\beta-\alpha-1}\, e^{\frac{b_n^\beta}{\beta}} \\
&\quad = c_{\alpha,\beta}^{-1} \left((\beta \log n)^{\frac{1}{\beta}} + \delta_n \right)^{\beta-\alpha-1} \exp\left(\frac{1}{\beta} \left((\beta \log n)^{\frac{1}{\beta}} + \delta_n \right)^\beta \right) \\
&\quad = n\, c_{\alpha,\beta}^{-1} \left((\beta \log n)^{\frac{\beta-1-\alpha}{\beta}} + (\beta - 1 - \alpha)\delta_n \left((\beta \log n)^{\frac{1}{\beta}} + C' \right)^{\beta-2-\alpha} \right) \\
&\qquad \times \exp\left((\beta \log n)^{\frac{\beta-1}{\beta}} \delta_n \right) \exp\left(\frac{\delta_n^2(\beta-1)}{2} \left((\beta \log n)^{\frac{1}{\beta}} + C \right)^{\beta-2} \right).
\end{aligned} \tag{3.5}$$

Now, we aim to determine δ_n in a way that all expressions above, except for n, are asymptotically equivalent to 1. To achieve this, we choose δ_n such that

$$\begin{aligned}
& c_{\alpha,\beta}^{-1} (\beta \log n)^{\frac{\beta-\alpha-1}{\beta}} \exp\left((\beta \log n)^{\frac{\beta-1}{\beta}} \delta_n \right) = 1 \\
\Leftrightarrow \quad & \exp\left((\beta \log n)^{\frac{\beta-1}{\beta}} \delta_n \right) = \frac{1}{c_{\alpha,\beta}^{-1} (\beta \log n)^{\frac{\beta-\alpha-1}{\beta}}} \\
\Leftrightarrow \quad & \exp\left((\beta \log n)^{\frac{\beta-1}{\beta}} \delta_n \right) = \frac{1}{\left(c_{\alpha,\beta}^{-\frac{\beta}{\beta-\alpha-1}} \beta \log n \right)^{\frac{\beta-\alpha-1}{\beta}}} \\
\Leftrightarrow \quad & (\beta \log n)^{\frac{\beta-1}{\beta}} \delta_n = -\frac{\beta-\alpha-1}{\beta} \log\left(c_{\alpha,\beta}^{-\frac{\beta}{\beta-\alpha-1}} \beta \log n \right) \\
\Leftrightarrow \quad & \delta_n = -\frac{\beta-\alpha-1}{\beta} \frac{\log\left(c_{\alpha,\beta}^{-\frac{\beta}{\beta-\alpha-1}} \beta \log n \right)}{(\beta \log n)^{\frac{\beta-1}{\beta}}}.
\end{aligned}$$

Evidently, δ_n converges to 0, as $n \to \infty$. Furthermore, since

$$\frac{\left((\beta \log n)^{\frac{1}{\beta}} + C' \right)^{\beta-2-\alpha}}{(\beta \log n)^{\frac{\beta-1}{\beta} + \frac{\beta-1-\alpha}{\beta}}} \sim \frac{(\beta \log n)^{\frac{\beta-2-\alpha}{\beta}}}{(\beta \log n)^{\frac{2\beta-2-\alpha}{\beta}}} = (\beta \log n)^{-\frac{\beta}{\beta}} = (\beta \log n)^{-1},$$

as $n \to \infty$, we achieve that

$$\begin{aligned}
&\frac{(\beta - 1 - \alpha)\delta_n \left((\beta \log n)^{\frac{1}{\beta}} + C' \right)^{\beta-2-\alpha}}{(\beta \log n)^{\frac{\beta-1-\alpha}{\beta}}} \\
&\qquad = -\frac{(\beta-1-\alpha)^2}{\beta} \log\left(c_{\alpha,\beta}^{-\frac{\beta}{\beta-\alpha-1}} \beta \log n \right) \frac{\left((\beta \log n)^{\frac{1}{\beta}} + C' \right)^{\beta-2-\alpha}}{(\beta \log n)^{\frac{2\beta-2-\alpha}{\beta}}}
\end{aligned}$$

$$\sim -\frac{(\beta-1-\alpha)^2}{\beta}\log\left(c_{\alpha,\beta}^{-\frac{\beta}{\beta-\alpha-1}}\,\beta\log n\right)(\beta\log n)^{-1},$$

as $n \to \infty$. As a result, we get that

$$(\beta\log n)^{\frac{\beta-1-\alpha}{\beta}} + (\beta-1-\alpha)\delta_n\left((\beta\log n)^{\frac{1}{\beta}} + C'\right)^{\beta-2-\alpha} \sim (\beta\log n)^{\frac{\beta-1-\alpha}{\beta}}, \tag{3.6}$$

as $n \to \infty$. By exploiting the definition of δ_n, we have similarly that

$$\frac{\delta_n^2(\beta-1)}{2}\left((\beta\log n)^{\frac{1}{\beta}} + C\right)^{\beta-2}$$

converges to 0, as $n \to \infty$, and, therefore,

$$\exp\left(\frac{\delta_n^2(\beta-1)}{2}\left((\beta\log n)^{\frac{1}{\beta}} + C\right)^{\beta-2}\right) \sim 1, \tag{3.7}$$

as $n \to \infty$. Summarizing (3.6) and (3.7) yields that our choice of δ_n ensures that the right hand side of (3.5) is indeed asymptotically equivalent to n. As a consequence, the sequence $(b_n)_{n\in\mathbb{N}}$ can be defined as

$$b_n := (\beta\log n)^{\frac{1}{\beta}} - \frac{(\beta-\alpha-1)\,\log\left(c_{\alpha,\beta}^{-\frac{\beta}{\beta-\alpha-1}}\,\beta\log n\right)}{\beta\,(\beta\log n)^{\frac{\beta-1}{\beta}}},$$

for all $n \in \mathbb{N}$. Moreover, we know from the second part of (3.3) that the sequence $(a_n)_{n\in\mathbb{N}}$ fulfills $a_n = g(b_n) = \frac{1}{b_n^{\beta-1}}$. Since

$$b_n \sim (\beta\log n)^{\frac{1}{\beta}},$$

as $n \to \infty$, we lastly obtain that a_n can be chosen like

$$a_n := \frac{1}{(\beta\log n)^{\frac{\beta-1}{\beta}}},$$

for all $n \in \mathbb{N}$. That proves the theorem. $\square$

3.1.2 Scaling transformation

Let us start this section by stating the general setup, needed to define the crucial scaling transformation, taken in the Gaussian case from [23, Equation (1.5)]. If u_0 is the north pole on the sphere $\mathbb{S}^{d-1}$ and $T_{u_0} := T_{u_0}(\mathbb{S}^{d-1})$ the tangent space at this point, we identify T_{u_0} with the $(d-1)$-dimensional Euclidean space $\mathbb{R}^{d-1}$. Besides, we define $\exp^{-1}$ as the inverse of the exponential map $\exp := \exp_{u_0} : T_{u_0} \to \mathbb{S}^{d-1}$. It maps a vector $v \in T_{u_0}$ to the point $u \in \mathbb{S}^{d-1}$ in such a way that u lies at the end of the unique geodesic ray with length $\|v\|$, emanating at u_0 and having direction v. Note that the exponential map is injective on $\mathbb{B}_{d-1}(\boldsymbol{o}, \pi) := \{v \in T_{u_0} : \|v\| < \pi\}$ and we have that $\exp(\mathbb{B}_{d-1}(\boldsymbol{o}, \pi)) = \mathbb{S}^{d-1} \setminus \{-u_0\}$. (Following [23], we prefer to write $\mathbb{B}_{d-1}(\boldsymbol{o}, r)$ for a centered ball of radius $r > 0$ in T_{u_0} instead of $\mathbb{B}^{d-1}(\boldsymbol{o}, r)$ to prevent confusions.) Since the inverse of the exponential map is well-defined on the whole sphere $\mathbb{S}^{d-1}$, except for the point $-u_0$, we put $\exp^{-1}(-u_0) := (\boldsymbol{o}, \pi)$.

In the previous section, we saw that for sufficiently large λ, the polytope K_λ can be expected to grow like the d-dimensional Euclidean ball centered at the origin with radius $(\beta \log \lambda)^{\frac{1}{\beta}}$. In order to reflect this behavior in our scaling transformation, define

$$R_\lambda := \left[\beta \log \lambda - \left(\frac{\beta(d+1) - 2d - 2\alpha}{2}\right) \log\left(c_{\alpha,\beta}^{-\frac{2\beta d}{\beta(d+1)-2d-2\alpha}} \beta \log \lambda\right)\right]^{\frac{1}{\beta}}, \tag{3.8}$$

for all $\lambda > 0$ such that $R_\lambda \geq 1$. In particular, R_λ is asymptotically equivalent to the critical radius $(\beta \log \lambda)^{\frac{1}{\beta}}$ itself. The reason for the explicit choice of R_λ will become clear in the proof of the upcoming Lemma 3.1.7. We are now in the position to define the scaling transformation, illustrated by Figure 3.1 in the planar setting.

Definition 3.1.6 The mapping $T_\lambda : \mathbb{R}^d \to \mathbb{R}^{d-1} \times \mathbb{R}$, defined by

$$T_\lambda(x) := \left(R_\lambda^{\frac{\beta}{2}} \exp^{-1}\left(\frac{x}{\|x\|}\right), R_\lambda^{\beta}\left(1 - \frac{\|x\|}{R_\lambda}\right)\right), \qquad x \in \mathbb{R}^d \setminus \{\boldsymbol{o}\}, \tag{3.9}$$

maps $\mathbb{R}^d \setminus \{\boldsymbol{o}\}$ into the region

$$W_\lambda := R_\lambda^{\frac{\beta}{2}} \, \mathbb{B}_{d-1}(\boldsymbol{o}, \pi) \times (-\infty, R_\lambda^{\beta}] \subseteq \mathbb{R}^{d-1} \times \mathbb{R}.$$

Putting $T_\lambda(\boldsymbol{o}) := (\boldsymbol{o}, R_\lambda^{\beta})$, the transformation T_λ is a bijection between $\mathbb{R}^d$ and W_λ.

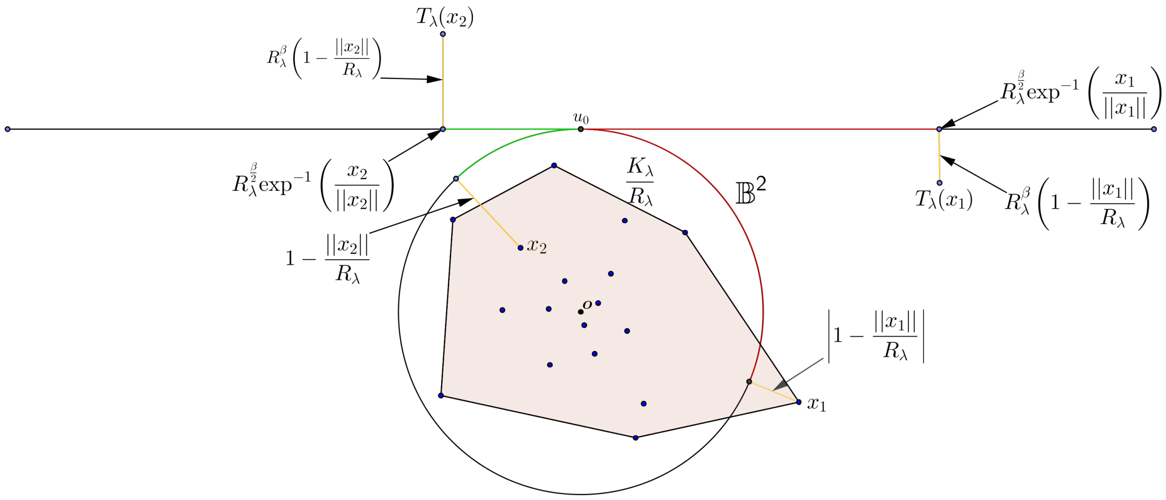

FIGURE 3.1: The scaling transformation T_λ.

Now, define the rescaled point process by

$$\mathcal{P}^{(\lambda)} := T_\lambda(\mathcal{P}_\lambda),$$

(see Figure 3.2). Due to the mapping property for Poisson point processes (see Theorem 2.6.2), the point process $\mathcal{P}^{(\lambda)}$ is actually also a Poisson point process in W_λ. Its distributional properties will be analyzed in the following two lemmas.

Lemma 3.1.7 *The intensity measure of $\mathcal{P}^{(\lambda)}$ has density*

$$\begin{aligned}(v,h) \mapsto & \frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}}\|v\|)}{\|R_\lambda^{-\frac{\beta}{2}}v\|^{d-2}} \frac{(\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}} \\ & \times \exp\left(h - \frac{h^2}{2R_\lambda^\beta}(\beta-1)(1-C)^{\beta-2}\right)\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathbf{1}((v,h)\in W_\lambda),\end{aligned} \tag{3.10}$$

with respect to the Lebesgue measure on $\mathbb{R}^{d-1}\times\mathbb{R}$, where $C \in (-\infty, 1)$ is an absolute constant.

Remark 3.1.8 Later, it turns out to be crucial to bound the exponential term in (3.10) uniformly by e^h, *for all* $h \in \mathbb{R}$. Examples are provided by the estimates presuming (3.46), (3.47), (3.53), (3.63) and (3.68). However, if $\beta < 1$, this is not achievable and, therefore, we may and will restrict to the condition $\beta \geq 1$. This natural condition was used also by Carnal [25, Page 171] and Eddy and Gale [41, Page 757].

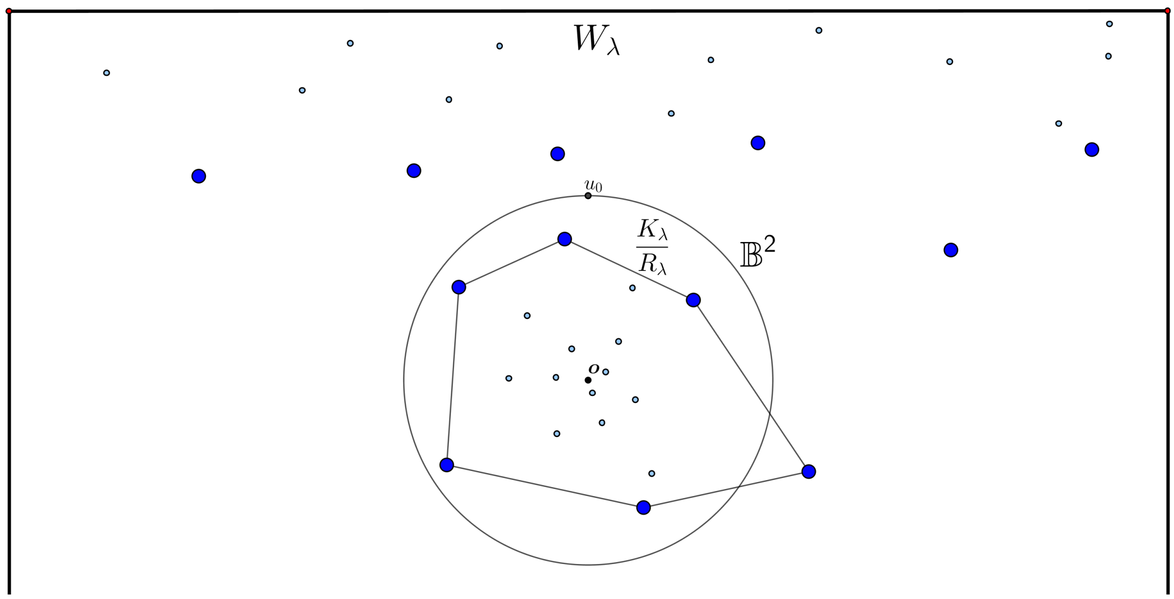

FIGURE 3.2: The rescaled Poisson point process $\mathcal{P}^{(\lambda)}$.

Due to the properties of the sine function and the definition of R_λ, the first two fractions in (3.10) converge to 1, as $\lambda \to \infty$, on compact subsets of W_λ. Moreover, for fixed $h \in \mathbb{R}$, the same holds true for the fourth expression, while the exponential term tends to e^h, as $\lambda \to \infty$. Summarizing, this implies the following important corollary.

Corollary 3.1.9 *As $\lambda \to \infty$, $\mathcal{P}^{(\lambda)}$ converges in distribution, in the sense of total variation convergence on compact sets, to a Poisson point process $\mathcal{P}$ on $\mathbb{R}^{d-1} \times \mathbb{R}$, whose intensity measure has density*

$$(v, h) \mapsto e^h, \qquad (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R},$$

with respect to the Lebesgue measure on $\mathbb{R}^{d-1} \times \mathbb{R}$, for all parameter α and β in the density $\phi_{\alpha,\beta}$.

Remark 3.1.10 The scaling transformation T_λ carries $\mathcal{P}_\lambda$ into a Poisson point process in the product space $\mathbb{R}^{d-1} \times \mathbb{R}$ that is stationary in the spatial coordinate, as $\lambda \to \infty$. On the one hand, this was to be expected in view of [41, Theorem 4.1] (generalizing a result obtained in [40]), where a transformation was constructed to carry the binomial counterpart of our $\mathcal{P}_\lambda$ into a point process in $\mathbb{R} \times \mathbb{R}^{d-1}$, whose height coordinate is determined by a Poisson point process with intensity $e^{-h}\mathrm{d}h$, $h \in \mathbb{R}$, while in the spatial regime a standard Gaussian process arises. On the other hand, the result in [41] clearly contrasts ours, in particular concerning the distribution in the spatial coordinate.

Proof of Lemma 3.1.7. Let us write $x \in \mathbb{R}^d$ as $x = ur$ with $u \in \mathbb{S}^{d-1}$ and $r \geq 0$. Thus, using polar coordinates, it follows that

$$\lambda \, \phi_{\alpha,\beta}(x) \, \mathrm{d}x = \lambda \, \phi_{\alpha,\beta}(ur) \, r^{d-1} \, \mathrm{d}r \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).$$

Following the proof of [23, Lemma 3.2], we achieve, by making the change of variables

$$v := R_\lambda^{\frac{\beta}{2}} \exp^{-1}(u) \qquad \text{and} \qquad h := R_\lambda^\beta \left(1 - \frac{r}{R_\lambda}\right) \Leftrightarrow r = R_\lambda \left(1 - \frac{h}{R_\lambda^\beta}\right),$$

that

$$\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) = \frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}} \|v\|)}{\|R_\lambda^{-\frac{\beta}{2}} v\|^{d-2}} \left(R_\lambda^{-\frac{\beta}{2}}\right)^{d-1} \mathrm{d}v. \tag{3.11}$$

Indeed, for all $v \in \mathbb{R}^{d-1} \setminus \{\boldsymbol{o}\}$, the exponential map can be expressed as

$$\exp(v) = \cos(||v||)(\boldsymbol{o}, 1) + \sin(||v||) \left(\frac{v}{||v||}, 0\right),$$

(see [23, Equation (3.14)]). Thus,

$$\begin{aligned} \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) &= \sin(|| \exp^{-1}(u)||)^{d-2} \, \mathrm{d}(|| \exp^{-1}(u)||) \, \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}} \left(\mathrm{d}\frac{\exp^{-1}(u)}{|| \exp^{-1}(u)||}\right) \\ &= \frac{\sin(|| \exp^{-1}(u)||)^{d-2}}{|| \exp^{-1}(u)||^{d-2}} \, \mathrm{d}(\exp^{-1}(u)), \end{aligned}$$

and the claim follows from $\exp^{-1}(u) = R_\lambda^{-\frac{\beta}{2}} v$. Moreover, by the choice of r, we achieve

$$r^{d-1} \, \mathrm{d}r = \left[R_\lambda \left(1 - \frac{h}{R_\lambda^\beta}\right)\right]^{d-1} R_\lambda^{-(\beta-1)} \, \mathrm{d}h. \tag{3.12}$$

Furthermore, we get

$$\begin{aligned} &\lambda \, \phi_{\alpha,\beta}(ur) \\ &\quad = (\beta \log \lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}} R_\lambda^\alpha \left(1 - \frac{h}{R_\lambda^\beta}\right)^\alpha \exp\left(h - \frac{h^2}{2R_\lambda^\beta}(\beta - 1)(1 - C)^{\beta-2}\right), \end{aligned} \tag{3.13}$$

for some absolute constant $C \in (-\infty, 1)$.

Indeed, using the Taylor-Lagrange expansion up to second order of the function $(1-x)^\beta$ at the point 0 yields that there exists an absolute constant $C \in (0, x)$ such that

$$(1-x)^\beta = 1 - \beta x + \frac{x^2}{2}\beta(\beta-1)(1-C)^{\beta-2}. \tag{3.14}$$

The definitions of r and $\phi_{\alpha,\beta}(x)$, as well as (3.14) applied to $x = h/R_\lambda^\beta$, imply that there is an absolute constant $C \in (-\infty, 1)$, satisfying

$$\begin{aligned}
&\phi_{\alpha,\beta}(ur)\\
&\quad= \phi_{\alpha,\beta}\left(u\,R_\lambda\left(1-\frac{h}{R_\lambda^\beta}\right)\right)\\
&\quad= c_{\alpha,\beta}^d\,R_\lambda^\alpha\left(1-\frac{h}{R_\lambda^\beta}\right)^\alpha \exp\left(-\frac{1}{\beta}\,R_\lambda^\beta\left(1-\frac{h}{R_\lambda^\beta}\right)^\beta\right)\\
&\quad= c_{\alpha,\beta}^d\,R_\lambda^\alpha\left(1-\frac{h}{R_\lambda^\beta}\right)^\alpha \exp\left(-\frac{1}{\beta}\,R_\lambda^\beta\left(1-\beta\frac{h}{R_\lambda^\beta}+\frac{h^2}{2R_\lambda^{2\beta}}\,\beta(\beta-1)(1-C)^{\beta-2}\right)\right)\\
&\quad= c_{\alpha,\beta}^d\,R_\lambda^\alpha\left(1-\frac{h}{R_\lambda^\beta}\right)^\alpha \exp\left(-\frac{R_\lambda^\beta}{\beta}+h-\frac{h^2}{2R_\lambda^\beta}\,(\beta-1)(1-C)^{\beta-2}\right)\\
&\quad= c_{\alpha,\beta}^d\,R_\lambda^\alpha\left(1-\frac{h}{R_\lambda^\beta}\right)^\alpha \exp\left(-\frac{R_\lambda^\beta}{\beta}\right)\exp\left(h-\frac{h^2}{2R_\lambda^\beta}\,(\beta-1)(1-C)^{\beta-2}\right)\\
&\quad= \frac{1}{\lambda}\,R_\lambda^\alpha\left(1-\frac{h}{R_\lambda^\beta}\right)^\alpha (\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}\exp\left(h-\frac{h^2}{2R_\lambda^\beta}\,(\beta-1)(1-C)^{\beta-2}\right).
\end{aligned}$$

Note that we used the explicit choice of R_λ in the last step to deduce

$$\begin{aligned}
&\exp\left(-\frac{R_\lambda^\beta}{\beta}\right)\\
&\quad= \exp\left(-\frac{1}{\beta}\left(\beta\log\lambda-\left(\frac{\beta(d+1)-2d-2\alpha}{2}\right)\log\left(c_{\alpha,\beta}^{-\frac{2\beta d}{\beta(d+1)-2d-2\alpha}}\,\beta\log\lambda\right)\right)\right)\\
&\quad= \exp(-\log\lambda)\,\exp\left(\left(\frac{\beta(d+1)-2d-2\alpha}{2\beta}\right)\log\left(c_{\alpha,\beta}^{-\frac{2\beta d}{\beta(d+1)-2d-2\alpha}}\,\beta\log\lambda\right)\right)\\
&\quad= \frac{1}{\lambda}\left(c_{\alpha,\beta}^{-\frac{2\beta d}{\beta(d+1)-2d-2\alpha}}\,\beta\log\lambda\right)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}\\
&\quad= \frac{1}{\lambda}\,(\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}\,c_{\alpha,\beta}^{-d}.
\end{aligned}$$

Combining (3.11), (3.12) and (3.13) with

$$R_\lambda^\alpha \, R_\lambda^{-\frac{\beta(d-1)}{2}} \, R_\lambda^{d-1} \, R_\lambda^{-(\beta-1)} = R_\lambda^{\frac{-\beta d+\beta+2d-2-2\beta+2+2\alpha}{2}} = R_\lambda^{\frac{-\beta(d+1)+2d+2\alpha}{2}}$$

finishes the proof. □

Let $i \in \{1, \ldots, d\}$. Similarly to the notation used in [23, Page 41], we denote by $\mathrm{vol}_i^{(\lambda)}$ the image of

$$R_\lambda^{\frac{\beta(d+1)-2i}{2}} \, \|x\|^{i-d} \, \mathrm{vol}_d$$

under the scaling transformation T_λ, where vol_d is the usual d-dimensional Lebesgue-measure on $\mathbb{R}^d$.

Lemma 3.1.11 *Let $i \in \{1, \ldots, d\}$. Then, the image measure $\mathrm{vol}_i^{(\lambda)}$ under the scaling transformation T_λ has density*

$$(v, h) \mapsto \frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}} \|v\|)}{\|R_\lambda^{-\frac{\beta}{2}} v\|^{d-2}} \left(1 - \frac{h}{R_\lambda^\beta}\right)^{i-1} \mathbf{1}((v, h) \in W_\lambda), \tag{3.15}$$

with respect to the Lebesgue measure on the product space $\mathbb{R}^{d-1} \times \mathbb{R}$.

Corollary 3.1.12 *It is readily seen that the density in* (3.15) *converges point wise to 1, as $\lambda \to \infty$, proving that the image measure $\mathrm{vol}_i^{(\lambda)}$, $i \in \{1, \ldots, d\}$, converges in distribution to vol_d, again in the sense of total variation convergence on compact sets.*

Proof of Lemma 3.1.11. Starting with $\|x\|^{i-d} \, \mathrm{d}x$ instead of $\lambda \, \phi_{\alpha,\beta}(x) \, \mathrm{d}x$ in the proof of Lemma 3.1.7 implies that the density of the image measure $\mathrm{vol}_i^{(\lambda)}$ is given by the product of the terms on the right hand sides of (3.11) and (3.12), times

$$R_\lambda^{\frac{\beta(d+1)-2i}{2}} \, R_\lambda^{i-d} \left(1 - \frac{h}{R_\lambda^\beta}\right)^{i-d},$$

with respect to the Lebesgue measure on $\mathbb{R}^{d-1} \times \mathbb{R}$. Multiplication of these three expressions yields the density stated in (3.15). □

3.1.3 Germ-grain processes

Let us start this section with two observations regarding K_λ, explained in detail for example in [23, Page 14]. A point $x' \in \mathcal{P}_\lambda$ is a vertex of K_λ, if and only if the ball

$$\mathbb{B}^d\left(\frac{x'}{2}, \frac{||x'||}{2}\right)$$

is not contained in the union of all balls corresponding to the other points of $\mathcal{P}_\lambda$, i.e, in

$$\bigcup_{\substack{y\in\mathcal{P}_\lambda \\ y\neq x}} \mathbb{B}^d\left(\frac{y}{2}, \frac{||y||}{2}\right).$$

Let

$$\theta := d_{\mathbb{S}^{d-1}}\left(\frac{x}{||x||}, \frac{x'}{||x'||}\right)$$

be the geodesic distance on the sphere. Then, we can rewrite the ball as

$$\begin{aligned} \mathbb{B}^d\left(\frac{x'}{2}, \frac{||x'||}{2}\right) &= \{x \in \mathbb{R}^d : ||x|| \leq ||x'||\, \cos\theta\} \\ &= \left\{x \in \mathbb{R}^d : R_\lambda^\beta \left(1 - \frac{||x||}{R_\lambda \cos\theta}\right) \geq R_\lambda^\beta \left(1 - \frac{||x'||}{R_\lambda}\right)\right\}. \end{aligned} \tag{3.16}$$

On the other hand, $\mathbb{R}^d \setminus K_\lambda$ is the union of half-spaces that do not contain points of $\mathcal{P}_\lambda$. For $x' \in \mathbb{R}^d$, consider the half-space

$$\begin{aligned} H(x') :&= \{x \in \mathbb{R}^d : ||x'|| \leq ||x||\, \cos\theta\} \\ &= \left\{x \in \mathbb{R}^d : R_\lambda^\beta \left(1 - \frac{||x'||}{R_\lambda \cos\theta}\right) \geq R_\lambda^\beta \left(1 - \frac{||x||}{R_\lambda}\right)\right\}, \end{aligned} \tag{3.17}$$

which is one of the main ingredients of the following lemma.

Lemma 3.1.13 *Putting $T_\lambda(x') := (v', h') \in \mathbb{R}^{d-1} \times \mathbb{R}$, the scaling transformation T_λ transforms the ball $\mathbb{B}^d\left(\frac{x'}{2}, \frac{||x'||}{2}\right)$ and the half-space $H(x')$ into the upward opening grain*

$$[\Pi^\uparrow(v', h')]^{(\lambda)} := \left\{(v, h) \in W_\lambda : h \geq R_\lambda^\beta(1 - \cos(d_\lambda(v', v))) + h' \cos(d_\lambda(v', v))\right\}, \tag{3.18}$$

and the downward opening grain

$$[\Pi^{\downarrow}(v',h')]^{(\lambda)} := \left\{(v,h) \in W_\lambda : h \le R_\lambda^\beta - \frac{R_\lambda^\beta - h'}{\cos(d_\lambda(v',v))}\right\}, \tag{3.19}$$

respectively, where

$$d_\lambda(v',v) := d_{\mathbb{S}^{d-1}}\Big(\exp(R_\lambda^{-\frac{\beta}{2}} v'), \exp(R_\lambda^{-\frac{\beta}{2}} v)\Big)$$

is the geodesic distance between the images of the rescaled points v' and v under the exponential map.

Proof. The second characterization of the ball in (3.16) implies that

$$\begin{aligned}
& R_\lambda^\beta\left(1 - \frac{||x||}{R_\lambda \cos\theta}\right) && \ge R_\lambda^\beta\left(1 - \frac{||x'||}{R_\lambda}\right) \\
\Leftrightarrow \quad & R_\lambda^\beta \cos\theta - R_\lambda^{\beta-1}\,||x|| \ge R_\lambda^\beta\left(1 - \frac{||x'||}{R_\lambda}\right)\cos\theta && \\
\Leftrightarrow \quad & R_\lambda^\beta - R_\lambda^{\beta-1}\,||x|| && \ge R_\lambda^\beta - R_\lambda^\beta\cos\theta + R_\lambda^\beta\left(1 - \frac{||x'||}{R_\lambda}\right)\cos\theta \\
\Leftrightarrow \quad & R_\lambda^\beta\left(1 - \frac{||x||}{R_\lambda}\right) && \ge R_\lambda^\beta(1-\cos\theta) + R_\lambda^\beta\left(1 - \frac{||x'||}{R_\lambda}\right)\cos\theta.
\end{aligned}$$

Therefore,

$$h \ge R_\lambda^\beta(1 - \cos(d_\lambda(v',v))) + h'\cos(d_\lambda(v',v)),$$

where we used

$$h' = R_\lambda^\beta\left(1 - \frac{||x'||}{R_\lambda}\right), \qquad h = R_\lambda^\beta\left(1 - \frac{||x||}{R_\lambda}\right), \qquad v' = R_\lambda^{\frac{\beta}{2}}\exp^{-1}\left(\frac{x'}{||x'||}\right),$$

and

$$v = R_\lambda^{\frac{\beta}{2}}\exp^{-1}\left(\frac{x}{||x||}\right),$$

in view of the scaling transformation T_λ.

Similarly, we get from (3.17) that

$$R_\lambda^\beta \left(1 - \frac{||x||}{R_\lambda}\right) \le R_\lambda^\beta - \frac{R_\lambda^\beta \frac{||x'||}{R_\lambda}}{\cos\theta} = R_\lambda^\beta - \frac{R_\lambda^\beta - R_\lambda^\beta \left(1 - \frac{||x'||}{R_\lambda}\right)}{\cos\theta},$$

and, thus,

$$h \le R_\lambda^\beta - \frac{R_\lambda^\beta - h'}{\cos(d_\lambda(v', v))}.$$

This proves the claim. □

Consequently, T_λ transforms the sets

$$\bigcup_{x \in \mathcal{P}_\lambda} \mathbb{B}^d \left(\frac{x}{2}, \frac{||x||}{2}\right) \qquad \text{and} \qquad \mathbb{R}^d \setminus K_\lambda$$

into the quasi-paraboloid germ-grain models

$$\Psi^{(\lambda)} := \Psi^{(\lambda)}(T_\lambda(\mathcal{P}_\lambda)) := \bigcup_{w \in \mathcal{P}^{(\lambda)}} [\Pi^\uparrow(w)]^{(\lambda)},$$

(see Figure 3.3), and

$$\Phi^{(\lambda)} := \Phi^{(\lambda)}(T_\lambda(\mathcal{P}_\lambda)) := \bigcup_{\substack{w \in W_\lambda \\ \mathcal{P}^{(\lambda)} \cap [\Pi^\downarrow(w)]^{(\lambda)} = \emptyset}} [\Pi^\downarrow(w)]^{(\lambda)},$$

(see Figure 3.4), respectively.

What is crucial about these germ-grain processes is that now, for sufficiently large λ, a point $x' \in \mathcal{P}_\lambda$ is a vertex of K_λ, if and only if the germ $[\Pi^\uparrow(T_\lambda(x'))]^{(\lambda)}$ is not covered by $\Psi^{(\lambda)}(T_\lambda(\mathcal{P}_\lambda \setminus \{x'\}))$. This observation has been used extensively in the Gaussian case in [23] and also our results exploit this fact. In this case, $T_\lambda(x')$ is called an extreme point of $\mathcal{P}^{(\lambda)}$, whose collection we denote by $\text{ext}(\mathcal{P}^{(\lambda)})$. Moreover, the boundary $\partial\Phi^{(\lambda)}$ of $\Phi^{(\lambda)}$ is build from piecewise quasi-parabolic facets, glued together at the extreme points of $\mathcal{P}^{(\lambda)}$. Recall that $T_\lambda(\mathcal{P}_\lambda)$ converges in distribution to a Poisson point process $\mathcal{P}$ in $\mathbb{R}^{d-1} \times \mathbb{R}$, whose intensity measure has density

$$(v, h) \mapsto e^h, \qquad (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R},$$

with respect to the Lebesgue measure on $\mathbb{R}^{d-1} \times \mathbb{R}$.

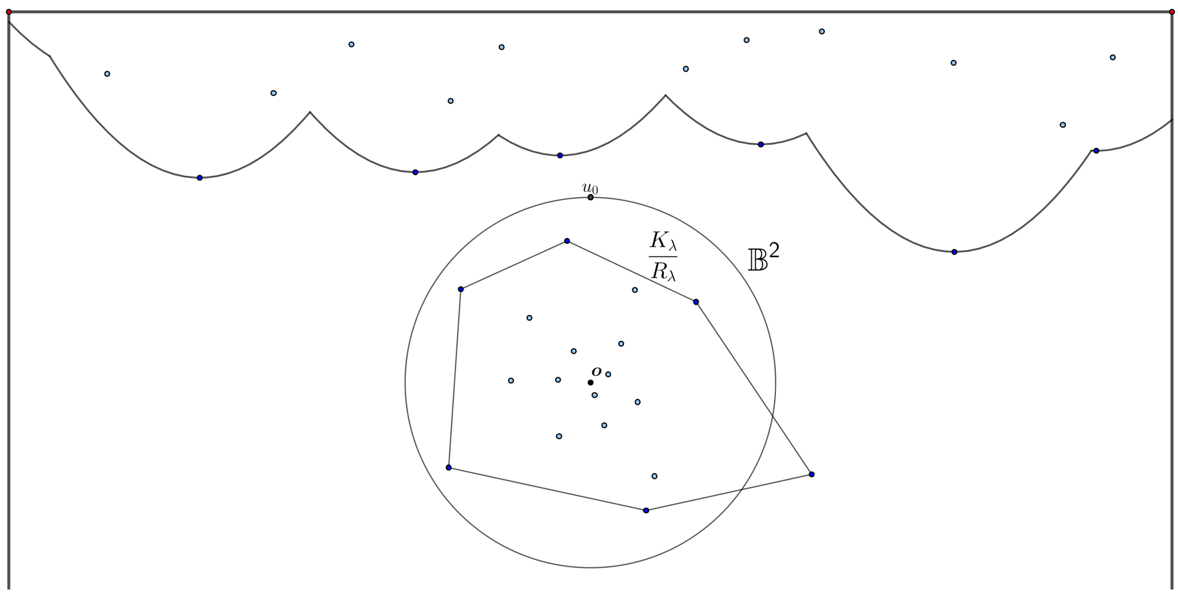

FIGURE 3.3: The germ-grain model $\Psi^{(\lambda)}$.

Thus, it seems natural that the boundaries of the quasi-paraboloid germ-grain models $\Psi^{(\lambda)}$ and $\Phi^{(\lambda)}$ converge to those of so-called limit paraboloid germ-grain models Ψ and Φ, corresponding to $\mathcal{P}$ and defined as follows. Putting

$$\Pi^{\uparrow} := \left\{ (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \geq \frac{\|v\|^2}{2} \right\},$$

and

$$\Pi^{\downarrow} := \left\{ (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \leq -\frac{\|v\|^2}{2} \right\},$$

to be the unit up- and downward paraboloids, respectively, define

$$\Psi := \Psi(\mathcal{P}) := \bigcup_{w \in \mathcal{P}} [\Pi^{\uparrow}(w)]^{(\infty)} \quad \text{and} \quad \Phi := \Phi(\mathcal{P}) := \bigcup_{\substack{w \in \mathbb{R}^{d-1} \times \mathbb{R} \\ \mathcal{P} \cap \mathrm{int}(\Pi^{\downarrow}(w)) = \emptyset}} [\Pi^{\downarrow}(w)]^{(\infty)}, \tag{3.20}$$

where, for $w := (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}$,

$$[\Pi^{\uparrow}(w)]^{(\infty)} := w \oplus \Pi^{\uparrow} \qquad \text{and} \qquad [\Pi^{\downarrow}(w)]^{(\infty)} := w \oplus \Pi^{\downarrow}.$$

All the points of $\mathcal{P}$ that belong to the boundary of Φ are summarized in the set of extreme points of $\mathcal{P}$, denoted by $\mathrm{ext}(\mathcal{P})$.

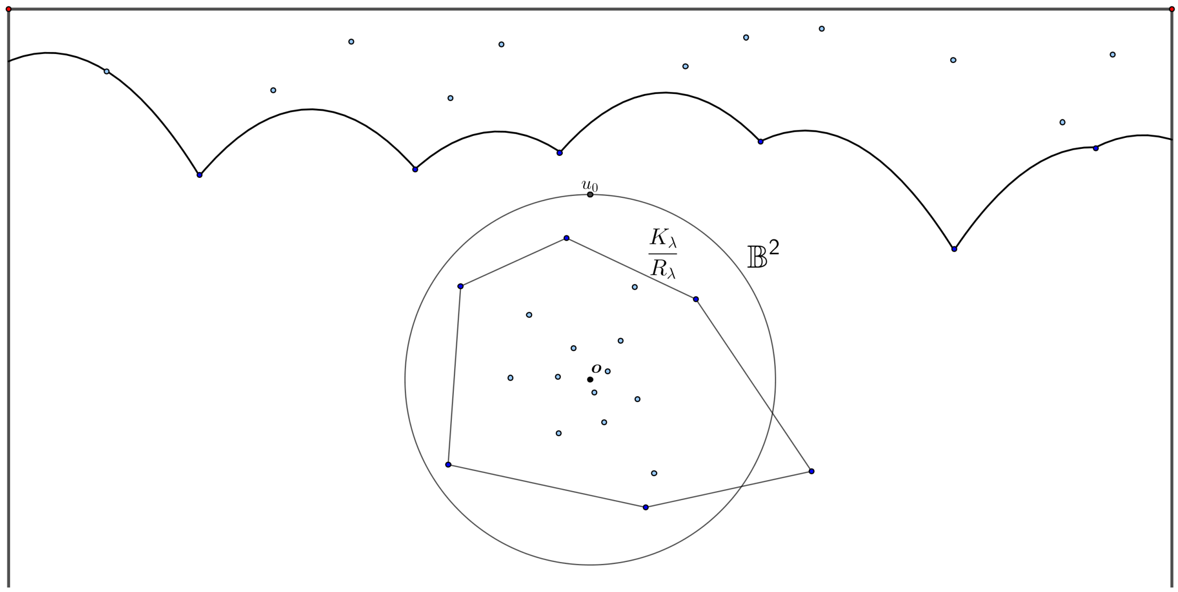

FIGURE 3.4: The germ-grain model $\Phi^{(\lambda)}$.

3.1.4 Functionals of interest

Recall that K_λ is the generalized Gamma polytope, arising as the convex hull of the Poisson point process $\mathcal{P}_\lambda$. If x is an extreme point of $\mathcal{P}_\lambda$, we denote by $\mathcal{F}_j(x, \mathcal{P}_\lambda)$, $j \in \{1, \dots, d-1\}$, the set of all j-dimensional faces of K_λ containing x, while $|\mathcal{F}_j(x, \mathcal{P}_\lambda)|$ indicates its cardinality. Moreover, we define by $\operatorname{cone}(x, \mathcal{P}_\lambda) := \{ry : r \geq 0, y \in \mathcal{F}_{d-1}(x, \mathcal{P}_\lambda)\}$ the cone corresponding to the facets $\mathcal{F}_{d-1}(x, \mathcal{P}_\lambda)$. Furthermore, recall the definition of the projection-avoidance functional θ_i from (2.2). We are now in the position to introduce the functionals of interest and start with those regarding to $\mathcal{P}_\lambda$.

Definition 3.1.14 (Intrinsic volume and face functionals) For $i \in \{1, \dots, d\}$, we define the defect intrinsic volume functional with respect to the ball $\mathbb{B}^d(\boldsymbol{o}, R_\lambda)$ by putting

$$\xi_{V_i}(x, \mathcal{P}_\lambda) := R_\lambda^{\frac{\beta(d+1)-2i}{2}} \frac{\binom{d-1}{i-1}}{d\,\kappa_{d-i}} \int\limits_{\operatorname{cone}(x,\mathcal{P}_\lambda)} [\theta_i(y, K_\lambda) - \theta_i(y, \mathbb{B}^d(\boldsymbol{o}, R_\lambda))] \frac{1}{||y||^{d-i}} \, \mathrm{d}y,$$

if x is an extreme point of $\mathcal{P}_\lambda$, and 0 for all other points of $\mathcal{P}_\lambda$. In particular, for $i = d$, we arrive at

$$\xi_{V_d}(x, \mathcal{P}_\lambda) = \frac{1}{d} R_\lambda^{\frac{\beta(d+1)-2d}{2}} [\operatorname{vol}_d(\operatorname{cone}(x, \mathcal{P}_\lambda) \cap \mathbb{B}^d(\boldsymbol{o}, R_\lambda)) - \operatorname{vol}_d(\operatorname{cone}(x, \mathcal{P}_\lambda) \cap K_\lambda)],$$

i.e., the rescaled defect volume with respect to the aforementioned ball.

Moreover, for $j \in \{0, \dots, d-1\}$, we define the j-face functional of the generalized Gamma polytope K_λ by putting

$$\xi_{f_j}(x, \mathcal{P}_\lambda) := \begin{cases} \frac{1}{j+1}|\mathcal{F}_j(x, \mathcal{P}_\lambda)| & : x \in \mathrm{ext}(\mathcal{P}_\lambda) \\ 0 & : x \notin \mathrm{ext}(\mathcal{P}_\lambda). \end{cases}$$

We shall write

$$\Xi := \{\xi_{V_1}, \dots, \xi_{V_d}, \xi_{f_0}, \dots, \xi_{f_{d-1}}\}$$

for the collection of the geometric functionals and use, for $\xi \in \Xi$, the abbreviation

$$H_\lambda^\xi := \sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda). \tag{3.21}$$

With these definitions, it follows that the total number of j-dimensional faces of K_λ, $j \in \{0, \dots, d-1\}$, almost surely satisfies

$$f_j(K_\lambda) = H_\lambda^{\xi_{f_j}},$$

while the total i-th defect intrinsic volume of K_λ, $i \in \{1, \dots, d\}$, with respect to the ball $\mathbb{B}^d(\boldsymbol{o}, R_\lambda)$, almost surely fulfills

$$V_i(\mathbb{B}^d(\boldsymbol{o}, R_\lambda)) - V_i(K_\lambda) = R_\lambda^{-\frac{\beta(d+1)-2i}{2}} H_\lambda^{\xi_{V_i}}, \tag{3.22}$$

conditioned on the event that $\boldsymbol{o} \in K_\lambda$. We notice that this event occurs with probability at least $1 - e^{-c\lambda}$, for some constant $c \in (0, \infty)$ only depending on d. To keep our presentation short, in all computations concerning the functional ξ_{V_i} that are carried out in this chapter, we implicitly condition on this event. In fact, this causes – up to constants – no changes in our results since conditioning on the complementary event only leads to terms that are negligible for sufficiently large λ. Also, implicitly this convention has already been used in [22, 23, 56].

If $w \in \mathrm{ext}(\mathcal{P}^{(\lambda)})$, recall the definition in Section 3.1.3, let $\mathrm{Cyl}^{(\lambda)}(w)$ indicate the set in $\mathbb{R}^{d-1}$, achieved by projecting the facets of $\Phi^{(\lambda)}$ that contain the point w onto $\mathbb{R}^{d-1}$. Furthermore, define $|\mathcal{F}_j(w, \mathcal{P}^{(\lambda)})|$ to be the number of j-dimensional faces of $\partial(\bigcup_{v \in \mathcal{P}^{(\lambda)}} [\Pi^{\downarrow}(v)]^{(\lambda)})$ that contain w.

Definition 3.1.15 (Rescaled intrinsic volume and face functionals) Let $\xi \in \Xi$ and λ be sufficiently large. The rescaled functional $\xi^{(\lambda)}$ under the scaling transformation T_λ is defined by

$$\xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := \xi(T_\lambda^{-1}(w), T_\lambda^{-1}(\mathcal{P}^{(\lambda)})), \qquad w \in W_\lambda.$$

In particular, for $i \in \{1, \ldots, d\}$, define

$$\xi_{V_i}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := \frac{1}{d} \int_{\mathrm{Cyl}^{(\lambda)}(w)} \int_0^{\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})(v)} \mathrm{vol}_i^{(\lambda)}(\mathrm{d}(v, h)),$$

if w belongs to the extreme points of $\mathcal{P}^{(\lambda)}$, and 0 otherwise. Additionally, for $j \in \{0, \ldots, d-1\}$, define

$$\xi_{f_j}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := \begin{cases} \frac{1}{j+1} \left|\mathcal{F}_j(w, \mathcal{P}^{(\lambda)})\right| & : w \in \mathrm{ext}(\mathcal{P}^{(\lambda)}) \\ 0 & : w \notin \mathrm{ext}(\mathcal{P}^{(\lambda)}). \end{cases}$$

Denote by

$$\Xi^{(\lambda)} := \{\xi_{V_1}^{(\lambda)}, \ldots, \xi_{V_d}^{(\lambda)}, \xi_{f_0}^{(\lambda)}, \ldots, \xi_{f_{d-1}}^{(\lambda)}\}$$

the family of rescaled geometric functionals.

Remark 3.1.16 Here and in the rest of this chapter, we adopt the following notational convention. If $w \in W_\lambda$ does not belong to the rescaled point process $\mathcal{P}^{(\lambda)}$, we understand $\xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$ as $\xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)} \cup \{w\})$ and, similarly, also $\xi(x, \mathcal{P}_\lambda)$ as $\xi(x, \mathcal{P}_\lambda \cup \{x\})$.

Now, if $w \in \mathrm{ext}(\mathcal{P})$, let $\mathrm{Cyl}(w)$ denote the set in $\mathbb{R}^{d-1}$ obtained by projecting the hyperfaces of $\partial(\Phi(\mathcal{P}))$ that contain w onto $\mathbb{R}^{d-1}$. Moreover, $|\mathcal{F}_j(w, \mathcal{P})|$, $j \in \{1, \ldots, d-1\}$, indicates the number of j-dimensional parabolic faces of $\partial(\Phi(\mathcal{P}))$ containing w.

In order to define the scaling limit intrinsic volume and face functionals, we first introduce some more necessary notation. For every $w = (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}$, we denote by $w^\updownarrow$ the set $\{v\} \times \mathbb{R}$ and by $\mu_i^{w^\updownarrow}$ the normalized Haar measure on the set $A(w^\updownarrow, i)$ of all i-dimensional affine spaces in $\mathbb{R}^d$ containing $w^\updownarrow$. Moreover, for every affine space L

containing $w^{\updownarrow}$, we define the corresponding orthogonal paraboloid $\Pi^{\perp}[w;L]$ as the set

$$\left\{ w' = (v', h') \in \mathbb{R}^{d-1} \times \mathbb{R} : (w - w') \perp L, h' \leq h - \frac{||v - v'||^2}{2} \right\},$$

where $(w - w') \perp L$ indicates that the vector $w - w'$ is orthogonal to L. In other words, $\Pi^{\perp}[w;L]$ is the set of points of $w \oplus L^{\perp}$, positioned 'under' the paraboloid surface $\partial \Pi^{\downarrow}(w)$ with apex at w.

Building on all this notation, we put

$$\vartheta_L^{(\infty)}(w) := \begin{cases} 1 & : \Pi^{\perp}[w;L] \cap T_{\lambda}(K_{\lambda}) = \emptyset \\ 0 & : \text{otherwise.} \end{cases}$$

In particular, when $L = w^{\updownarrow}$, we get that $\vartheta_{w^{\updownarrow}}^{(\infty)}(w) = \mathbf{1}(\Pi^{\perp}[w;L] \cap \Phi = \emptyset)$.

Definition 3.1.17 (Scaling limit intrinsic volume and face functionals) For $i \in \{1, \ldots, d\}$, the scaling limit defect intrinsic volume functional is given by

$$\xi_{V_i}^{(\infty)}(w, \mathcal{P}) := \frac{1}{d} \frac{\binom{d-1}{i-1}}{\kappa_{d-i}} \int\limits_{\mathrm{Cyl}(w)} \left[\vartheta_i^{(\infty)}(w') - \mathbf{1}(\{w' \in \mathbb{R}^{d-1} \times \mathbb{R}_-\}) \right] \mathrm{d}w',$$

if w belongs to the extreme points of $\mathcal{P}$, and 0 otherwise. Here, for every $w \in \mathbb{R}^d$,

$$\vartheta_i^{(\infty)}(w) := \int\limits_{A(w^{\updownarrow}, i)} \vartheta_L^{(\infty)}(w') \, \mathrm{d}\mu_i^{w^{\updownarrow}}(L).$$

Moreover, for $j \in \{0, \ldots, d-1\}$,

$$\xi_{f_j}^{(\infty)}(w, \mathcal{P}) := \begin{cases} \frac{1}{j+1} |\mathcal{F}_j(w, \mathcal{P})| & : w \in \mathrm{ext}(\mathcal{P}) \\ 0 & : w \notin \mathrm{ext}(\mathcal{P}) \end{cases}$$

is the scaling limit j-face functional. Similarly as before, let

$$\Xi^{(\infty)} := \{\xi_{V_1}^{(\infty)}, \ldots, \xi_{V_d}^{(\infty)}, \xi_{f_0}^{(\infty)}, \ldots, \xi_{f_{d-1}}^{(\infty)}\}$$

indicate the family of scaling limit functionals.

Based on the rescaled and scaling limit functionals, we introduce so-called second order correlation functions, describing the limiting constants in the expectation and variance asymptotics stated in Theorem 3.4.1.

Definition 3.1.18 (Second order correlation functions) Let $\xi^{(\lambda)} \in \Xi^{(\lambda)}$ and $\xi^{(\infty)} \in \Xi^{(\infty)}$. For all $h_0 \in \mathbb{R}$, $(v_1, h_1) \in W_\lambda$ and $x, y \in \mathbb{R}^d$, define

$$\begin{aligned} c^{\xi^{(\lambda)}}&((\boldsymbol{o}, h_0), (v_1, h_1), \mathcal{P}^{(\lambda)}) \\ &:= \mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o}, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\})\, \xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(\boldsymbol{o}, h_0)\})] \\ &\quad - \mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o}, h_0), \mathcal{P}^{(\lambda)})]\, \mathbb{E}[\xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)})], \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} c^{\xi^{(\infty)}}&(x, y, \mathcal{P}) \\ &:= \mathbb{E}[\xi^{(\infty)}(x, \mathcal{P} \cup \{y\})\, \xi^{(\infty)}(y, \mathcal{P} \cup \{x\})] - \mathbb{E}[\xi^{(\infty)}(x, \mathcal{P})]\, \mathbb{E}[\xi^{(\infty)}(y, \mathcal{P})]. \end{aligned} \tag{3.24}$$

Furthermore, put

$$\begin{aligned} \sigma^2(\xi^{(\infty)}) := \int_{-\infty}^{\infty} &\mathbb{E}[\xi^{(\infty)}((\boldsymbol{o}, h_0), \mathcal{P})^2]\, e^{h_0}\, \mathrm{d}h_0 \\ &+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} c^{\xi^{(\infty)}}((\boldsymbol{o}, h_0), (v_1, h_1), \mathcal{P})\, e^{h_0 + h_1}\, \mathrm{d}h_0\, \mathrm{d}h_1\, \mathrm{d}v_1. \end{aligned} \tag{3.25}$$

3.1.5 Empirical measures and their cumulants

It is crucial in the proofs of our main results to have very precise control on the growth of the cumulants of the geometric characteristics H_λ^ξ, given in (3.21). For that purpose, it turns out to be more convenient to work with the measure-valued versions of H_λ^ξ. For this reason, for all $\lambda > 0$ with $R_\lambda \geq 1$, we define the empirical random measures

$$\mu_\lambda^\xi := \sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda)\, \delta_x = \sum_{w \in \mathcal{P}^{(\lambda)}} \xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)})\, \delta_{T_\lambda^{-1}(w)}, \qquad \xi \in \Xi, \tag{3.26}$$

where δ_x is the Dirac measure at x. The corresponding centered versions are given by

$$\bar{\mu}_\lambda^\xi := \mu_\lambda^\xi - \mathbb{E}[\mu_\lambda^\xi].$$

The method of expanding the cumulant measures associated with μ_λ^ξ in terms of cluster measures has been developed and successfully applied in [12] in the context of proving a central limit theorem. For a function $f \in \mathcal{B}(\mathbb{R}^d)$ and $r \in \mathbb{R}\backslash\{0\}$, define $f_r(x) := f(x/r)$. We use a refined version from [43, 56] to deduce sharp bounds for the cumulants of $\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle = \int_{\mathbb{R}^d} f_{R_\lambda}(x)\,\mathrm{d}\mu_\lambda^\xi$. To present the main formulas, let us write M_λ^k, $k \in \mathbb{N}$, for the k-th order moment measure of μ_λ^k, defined by the relation

$$\mathbb{E}[\exp(\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle)] = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \langle f_{R_\lambda}^k, M_\lambda^k \rangle,$$

in which we write f^k for the k-th tensor power of a function $f \in \mathcal{B}(\mathbb{R}^d)$, given by $f^k(x_1, \ldots, x_k) := f(x_1) \cdots f(x_k)$. (Here and in what follows, we think of $\xi \in \Xi$ being fixed and, hence, suppress the dependence on ξ in our notation.) To appropriately handle the moment measures, for $g \in \mathcal{B}(\mathbb{R}^d)$ and $F \in \mathcal{B}((\mathbb{R}^d)^k)$, we define the singular differential $\bar{\mathrm{d}}[g]$ by the relation

$$\int_{(\mathbb{R}^d)^k} F(x_1, \ldots, x_k)\,\bar{\mathrm{d}}[g](x_1, \ldots, x_k) := \int_{\mathbb{R}^d} F(y, \ldots, y)\ g(y)\,\mathrm{d}y, \tag{3.27}$$

and, for $\mathbf{x} := (x_1, \ldots, x_k) \in (\mathbb{R}^d)^k$, put

$$\tilde{\mathrm{d}}[g](\mathbf{x}) := \sum_{L_1, \ldots, L_p \preceq [\![k]\!]} \bar{\mathrm{d}}[g](\mathbf{x}_{L_1}) \ldots \bar{\mathrm{d}}[g](\mathbf{x}_{L_p}), \tag{3.28}$$

where $\mathbf{x}_{L_i} := (x_\ell)_{\ell \in L_i}$, for $i \in \{1, \ldots, p\}$. Now, from [43, Proposition 3.1], it follows that the density of M_λ^k with respect to $\tilde{\mathrm{d}}[\lambda\phi_{\alpha,\beta}]$ equals

$$m_\lambda(\mathbf{x}) = m_\lambda(x_1, \ldots, x_k) := \mathbb{E}\left[\prod_{i=1}^{k} \xi^{(\lambda)} \left(T_\lambda(x_i), \mathcal{P}^{(\lambda)} \cup \bigcup_{i=1}^{k} \{T_\lambda(x_i)\}\right)\right]. \tag{3.29}$$

Moreover, the k-th cumulant measure c_λ^k, associated with μ_λ^ξ, is defined as

$$c_\lambda^k := \sum_{L_1, \ldots, L_p \preceq [\![k]\!]} (-1)^{p-1}(p-1)!\ M_\lambda^{|L_1|} \otimes \ldots \otimes M_\lambda^{|L_p|}, \tag{3.30}$$

where $M_\lambda^{|L_1|} \otimes \ldots \otimes M_\lambda^{|L_p|}$ denotes the product measure of $M_\lambda^{|L_1|}, \ldots, M_\lambda^{|L_p|}$. The cumulant measures can alternatively be expressed as a sum of cluster measures.

Indeed, for non-empty and disjoint sets $S, T \subseteq \mathbb{N}$, the cluster measure $U_\lambda^{S,T}$ on $(\mathbb{R}^d)^{|S|} \times (\mathbb{R}^d)^{|T|}$ is defined by

$$U_\lambda^{S,T}(A \times B) := M_\lambda^{|S \cup T|}(A \times B) - M_\lambda^{|S|}(A)\, M_\lambda^{|T|}(B),$$

for Borel sets $A \subseteq (\mathbb{R}^d)^{|S|}$ and $B \subseteq (\mathbb{R}^d)^{|T|}$. Loosely speaking, the cluster measures will capture the spatial correlations of the rescaled functionals $\xi^{(\lambda)}$ and their measure-valued counterparts (see Lemma 3.2.15). To proceed, for $\mathbf{x} = (x_1, \ldots, x_k) \in (\mathbb{R}^d)^k$ and their rescaled images $(v_i, h_i) := T_\lambda(x_i)$, $i \in \{1, \ldots, k\}$, define the quantity

$$\delta(\mathbf{x}) := \delta\left(v_1, \ldots, v_k\right) := \max\left\{d(\mathbf{v}_S, \mathbf{v}_T) : \{S, T\} \preceq [\![k]\!]\right\}, \tag{3.31}$$

where $\mathbf{v}_S = (v_s)_{s \in S}$, $\mathbf{v}_T = (v_t)_{t \in T}$, and

$$d(\mathbf{v}_S, \mathbf{v}_T) := \min_{s \in S, t \in T} \|v_s - v_t\|$$

is the separation for the partition $\{S, T\}$ of $\{1, \ldots, k\}$. Moreover, let

$$\Delta := \{(x, \ldots, x) \in (\mathbb{R}^d)^k : x \in \mathbb{R}^d\}$$

be the diagonal in $(\mathbb{R}^d)^k$. Similarly to what has been explained in [12, 43, 56], one can decompose the space $(\mathbb{R}^d)^k \setminus \Delta$ into a disjoint union of sets $\delta(\{S, T\})$ with non-trivial partitions $\{S, T\} \preceq [\![k]\!]$, such that $\mathbf{x} \in \delta(\{S, T\})$ implies that $d(\mathbf{v}_S, \mathbf{v}_T) = \delta(\mathbf{x})$. This leads to the following cluster measure representation.

Lemma 3.1.19 *Fix $k \in \{2, 3, \ldots\}$ and $f \in \mathcal{B}(\mathbb{R}^d)$. Then, it holds that*

$$\begin{aligned}\langle f_{R_\lambda}^k, c_\lambda^k \rangle = \int_\Delta f_{R_\lambda}^k \, \mathrm{d}c_\lambda^k \; + \sum_{S,T \preceq [\![k]\!]} \int_{\delta(\{S,T\})} \sum_{S',T',K_1,\ldots,K_s \preceq [\![k]\!]} a_{S',T',K_1,\ldots,K_s} \\ \times f_{R_\lambda}^k \, \mathrm{d}(U_\lambda^{S',T'} \otimes M_\lambda^{|K_1|} \otimes \cdots \otimes M_\lambda^{|K_s|}),\end{aligned} \tag{3.32}$$

where in every summand, $S', T', K_1, \ldots, K_s$ is a partition of $\{1, \ldots, k\}$ with $S' \subseteq S$, $T' \subseteq T$, and the constants $a_{S',T',K_1,\ldots,K_s}$ satisfy the estimate

$$\sum_{S',T',K_1,\ldots,K_s \preceq [\![k]\!]} |a_{S',T',K_1,\ldots,K_s}| \leq 2^k\, k!. \tag{3.33}$$

Remark 3.1.20 The proof in [43] shows that the bound (3.33) cannot be improved.

3.2 Properties of the functionals of interest and the germ-grain processes

In the following, $C, C_1, C_2 \in (0, \infty)$ will always denote absolute constants that may change from line to line. The same holds for $c, c_1, c_2, \ldots \in (0, \infty)$, which are, unless specified differently, allowed to depend on the dimension d, the parameter α and β in the underlying distribution, and the functional $\xi \in \Xi$. Moreover, writing that a statement holds *for sufficiently large* λ means that there exists a $\lambda_0 > 0$, depending on d, α, β and the geometric functional under consideration, such that the statement is valid for all $\lambda \geq \lambda_0$.

3.2.1 Theory of localization

In this section, we prove that the rescaled functionals $\xi^{(\lambda)} \in \Xi^{(\lambda)}$ and the scaling limit functionals $\xi^{(\infty)} \in \Xi^{(\infty)}$, defined on points $w := (v, h) \in W_\lambda$ and $w' := (v', h') \in \mathbb{R}^{d-1} \times \mathbb{R}$, respectively, 'localize' in their spatial coordinates v and v', as well as their height coordinates h and h', respectively.

In order to do this for the height coordinates, we introduce the following characteristic of the germ-grain processes. If $w \in \text{ext}(\mathcal{P}^{(\lambda)})$, let $H(w) := H(w, \mathcal{P}^{(\lambda)})$ be the maximal height of an apex of a downward paraboloid which contains a parabolic facet in the boundary of $\Phi^{(\lambda)}$ that contains w, and 0 otherwise. Figure 3.5 illustrates the functional in the planar setting. $H'(w') := H'(w', \mathcal{P})$ is defined analogously with respect to the processes $\mathcal{P}$ and Φ. On the other hand, to deal with the spatial coordinates, we need the notion of the so-called radius of localization. It has been introduced and used heavily in the context of random polytopes before (see, for example, [22, 23, 56, 120]).

Definition 3.2.1 (Radius of localization) Let $\xi^{(\lambda)} \in \Xi^{(\lambda)}$ and $r > 0$. Given a point $w := (v, h) \in W_\lambda$, define

$$\xi^{(\lambda)}_{[r]}(w, \mathcal{P}^{(\lambda)}) := \xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)} \cap C_{d-1}(v, r))$$

to be the restriction of the functional to the cylinder $C_{d-1}(v, r) := \mathbb{B}^{d-1}(v, r) \times \mathbb{R}$. Similarly, for $\xi^{(\infty)} \in \Xi^{(\infty)}$, $r > 0$ and $w' := (v', h') \in \mathbb{R}^{d-1} \times \mathbb{R}$, put

$$\xi^{(\infty)}_{[r]}(w', \mathcal{P}) := \xi^{(\infty)}(w', \mathcal{P} \cap C_{d-1}(v', r)).$$

If there exist random variables $L(w) := L(\xi^{(\lambda)}, w)$ and $L'(w') := L'(\xi^{(\infty)}, w')$ that almost surely fulfill

$$\xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \xi^{(\lambda)}_{[L(w)]}(w, \mathcal{P}^{(\lambda)}) \qquad \text{and} \qquad \xi^{(\lambda)}_{[L(w)]}(w, \mathcal{P}^{(\lambda)}) = \xi^{(\lambda)}_{[s]}(w, \mathcal{P}^{(\lambda)}),$$

for all $s \geq L(w)$, and

$$\xi^{(\infty)}(w', \mathcal{P}) = \xi^{(\infty)}_{[L'(w')]}(w', \mathcal{P}) \qquad \text{and} \qquad \xi^{(\infty)}_{[L'(w')]}(w', \mathcal{P}) = \xi^{(\infty)}_{[s]}(w', \mathcal{P}),$$

for all $s \geq L'(w')$, respectively, then, the functionals $\xi^{(\lambda)}$ and $\xi^{(\infty)}$ are said to localize. The infima over all such random variables satisfying the above conditions are called the radii of localization of the corresponding functional. To simplify the notation, let us refer to them also as $L(w)$ and $L'(w')$ in what follows.

Theorem 3.2.2 *Let $\xi^{(\lambda)} \in \Xi^{(\lambda)}$ and $\xi^{(\infty)} \in \Xi^{(\infty)}$. Then, for all $w = (v, h) \in W_\lambda$, $w' = (v', h') \in \mathbb{R}^{d-1} \times \mathbb{R}$ and sufficiently large λ,*

(a) *it holds that*

$$\mathbb{P}(H(w) \geq t) \leq c_1 \exp\left(-\frac{e^t}{c_2}\right) \quad \text{and} \quad \mathbb{P}(H'(w') \geq t) \leq c_3 \exp\left(-\frac{e^t}{c_4}\right), \tag{3.34}$$

for all $t \geq h \vee 0$ and $t \geq h' \vee 0$, respectively, and

(b) *the radii of localization $L(w)$ and $L'(w')$ satisfy*

$$\mathbb{P}(L(w) \geq t) \leq c_1 \exp\left(-\frac{t^2}{c_2}\right) \quad \text{and} \quad \mathbb{P}(L'(w') \geq t) \leq c_3 \exp\left(-\frac{t^2}{c_4}\right), \tag{3.35}$$

as well as the weaker estimates

$$\mathbb{P}(L(w) \geq t) \leq c_5 \exp\left(-\frac{t}{c_6}\right) \quad \text{and} \quad \mathbb{P}(L'(w') \geq t) \leq c_7 \exp\left(-\frac{t}{c_9}\right), \tag{3.36}$$

for all $t \geq |h|$ and $t \geq |h'|$, respectively.

As a direct consequence, we achieve an exponential decay for the probability that a point belongs to the set of extreme points of $\mathcal{P}^{(\lambda)}$, respectively $\mathcal{P}$, with respect to their height coordinates.

If $w \notin \mathcal{P}^{(\lambda)}$ or $w' \notin \mathcal{P}$, we use the notation $w \in \mathrm{ext}(\mathcal{P}^{(\lambda)})$ and $w' \in \mathrm{ext}(\mathcal{P})$ for

$$w \in \mathrm{ext}\Bigg(\bigcup_{z \in \mathcal{P}^{(\lambda)} \cup \{w\}} [\Pi^{\uparrow}(z)]^{(\lambda)}\Bigg) \qquad \text{and} \qquad w' \in \mathrm{ext}\Bigg(\bigcup_{z \in \mathcal{P} \cup \{w'\}} [\Pi^{\uparrow}(z)]\Bigg).$$

Corollary 3.2.3 *Let $w := (v,h) \in W_\lambda$ and $w' := (v',h') \in \mathbb{R}^{d-1} \times \mathbb{R}$. Then, for sufficiently large λ, it holds that*

$$\mathbb{P}(w \in \mathrm{ext}(\mathcal{P}^{(\lambda)})) \le c_1 \exp\left(-\frac{e^{h\vee 0}}{c_2}\right) \quad \textit{and} \quad \mathbb{P}(w' \in \mathrm{ext}(\mathcal{P})) \le c_3 \exp\left(-\frac{e^{h'\vee 0}}{c_4}\right).$$

After having investigated the localization properties of the functionals $\xi^{(\lambda)}$ and $\xi^{(\infty)}$, we turn to the germ-grain processes $\Psi^{(\lambda)}$, $\Phi^{(\lambda)}$, Ψ and Φ.

Theorem 3.2.4 *For all $M \in (0,\infty)$ and sufficiently large λ, it holds that*

$$\mathbb{P}(\|\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C_{d-1}(v,M)\|_\infty \ge t) \le c_1\, M^{2(d-1)} \exp\left(-\frac{t}{c_2}\right),$$

and

$$\mathbb{P}(\|\partial\Psi(\mathcal{P}) \cap C_{d-1}(v,M)\|_\infty \ge t) \le c_3\, M^{2(d-1)} \exp\left(-\frac{t}{c_4}\right),$$

for all $t > 0$. The two bounds also hold for the dual processes $\Phi^{(\lambda)}$ and Φ.

Let us briefly comment on the previous statements. First, we emphasize that the tail estimates (3.35) and (3.36) are valid only for arguments $t \ge |h|$. Next, also the probability for a point $w = (v,h) \in W_\lambda$ to belong to the extreme points of $\mathcal{P}^{(\lambda)}$ separates into two cases. Namely, if the height h exceeds 0, then, $\mathbb{P}(w \in \mathrm{ext}(\mathcal{P}^{(\lambda)}))$ decays super-exponentially fast, while, if $h \le 0$, one only has an estimate independently of h (which is in some sense trivial). Similarly, also the probability for the event that $H(w) \ge t$ can only be estimated in a meaningful way if t or h are not too small. This underlines the effect already discussed in Section 1.2, that the spatial localization property of the rescaled geometric functionals $\xi^{(\lambda)} \in \Xi^{(\lambda)}$ we consider can only be handled effectively in the upper half-space $\mathbb{R}^{d-1} \times [0,\infty)$, while in the lower half-space no such spatial localization is available. This phenomenon is new compared to the theory of random polytopes in the unit ball developed in [22, 56, 120] and is in fact the leading cause for the technical complications that arise in the context of our class of random polytopes.

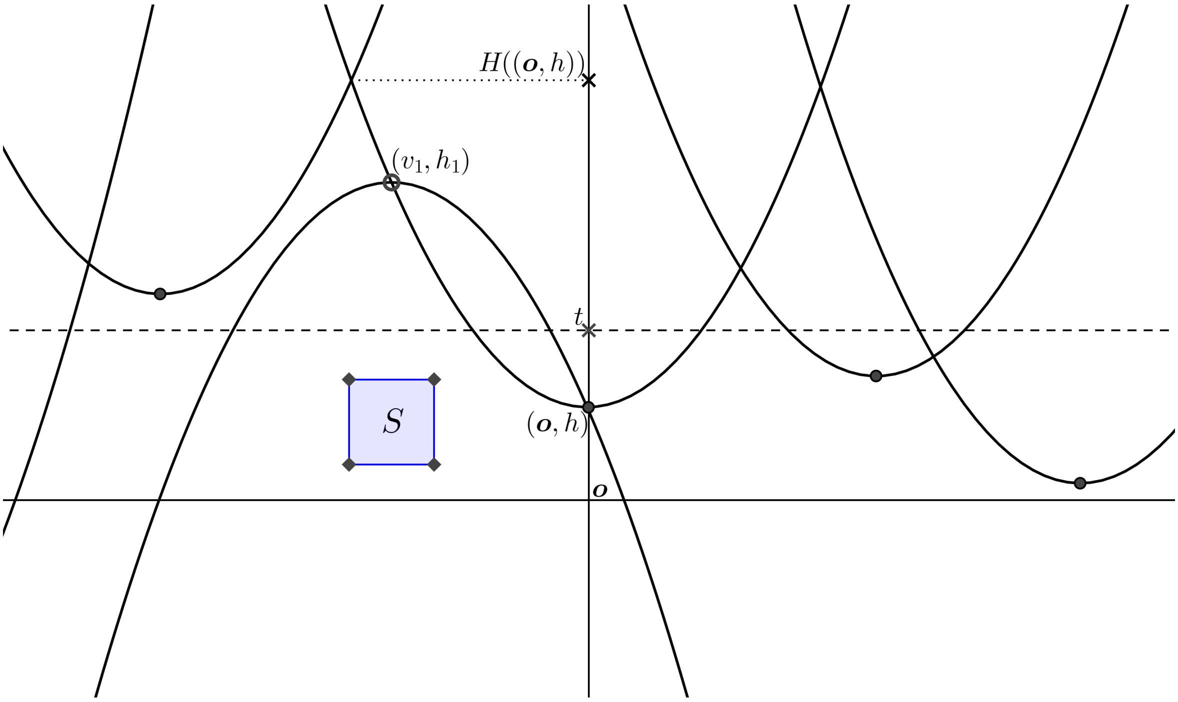

FIGURE 3.5: The event $\{H(w) \geq t\}$ and the unit volume cube S.

Remark 3.2.5 As aforementioned and proven in Corollary 3.1.9, the limiting Poisson point process $\mathcal{P}$, as well as the corresponding germ-grain models Ψ and Φ, do *not* depend on the parameter α and β in the underlying distribution. Hence, the proofs of the assertions for these three limit processes stated in Theorem 3.2.2, Corollary 3.2.3 and Theorem 3.2.4 stay absolutely the same compared with the ones derived in the Gaussian case in [23] and can be omitted. Thus, it remains to derive the above stated assertions connected with $\mathcal{P}^{(\lambda)}$, $\Psi^{(\lambda)}$ and $\Phi^{(\lambda)}$, which of course depend on α and β.

Due to the rotational invariance of the underlying Poisson point process $\mathcal{P}_\lambda$, it is enough to prove all these results for points $w = (\boldsymbol{o}, h) \in W_\lambda$ with $h \in (-\infty, R_\lambda^\beta]$. Let λ be sufficiently large. Similarly to what has been done in [23, Page 25], let us investigate the event $\{H(w) \geq t\}$, which can be rewritten in the form

$$\{H(w) \geq t\} = \{\exists w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow(w)]^{(\lambda)} : h_1 \geq t, [\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset\},$$

(see Figure 3.5). Now, consider such a $w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow(w)]^{(\lambda)}$ and define the inverse of the scaling transformation of w by $\rho u_0 := T_\lambda^{-1}(w)$, $\rho > 0$, where we recall that u_0 indicates the north pole on the sphere $\mathbb{S}^{d-1}$. The parameter ρ is positive since otherwise, the spatial coordinate of w would be $\pi R_\lambda^{\frac{\beta}{2}}$ instead of $\boldsymbol{o}$, by definition of T_λ.

Lemma 3.2.6 *Denote by S the unit volume cube centered in $(v_1, \frac{h_1}{(\beta+1)^\beta} - 1)$, illustrated in Figure 3.5. For sufficiently large λ, it fulfills*

$$S \subseteq [\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap C_{d-1}\Big(\boldsymbol{o}, \frac{3\pi R_\lambda^{\frac{\beta}{2}}}{4}\Big). \tag{3.37}$$

Proof. For sufficiently large λ, the cube S is included in $[\Pi^{\downarrow}(w_1)]^{(\lambda)}$. Indeed, due to the upcoming estimate in (3.76), the boundaries of $[\Pi^{\downarrow}(w_1)]^{(\lambda)}$ and $[\Pi^{\downarrow}(w_1)]^{(\infty)}$ are not 'far' from each other, and the latter downward germ contains the cube S by definition. Furthermore, the ball $\mathbb{B}^d(\frac{\rho u_0}{2}, \frac{\rho}{2})$, that is mapped into the germ $[\Pi^{\uparrow}(w)]^{(\lambda)}$ by the scaling transformation T_λ (see Lemma 3.1.13), is a subspace of $\mathbb{R}^{d-1} \times (0, \infty)$, since $\rho > 0$. Additionally, T_λ transforms this upper half space into the cylinder

$$C_{d-1}\Big(\boldsymbol{o}, \pi\frac{R_\lambda^{\frac{\beta}{2}}}{2}\Big).$$

This leads to the relation

$$[\Pi^{\uparrow}(w)]^{(\lambda)} = T_\lambda\left(\mathbb{B}^d\left(\frac{\rho u_0}{2}, \frac{\rho}{2}\right)\right) \subseteq C_{d-1}\Big(0, \frac{\pi R_\lambda^{\frac{\beta}{2}}}{2}\Big),$$

which implies $||v_1|| \leq \frac{\pi R_\lambda^{\frac{\beta}{2}}}{2}$ and, therefore, $S \subseteq C_{d-1}\Big(0, \frac{3\pi R_\lambda^{\frac{\beta}{2}}}{4}\Big)$. □

The cube S is the main ingredient when proving the next assertion.

Lemma 3.2.7 *For sufficiently large λ, it holds that*

$$\mathbb{P}([\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset) \leq \exp(-c_1\, e^{c_2 h_1}).$$

Proof. Let $(v, h) \in S$. From the definition of the cube S, we get that

$$h \in \left[\frac{h_1}{(\beta+1)^\beta} - \frac{3}{2}, \frac{h_1}{(\beta+1)^\beta} - \frac{1}{2}\right],$$

and, thus,

$$\frac{h}{R_\lambda^\beta} \in \left[\frac{h_1}{(\beta+1)^\beta R_\lambda^\beta} - \frac{3}{2R_\lambda^\beta}, \frac{h_1}{(\beta+1)^\beta R_\lambda^\beta} - \frac{1}{2R_\lambda^\beta}\right] \subseteq \left[-\frac{3}{2}, \frac{1}{2}\right], \tag{3.38}$$

since $h_1/R_\lambda^\beta \in [0,1]$ and $\beta \geq 1$.

Hence, in view of (3.10), the density of the intensity measure of $\mathcal{P}^{(\lambda)}$ in each point $(v,h) \in S$ looks like

$$\frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}}\|v\|)}{\|R_\lambda^{-\frac{\beta}{2}}v\|^{d-2}} \frac{(\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}} \times \exp\left(h - \frac{h^2}{2R_\lambda^\beta}(\beta-1)(1-C)^{\beta-2}\right)\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}, \tag{3.39}$$

for some $C \in \left[-\frac{3}{2}, \frac{1}{2}\right]$. Besides, the preparation (3.37) implies that

$$R_\lambda^{-\frac{\beta}{2}}\|v\| \leq R_\lambda^{-\frac{\beta}{2}} \frac{3\pi R_\lambda^{\frac{\beta}{2}}}{4} = \frac{3\pi}{4}.$$

Therefore, for sufficiently large λ, the first fraction is bounded from below by a positive constant. Moreover, if the exponent $\frac{\beta(d+1)-2d-2\alpha}{2\beta}$ is positive, the definition of R_λ yields that

$$\frac{(\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}} = \left(\frac{\beta\log\lambda}{\beta\log\lambda - \underbrace{\left(\frac{\beta(d+1)-2d-2\alpha}{2}\right)}_{>0}\log\left(c_{\alpha,\beta}^{-\frac{2\beta d}{\beta(d+1)-2d-2\alpha}}\beta\log\lambda\right)}\right)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}} > 1,$$

since the term in the bracket is larger than 1. On the other hand, if the same exponent is negative, we also achieve

$$\frac{(\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}} = \left(\frac{\beta\log\lambda}{\beta\log\lambda - \underbrace{\left(\frac{\beta(d+1)-2d-2\alpha}{2}\right)}_{<0}\log\left(c_{\alpha,\beta}^{-\frac{2\beta d}{\beta(d+1)-2d-2\alpha}}\beta\log\lambda\right)}\right)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}} > 1.$$

Here, the inner fraction is smaller than 1, but since we have a negative exponent, we nevertheless achieve the statement. Summarizing, the second fraction in (3.39) is larger than 1.

Now, let us switch to the height coordinate h. First, notice that $d-1+\alpha>0$, since $\alpha>-1$. If $h\leq 0$, the fourth term in (3.39) is larger than 1. In the other case, the estimate derived in (3.38) yields that

$$\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}\geq\left(\frac{1}{2}\right)^{d-1+\alpha}>0.$$

Moreover, the third expression in (3.39) is bounded from below by $c_1\exp(c_2h_1)$. Indeed, we have

$$h\in\left[\frac{h_1}{(\beta+1)^\beta}-\frac{3}{2},\frac{h_1}{(\beta+1)^\beta}-\frac{1}{2}\right]\subseteq\left[\frac{h_1}{(\beta+1)^\beta}-\frac{3}{2},\frac{h_1}{(\beta+1)^\beta}\right].$$

On these grounds, since $h_1/R_\lambda^\beta\in[0,1]$,

$$\begin{aligned}
&\exp\left(h-\frac{h^2}{2R_\lambda^\beta}(\beta-1)(1-C)^{\beta-2}\right)\\
&\quad\geq\exp\left(-\frac{3}{2}\right)\exp\left(\frac{h_1}{(\beta+1)^\beta}-\frac{h_1^2}{2(\beta+1)^{2\beta}R_\lambda^\beta}(\beta-1)(1-C)^{\beta-2}\right)\\
&\quad\geq\exp\left(-\frac{3}{2}\right)\exp\left(\frac{h_1}{(\beta+1)^\beta}-\frac{h_1}{2(\beta+1)^{2\beta}}(\beta-1)(1-C)^{\beta-2}\right)\\
&\quad=\exp\left(-\frac{3}{2}\right)\exp\left(h_1\frac{2(\beta+1)^\beta-(\beta-1)(1-C)^{\beta-2}}{2(\beta+1)^{2\beta}}\right)\\
&\quad\geq\exp\left(-\frac{3}{2}\right)\exp\left(h_1\frac{(\beta+1)^\beta}{2(\beta+1)^{2\beta}}\right)\\
&\quad=\exp\left(-\frac{3}{2}\right)\exp\left(\frac{h_1}{2(\beta+1)^\beta}\right),
\end{aligned}$$

where in the last inequality we have used that

$$2(\beta+1)^\beta-(\beta-1)(1-C)^{\beta-2}\geq(\beta+1)^\beta,$$

since $C\in\left[-\frac{3}{2},\frac{1}{2}\right]$. This proves the claim.

Summarizing the last calculations, we obtain that the density of the intensity measure of $\mathcal{P}^{(\lambda)}$, evaluated in an arbitrary point $(v,h)\in S$, can be bounded from below by $c_1\exp(c_2h_1)$.

Since the cube S has by construction unit volume, we obtain, writing ν_λ for the intensity measure of the rescaled Poisson point process $\mathcal{P}^{(\lambda)}$, that

$$\nu_\lambda(S) \geq c_1 \exp(c_2 h_1).$$

Therefore,

$$\mathbb{P}([\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset) = \exp(-\nu_\lambda([\Pi^{\downarrow}(w_1)]^{(\lambda)})) \leq \exp(-\nu_\lambda(S)) \leq \exp(-c_1\, e^{c_2 h_1}).$$

This completes the proof. □

The last two lemmas assumed a fixed $w_1 := (v_1, h_1) \in \partial[\Pi^{\uparrow}(w)]^{(\lambda)}$ with $h_1 \geq t$. In the next step, we generalize this to all possible w_1, i.e., the region

$$\partial[\Pi^{\uparrow}(w)]^{(\lambda)} \cap \mathbb{R}^{d-1} \times [t, \infty).$$

In order to do this, it is crucial to derive the following observation concerning the spatial coordinates. Recall that we have $w = (\boldsymbol{o}, h)$ with $h \in (-\infty, R_\lambda^\beta]$.

Lemma 3.2.8 *For all $(v_1, h_1) \in \partial[\Pi^{\uparrow}(w)]^{(\lambda)}$, it holds that*

$$||v_1|| \leq C\, \frac{R_\lambda^{\frac{\beta}{2}} \sqrt{h_1 - h}}{\sqrt{R_\lambda^\beta - h}}.$$

Proof. From (3.18), we get that

$$h_1 = R_\lambda^\beta (1 - \cos(d_\lambda(\boldsymbol{o}, v_1))) + h\, \cos(d_\lambda(\boldsymbol{o}, v_1)),$$

and the inequality $1 - \cos\theta \geq C\theta^2$, valid for all $\theta \in [0, \pi]$, together with the definition of $d_\lambda(\boldsymbol{o}, v_1)$, implies that

$$1 - \cos(d_\lambda(\boldsymbol{o}, v_1)) \geq C\, d_\lambda(\boldsymbol{o}, v_1)^2 = C\, (R_\lambda^{-\frac{\beta}{2}} ||v_1||)^2 = C\, R_\lambda^{-\beta}\, ||v_1||^2.$$

Thus,

$$C\, ||v_1||^2 \leq R_\lambda^\beta\, (1 - \cos(d_\lambda(\boldsymbol{o}, v_1))) = h_1 - h \cos(d_\lambda(\boldsymbol{o}, v_1)) = \frac{R_\lambda^\beta (h_1 - h)}{R_\lambda^\beta - h}.$$

Indeed, in last step we used that

$$\begin{aligned}
& h_1 - h\cos(d_\lambda(\boldsymbol{o}, v_1)) && = \frac{R_\lambda^\beta(h_1-h)}{R_\lambda^\beta - h} \\
\Leftrightarrow \quad & (R_\lambda^\beta - h)(h_1 - h\cos(d_\lambda(\boldsymbol{o}, v_1))) && = R_\lambda^\beta(h_1-h) \\
\Leftrightarrow \quad & R_\lambda^\beta h_1 - R_\lambda^\beta h\cos(d_\lambda(\boldsymbol{o}, v_1)) - hh_1 + h^2\cos(d_\lambda(\boldsymbol{o}, v_1)) && = R_\lambda^\beta h_1 - R_\lambda^\beta h \\
\Leftrightarrow \quad & R_\lambda^\beta h(1-\cos(d_\lambda(\boldsymbol{o}, v_1))) && = h(h_1 - h\cos(d_\lambda(\boldsymbol{o}, v_1))) \\
\Leftrightarrow \quad & R_\lambda^\beta(1-\cos(d_\lambda(\boldsymbol{o}, v_1))) && = h_1 - h\cos(d_\lambda(\boldsymbol{o}, v_1)) \\
\Leftrightarrow \quad & R_\lambda^\beta(1-\cos(d_\lambda(\boldsymbol{o}, v_1))) + h\cos(d_\lambda(\boldsymbol{o}, v_1)) && = h_1,
\end{aligned}$$

which again holds in view of (3.18). Extracting the roots implies the claim. □

Proof of Theorem 3.2.2 (a). Recall that $w = (\boldsymbol{o}, h)$ with $h \in (-\infty, R_\lambda^\beta]$ and that the event $\{H(w) \geq t\}$ can be rewritten in the form

$$\{H(w) \geq t\} = \{\exists w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow(w)]^{(\lambda)} : h_1 \geq t, [\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset\}.$$

The possible range for the height coordinate h_1 is determined by $[t \vee h, R_\lambda^\beta]$. Moreover, the previous lemma gives a condition on the spatial coordinate v_1. Using the bound derived in Lemma 3.2.7, this leads to

$$\begin{aligned}
\mathbb{P}(H(w) \geq t) &= \int_{t\vee h}^{R_\lambda^\beta} \int_{\substack{\mathbb{B}^{d-2}(\boldsymbol{o}, v);\\ (v,h_1)\in\partial[\Pi^\uparrow(w)]^{(\lambda)}}} \mathbb{P}([\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset) \,\mathrm{d}v_1 \,\mathrm{d}h_1 \\
&\leq c_1 \int_{t\vee h}^{R_\lambda^\beta} \left(\frac{R_\lambda^{\frac{\beta}{2}}\sqrt{h_1-h}}{\sqrt{R_\lambda^\beta - h}} \right)^{d-2} \exp(-c_2\, e^{c_3 h_1}) \,\mathrm{d}h_1.
\end{aligned}$$

In order to finish the proof, we consider two different cases. If $h \in \left(-\infty, \frac{R_\lambda^\beta}{2}\right]$, we achieve that

$$\frac{R_\lambda^{\frac{\beta}{2}}}{\sqrt{R_\lambda^\beta - h}} \leq \sqrt{2}.$$

Indeed,

$$\frac{R_\lambda^{\frac{\beta}{2}}}{\sqrt{R_\lambda^\beta - h}} \le \sqrt{2} \quad \Leftrightarrow \quad R_\lambda^\beta \le 2(R_\lambda^\beta - h) \quad \Leftrightarrow \quad \frac{R_\lambda^\beta}{2} \le R_\lambda^\beta - h \quad \Leftrightarrow \quad h \le \frac{R_\lambda^\beta}{2}$$

holds by assumption. As a result, we get that

$$\mathbb{P}(H(w) \ge t) \le c_1 \, \exp(-c_2 \, e^{c_3 t}). \tag{3.40}$$

Truly, it holds that

$$\begin{aligned}
\mathbb{P}(H(w) \ge t) &\le c_1 \int_{t \vee h}^{R_\lambda^\beta} (h_1 - h)^{\frac{d-2}{2}} \, \exp(-c_2 e^{c_3 h_1}) \, \mathrm{d}h_1 \\
&\le c_1 \int_{t \vee h}^{\infty} \exp(c_2 \log(h_1 - h) \, - c_3 e^{c_4 h_1}) \, \mathrm{d}h_1 \\
&\le c_1 \int_{t \vee h}^{\infty} \exp(c_2 \log h_1 \, - c_3 e^{c_4 h_1}) \, \mathrm{d}h_1 \\
&\le c_1 \int_{t}^{\infty} \exp(-c_2 e^{c_3 h_1}) \, \mathrm{d}h_1 \\
&\le c_1 \, e^{c_2 t} \, \exp(-c_3 e^{c_4 t}) \\
&\le c_1 \, \exp(-c_2 \, e^{c_3 t}),
\end{aligned}$$

since for all $h_1 \ge 0$ we have that

$$\exp(c_1 \log h_1 \, - c_2 e^{c_3 h_1}) \le \exp(-c_4 e^{c_5 h_1}).$$

If else-wise $h \in \left(\frac{R_\lambda^\beta}{2}, R_\lambda^\beta\right]$, we have that

$$\frac{h_1 - h}{R_\lambda^\beta - h} \le 1 \qquad \text{and} \qquad \exp(-c \, e^{c \, h_1}) \le \exp\left(-\frac{c \, e^{c \, h_1}}{2} - \frac{c \, e^{\frac{c \, R_\lambda^\beta}{2}}}{2}\right).$$

The first inequality holds since $h_1 \leq R_\lambda^\beta$ and the second follows because

$$\begin{aligned}
& \exp(-c\, e^{c h_1}) \leq \exp\left(-\frac{c\, e^{c\, h_1}}{2} - \frac{c\, e^{\frac{c\, R_\lambda^\beta}{2}}}{2}\right) \\
\Leftrightarrow \quad & -\frac{c\, e^{c\, h_1}}{2} \leq -\frac{c\, e^{\frac{c\, R_\lambda^\beta}{2}}}{2} \\
\Leftrightarrow \quad & h_1 \geq \frac{R_\lambda^\beta}{2}
\end{aligned}$$

is true due to the range of h and the fact that $h_1 \geq h$ in the integral under investigation. In this case, we obtain that

$$\begin{aligned}
\mathbb{P}(H(w) \geq t) &\leq c_1 \int_{t \vee h}^{R_\lambda^\beta} R_\lambda^{\frac{\beta(d-2)}{2}} \exp\left(-\frac{c_2\, e^{c_3 h_1}}{2} - \frac{c_4\, e^{\frac{c_5 R_\lambda^\beta}{2}}}{2}\right) \mathrm{d}h_1 \\
&\leq c_1\, R_\lambda^{\frac{\beta(d-2)}{2}} \exp\left(-\frac{c_2\, e^{\frac{c_3\, R_\lambda^\beta}{2}}}{2}\right) \int_t^\infty \exp\left(-\frac{c_4\, e^{c_5\, h_1}}{2}\right) \mathrm{d}h_1 \\
&\leq c_1 \int_t^\infty \exp\left(-\frac{c_2\, e^{c_3\, h_1}}{2}\right) \mathrm{d}h_1 \\
&\leq c_1 \exp(-c_2\, e^{c_3 t}),
\end{aligned} \tag{3.41}$$

where in the third step we used that for sufficiently large λ,

$$R_\lambda^{\frac{\beta(d-2)}{2}} \exp\left(-c_1\, e^{c_2\, R_\lambda^\beta}\right)$$

can be bounded by an absolute constant. Combining (3.40) and (3.41) yields the result. □

Part (a) of the theorem concerning $H(w)$ is heavily used to derive a proof for part (b), dealing with the localization in the spatial regime. In order to develop the proof, let $\xi^{(\lambda)} \in \Xi^{(\lambda)}$, $w = (\boldsymbol{o}, h) \in W_\lambda$ and $t \geq |h|$. In contrast to the estimates presented before, we analyze the event $\{H(w) \leq t\}$. The motivation for this choice will be provided later.

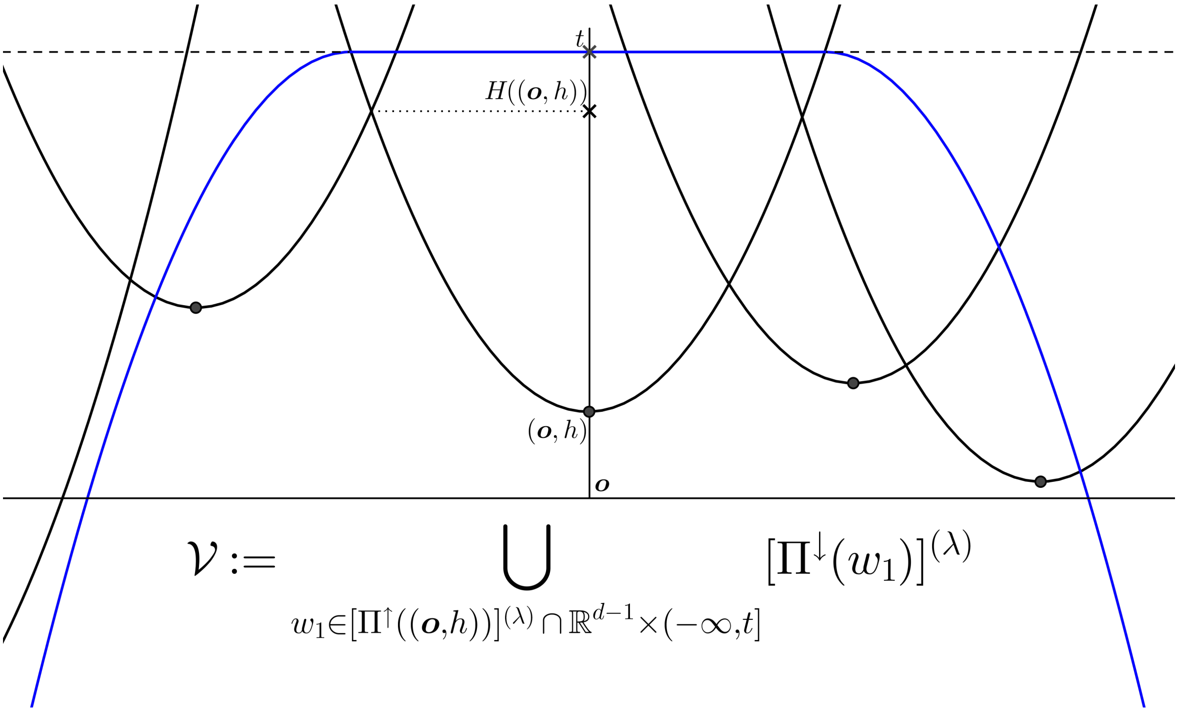

FIGURE 3.6: The set $\mathcal{V}$.

Similarly to the considerations in [23, Page 26], we start with the observation that in the case that $H(w) \leq t$, the functional $\xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$ only depends on the points of the Poisson point process $\mathcal{P}^{(\lambda)}$ contained in the set

$$\mathcal{V} := \bigcup_{w_1 \in [\Pi^{\uparrow}(w)]^{(\lambda)} \cap \mathbb{R}^{d-1} \times (-\infty,t]} [\Pi^{\downarrow}(w_1)]^{(\lambda)}. \tag{3.42}$$

Indeed, by definition of the rescaled functionals, at least all the points from $\mathcal{P}^{(\lambda)}$ sharing a quasi-parabolic facet of $\Psi^{(\lambda)}$ with w, and the paraboloids that determine $\xi^{(\lambda)}$, are contained in $\mathcal{V}$. Figure 3.6 illustrates this phenomenon in the planar setting.

Lemma 3.2.9 *Fix $w' := (v', h') \in \mathcal{V}$ and $w_1 := (v_1, h_1) \in [\Pi^{\uparrow}(w)]^{(\lambda)}$, $h_1 \leq t$, in a way that both w' and w lie on the boundary of the downward grain $[\Pi^{\downarrow}(w_1)]^{(\lambda)}$ corresponding to w_1. In Figure 3.7, we provide an example of this setup. Then, there exists a constant $C \in (0, \infty)$ such that $h' \in [-C_1 t^2, \infty)$ implies that $||v'|| \leq t$.*

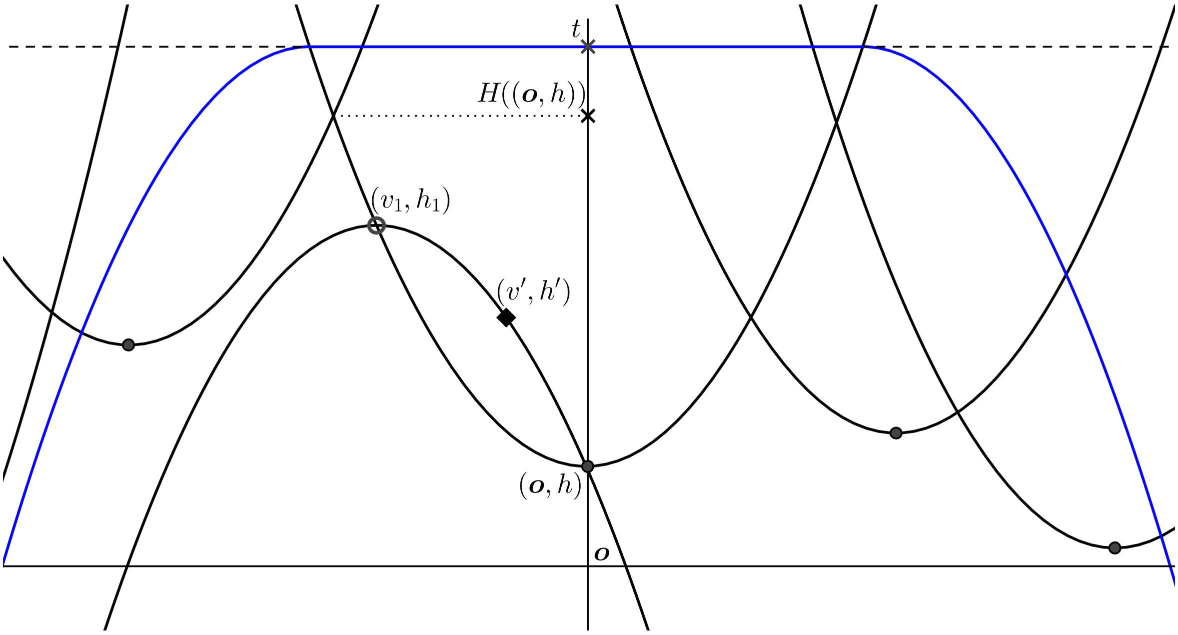

FIGURE 3.7: The situation in Lemma 3.2.9.

Proof. In order to prove the assertion, we recall the definition of the upward quasi-paraboloid grain $[\Pi^{\uparrow}(w)]^{(\lambda)}$, i.e.,

$$[\Pi^{\uparrow}(w)]^{(\lambda)} = \Big\{(v_1,h_1) \in W_\lambda : h_1 \geq R_\lambda^{\beta}(1-\cos(d_\lambda(\boldsymbol{o},v_1))) + h\,\cos(d_\lambda(\boldsymbol{o},v_1))\Big\}.$$

We have that $t \geq h_1$ and $t \geq |h|$ by definition of the point w_1 and the assumption made above, respectively. Thus, we obtain

$$R_\lambda^{\beta}(1-\cos(d_\lambda(\boldsymbol{o},v_1))) \leq h_1 - h\,\cos(d_\lambda(\boldsymbol{o},v_1)) \leq t - h\,\cos(d_\lambda(\boldsymbol{o},v_1)) \leq 2t,$$

since $\cos(d_\lambda(\boldsymbol{o},v_1)) \in [-1,1]$. Again, using the inequality $1-\cos\theta \geq C\theta^2$, it follows that

$$d_\lambda(\boldsymbol{o},v_1)^2 \leq C(1-\cos(d_\lambda(\boldsymbol{o},v_1))) \leq 2\,C\,R_\lambda^{-\beta}\,t,$$

and, thus,

$$d_\lambda(\boldsymbol{o},v_1) \leq C\,\sqrt{2t}\,R_\lambda^{-\frac{\beta}{2}}. \tag{3.43}$$

On the other hand, w' belongs to the boundary of

$$[\Pi^{\downarrow}(w_1)]^{(\lambda)} := \left\{ (v', h') \in W_\lambda : h' \leq R_\lambda^\beta - \frac{R_\lambda^\beta - h_1}{\cos(d_\lambda(v', v_1))} \right\}.$$

Based on $1 - \cos\theta \geq C\theta^2$, the equivalence

$$\begin{aligned}
& h' && = R_\lambda^\beta - \frac{R_\lambda^\beta - h_1}{\cos(d_\lambda(v', v_1))} \\
\Leftrightarrow \quad & h' \cos(d_\lambda(v', v_1)) && = R_\lambda^\beta \cos(d_\lambda(v', v_1)) - R_\lambda^\beta + h_1 \\
\Leftrightarrow \quad & R_\lambda^\beta - R_\lambda^\beta \cos(d_\lambda(v', v_1)) + h' \cos(d_\lambda(v', v_1)) - h' && = h_1 - h' \\
\Leftrightarrow \quad & (R_\lambda^\beta - h')(1 - \cos(d_\lambda(v', v_1))) && = h_1 - h' \\
\Leftrightarrow \quad & 1 - \cos(d_\lambda(v', v_1)) && = \frac{h_1 - h'}{R_\lambda^\beta - h'},
\end{aligned}$$

the fact that $h', h_1 \leq t$ by construction, $t \leq 2\pi R_\lambda^{\frac{\beta}{2}}$ (since the radius of localization never exceeds the spatial diameter of W_λ), and

$$R_\lambda^\beta - 2\pi R_\lambda^{\frac{\beta}{2}} \geq \frac{1}{2} R_\lambda^\beta,$$

which is true for sufficiently large λ, we obtain that

$$\begin{aligned}
(d_\lambda(v', v_1))^2 &\leq C\,(1 - \cos(d_\lambda(v', v_1))) = C\,\frac{h_1 - h'}{R_\lambda^\beta - h'} \\
&\leq C\,\frac{t - h'}{R_\lambda^\beta - t} \leq C\,\frac{t - h'}{R_\lambda^\beta - 2\pi R_\lambda^{\frac{\beta}{2}}} \leq C\,\frac{t - h'}{R_\lambda^\beta}.
\end{aligned}$$

Therefore, it holds that

$$d_\lambda(v', v_1) \leq C\,\frac{\sqrt{t - h'}}{R_\lambda^{\frac{\beta}{2}}}. \tag{3.44}$$

Finally, putting together (3.43) and (3.44) with $||v'|| = R_\lambda^{\frac{\beta}{2}} d_\lambda(v', \boldsymbol{o})$ yields that

$$||v'|| = R_\lambda^{\frac{\beta}{2}} d_\lambda(v', \boldsymbol{o}) \leq R_\lambda^{\frac{\beta}{2}} (d_\lambda(\boldsymbol{o}, v_1) + d_\lambda(v_1, v')) \leq C\,(\sqrt{t} + \sqrt{t - h'}). \tag{3.45}$$

Beyond that, if $h' \in [-Ct^2, \infty)$, we even get $||v'|| \leq t$ since it holds that

$$\begin{aligned}
& C\,(\sqrt{t} + \sqrt{t-h'}) \leq t \\
\Leftrightarrow \quad & C\,\sqrt{t-h'} \leq t - C\,\sqrt{t} \\
\Leftrightarrow \quad & C^2\,(t-h') \leq t^2 - 2tC\sqrt{t} + C^2 t \\
\Leftrightarrow \quad & t - h' \leq \frac{t^2}{C^2} - \frac{2t\sqrt{t}}{C} + t \\
\Leftrightarrow \quad & h' \geq -\frac{t^2}{C^2} + \frac{2t\sqrt{t}}{C} \geq -\frac{t^2}{C^2} = -C_1\,t^2
\end{aligned}$$

by the assumption of the lemma with $C_1 := \frac{1}{C^2}$. This proves the result. □

Proof of Theorem 3.2.2 (b). Let $\xi^{(\lambda)} \in \Xi^{(\lambda)}$, $w = (\boldsymbol{o}, h) \in W_\lambda$ and $t \geq |h|$. Using the result from part (a) of the theorem implies that

$$\begin{aligned}
\mathbb{P}(L(w) \geq t) &= \mathbb{P}(L(w) \geq t, H(w) \geq t) + \mathbb{P}(L(w) \geq t, H(w) \leq t) \\
&\leq \mathbb{P}(H(w) \geq t) + \mathbb{P}(L(w) \geq t, H(w) \leq t) \\
&\leq c_1\,\exp(-c_2\,e^{c_3 t}) + \mathbb{P}(L(w) \geq t, H(w) \leq t).
\end{aligned}$$

Thus, it remains to bound $\mathbb{P}(L(w) \geq t, H(w) \leq t)$ in an appropriate way. Recall, if $H(w) \leq t$, the functional $\xi^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$ only depends on the region given by $\mathcal{V}$, defined in (3.42). Moreover, the previous lemma states that the spatial coordinate of all points $w' = (v', h') \in \mathcal{V}$ is bounded by t, as long as its height coordinate fulfills $h' \in [-Ct^2, \infty)$. As a consequence, it is enough to consider the region

$$\mathcal{V} \cap \mathbb{R}^{d-1} \times (-\infty, -Ct^2).$$

If it is devoid of points from $\mathcal{P}^{(\lambda)}$, then, the analyzed functional only depends on points whose spatial coordinates are bounded by t, i.e., the radius of localization is smaller than t. Thus,

$$\mathbb{P}(L(w) \geq t, H(w) \leq t) \leq \mathbb{P}(\mathcal{P}^{(\lambda)} \cap \mathcal{V} \cap \mathbb{R}^{d-1} \times (-\infty, -Ct^2) \neq \emptyset).$$

Writing again ν_λ for the intensity measure of the rescaled Poisson point process $\mathcal{P}^{(\lambda)}$, by using (3.10), we obtain that

$$\begin{aligned}&\nu_\lambda(\mathcal{V}\cap\mathbb{R}^{d-1}\times(-\infty,-Ct^2))\\&=\int_{-\infty}^{-Ct^2}\int_{\substack{\mathbb{B}^{d-1}(\boldsymbol{o},v);\\(v,h_1)\in\mathcal{V}}}\frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}}\|v'\|)}{\|R_\lambda^{-\frac{\beta}{2}}v'\|^{d-2}}\frac{(\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}}\\&\qquad\times\exp\left(h'-\frac{h'^2}{2R_\lambda^\beta}(\beta-1)(1-C_1)^{\beta-2}\right)\left(1-\frac{h'}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}v'\,\mathrm{d}h',\end{aligned}\tag{3.46}$$

for some $C_1\in(-\infty,1)$. Now, we have that, for sufficiently large λ, the sine expression, the second fraction and the exponential term are bounded from above by 1, a positive constant and $e^{h'}$, respectively. Moreover, we use (3.45) to bound the spatial region. This implies that

$$\begin{aligned}\nu_\lambda(\mathcal{V}\cap\mathbb{R}^{d-1}\times(-\infty,-Ct^2))&\le c_1\int_{-\infty}^{-Ct^2}(\sqrt{t}+\sqrt{t-h'})^{d-1}\,e^{h'}\left(1-\frac{h'}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}h'\\&=c_1\int_{Ct^2}^{\infty}(\sqrt{t}+\sqrt{t+h'})^{d-1}\,e^{-h'}\left(1+\frac{h'}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}h'\\&\le c_1\int_{Ct^2}^{\infty}(\sqrt{t}+\sqrt{t+h'})^{d-1}\;e^{-c_2h'}\,\mathrm{d}h'\\&\le c_1\int_{Ct^2}^{\infty}\exp(c_2\log(\sqrt{t}+\sqrt{t+h'})-c_3h')\,\mathrm{d}h'\\&\le c_1\,\exp(-c_2t^2),\end{aligned}$$

since we have for all $h'\ge 0$ that

$$e^{-h'}\left(1+\frac{h'}{R_\lambda^\beta}\right)^{d-1+\alpha}\le e^{-ch'},$$

and, similarly,

$$\exp(c_1\log(\sqrt{t}+\sqrt{t+h'})-c_2h')\le\exp(-c_3h').$$

As a conclusion, the inequality $1 - e^{-x} \leq x$, valid for all $x \in \mathbb{R}$, leads to

$$\begin{aligned}\mathbb{P}(\mathcal{P}^{(\lambda)} \cap \mathcal{V} \cap \mathbb{R}^{d-1} \times (-\infty, -ct^2) \neq \emptyset) &= 1 - \mathbb{P}(\mathcal{P}^{(\lambda)} \cap \mathcal{V} \cap \mathbb{R}^{d-1} \times (-\infty, -ct^2) = \emptyset) \\ &= 1 - \exp(-\nu_\lambda(\mathcal{V} \cap \mathbb{R}^{d-1} \times (-\infty, -ct^2))) \\ &\leq \nu_\lambda(\mathcal{V} \cap \mathbb{R}^{d-1} \times (-\infty, -ct^2)) \\ &\leq c_1 \exp(-c_2 t^2).\end{aligned}$$

Finally, this implies that

$$\mathbb{P}(L(w) \geq t) \leq c_1 \exp(-c_2 e^{c_3 t}) + c_4 \exp(-c_5 t^2) \leq c_6 \exp(-c_7 t^2) \leq c_8 \exp(-c_9 t),$$

and the theorem is proved. □

Proof of Corollary 3.2.3. Let $w = (\boldsymbol{o}, h) \in W_\lambda$. In the case that $h \in (0, R_\lambda^\beta]$, we apply (3.34) for $t = h$, which yields

$$\mathbb{P}(w \in \text{ext}(\mathcal{P}^{(\lambda)})) = \mathbb{P}(H(w) \geq h) \leq c_1 \exp\left(-\frac{e^h}{c_2}\right).$$

Indeed, if $H(w)$ is supposed to be bigger than 0, then, by definition, w has to belong to the extreme points of $\mathcal{P}^{(\lambda)}$. In the second case, i.e., $h \in (-\infty, 0]$, one can bound the probability in a trivial way by $C \exp(-\frac{e^0}{C})$ with a sufficiently large constant C. Combining the two cases yields the result. □

In order to prove the assertion stated in Theorem 3.2.4, let $M \in (0, \infty)$, $t \geq 0$, λ be sufficiently large and define the events

$$T_1 := \{\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap \{(v, h) : ||v|| \leq M, h > t\} \neq \emptyset\},$$

and

$$T_2 := \{\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap \{(v, h) : ||v|| \leq M, h < -t\} \neq \emptyset\}.$$

Lemma 3.2.10 *For sufficiently large λ, it holds that*

$$\mathbb{P}(T_1) \leq c_1 M^{d-1} \exp(-c_2 e^t).$$

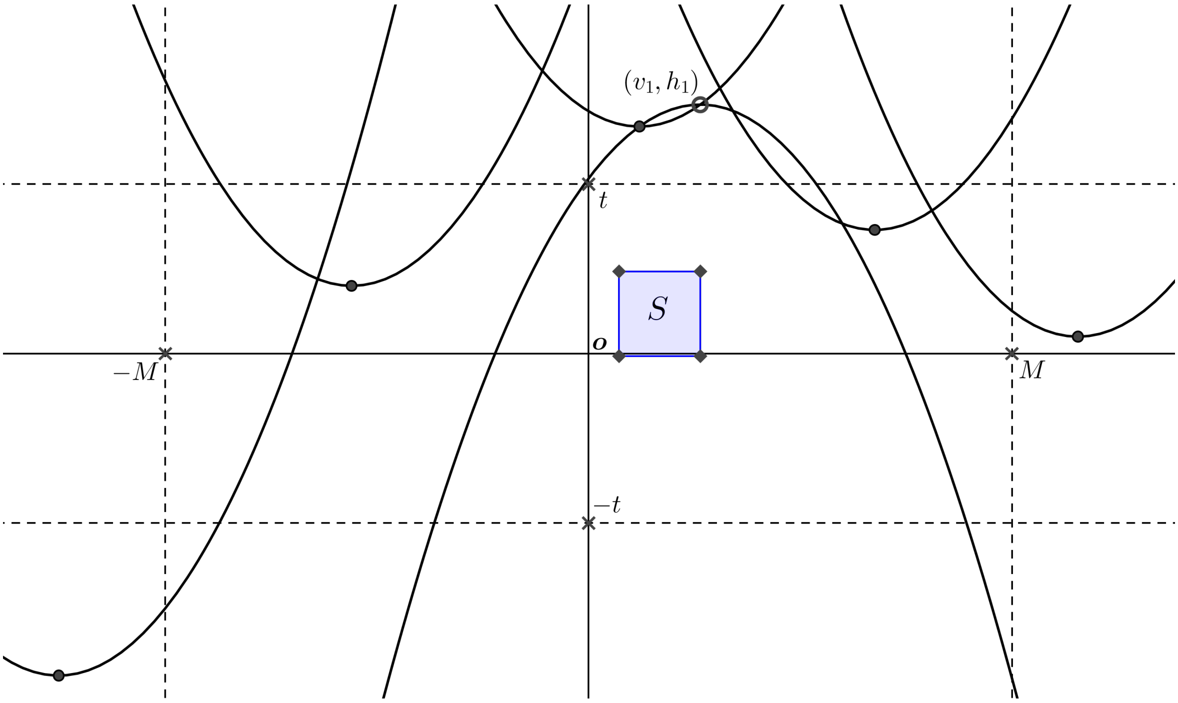

FIGURE 3.8: The event T_1 and the unit volume cube S.

Proof. Similarly to the considerations concerning the event $\{H(w) \geq t\}$, we have that

$$T_1 = \{\exists w_1 := (v_1, h_1) \in \partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) : h_1 \geq t, ||v_1|| \leq M, [\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset\},$$

(see Figure 3.8). The only difference is that there is now also a condition on the spatial coordinate v_1. Fix $w_1 := (v_1, h_1) \in \partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$. Analogously as in the proof of Lemma 3.2.6, we construct a unit volume cube S centered in

$$\left(v_1 - \frac{\sqrt{d-1}v_1}{2||v_1||}, \frac{h_1}{(\beta+1)^{\beta}} - 1 \right),$$

(see again Figure 3.8), to obtain that

$$S \subseteq [\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap C_{d-1}\left(\boldsymbol{o}, M \wedge \frac{3\pi R_{\lambda}^{\frac{\beta}{2}}}{4} \right),$$

for all sufficiently large λ. The shift in the spatial coordinate of the center of S is necessary to ensure that $S \subseteq C_{d-1}(\boldsymbol{o}, M)$. Now, by using this cube S, we achieve as in the proof of Lemma 3.2.7 that

$$\mathbb{P}([\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset) \leq \exp(-c_1\, e^{c_2 h_1}),$$

for sufficiently large λ. Since the Euclidean norm of the spatial coordinate of w_1 is bounded by M and the height coordinate is larger than t, we get, for sufficiently large λ,

$$\begin{aligned}\mathbb{P}(T_1) &\leq c \int_t^{\infty} M^{d-1}\, \mathbb{P}([\Pi^{\downarrow}(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset)\, \mathrm{d}h_1 \\ &\leq c_1\, M^{d-1} \int_t^{\infty} \exp(-c_2\, e^{c_3 h_1}) \\ &\leq c_1\, M^{d-1}\, \exp(-c_2 e^{c_3 t}).\end{aligned}$$

This completes the proof. □

Lemma 3.2.11 *For sufficiently large λ, it holds that*

$$\mathbb{P}(T_2) \leq c_1\, M^{2(d-1)}\, e^{-c_2 t}.$$

Proof. If T_2 occurs, then, there must be an explicit point $x \in \mathcal{P}^{(\lambda)}$ with

$$T_2 = \{\exists w_1 := (v_1, h_1) \in \partial\Psi^{(\lambda)}(x) : h_1 \in (-\infty, -t], ||v_1|| \leq M\},$$

and

$$x \in \mathcal{U} := \bigcup_{w \in \mathbb{B}^{d-1}(\boldsymbol{o}, M) \times \{h_1\}} [\Pi^{\downarrow}(w)]^{(\lambda)}.$$

Figure 3.9 illustrates the set $\mathcal{U}$ in the plane. By using (3.10) and the fact that the spatial region is bounded by M, we get similarly as before that

$$\nu_\lambda(\mathcal{U}) \leq c \int_{-\infty}^{h_1} M^{d-1}\, e^h \left(1 - \frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h = c_1\, M^{d-1}\, e^{c_2 h_1}. \qquad (3.47)$$

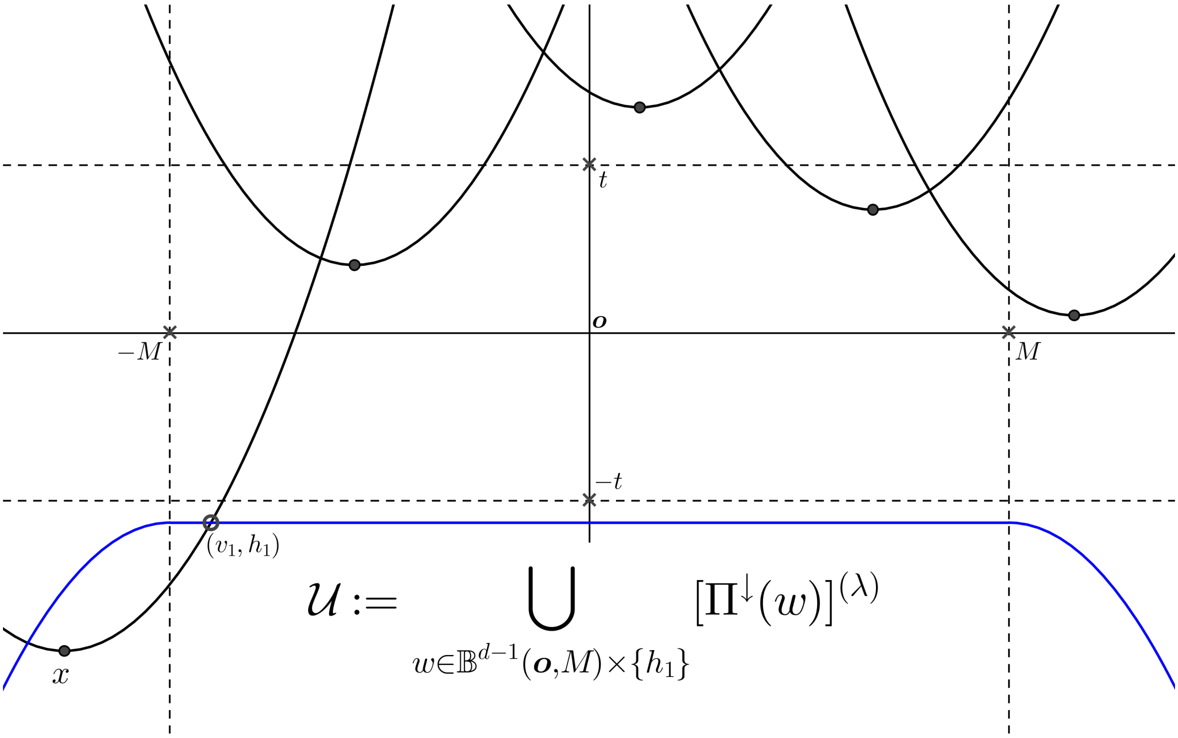

FIGURE 3.9: The set $\mathcal{U}$ and the event T_2.

This implies that

$$\mathbb{P}(\mathcal{U} \cap \mathcal{P}^{(\lambda)} \neq \emptyset) = 1 - \mathbb{P}(\mathcal{U} \cap \mathcal{P}^{(\lambda)} = \emptyset) = 1 - \exp(-\nu_\lambda(\mathcal{U})) \leq \nu_\lambda(\mathcal{U}) \leq c_1 \, M^{d-1} \, e^{c_2 h_1}.$$

Finally, this yields that

$$\mathbb{P}(T_2) \leq c_1 \int_{-\infty}^{-t} M^{d-1} \, \mathbb{P}(\mathcal{U} \cap \mathcal{P}^{(\lambda)} \neq \emptyset) \, \mathrm{d}h_1 \leq c_2 \, M^{2(d-1)} \int_{-\infty}^{-t} e^{c_3 h_1} \, \mathrm{d}h_1 = c_4 \, M^{2(d-1)} \, e^{-c_5 t},$$

completing the proof. □

Proof of Theorem 3.2.4. Recalling the definition of the events T_1 and T_2 in combination with the results from Lemma 3.2.10 and Lemma 3.2.11 gives that

$$\begin{aligned}
\mathbb{P}(\|\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C_{d-1}(v, M)\|_\infty \geq t) &= \mathbb{P}(T_1) + \mathbb{P}(T_2) \\
&\leq c_1 \, M^{d-1} \, \exp(-c_2 e^t) + c_3 \, M^{2(d-1)} \, e^{-c_4 t} \\
&\leq c_1 \, M^{2(d-1)} \, \exp(-c_2 t).
\end{aligned}$$

This finishes the proof. □

3.2.2 Moment estimates

This section contains the first step of the proof of the cumulant estimate presented in Theorem 3.3.1 and shows another crucial property of the functionals $\xi \in \Xi$, namely, a moment estimate, that considerably refines the existing one from [23, Page 26] in the Gaussian setting. As already discussed above, deriving such bounds in the context of our class of generalized Gamma polytopes is a much more delicate task compared to random polytopes in the unit ball studied in [56], although at the beginning we could follow the principal idea from [23, Page 27 and 28] in the Gaussian case. To present our results in a unified way, let us define for $\xi \in \Xi$ the weights

$$u[\xi] := \begin{cases} i & : \xi = \xi_{V_i} \\ 2j & : \xi = \xi_{f_j}, \end{cases} \qquad v[\xi] := \begin{cases} 1 & : \xi = \xi_{V_i} \\ j & : \xi = \xi_{f_j}, \end{cases}$$

and

$$w[\xi] := \begin{cases} 2 & : \xi = \xi_{V_i} \\ j & : \xi = \xi_{f_j}, \end{cases}$$

where $i \in \{1, \dots, d\}$ and $j \in \{0, \dots, d-1\}$.

Theorem 3.2.12 *Let $\xi \in \Xi$, $p \in \mathbb{N}$, $x_1 = (v_1, h_1), \dots, x_p = (v_p, h_p) \in W_\lambda$, and put $\delta := \min_{i,j=1,\dots,p} \|v_i - v_j\|$.*

(i) *For sufficiently large λ, it holds that*

$$\mathbb{E}\big|\xi^{(\lambda)}(x_1, \mathcal{P}^{(\lambda)})\big|^p \le c_1\, c_2^p\, (p!)^{u[\xi]}\, (pdv[\xi])!\, (1+|h|)^{p(d-1)v[\xi]+d} \exp\left(-\frac{e^{h\vee 0}}{c_3}\right),$$

and

$$\mathbb{E}\big|\xi^{(\lambda)}(x_1, \mathcal{P}^{(\lambda)} \cap C_{d-1}\Big(x_1, \frac{\delta}{2}\Big))\big|^p$$
$$\le c_4\, c_5^p\, (p!)^{u[\xi]}\, (pdv[\xi])!\, (1+|h|)^{p(d-1)v[\xi]+d} \exp\left(-\frac{e^{h\vee 0}}{c_6}\right).$$

(ii) *Moreover, for sufficiently large λ, it holds that*

$$\mathbb{E}\left[\left(\prod_{i=1}^{p}\xi^{(\lambda)}\left(x_i,\mathcal{P}^{(\lambda)}\cup\bigcup_{i=1}^{p}\{x_i\}\right)\right)^2\right] \le c_1\,c_2^p\,(p!)^{2u[\xi]}\,((pdv[\xi])!)^2\prod_{i=1}^{p}\left[(1+|h_i|)^{2dw[\xi]}\,\exp\left(-\frac{e^{h_i\vee 0}}{c_3\,k}\right)\right],$$

and

$$\mathbb{E}\left[\left(\prod_{i=1}^{p}\xi^{(\lambda)}\left(x_i,\left(\mathcal{P}^{(\lambda)}\cup\bigcup_{i=1}^{k}\{x_i\}\right)\cap C_{d-1}\left(x_i,\frac{\delta}{2}\right)\right)\right)^2\right] \le c_4\,c_5^p\,(p!)^{2u[\xi]}\,((pdv[\xi])!)^2\prod_{i=1}^{p}\left[(1+|h_i|)^{2dw[\xi]}\,\exp\left(-\frac{e^{h_i\vee 0}}{c_6\,k}\right)\right].$$

Remark 3.2.13 It is enough to prove the bound for

$$\mathbb{E}\big|\xi^{(\lambda)}(x_1,\mathcal{P}^{(\lambda)})\big|^p,$$

since the one for

$$\mathbb{E}\big|\xi^{(\lambda)}(x_1,\mathcal{P}^{(\lambda)}\cap C_{d-1}\Big(x_1,\frac{\delta}{2}\Big))\big|^p$$

is completely similar. The same holds true in part (ii) of the theorem.

Proof of Theorem 3.2.12 (i) for $\xi=\xi_{V_i}$. We start with the first assertion and choose $\xi=\xi_{V_i}$, $i\in\{1,\dots,d\}$. Because of rotational invariance of the underlying point process, we may assume that the point $x:=x_1$ has representation $(\boldsymbol{o},h)$ with $h\in(-\infty,R_\lambda^\beta]$. Now, for all $M\in(0,\infty)$ and sufficiently large λ, put

$$D^{(\lambda)}(M):=\|\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})\cap C_{d-1}(\boldsymbol{o},M)\|_\infty,$$

and let

$$L:=L(\xi^{(\lambda)},(\boldsymbol{o},h))$$

be the radius of localization of the functional $\xi^{(\lambda)}$, evaluated at $(\boldsymbol{o},h)$.

Then, for sufficiently large λ, in view of (3.15), $\left|\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})\right|$ is bounded by the Lebesgue measure of the set

$$\mathbb{B}_{d-1}(\boldsymbol{o},L)\times[-D^{(\lambda)}(L),D^{(\lambda)}(L)],$$

times

$$c_1\left(1+\frac{D^{(\lambda)}(L)}{R_\lambda^\beta}\right)^{i-1}\leq c_1\,(1+D^{(\lambda)}(L))^{i-1}\leq c_1\,2^{i-1}\,D^{(\lambda)}(L)^{i-1}=c_2\,D^{(\lambda)}(L)^{i-1}.$$

Hence, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}\left|\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})\right|^p&\leq c_1\,c_2^p\,\mathbb{E}\left|L^{d-1}D^{(\lambda)}(L)^i\right|^p\\&\leq c_1\,c_2^p\left(\mathbb{E}\left[L^{2p(d-1)}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[D^{(\lambda)}(L)^{2pi}\right]\right)^{\frac{1}{2}}.\end{aligned}\tag{3.48}$$

Using (3.36) and the definition of the Gamma function implies that

$$\begin{aligned}\mathbb{E}[L^r]&=r\int_0^\infty\mathbb{P}(L>t)\,t^{r-1}\,\mathrm{d}t\\&\leq r\,c_1\int_0^\infty\exp\left(-\frac{t}{c_2}\right)\,t^{r-1}\,\mathrm{d}t+r\int_0^{|h|}t^{r-1}\,\mathrm{d}t\\&\leq c_1\,c_2^r\,r!+|h|^r,\end{aligned}$$

for all $r\in\mathbb{N}$. Hence, in view of (2.6),

$$\begin{aligned}\mathbb{E}\left[L^{2p(d-1)}\right]&\leq c_1\,c_2^p\,(2p(d-1))!+|h|^{2p(d-1)}\leq c_3\,c_4^p\,(2pd)!+|h|^{2p(d-1)}\\&\leq c_1\,c_2^p\,((pd)!)^2+|h|^{2p(d-1)}\leq c_3\,c_4^p\,((pd)!)^2\,(1+|h|^{2p(d-1)})\\&\leq c_1\,c_2^p\,((pd)!)^2\,(1+|h|)^{2p(d-1)}.\end{aligned}$$

Thus,

$$\begin{aligned}\left(\mathbb{E}\left[L^{2p(d-1)}\right]\right)^{\frac{1}{2}}&\leq\left(c_1\,c_2^p\,((pd)!)^2\,(1+|h|)^{2p(d-1)}\right)^{\frac{1}{2}}\\&\leq c_1\,c_2^p\,(pd)!\,(1+|h|)^{p(d-1)}.\end{aligned}\tag{3.49}$$

On the other hand, we have that

$$\begin{aligned}\mathbb{E}\big[D^{(\lambda)}(L)^r\big] &= \sum_{i=0}^{\infty} \mathbb{E}\big[D^{(\lambda)}(L)^r\,\mathbf{1}(i \le L < i+1)\big] \\ &\le \sum_{i=0}^{\infty} \mathbb{E}\big[D^{(\lambda)}(i+1)^r\,\mathbf{1}(L \ge i)\big] \\ &\le \sum_{i=0}^{\infty} \big(\mathbb{E}\big[D^{(\lambda)}(i+1)^{2r}\big]\big)^{\frac{1}{2}}\,\mathbb{P}(L > i)^{\frac{1}{2}},\end{aligned} \tag{3.50}$$

for all $r \in \mathbb{N}$, by using the Cauchy-Schwarz inequality in the last step. Using Theorem 3.2.4 and (2.6) leads to

$$\begin{aligned}\mathbb{E}\big[D^{(\lambda)}(i+1)^{2r}\big] &= 2r \int_0^\infty \mathbb{P}(D^{(\lambda)}(i+1) > t)\, t^{2r-1}\mathrm{d}t \\ &\le 2r\, c_1\, (i+1)^{2(d-1)} \int_0^\infty \exp\left(-\frac{t}{c_2}\right)\, t^{2r-1}\,\mathrm{d}t \\ &= c_1\, c_2^r\, (i+1)^{2(d-1)}\,(2r)! \\ &\le c_1\, c_2^r\, (i+1)^{2(d-1)}\,(r!)^2.\end{aligned}$$

Combining this with (3.36) and the fact that $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, for all $a, b \ge 0$, it follows from (3.50) that

$$\begin{aligned}\mathbb{E}\big[D^{(\lambda)}(L)^r\big] &\le \sum_{i=0}^{\infty} c_1\, c_2^r\,(i+1)^{d-1}\, r!\, \left(c_3\, \exp\left(-\frac{i}{c_4}\right) + \mathbf{1}(i \le |h|)\right)^{\frac{1}{2}} \\ &\le c_1\, c_2^r\, r!\, \left(\sum_{i=0}^{\infty}(i+1)^{d-1}\,\exp\left(-\frac{i}{c_3}\right) + \sum_{i=0}^{\infty}(i+1)^{d-1}\,\mathbf{1}(i \le |h|)\right) \\ &\le c_1\, c_2^r\, r!\,(1+|h|)^d,\end{aligned}$$

since the first sum is bounded by a constant only depending on d, and for the second one we have that

$$\sum_{i=0}^{\infty}(i+1)^{d-1}\,\mathbf{1}(i \le |h|) \le (1+|h|)^{d-1} \sum_{i=0}^{\infty}\mathbf{1}(i \le |h|) \le (1+|h|)^d.$$

Again, by (2.6), this shows that

$$\big(\mathbb{E}\big[D^{(\lambda)}(L)^{2pi}\big]\big)^{\frac{1}{2}} \le \big(c_1\, c_2^p\, (2pi)!\, (1+|h|)^d\big)^{\frac{1}{2}} \le c_3\, c_4^p\, (pi)!\, (1+|h|)^d. \tag{3.51}$$

Summarizing, we conclude from (3.48), (3.49) and (3.51) the bound

$$\mathbb{E}\big|\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})\big|^p \le c_1\, c_2^p\, (p!)^i\, (pd)!\, (1+|h|)^{p(d-1)+d}.$$

In the next step, we improve this by an exponential term. Namely, if the point $(\boldsymbol{o},h)$ does not belong to the extreme points of $\mathcal{P}^{(\lambda)}$, the functional $\xi^{(\lambda)}$, evaluated at this point, is automatically equal to 0. This means that we can condition on this event without changing the value of the expression. By Corollary 3.2.3 and the Cauchy-Schwarz inequality, this leads to

$$\begin{aligned}
\mathbb{E}\big|\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})\big|^p &= \mathbb{E}\big|\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})\,\mathbf{1}((\boldsymbol{o},h)\in \operatorname{ext}(\mathcal{P}^{(\lambda)}))\big|^p \\
&\le \big(c_1\, c_2^p\, (2pd)!\, ((2p)!)^i\, (1+|h|)^{2p(d-1)+d}\big)^{\frac{1}{2}} \left(\exp\left(-\frac{e^{h\vee 0}}{c_3}\right)\right)^{\frac{1}{2}} \\
&\le c_1\, c_2^p\, (pd)!\, (p!)^i\, (1+|h|)^{p(d-1)+d} \exp\left(-\frac{e^{h\vee 0}}{c_3}\right),
\end{aligned}$$

where we used (2.6) in the last step. This completes the proof of (i) for the intrinsic volume functionals ξ_{V_i}, since $u[\xi_{V_i}] = i$ and $v[\xi_{V_i}] = 1$, where we recall their definitions at the beginning of this section. □

Proof of Proposition 3.2.12 (i) for $\xi = \xi_{f_j}$. Next, we turn to the j-face functional $\xi = \xi_{f_j}$ with $j \in \{0,\dots,d-1\}$. Because of rotational invariance, it is again enough to prove the assertion for the point $(\boldsymbol{o},h)$ with $h \in (-\infty, R_\lambda^\beta]$. Let $N^{(\lambda)}$ be the number of extreme points of $\mathcal{P}^{(\lambda)}$ contained in the cylinder $C_{d-1}(\boldsymbol{o},L)$, where L is the radius of localization of $\xi^{(\lambda)}$, evaluated at $(\boldsymbol{o},h)$. If $j=0$, then, $\xi_{f_0} \le 1$ and, hence,

$$\mathbb{E}\big|\xi_{f_0}^{(\lambda)}(x,\mathcal{P}^{(\lambda)})\big|^p \le 1, \tag{3.52}$$

for all $p \in \mathbb{N}$. Therefore, it remains to consider the case that $j \in \{1,\dots,d-1\}$.

We have that

$$\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)}) \le \frac{1}{j+1}\binom{N^{(\lambda)}}{j} \le (N^{(\lambda)})^j,$$

and, hence, it follows that

$$\mathbb{E}|\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})|^p \le \mathbb{E}[(N^{(\lambda)})^{pj}].$$

Thus, it is enough to find a bound for $\mathbb{E}[(N^{(\lambda)})^{pj}]$. Writing once more ν_λ for the intensity measure of the rescaled Poisson point process $\mathcal{P}^{(\lambda)}$, we observe that in view of (3.10), we have for sufficiently large λ, $r \in [0, \pi R_\lambda^{\frac{\beta}{2}}]$ and $\ell \in (-\infty, R_\lambda^\beta]$ that

$$\nu_\lambda(C_{d-1}(\boldsymbol{o},r)\cap(-\infty,\ell)) \le c\, r^{d-1}\,(e^\ell \vee 1), \tag{3.53}$$

slightly different from [23, Equation (4.18)]. Indeed, to verify this inequality, we notice first that for all such r,ℓ and sufficiently large λ, we get by using the density function in (3.10) that there is an absolute constant $C \in (-\infty, 1)$ such that

$$\begin{aligned}
&\nu_\lambda(C_{d-1}(\boldsymbol{o},r)\cap(-\infty,\ell))\\
&\quad= \int\limits_{C_{d-1}(\boldsymbol{o},r)\cap(-\infty,\ell)} \frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}}\|v\|)}{\|R_\lambda^{-\frac{\beta}{2}}v\|^{d-2}}\,\frac{(\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}}\\
&\qquad\quad\times\exp\left(h-\frac{h^2}{2R_\lambda^\beta}(\beta-1)(1-C)^{\beta-2}\right)\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}v\mathrm{d}h\\
&\quad\le c\int\limits_{C_{d-1}(\boldsymbol{o},r)\cap(-\infty,\ell)} e^h\left(1-\frac{h}{R_\lambda^2}\right)^{d-1+\alpha}\mathrm{d}v\mathrm{d}h\\
&\quad= c\int\limits_{-\infty}^{\ell} e^h\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}h\int\limits_{\mathbb{B}^{d-1}(\boldsymbol{o},r)}\mathrm{d}v\\
&\quad= c\,r^{d-1}\int\limits_{-\infty}^{\ell} e^h\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}h,
\end{aligned}$$

where we used for the inequality that the fractions involving the sine term and the critical radius R_λ are bounded from above by 1 and a constant $C \in (0,\infty)$, respectively.

Applying the fact that

$$\left(1 - \frac{h}{R_\lambda^\beta}\right) \le (-(h-1) \vee 1),$$

for all $h \in (-\infty, R_\lambda^\beta]$, whenever $R_\lambda \ge 1$, yields that

$$\begin{aligned}
\int_{-\infty}^{\ell} e^h \left(1 - \frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h &\le \int_{-\infty}^{\ell} e^h \left(-(h-1) \vee 1\right)^{d-1+\alpha} \mathrm{d}h \\
&= \int_{-\infty}^{0} e^h \left(-(h-1)\right)^{d-1+\alpha} \mathrm{d}h + \int_{0}^{\ell} e^h \, \mathrm{d}h \\
&= \int_{0}^{\infty} e^{-h} \left(1+h\right)^{d-1+\alpha} \mathrm{d}h + \int_{0}^{\ell} e^h \, \mathrm{d}h \\
&\le c\,\Gamma(d-1+\alpha) + e^\ell \\
&\le c\,(e^\ell \vee 1).
\end{aligned}$$

Combining the last two calculations shows the bound claimed in (3.53).

Thus, writing $\mathrm{Po}(\alpha)$ for a Poisson distributed random variable with mean $\alpha > 0$ and recalling the definition of the random variable H from the paragraph at the beginning of Section 3.2.1, we get that, for sufficiently large λ,

$$\begin{aligned}
\mathbb{E}[(N^{(\lambda)})^{pj}] &\le \mathbb{E}|\mathcal{P}^{(\lambda)} \cap (C_{d-1}(\boldsymbol{o}, L) \cap (-\infty, H))|^{pj} \\
&\le \sum_{i=0}^{\infty} \sum_{m=h}^{\infty} \mathbb{E}[\mathrm{Po}(\nu_\lambda(C_{d-1}(\boldsymbol{o}, i+1) \cap (-\infty, m+1)))^{pj} \\
&\qquad\qquad \times \mathbf{1}(i \le L < i+1, m \le H < m+1)] \\
&\le \sum_{i=0}^{\infty} \sum_{m=h}^{\infty} \mathbb{E}[\mathrm{Po}(c\,(i+1)^{d-1}\,(e^{m+1} \vee 1))^{pj}\, \mathbf{1}(L \ge i, H \ge m)].
\end{aligned}$$

(Here and below, h has to be interpreted as the integer $\lfloor h \rfloor$, but we refrain from such a notation for simplicity. Moreover, from now on we interpret sums like $\sum_{i=h}^{0} a_i$ as 0, if $h > 0$.)

The moments of $\mathrm{Po}(\alpha)$ are given by the so-called Touchard polynomials. More precisely,

$$\mathbb{E}[\mathrm{Po}(\alpha)^k] = \sum_{i=1}^{k} \alpha^i \left\{ \begin{matrix} k \\ i \end{matrix} \right\},$$

where $\left\{ \begin{matrix} k \\ i \end{matrix} \right\}$ denotes the Stirling number of second kind (see [129]). Since

$$\sum_{i=1}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} = \sum_{L_1,\dots,L_p \preceq [\![k]\!]} 1 \leq k!, \tag{3.54}$$

we have that

$$\mathbb{E}[\mathrm{Po}(\alpha)^k] \leq \alpha^k \sum_{i=1}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} \mathbf{1}(\alpha \geq 1) + \sum_{i=1}^{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} \mathbf{1}(\alpha < 1) \leq \alpha^k\, k! + k!. \tag{3.55}$$

In the last step we used that the number of unordered partitions of $\{1, \dots, k\}$ is known as the k-th Bell number, which can be optimally bounded of order $k!$ (see [34]).

Now, Hölder's inequality, (3.55), (3.53), (2.6) and the fact that $(a+b)^{\frac{1}{3}} \leq a^{\frac{1}{3}} + b^{\frac{1}{3}}$, for $a, b \geq 0$, imply that

$$\begin{aligned}
&\mathbb{E}[(N^{(\lambda)})^{pj}] \\
&\quad \leq \sum_{i=0}^{\infty} \sum_{m=h}^{\infty} (\mathbb{E}[\mathrm{Po}(c\,(i+1)^{d-1}\,(e^{m+1} \vee 1))^{3pj}])^{\frac{1}{3}}\, \mathbb{P}(L \geq i)^{\frac{1}{3}}\, \mathbb{P}(H \geq m)^{\frac{1}{3}} \\
&\quad \leq c_1\, c_2^p \sum_{i=0}^{\infty} \sum_{m=h}^{\infty} ((3pj)!\,(i+1)^{3p(d-1)j}\,(e^{m+1} \vee 1)^{3pj} + (3pj)!)^{\frac{1}{3}} \\
&\qquad\qquad\qquad\qquad \times \mathbb{P}(L \geq i)^{\frac{1}{3}}\, \mathbb{P}(H \geq m)^{\frac{1}{3}} \\
&\quad \leq c_1\, c_2^p \sum_{i=0}^{\infty} \sum_{m=h}^{\infty} (pj)!\,(i+1)^{p(d-1)j}\,(e^{m+1} \vee 1)^{pj}\, \mathbb{P}(L \geq i)^{\frac{1}{3}}\, \mathbb{P}(H \geq m)^{\frac{1}{3}} \\
&\qquad + c_3\, c_4^p \sum_{i=0}^{\infty} \sum_{m=h}^{\infty} (pj)!\, \mathbb{P}(L \geq i)^{\frac{1}{3}}\, \mathbb{P}(H \geq m)^{\frac{1}{3}} \\
&\quad =: T_1 + T_2.
\end{aligned}$$

We bound both terms T_1 and T_2 separately.

For T_2, we get by splitting the summation over i into $i \le |h|$ and $i > |h|$, and by using (3.35), that it equals

$$\begin{aligned}
&c_1\, c_2^p\, (pj)! \sum_{i=0}^{|h|} \underbrace{\mathbb{P}(L \ge i)^{\frac{1}{3}}}_{\le 1} \sum_{m=h}^{\infty} \mathbb{P}(H \ge m)^{\frac{1}{3}} \\
&\qquad + c_3\, c_4^p\, (pj)! \sum_{i=|h|}^{\infty} \mathbb{P}(L \ge i)^{\frac{1}{3}} \sum_{m=h}^{\infty} \mathbb{P}(H \ge m)^{\frac{1}{3}} \\
&\le c_1\, c_2^p\, (pj)! \sum_{i=0}^{|h|} \sum_{m=h}^{\infty} \mathbb{P}(H \ge m)^{\frac{1}{3}} + c_3\, c_4^p\, (pj)! \sum_{i=|h|}^{\infty} \exp(-c_5 i^2) \sum_{m=h}^{\infty} \mathbb{P}(H \ge m)^{\frac{1}{3}} \\
&\le c_1\, c_2^p\, (pj)!\, |h|\, (|h| + c_3) + c_4\, c_5^p\, (pj)!\, (|h| + c_6) \\
&\le c_1\, c_2^p\, (pj)!\, (1 + |h|)^2,
\end{aligned}$$

since, by (3.34),

$$\begin{aligned}
\sum_{m=h}^{\infty} \mathbb{P}(H \ge m)^{\frac{1}{3}} &= \sum_{m=h}^{0} \mathbb{P}(H \ge m)^{\frac{1}{3}} + \sum_{m=0}^{\infty} \mathbb{P}(H \ge m)^{\frac{1}{3}} \\
&\le \sum_{m=h}^{0} 1 + \sum_{m=0}^{\infty} c_1 \exp(-c_2 e^m) \\
&\le |h| + c_1.
\end{aligned}$$

Now, we turn to T_1, where we first notice that

$$\begin{aligned}
\sum_{i=0}^{|h|} (i+1)^{p(d-1)j} \underbrace{\mathbb{P}(L \ge i)^{\frac{1}{3}}}_{\le 1} &\le \sum_{i=0}^{|h|} (1 + |h|)^{p(d-1)j} \\
&\le |h|\, (1 + |h|)^{p(d-1)j} \\
&\le (1 + |h|)^{p(d-1)j+1}.
\end{aligned} \tag{3.56}$$

Using (2.5) with $n = pj + 1$ in conjunction with (3.34) and the observation that, for all $m \ge 0$,

$$(e^{m+1} \vee 1)^{pj} = e^{(m+1)pj},$$

implies that

$$\begin{aligned}\sum_{m=0}^{\infty}(e^{m+1}\vee 1)^{pj}\,\mathbb{P}(H\geq m)^{\frac{1}{3}} &= e^{pj}\sum_{m=0}^{\infty}e^{mpj}\,\mathbb{P}(H\geq m)^{\frac{1}{3}}\\ &\leq c_1\,c_2^p\sum_{m=0}^{\infty}e^{mpj}\,\exp\left(-\frac{e^m}{c_3}\right)\\ &\leq c_1\,c_2^p\,(pj+1)!\sum_{m=0}^{\infty}e^{mpj}e^{-m(pj+1)}\\ &\leq c_1\,c_2^p\,(pj+1)!\sum_{m=0}^{\infty}e^{-m}\\ &\leq c_1\,c_2^p\,(pj+1)!,\end{aligned}$$

and

$$\sum_{m=h}^{-1}\underbrace{(e^{m+1}\vee 1)^{pj}}_{\leq 1}\,\underbrace{\mathbb{P}(H\geq m)^{\frac{1}{3}}}_{\leq 1}\leq\sum_{m=1}^{|h|}1=|h|\leq(1+|h|).$$

Together with (2.7), this leads to

$$\begin{aligned}&\sum_{m=h}^{\infty}(e^{m+1}\vee 1)^{pj}\,\mathbb{P}(H\geq m)^{\frac{1}{3}}\\ &\quad=\sum_{m=h}^{-1}(e^{m+1}\vee 1)^{pj}\,\mathbb{P}(H\geq m)^{\frac{1}{3}}+\sum_{m=0}^{\infty}(e^{m+1}\vee 1)^{pj}\,\mathbb{P}(H\geq m)^{\frac{1}{3}}\\ &\quad\leq(1+|h|)+c_1\,c_2^p\,(pj+1)!\\ &\quad\leq c_1\,c_2^p\,(pj)!\,(1+|h|).\end{aligned}\tag{3.57}$$

Moreover, with (3.35), (2.7), (2.5) applied at $n=p(d-1)j+2$ and the fact that $i^2\geq i+1$, for all $i\geq 2$, we get

$$\sum_{i=|h|}^{\infty}(i+1)^{p(d-1)j}\,\mathbb{P}(L\geq i)^{\frac{1}{3}}\leq c_1\,c_2^p\,(pdj)!.\tag{3.58}$$

Indeed, it holds that

$$\begin{aligned}\sum_{i=|h|}^{\infty}(i+1)^{p(d-1)j}\,\mathbb{P}(L\geq i)^{\frac{1}{3}} &\leq c_1\sum_{i=|h|}^{\infty}(i+1)^{p(d-1)j}\,\exp\left(-\frac{i^2}{c_2}\right)\\ &\leq c_1\sum_{i=0}^{\infty}(i+1)^{p(d-1)j}\,\exp\left(-\frac{i+1}{c_2}\right)\\ &\leq c_1\,c_2^p\,(p(d-1)j+2)!\sum_{i=0}^{\infty}(i+1)^{p(d-1)j-p(d-1)j-2}\\ &\leq c_1\,c_2^p\,(p(d-1)j+2)!\sum_{i=0}^{\infty}(i+1)^{-2}\\ &= c_1\,c_2^p\,(p(d-1)j+2)!\\ &\leq c_1\,c_2^p\,(pdj)!.\end{aligned}$$

Combining (3.56), (3.57) and (3.58), we see that T_1 is bounded as follows:

$$\begin{aligned}T_1 &= c_1 c_2^p\sum_{i=0}^{\infty}\sum_{m=h}^{\infty}(pj)!\,(i+1)^{p(d-1)j}\,(e^{m+1}\vee 1)^{pj}\,\mathbb{P}(L\geq i)^{\frac{1}{3}}\,\mathbb{P}(H\geq m)^{\frac{1}{3}}\\ &= c_1\,c_2^p\,(pj)!\sum_{i=0}^{|h|}(i+1)^{p(d-1)j}\,\mathbb{P}(L\geq i)^{\frac{1}{3}}\sum_{m=h}^{\infty}(e^{m+1}\vee 1)^{pj}\,\mathbb{P}(H\geq m)^{\frac{1}{3}}\\ &\quad + c_3\,c_4^p\,(pj)!\sum_{i=|h|}^{\infty}(i+1)^{p(d-1)j}\,\mathbb{P}(L\geq i)^{\frac{1}{3}}\sum_{m=h}^{\infty}(e^{m+1}\vee 1)^{pj}\,\mathbb{P}(H\geq m)^{\frac{1}{3}}\\ &\leq c_1\,c_2^p\,(pj)!\,(1+|h|)^{p(d-1)j+1}\,(pj)!\,(1+|h|)+c_3\,c_4^p\,(pj)!\,(pdj)!\,(pj)!\,(1+|h|)\\ &\leq c_1\,c_2^p\,(pdj)!\,((pj)!)^2\,(1+|h|)^{p(d-1)j+2}.\end{aligned}$$

Combining the estimates for T_1 and T_2 yields that

$$\begin{aligned}\mathbb{E}[(N^{(\lambda)})^{pj}] &\leq c_1\,c_2^p\,(pdj)!\,((pj)!)^2\,(1+|h|)^{p(d-1)j+2}+c_3\,c_4^p\,(pj)!\,(1+|h|)^2\\ &\leq c_1\,c_2^p\,(pdj)!\,(p!)^{2j}\,(1+|h|)^{p(d-1)j+d}.\end{aligned} \tag{3.59}$$

In view of (3.52), this bound clearly holds for the functional ξ_{f_0}, too. Finally, the additional exponential term appears in the same way as for the defect intrinsic volume functionals by conditioning on the event that the point $x=(\boldsymbol{o},h)$ belongs to the extreme points of $\mathcal{P}^{(\lambda)}$. This completes the proof of the first part of the theorem. □

Proof of Proposition 3.2.12 (ii). Next, we turn to the second assertion and consider the defect intrinsic volume functional ξ_{V_i}, $i \in \{1, \dots, d\}$, first. With Hölder's inequality and (2.6), we get that the expectation is bounded from above by

$$\begin{aligned}
&\prod_{j=1}^{p} \left(\mathbb{E} \left[\xi_{V_i}^{(\lambda)} \left(x_j, \mathcal{P}^{(\lambda)} \right) \right]^{2p} \right)^{\frac{1}{p}} \\
&\quad \le \prod_{j=1}^{p} \left[c_1\, c_2^p\, (2pd)!\, ((2p)!)^i\, (1 + |h_j|)^{2p(d-1)+d} \exp\left(-\frac{e^{h_j \vee 0}}{c_3} \right) \right]^{\frac{1}{p}} \\
&\quad \le \prod_{j=1}^{p} \left[c_1\, c_2^p\, ((pd)!)^2\, (p!)^{2i}\, (1 + |h_j|)^{2p(d-1)+d} \exp\left(-\frac{e^{h_j \vee 0}}{c_3} \right) \right]^{\frac{1}{p}} \\
&\quad \le c_1\, c_2^p\, ((pd)!)^2\, (p!)^{2i} \prod_{j=1}^{p} \left[(1 + |h_j|)^{2(d-1)+\frac{d}{p}} \exp\left(-\frac{e^{h_j \vee 0}}{c_3\, p} \right) \right] \\
&\quad \le c_1\, c_2^p\, ((pd)!)^2\, (p!)^{2i} \prod_{j=1}^{p} \left[(1 + |h_j|)^{2(d-1)+d} \exp\left(-\frac{e^{h_j \vee 0}}{c_3\, p} \right) \right] \\
&\quad \le c_1\, c_2^p\, ((pd)!)^2\, (p!)^{2i} \prod_{j=1}^{p} \left[(1 + |h_j|)^{4d} \exp\left(-\frac{e^{h_j \vee 0}}{c_3\, p} \right) \right].
\end{aligned}$$

The trivial estimate $2(d-1) + d \le 4d$ implies the last inequality. Finally, we consider the functional ξ_{f_j} with $j \in \{0, \dots, d-1\}$. Instead of using the bound (3.59), it is more convenient to work with

$$\begin{aligned}
\mathbb{E}[(N^{(\lambda)})^{pj}] &\le c_1\, c_2^p\, (pdj)!\, ((pj)!)^2\, (1 + |h|)^{p(d-1)j+2} + c_3\, (pj)!\, (1 + |h|)^2 \\
&\le c_1\, c_2^p\, (pdj)!\, ((pj)!)^2\, (1 + |h|)^{pdj},
\end{aligned} \tag{3.60}$$

which holds because of $p(d-1)j + 1 \le pdj$. Then, a similar computation as for the intrinsic volume functional completes the proof. □

Remark 3.2.14 In the proof of the previous theorem, one could also use the better estimate $2(d-1)+d \le 3d$. However, since later we need an exponent from the natural numbers after taking the square-root, we directly work with the upper bound $4d$.

The next clustering lemma is the analogue of [56, Lemma 5.4]. The main difference and what makes it more complicated is that in view of (3.36), we do not have a localization property on the whole space for our class of generalized Gamma polytopes. This leads to an additional indicator function in the bound for the correlation function, which also makes the analysis later more involved.

Lemma 3.2.15 *Let $\{S, T\}$ be a non-trivial partition of $\{1, \dots, k\}$ and $\xi \in \Xi$. Then, for all $x_1 = (v_1, h_1), \dots, x_k = (v_k, h_k) \in W_\lambda$ and sufficiently large λ, it holds that*

$$\begin{aligned}
&|m_\lambda(\mathbf{x}_{S\cup T}) - m_\lambda(\mathbf{x}_S)m_\lambda(\mathbf{x}_T)| \\
&\qquad \le c_1\, c_2^k\, k\, (k!)^{u[\xi]}\, (kdv[\xi])!\, \Big(\exp(-c_3\,\delta) + \mathbf{1}(\delta \le 2 \max_{r\in S\cup T}\{|h_r|\})\Big) \\
&\qquad\qquad \times \prod_{r\in S\cup T} \left[(1+|h_r|)^{dw[\xi]} \exp\left(-\frac{e^{h_r \vee 0}}{c_4\, k}\right)\right],
\end{aligned}$$

with $k = |S\cup T|$, $\delta := d(\mathbf{v}_S, \mathbf{v}_T) := \min_{s\in S, t\in T} \|v_s - v_t\|$,

$$m_\lambda(\mathbf{x}_S) := \mathbb{E}\left[\prod_{s\in S} \xi^{(\lambda)}\left(x_s, \mathcal{P}^{(\lambda)} \cup \bigcup_{s\in S}\{x_s\}\right)\right],$$

and $m_\lambda(\mathbf{x}_T)$ and $m_\lambda(\mathbf{x}_{S\cup T})$ defined similarly.

Corollary 3.2.16 *Let $\xi^{(\lambda)} \in \Xi^{(\lambda)}$, $h_0 \in (-\infty, R_\lambda^\beta]$ and $(v_1, h_1) \in W_\lambda$. For all sufficiently large λ, it holds that*

$$\begin{aligned}
|c^{\xi^{(\lambda)}}((\boldsymbol{o}, h_0), (v_1, h_1), \mathcal{P}^{(\lambda)})| &\le c_1\,(1+|h_0|)^{dw[\xi]}\,(1+|h_1|)^{dw[\xi]}\, \exp\left(-\frac{1}{c_2}(e^{h_0\vee 0} + e^{h_1 \vee 0})\right) \\
&\qquad \times (\exp(-c_3\,|v_1|) + \mathbf{1}(|v_1| \le 2\max\{|h_0|, |h_1|\}))\,,
\end{aligned}$$

where $c^{\xi^{(\lambda)}}((\boldsymbol{o}, h_0), (v_1, h_1), \mathcal{P}^{(\lambda)})$ is the second order correlation function from (3.23).

Proof of Lemma 3.2.15. Let us define the random variables

$$X := \prod_{s\in S} \xi^{(\lambda)}\Big(x_s, \mathcal{P}^{(\lambda)} \cup \bigcup_{j\in S}\{x_j\}\Big), \qquad Y := \prod_{t\in T} \xi^{(\lambda)}\Big(x_t, \mathcal{P}^{(\lambda)} \cup \bigcup_{j\in T}\{x_j\}\Big),$$
$$W := \prod_{r\in S\cup T} \xi^{(\lambda)}\Big(x_r, \mathcal{P}^{(\lambda)} \cup \bigcup_{j\in S\cup T}\{x_j\}\Big),$$

and

$$X_\delta := \prod_{s\in S} \xi^{(\lambda)}\Big(x_s, \Big(\mathcal{P}^{(\lambda)} \cup \bigcup_{j\in S}\{x_j\}\Big) \cap C_{d-1}\Big(x_s, \frac{\delta}{2}\Big)\Big),$$
$$Y_\delta := \prod_{t\in T} \xi^{(\lambda)}\Big(x_t, \Big(\mathcal{P}^{(\lambda)} \cup \bigcup_{j\in T}\{x_j\}\Big) \cap C_{d-1}\Big(x_t, \frac{\delta}{2}\Big)\Big),$$

$$W_\delta := \prod_{r\in S\cup T} \xi^{(\lambda)}\Big(x_r, \Big(\mathcal{P}^{(\lambda)} \cup \bigcup_{j\in S\cup T}\{x_j\}\Big) \cap C_{d-1}\Big(x_r, \frac{\delta}{2}\Big)\Big).$$

For $s \in S$ and $t \in T$, the cylinder $C_{d-1}\left(x_s, \frac{\delta}{2}\right)$ and $C_{d-1}\left(x_t, \frac{\delta}{2}\right)$ have empty intersection by definition of δ. As a consequence of the independence of X_δ and Y_δ, we get

$$\begin{aligned} &m_\lambda(\mathbf{x}_{S\cup T}) - m_\lambda(\mathbf{x}_S)m_\lambda(\mathbf{x}_T) \\ &\quad= \mathbb{E}[W] - \mathbb{E}[X]\mathbb{E}[Y] \\ &\quad= \mathbb{E}[W_\delta] - \mathbb{E}[X_\delta]\mathbb{E}[Y_\delta] + \mathbb{E}[W - W_\delta] \\ &\qquad\quad - \mathbb{E}[X_\delta]\mathbb{E}[Y - Y_\delta] - \mathbb{E}[Y]\mathbb{E}[X - X_\delta] \\ &\quad= \mathbb{E}[W - W_\delta] - \mathbb{E}[X_\delta]\mathbb{E}[Y - Y_\delta] - \mathbb{E}[Y]\mathbb{E}[X - X_\delta]. \end{aligned} \tag{3.61}$$

Now, observe that Theorem 3.2.12 implies the estimates

$$\mathbb{E}|X_\delta| \le c_1\, c_2^k\, (|S|!)^{u[\xi]}\, (|S|dv[\xi])! \prod_{s\in S}\left[(1+|h_s|)^{dw[\xi]} \exp\left(-\frac{e^{h_s\vee 0}}{c_3\, k}\right)\right],$$

and

$$\mathbb{E}|Y| \le c_1\, c_2^k\, (|T|!)^{u[\xi]}\, (|T|dv[\xi])! \prod_{t\in T}\left[(1+|h_t|)^{dw[\xi]} \exp\left(-\frac{e^{h_t\vee 0}}{c_3\, k}\right)\right],$$

where we also used that $|S|, |T| \le k$. Next, let N_S be the event that at least one x_s, $s \in S$, has a radius of localization bigger than or equal to $\frac{\delta}{2}$. On the complement of N_S, we have that $X_\delta = X$. Thus, the Cauchy-Schwarz inequality, the fact that $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, for all $a, b \ge 0$, Theorem 3.2.12 and (3.36) imply that

$$\begin{aligned} \mathbb{E}|X - X_\delta| &= \mathbb{E}|X_\delta\, \mathbf{1}(N_S)| \le (\mathbb{E}[X_\delta^2])^{\frac{1}{2}} (\mathbb{P}(N_S))^{\frac{1}{2}} \\ &\le \left(c_1\, c_2^k\, (|S|!)^{2u[\xi]}\, ((|S|dv[\xi])!)^2 \prod_{s\in S}\left[(1+|h_s|)^{2dw[\xi]} \exp\left(-\frac{e^{h_s\vee 0}}{c_3\, |S|}\right)\right]\right)^{\frac{1}{2}} \\ &\qquad \times \left(c_4\, |S|\, (\exp(-c_5\, \delta) + \mathbf{1}(\delta \le 2\max_{s\in S}\{|h_s|\}))\right)^{\frac{1}{2}} \\ &\le c_1\, c_2^k\, (|S|!)^{u[\xi]}\, (|S|dv[\xi])!\, k\, \left(\exp(-c_3\, \delta) + \mathbf{1}(\delta \le 2\max_{r\in S\cup T}\{|h_r|\})\right) \\ &\qquad \times \prod_{s\in S}\left[(1+|h_s|)^{dw[\xi]} \exp\left(-\frac{e^{h_s\vee 0}}{c_4\, k}\right)\right], \end{aligned}$$

since

$$\mathbf{1}(\delta \leq 2\max_{s\in S}\{|h_s|\}) \leq \mathbf{1}(\delta \leq 2\max_{r\in S\cup T}\{|h_r|\}) \qquad \text{and} \qquad |S| \leq k.$$

Moreover, from (2.8), we have that

$$(|S|dv[\xi])!\,(|T|dv[\xi])! \leq (kdv[\xi])! \qquad \text{and} \qquad |T|!\,|S|! \leq k!,$$

which leads to

$$\begin{aligned}\mathbb{E}|Y|\,\mathbb{E}|X - X_\delta| \leq c_1\, c_2^k\,(k!)^{u[\xi]}\,(kdv[\xi])!\,k\,\Big(\exp(-c_3\,\delta) + \mathbf{1}(\delta \leq 2\max_{r\in S\cup T}\{|h_r|\})\Big)\\ \times \prod_{r\in S\cup T}\left[(1+|h_r|)^{dw[\xi]}\exp\left(-\frac{e^{h_r\vee 0}}{c_4\,k}\right)\right],\end{aligned}$$

for sufficiently large λ. Similar estimates hold for $\mathbb{E}|X_\delta|\,\mathbb{E}|Y - Y_\delta|$ and $\mathbb{E}|W - W_\delta|$. This completes the proof in view of (3.61). □

3.3 Proof of the cumulant bound

This section contains the most technical part of the proof of our main results, that is, the proof of the cumulant bound, content of Theorem 3.3.1. Before going into the details, let us briefly describe the main steps. The starting point is the cluster measure representation of the cumulant measures presented in Lemma 3.1.19. In a first step, we deal with the diagonal term (see Lemma 3.3.2). By using (3.30), we get that for this we just need to control the moments of ξ. We have prepared such a bound in the first part of Theorem 3.2.12.

In a second and considerably more involved step of the proof, we deal with the off-diagonal term. The cluster measure representation of the cumulant measures presented in Lemma 3.1.19, the description of spatial correlations from the clustering Lemma 3.2.15, as well as the second part of our moment estimates in Theorem 3.2.12, allow us to derive a first integral representation for the individual terms (see Lemma 3.3.3). These integrals are then estimated further, starting with the inner integral (see Lemma 3.3.4). The fact that the geometric functionals $\xi \in \Xi$ are not globally localizing, implies that this estimate has two terms that need to be investigated next (see Lemma 3.3.5 and Lemma 3.3.6). Combining all bounds, finally, results in a bound for the off-diagonal term (see Lemma 3.3.7).

Recall the definition of the weights $u[\xi]$, $v[\xi]$ and $w[\xi]$ from the beginning of the foregoing section.

Theorem 3.3.1 (Cumulant bound) *Let $k \in \{3, 4, \ldots\}$, $\xi \in \Xi$ and $f \in \mathcal{B}(\mathbb{R}^d)$. Then, for sufficiently large λ, it holds that*

$$|\langle f_{R_\lambda}^k, c_\lambda^k \rangle| \leq \begin{cases} c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (k!)^{2d+i+5} & : \xi = \xi_{V_i}\,, i \in \{1, \ldots, d\} \\ c_3\, c_4^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (k!)^{d+5} & : \xi = \xi_{f_0} \\ c_5\, c_6^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (k!)^{2d+7} & : \xi = \xi_{f_1} \\ c_7\, c_8^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (k!)^{4d+7} & : \xi = \xi_{f_2} \\ c_9\, c_{10}^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (k!)^{6d+7} & : \xi = \xi_{f_3} \\ c_{11}\, c_{12}^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (k!)^{2j(d+1)} & : \xi = \xi_{f_j}\,, j \in \{4, \ldots, d-1\}, \end{cases}$$

where $c_1, \ldots, c_{12} \in (0, \infty)$ are constants only depending on d, ξ, α and β. In a unified form, this means that

$$|\langle f_{R_\lambda}^k, c_\lambda^k \rangle| \leq c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (k!)^{u[\xi]+2dv[\xi]+z[\xi]},$$

for all $\xi \in \Xi$, where

$$z[\xi] := \begin{cases} d+5 & : \xi = \xi_{f_0} \\ 5 & : \xi \in \{\xi_{V_1}, \ldots, \xi_{V_d}, \xi_{f_1}\} \\ 3 & : \xi = \xi_{f_2} \\ 1 & : \xi = \xi_{f_3} \\ 0 & : \xi \in \{\xi_{f_4}, \ldots \xi_{f_{d-1}}\}. \end{cases} \tag{3.62}$$

As anticipated above, we start by dealing with the diagonal term. From now on, we fix $k \in \{3, 4, \ldots\}$.

Lemma 3.3.2 *Let $\xi \in \Xi$ and $f \in \mathcal{B}(\mathbb{R}^d)$. Then, it holds that*

$$\left| \int_\Delta f_{R_\lambda}^k\, \mathrm{d}c_\lambda^k \right| \leq c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (k!)^{u[\xi]}\, ((kdv[\xi])!)^2,$$

for sufficiently large λ.

Proof. From the definition (3.30) of the cumulant measure, we obtain

$$\int_{\Delta} f_{R_\lambda}^k \,\mathrm{d}c_\lambda^k = \sum_{L_1,\dots,L_p \preceq [\![k]\!]} (-1)^{(p-1)}\,(p-1)! \int_{\Delta} f_{R_\lambda}^k \,\mathrm{d}(M_\lambda^{|L_1|}\cdots M_\lambda^{|L_p|}).$$

Now, property (3.29) of the moment measures implies

$$\int_{\Delta} f_{R_\lambda}^k \,\mathrm{d}c_\lambda^k = \sum_{L_1,\dots,L_p \preceq [\![k]\!]} (-1)^{(p-1)}\,(p-1)! \int_{\Delta} f_{R_\lambda}^k(\mathbf{x})\; m_\lambda(\mathbf{x}_{L_1})\cdots m_\lambda(\mathbf{x}_{L_p})\,\tilde{\mathrm{d}}[\lambda\phi_{\alpha,\beta}](\mathbf{x}),$$

with $m_\lambda(\mathbf{x}_{L_i})$ as before. Since we integrate over the diagonal Δ, the vector $\mathbf{x}$ is of the form $(x,\dots,x)$, for some $x \in \mathbb{R}^d$. Now, because there is just one way to partition such a vector, namely, into one complete block, we can only have $p=1$ in the above sum. Therefore, we get with the definition of the singular differentials in (3.27) that

$$\begin{aligned}
\left|\int_{\Delta} f_{R_\lambda}^k \,\mathrm{d}c_\lambda^k\right| &\le \int_{\Delta} |f_{R_\lambda}^k(\mathbf{x})|\;\; |m_\lambda(\mathbf{x}_{L_1})|\,\bar{\mathrm{d}}[\lambda\phi_{\alpha,\beta}](\mathbf{x}) \\
&\le ||f||_\infty^k \int_{\Delta} |m_\lambda(\mathbf{x}_{L_1})|\,\bar{\mathrm{d}}[\lambda\phi_{\alpha,\beta}](\mathbf{x}) \\
&= ||f||_\infty^k \int_{\mathbb{R}^d} |m_\lambda(x,\dots,x)|\,\lambda\,\phi_{\alpha,\beta}(x)\,\mathrm{d}x \\
&\le \|f\|_\infty^k \int_{\mathbb{R}^d} \mathbb{E}\big|\xi(x,\mathcal{P}_\lambda)\big|^k\,\lambda\,\phi_{\alpha,\beta}(x)\,\mathrm{d}x.
\end{aligned}$$

By rotational invariance of the Poisson point process $\mathcal{P}_\lambda$, we have that

$$\mathbb{E}\big|\xi(x,\mathcal{P}_\lambda)\big|^k = \mathbb{E}\big|\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})\big|^k,$$

where h is defined by $\|x\| = R_\lambda(1-h/R_\lambda^\beta)$ in view of the scaling transformation T_λ. Writing $u = x/\|x\|$, we can rewrite $\mathrm{d}x$ as

$$\mathrm{d}x = \left[R_\lambda\left(1-\frac{h}{R_\lambda^\beta}\right)\right]^{d-1} R_\lambda^{-(\beta-1)}\,\mathrm{d}h\,\mathcal{H}_{\mathbb{S}^{d-1}}^{d-1}(\mathrm{d}u).$$

For the latter step, we also refer to (3.12).

Since

$$d+\alpha-\beta+\frac{\beta(d+1)-2d-2\alpha}{2}=\frac{2d+2\alpha-2\beta+\beta d+\beta-2d-2\alpha}{2}=\frac{\beta(d-1)}{2},$$

the above integral is bounded by

$$\|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}} \int\limits_{\mathbb{S}^{d-1}} \int\limits_{-\infty}^{R_\lambda^\beta} \mathbb{E}\left|\xi^{(\lambda)}\left((\boldsymbol{o},h),\mathcal{P}^{(\lambda)}\right)\right|^k \phi_\lambda(u,h)\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),$$

with

$$\phi_\lambda(u,h):=\frac{(\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}}\exp\left(h-\frac{h^2}{2R_\lambda^\beta}(\beta-1)(1-C)^{\beta-2}\right), \tag{3.63}$$

where $C\in(-\infty,1)$, see also the proof of formula (3.10) for further details. For sufficiently large λ, $\phi_\lambda(u,h)$ is bounded from above by a constant times e^h, for all $h\in\mathbb{R}$ and $u\in\mathbb{S}^{d-1}$. Furthermore, from Theorem 3.2.12 (i), we deduce that

$$\mathbb{E}\left|\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})\right|^k\le c_1\,c_2^k\,(k!)^{u[\xi]}\,(kdv[\xi])!\,(1+|h|)^{k(d-1)v[\xi]+d}\exp\left(-\frac{e^{h\vee 0}}{c_3}\right).$$

Thus, the integral we started with is bounded by

$$\begin{aligned}c_1\,c_2^k\,\|f\|_\infty^k\,R_\lambda^{\frac{\beta(d-1)}{2}}\,(k!)^{u[\xi]}\,(kdv[\xi])! \int\limits_{\mathbb{S}^{d-1}}\int\limits_{-\infty}^{R_\lambda^\beta}(1+|h|)^{k(d-1)v[\xi]+d}\exp\left(-\frac{e^{h\vee 0}}{c_3}\right)\\ \times e^h\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).\end{aligned}$$

We decompose the inner integral into

$$\begin{aligned}&\int\limits_{-\infty}^{0}(1+|h|)^{k(d-1)v[\xi]+d}\exp\left(-\frac{e^{h\vee 0}}{c_1}\right)e^h\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}h\\ &\quad+\int\limits_{0}^{R_\lambda^\beta}(1+|h|)^{k(d-1)v[\xi]+d}\exp\left(-\frac{e^{h\vee 0}}{c_2}\right)e^h\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}h=:T_3+T_4,\end{aligned}$$

which will be treated separately.

By using that $R_\lambda^\beta \geq 1$, for sufficiently large λ, and the inequality

$$k(d-1)v[\xi] + d + (d-1+\alpha) \leq kdv[\xi] + 2d + \alpha,$$

we obtain that

$$\begin{aligned}
T_3 &= \int_{-\infty}^{0} (1+|h|)^{k(d-1)v[\xi]+d} \underbrace{\exp\left(-\frac{e^{h\vee 0}}{c_1}\right)}_{\leq 1} e^h \left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h \\
&\leq \int_{-\infty}^{0} (1+|h|)^{k(d-1)v[\xi]+d} e^h \left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h \\
&= \int_{0}^{\infty} (1+h)^{k(d-1)v[\xi]+d} e^{-h} \left(1+\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h \\
&\leq \int_{0}^{\infty} (1+h)^{kdv[\xi]+2d+\alpha} e^{-h} \,\mathrm{d}h \\
&\leq c_1 \left(kdv[\xi] + 2d + \lceil\alpha\rceil\right)! \\
&\leq c_1\, c_2^k \left(kdv[\xi]\right)!,
\end{aligned}$$

where we used (2.7) in the last step.
To bound the term T_4, we apply (2.5) with $n = 2$ and (2.7) to achieve that

$$\begin{aligned}
T_4 &= \int_{0}^{R_\lambda^\beta} (1+|h|)^{k(d-1)v[\xi]+d} \exp\left(-\frac{e^h}{c}\right) e^h \underbrace{\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}}_{\leq 1} \mathrm{d}h \\
&\leq \int_{0}^{R_\lambda^\beta} (1+h)^{kdv[\xi]+d} \frac{2c^2}{e^{2h}} e^h \,\mathrm{d}h \\
&\leq c_1 \int_{0}^{\infty} (1+h)^{kdv[\xi]+d} e^{-h} \,\mathrm{d}h \\
&\leq c_1 \left(kdv[\xi] + d\right)! \\
&\leq c_1\, c_2^k \left(kdv[\xi]\right)!.
\end{aligned}$$

Putting together the bounds for T_3 and T_4, and using (2.3), implies that

$$\begin{aligned}\left|\int\limits_{\Delta} f_{R_\lambda}^k \, \mathrm{d}c_\lambda^k\right| &\le c_1\, c_2^k \, \|f\|_\infty^k \, R_\lambda^{\frac{\beta(d-1)}{2}} \, (k)^{u[\xi]} \, (kdv[\xi])! \, (kdv[\xi])! \int\limits_{\mathbb{S}^{d-1}} \mathcal{H}_{\mathbb{S}^{d-1}}^{d-1}(\mathrm{d}u) \\ &\le c_1\, c_2^k \, \|f\|_\infty^k \, R_\lambda^{\frac{\beta(d-1)}{2}} \, (k!)^{u[\xi]} \, ((kdv[\xi])!)^2,\end{aligned}$$

and, thus, proves the claim. □

An upper bound for the off-diagonal term in (3.32) is derived along the following four lemmas.

Lemma 3.3.3 *Let $\xi \in \Xi$ and $f \in \mathcal{B}(\mathbb{R}^d)$. Then, we have that for sufficiently large λ,*

$$\begin{aligned}&\left|\sum_{S,T \preceq [\![k]\!]} \int\limits_{\sigma(\{S,T\})} f_{R_\lambda}^k \, \mathrm{d}c_\lambda^k\right| \\ &\le c_1\, c_2^k \, \|f\|_\infty^k \, R_\lambda^{\frac{\beta(d-1)}{2}} \, k\, k! \, (k!)^{u[\xi]} \, (kdv[\xi])! \\ &\times \sum_{L_1,\dots,L_p \preceq [\![k]\!]} \int\limits_{\mathbb{S}^{d-1}} \int\limits_{-\infty}^{R_\lambda^\beta} \dots \int\limits_{-\infty}^{R_\lambda^\beta} \int\limits_{(\mathbb{R}^{d-1})^{p-1}} \Big(\exp(-c_3\, \delta(\boldsymbol{o}, \mathbf{v})) + \mathbf{1}(\delta(\boldsymbol{o},\mathbf{v}) \le 2 \max_{i=1,\dots,p} \{|h_i|\})\Big) \\ &\times \prod_{i=1}^p \left[(1+|h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_4\, k}\right) e^{h_i} \left(1 - \frac{h_i}{R_\lambda^\beta}\right)^{d-1}\right] \mathrm{d}\mathbf{v} \mathrm{d}h_1 \dots \mathrm{d}h_p \, \mathcal{H}_{\mathbb{S}^{d-1}}^{d-1}(\mathrm{d}u),\end{aligned}$$

where $\mathbf{v} := (v_2, \dots, v_p)$.

Proof. The definition of the cluster measures and the description of the densities of the moment measures, explained in Section 3.1.5, imply that for a fixed and non-trivial partition $\{S,T\}$ of $\{1,\dots,k\}$ and for fixed $S', T', K_1, \dots, K_s$ as in Lemma 3.1.19, it holds that

$$\begin{aligned}&\int\limits_{\delta(\{S,T\})} f_{R_\lambda}^k \, \mathrm{d}(U_\lambda^{S',T'} \otimes M_\lambda^{|K_1|} \otimes \dots \otimes M_\lambda^{|K_s|}) \\ &= \int\limits_{\delta(\{S,T\})} f_{R_\lambda}^k(\mathbf{x}) \, (m_\lambda(\mathbf{x}_{S' \cup T'}) - m_\lambda(\mathbf{x}_{S'}) m_\lambda(\mathbf{x}_{T'})) \\ &\quad \times m_\lambda(\mathbf{x}_{K_1}) \dots m_\lambda(\mathbf{x}_{K_s}) \, \tilde{\mathrm{d}}[\lambda \phi_{\alpha,\beta}](\mathbf{x}).\end{aligned} \tag{3.64}$$

In what follows, we use the parametrization $T_\lambda(x_i) = (v_i, h_i)$, for all $i \in \{1, \dots, k\}$. Using this notation, Lemma 3.2.15 shows that

$$
\begin{aligned}
&|m_\lambda(\mathbf{x}_{S' \cup T'}) - m_\lambda(\mathbf{x}_{S'}) m_\lambda(\mathbf{x}_{T'})| \\
&\quad \le c_1\, c_2^k\, k\, (|S^{'} \cup T^{'}|!)^{u[\xi]}\, (|S^{'} \cup T^{'}| dv[\xi])! \\
&\qquad \times \left(\exp(-c_3\, d(\mathbf{v}_{S'}, \mathbf{v}_{T'})) + \mathbf{1}(d(\mathbf{v}_{S'}, \mathbf{v}_{T'}) \le 2 \max_{r \in S' \cup T'} \{|h_r|\}) \right) \\
&\qquad \times \prod_{r \in S' \cup T'} \left[(1 + |h_r|)^{dw[\xi]} \exp\left(-\frac{e^{h_r \vee 0}}{c_4\, k} \right) \right].
\end{aligned}
$$

Furthermore, for all $i \in \{1, \dots, s\}$, Theorem 3.2.12 (ii) delivers the bound

$$
|m_\lambda(\mathbf{x}_{K_i})| \le c_1\, c_2^k\, (|K_i|!)^{u[\xi]}\, (|K_i| dv[\xi])! \prod_{i \in K_i} \left[(1 + |h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_3\, k} \right) \right],
$$

since $|K_i| \le k$. Now, we notice that $d(\mathbf{v}_{S'}, \mathbf{v}_{T'}) \ge d(\mathbf{v}_S, \mathbf{v}_T)$. Together with the observation that

$$
\max_{r \in S' \cup T'} \{|h_r|\} \le \max_{i = 1, \dots, k} \{|h_i|\},
$$

this yields that

$$
\begin{aligned}
&\exp(-c\, d(\mathbf{v}_{S'}, \mathbf{v}_{T'})) + \mathbf{1}(d(\mathbf{v}_{S'}, \mathbf{v}_{T'}) \le 2 \max_{r \in S' \cup T'} \{|h_r|\}) \\
&\quad \le \exp(-c\, d(\mathbf{v}_S, \mathbf{v}_T)) + \mathbf{1}(d(\mathbf{v}_S, \mathbf{v}_T) \le 2 \max_{i = 1, \dots, k} \{|h_i|\}),
\end{aligned}
$$

which in turn implies that

$$
\begin{aligned}
&|\, (m_\lambda(\mathbf{x}_{S' \cup T'}) - m_\lambda(\mathbf{x}_{S'}) m_\lambda(\mathbf{x}_{T'}))\, m_\lambda(\mathbf{x}_{K_1}) \cdots m_\lambda(\mathbf{x}_{K_s})| \\
&\le c_1\, c_2^k\, k \left[(|S^{'} \cup T^{'}|)!\, (|K_1|)! \cdots (|K_s|)! \right]^{u[\xi]} (|S^{'} \cup T^{'}| dv[\xi])! (|K_1| dv[\xi])! \cdots (|K_s| dv[\xi])! \\
&\quad \times \left(\exp(-c_3\, d(\mathbf{v}_S, \mathbf{v}_T)) + \mathbf{1}(d(\mathbf{v}_S, \mathbf{v}_T) \le 2 \max_{i = 1, \dots, k} \{|h_i|\}) \right) \\
&\quad \times \prod_{i=1}^{k} \left[(1 + |h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_4\, k} \right) \right].
\end{aligned}
$$

Now, we use the estimate (2.8) to achieve that

$$(|S^{'} \cup T^{'}|dv[\xi])!\,(|K_1|dv[\xi])! \cdots (|K_s|dv[\xi])! \leq (kdv[\xi])!,$$

and

$$(|S^{'} \cup T^{'}|)!\,(|K_1|)! \cdots (|K_s|)! \leq (k)!.$$

Thus, it follows that

$$\begin{aligned} &|\,(m_\lambda(\mathbf{x}_{S'\cup T'}) - m_\lambda(\mathbf{x}_{S'})m_\lambda(\mathbf{x}_{T'}))\,m_\lambda(\mathbf{x}_{K_1})\cdots m_\lambda(\mathbf{x}_{K_s})| \\ &\quad \leq c_1\,c_2^k\,k\,(k!)^{u[\xi]}\,(kdv[\xi])!\left(\exp(-c_3\,d(\mathbf{v}_S,\mathbf{v}_T)) + \mathbf{1}(d(\mathbf{v}_S,\mathbf{v}_T) \leq 2\max_{i=1,\dots,k}\{|h_i|\})\right) \\ &\qquad \times \prod_{i=1}^{k}\left[(1+|h_i|)^{dw[\xi]}\exp\left(-\frac{e^{h_i\vee 0}}{c_4\,k}\right)\right]. \end{aligned}$$

Recalling Lemma 3.1.19 and (3.33), summing (3.64) over all $S', T', K_1, \dots, K_s \preceq [\![k]\!]$ and observing that $d(\mathbf{v}_S, \mathbf{v}_T) = \delta(\mathbf{x})$, whenever we are integrating over $\delta(\{S,T\})$, we get that

$$\begin{aligned} \left|\int_{\delta(\{S,T\})} f_{R_\lambda}^k\,\mathrm{d}c_\lambda^k\right| &\leq c_1\,c_2^k\,\|f\|_\infty^k\,k\,k!\,(k!)^{u[\xi]}\,(kdv[\xi])! \\ &\quad \times \int_{\delta(\{S,T\})}\left(\exp(-c_3\,\delta(\mathbf{x})) + \mathbf{1}(\delta(\mathbf{x}) \leq 2\max_{i=1,\dots,k}\{|h_i|\})\right) \\ &\quad \times \prod_{i=1}^{k}\left[(1+|h_i|)^{dw[\xi]}\exp\left(-\frac{e^{h_i\vee 0}}{c_4\,k}\right)\right]\tilde{\mathrm{d}}[\lambda\phi_{\alpha,\beta}](\mathbf{x}). \end{aligned}$$

Finally, this leads to

$$\begin{aligned} \left|\sum_{S,T\preceq[\![k]\!]}\int_{\delta(\{S,T\})} f_{R_\lambda}^k\,\mathrm{d}c_\lambda^k\right| &\leq c_1\,c_2^k\,\|f\|_\infty^k\,k\,k!\,(k!)^{u[\xi]}\,(kdv[\xi])! \\ &\quad \times \int_{(\mathbb{R}^d)^k}\left(\exp(-c_3\,\delta(\mathbf{x})) + \mathbf{1}(\delta(\mathbf{x}) \leq 2\max_{i=1,\dots,k}\{|h_i|\})\right) \\ &\quad \times \prod_{i=1}^{k}\left[(1+|h_i|)^{dw[\xi]}\exp\left(-\frac{e^{h_i\vee 0}}{c_4\,k}\right)\right]\tilde{\mathrm{d}}[\lambda\phi_{\alpha,\beta}](\mathbf{x}). \end{aligned}$$

In the next step, a bound for the integral over $(\mathbb{R}^d)^k$ is derived. We can assume without loss of generality that, after a suitable rotation of the underlying point process, the point x_1 is mapped to $(\boldsymbol{o}, h_1) \in W_\lambda$ under the scaling transformation T_λ. Here, the height coordinate h_1 is determined by $\|x_1\| = R_\lambda(1 - h_1/R_\lambda^\beta)$, as in the previous lemma. Together with the definition of the singular differential $\tilde{\mathrm{d}}[\lambda\phi_{\alpha,\beta}](\mathbf{x})$, we conclude that

$$
\begin{aligned}
&\int\limits_{(\mathbb{R}^d)^k} \left(\exp(-c_1\,\delta(\mathbf{x})) + \mathbf{1}(\delta(\mathbf{x}) \leq 2 \max_{i=1,\dots,k}\{|h_i|\})\right) \\
&\qquad \times \prod_{i=1}^{k} \left[(1+|h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_2\,k}\right)\right] \tilde{\mathrm{d}}[\lambda\phi_{\alpha,\beta}](\mathbf{x}) \\
&= \sum_{L_1,\dots,L_p \preceq [\![k]\!]} \int\limits_{(\mathbb{R}^d)^k} \left(\exp(-c_1\,\delta(\mathbf{x})) + \mathbf{1}(\delta(\mathbf{x}) \leq 2 \max_{i=1,\dots,k}\{|h_i|\})\right) \\
&\qquad \times \prod_{i=1}^{k} \left[(1+|h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_2\,k}\right)\right] \bar{\mathrm{d}}[\lambda\phi_{\alpha,\beta}](\mathbf{x}_{L_1}) \dots \bar{\mathrm{d}}[\lambda\phi_{\alpha,\beta}](\mathbf{x}_{L_p}) \\
&= \sum_{L_1,\dots,L_p \preceq [\![k]\!]} \lambda^p \int\limits_{(\mathbb{R}^d)^p} \left(\exp(-c_1\,\delta(\boldsymbol{o}, v_2, \dots, v_p)) + \mathbf{1}(\delta(\boldsymbol{o}, v_2, \dots, v_p) \leq 2 \max_{i=1,\dots,p}\{|h_i|\})\right) \\
&\qquad \times \prod_{i=1}^{p} \left[(1+|h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_2\,k}\right)\right] \phi_{\alpha,\beta}(x_1) \dots \phi_{\alpha,\beta}(x_p)\,\mathrm{d}x_1 \dots \mathrm{d}x_p.
\end{aligned}
$$

Now, we re-parameterize as in the proof of Lemma 3.3.2 and notice that the differential elements transform into

$$
\lambda\,\phi_{\alpha,\beta}(x_1)\,\mathrm{d}x_1 = R_\lambda^{\frac{\beta(d-1)}{2}}\,\phi_\lambda(u, h_1) \left(1 - \frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h_1\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),
$$

and, for $i \in \{2, \dots, p\}$,

$$
\begin{aligned}
\lambda\,\phi_{\alpha,\beta}(x_i)\,\mathrm{d}x_i &= \frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}}\|v_i\|)}{\|R_\lambda^{-\frac{\beta}{2}} v_i\|^{d-2}}\, \frac{(\beta \log \lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}} \\
&\qquad \times \exp\left(h_i - \frac{h_i^2}{2R_\lambda^\beta}(\beta-1)(1-C_i)^{\beta-2}\right) \left(1 - \frac{h_i}{R_\lambda^\beta}\right)^{d-1-\alpha} \mathrm{d}v_i \mathrm{d}h_i \\
&= \frac{\sin^{d-2}(R_\lambda^{-\beta/2}\|v_i\|)}{\|R_\lambda^{-\beta/2} v_i\|^{d-2}}\,\phi_\lambda(u, h_i) \left(1 - \frac{h_i}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}v_i \mathrm{d}h_i,
\end{aligned}
$$

where $C_i \in (-\infty, 1)$ and $\phi_\lambda(u, h_i)$ is defined at (3.63).

The fractions involving the sine term are bounded from above by 1. Moreover, we have that $\phi_\lambda(u,h_i) \le c\, e^{h_i}$, for all $i \in \{1,\dots,p\}$ and sufficiently large λ, as in the proof of Lemma 3.3.2. This implies that

$$\begin{aligned}
&\lambda^p\, \phi_{\alpha,\beta}(x_1)\dots\phi_{\alpha,\beta}(x_p)\,\mathrm{d}x_1\dots\mathrm{d}x_p \\
&\qquad \le c^p\, R_\lambda^{\frac{\beta(d-1)}{2}} \prod_{i=1}^{p}\left[e^{h_i}\left(1-\frac{h_i}{R_\lambda^\beta}\right)^{d-1+\alpha}\right]\mathrm{d}h_1\dots\mathrm{d}h_p\mathrm{d}v_2\dots\mathrm{d}v_p\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),
\end{aligned}$$

and, therefore,

$$\begin{aligned}
&\lambda^p \int\limits_{(\mathbb{R}^d)^p}\Big(\exp(-c_1\,\delta(\boldsymbol{o},v_2,\dots,v_p)) + \mathbf{1}(\delta(\boldsymbol{o},v_2,\dots,v_p)\le 2\max_{i=1,\dots,p}\{|h_i|\})\Big)\\
&\quad\times\prod_{i=1}^{p}\left[(1+|h_i|)^{dw[\xi]}\exp\left(-\frac{e^{h_i\vee 0}}{c_2\,k}\right)\right]\phi_{\alpha,\beta}(x_1)\dots\phi_{\alpha,\beta}(x_p)\,\mathrm{d}x_1\dots\mathrm{d}x_p\\
&\le c_1^p\, R_\lambda^{\frac{\beta(d-1)}{2}}\int\limits_{\mathbb{S}^{d-1}}\int\limits_{-\infty}^{R_\lambda^\beta}\dots\int\limits_{-\infty}^{R_\lambda^\beta}\int\limits_{T_\lambda(\mathbb{S}^{d-1})}\dots\int\limits_{T_\lambda(\mathbb{S}^{d-1})}\\
&\quad\times\Big(\exp(-c_2\,\delta(\boldsymbol{o},v_2,\dots,v_p)) + \mathbf{1}(\delta(\boldsymbol{o},v_2,\dots,v_p)\le 2\max_{i=1,\dots,p}\{|h_i|\})\Big)\\
&\quad\times\prod_{i=1}^{p}\left[(1+|h_i|)^{dw[\xi]}\exp\left(-\frac{e^{h_i\vee 0}}{c_3\,k}\right)e^{h_i}\left(1-\frac{h_i}{R_\lambda^\beta}\right)^{d-1+\alpha}\right]\\
&\quad\times\mathrm{d}v_2\dots\mathrm{d}v_p\mathrm{d}h_1\dots\mathrm{d}h_p\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)\\
&\le c_1^p\, R_\lambda^{\frac{\beta(d-1)}{2}}\int\limits_{\mathbb{S}^{d-1}}\int\limits_{-\infty}^{R_\lambda^\beta}\dots\int\limits_{-\infty}^{R_\lambda^\beta}\int\limits_{(\mathbb{R}^d)^{p-1}}\Big(\exp(-c_2\,\delta(\boldsymbol{o},\mathbf{v})) + \mathbf{1}(\delta(\boldsymbol{o},\mathbf{v})\le 2\max_{i=1,\dots,p}\{|h_i|\})\Big)\\
&\quad\times\prod_{i=1}^{p}\left[(1+|h_i|)^{dw[\xi]}\exp\left(-\frac{e^{h_i\vee 0}}{c_3\,k}\right)e^{h_i}\left(1-\frac{h_i}{R_\lambda^\beta}\right)^{d-1+\alpha}\right]\mathrm{d}\mathbf{v}\mathrm{d}h_1\dots\mathrm{d}h_p\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),
\end{aligned}$$

with $\mathbf{v} := (v_2,\dots,v_p)$. This yields the desired result. □

The previous lemma shows that we already have separated the crucial factor $R_\lambda^{\frac{\beta(d-1)}{2}}$. In the next steps, we appropriately bound the remaining integrals. We start with the inner integral concerning the integration with respect to the vector $\mathbf{v}$.

Lemma 3.3.4 *In the situation of Lemma 3.3.3, it holds that*

$$\int_{(\mathbb{R}^d)^{p-1}} \left(\exp(-c_1\,\delta(\boldsymbol{o},\mathbf{v})) + \mathbf{1}(\delta(\boldsymbol{o},\mathbf{v}) \leq 2 \max_{i=1,\dots,p}\{|h_i|\})\right) \mathrm{d}\mathbf{v}$$
$$\leq c_2\, c_3^p\, p^{p-2} \left((dp)! + \left(\max_{i=1,\dots,p}\{|h_i|\}\right)^{(d-1)(p-1)}\right).$$

Proof. We divide the proof into two parts. First, we consider the integral regarding to the exponential function. By using that

$$\int_{\delta(\boldsymbol{o},\mathbf{v})}^{\infty} \exp(-c\,t)\,\mathrm{d}t = \frac{1}{c}\exp(-c\,\delta(\boldsymbol{o},\mathbf{v})),$$

together with Fubini‘s theorem, we obtain

$$\begin{aligned}\int_{(\mathbb{R}^d)^{p-1}} \exp(-c\,\delta(\boldsymbol{o},\mathbf{v}))\,\mathrm{d}\mathbf{v} &= c \int_{(\mathbb{R}^d)^{p-1}} \int_{\delta(\boldsymbol{o},\mathbf{v})}^{\infty} \exp(-c\,t)\,\mathrm{d}t\mathrm{d}\mathbf{v} \\ &= c\int_0^{\infty} \int_{\{\delta(\boldsymbol{o},\mathbf{v})<t\}} \mathrm{d}\mathbf{v}\exp(-c\,t)\,\mathrm{d}t.\end{aligned}$$

Now, suppose that $\delta(\boldsymbol{o},\mathbf{v}) < t$. Then, there is no partition $\{S,T\}$ of $\{\boldsymbol{o},\mathbf{v}\}$ such that the corresponding separation $d(\mathbf{v}_S,\mathbf{v}_T)$ is bigger than t. This implies that there exists a tree $\mathcal{T}$ on $\{1,\dots,p\}$ such that all adjacent vertices v_i, v_j in $\mathcal{T}$ satisfy $\|v_i - v_j\| < t$. We indicate this property by writing $(\boldsymbol{o},\mathbf{v}) \prec (t,\mathcal{T})$. Thus, we have that

$$\begin{aligned}\int_{\{\delta(\boldsymbol{o},\mathbf{v})<t\}} \mathrm{d}\mathbf{v} &\leq \sum_{\mathcal{T}} \int_{(\boldsymbol{o},\mathbf{v})\prec(t,\mathcal{T})} \mathrm{d}\mathbf{v} \\ &= \sum_{\mathcal{T}} \mathrm{vol}_{(p-1)(d-1)}\left(\left\{\mathbf{v} \in (\mathbb{R}^{d-1})^{p-1}\right\} : (\boldsymbol{o},\mathbf{v}) \prec (t,\mathcal{T})\right),\end{aligned}$$

where the sum ranges over all trees $\mathcal{T}$ on the edges $\{1,\dots,p\}$. By the geometry of these trees, it follows that

$$\mathrm{vol}_{(p-1)(d-1)}\left(\left\{\mathbf{v} \in (\mathbb{R}^{d-1})^{p-1}\right\} : (\boldsymbol{o},\mathbf{v}) \prec (t,\mathcal{T})\right) \leq \left(t^{d-1}\kappa_{d-1}\right)^{p-1} = t^{(d-1)(p-1)}\kappa_{d-1}^{p-1}.$$

Moreover, with Caley's theorem [4, Page 201] there are exactly p^{p-2} trees on $\{1,\dots,p\}$.

Thus,

$$\int\limits_{\{\delta(\boldsymbol{o},\mathbf{v})<t\}} \mathrm{d}\mathbf{v} \le p^{p-2}\,\kappa_{d-1}^{p-1}\,t^{(d-1)(p-1)}. \tag{3.65}$$

This leads to

$$\begin{aligned}\int\limits_{(\mathbb{R}^d)^{p-1}} \exp(-c_1\,\delta(\boldsymbol{o},\mathbf{v}))\,\mathrm{d}\mathbf{v} &\le c_2\,p^{p-2}\,\kappa_{d-1}^{p-1}\int\limits_0^\infty t^{(d-1)(p-1)}\exp(-c_3\,t)\,\mathrm{d}t\\ &\le c_1\,c_2^p\,\kappa_{d-1}^{p-1}\,p^{p-2}\,(dp)!.\end{aligned}$$

For the second part of the integral, again by (3.65), we have

$$\begin{aligned}\int\limits_{(\mathbb{R}^d)^{p-1}} \mathbf{1}(\delta(\boldsymbol{o},\mathbf{v})\le 2\max_{i=1,\dots,p}\{|h_i|\})\,\mathrm{d}\mathbf{v} &= \int\limits_{\{\delta(\boldsymbol{o},\mathbf{v})\le 2\max_{i=1,\dots,p}\{|h_i|\}\}} \mathrm{d}\mathbf{v}\\ &\le p^{p-2}\,\kappa_{d-1}^{p-1}\left(2\max_{i=1,\dots,p}\{|h_i|\}\right)^{(d-1)(p-1)}\\ &\le c_1^p\,p^{p-2}\,\kappa_{d-1}^{p-1}\left(\max_{i=1,\dots,p}\{|h_i|\}\right)^{(d-1)(p-1)}.\end{aligned}$$

Combining both estimates gives the result. □

The two last lemmas show that we are left with the bound

$$\left|\sum_{S,T\preceq[\![k]\!]}\ \int\limits_{\delta(\{S,T\})} f_{R_\lambda}^k\ \mathrm{d}c_\lambda^k\right| \le T_5+T_6, \tag{3.66}$$

where the terms T_5 and T_6 are given by

$$\begin{aligned}T_5 :=& c_1\,c_2^k\,\|f\|_\infty^k\,R_\lambda^{\frac{\beta(d-1)}{2}}\,k\,k!\,(dk)!\,(k!)^{u[\xi]}\,(kdv[\xi])!\,k^{k-2}\sum_{L_1,\dots,L_p\preceq[\![k]\!]}\ \int\limits_{\mathbb{S}^{d-1}}\int\limits_{-\infty}^{R_\lambda^\beta}\cdots\int\limits_{-\infty}^{R_\lambda^\beta}\\ &\times\prod_{i=1}^p\left[(1+|h_i|)^{dw[\xi]}\exp\left(-\frac{e^{h_i\vee 0}}{c_3\ k}\right)e^{h_i}\left(1-\frac{h_i}{R_\lambda^\beta}\right)^{d-1+\alpha}\right]\mathrm{d}h_1\dots\mathrm{d}h_p\,\mathcal{H}_{\mathbb{S}^{d-1}}^{d-1}(\mathrm{d}u),\end{aligned}$$

and

$$
\begin{aligned}
T_6 :=& c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, k\, k!\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{k-2} \\
&\times \sum_{L_1,\dots,L_p \preceq [\![k]\!]} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} \dots \int_{-\infty}^{R_\lambda^\beta} \left(\max_{i=1,\dots,p} \{|h_i|\} \right)^{(d-1)(p-1)} \\
&\times \prod_{i=1}^{p} \left[(1+|h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_3\, k}\right) e^{h_i} \left(1-\frac{h_i}{R_\lambda^\beta}\right)^{d-1+\alpha} \right] \mathrm{d}h_1 \dots \mathrm{d}h_p\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),
\end{aligned}
$$

respectively. Here, we used in several places that $p \leq k$.

Lemma 3.3.5 *For T_5, we have that*

$$
|T_5| \leq c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (dk)!\, (k!)^2\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{3k}.
$$

Proof. We start with the integral concerning the coordinate h_1. Similarly to the computations performed in the proof of Lemma 3.3.2, and by using (2.5) with $n=2$, we see that

$$
\begin{aligned}
&\int_{-\infty}^{R_\lambda^\beta} (1+|h_1|)^{dw[\xi]}\, \exp\left(-\frac{e^{h_1 \vee 0}}{c_1\, k}\right) e^{h_1} \left(1-\frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h_1 \\
&\quad = \int_{-\infty}^{0} (1+|h_1|)^{dw[\xi]}\, \underbrace{\exp\left(-\frac{e^{h_1 \vee 0}}{c_1\, k}\right)}_{\leq 1} e^{h_1} \left(1-\frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h_1 \\
&\qquad + \int_{0}^{R_\lambda^\beta} (1+|h_1|)^{dw[\xi]} \exp\left(-\frac{e^{h_1 \vee 0}}{c_2\, k}\right) e^{h_1} \underbrace{\left(1-\frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha}}_{\leq 1} \mathrm{d}h_1 \\
&\quad \leq \int_{-\infty}^{0} (1+|h_1|)^{dw[\xi]}\, e^{h_1} \left(1-\frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h_1 \\
&\qquad + \int_{0}^{R_\lambda^\beta} (1+h_1)^{dw[\xi]}\, \exp\left(-\frac{e^{h_1}}{c_1\, k}\right) e^{h_1} \mathrm{d}h_1 \\
&\quad \leq \int_{0}^{\infty} (1+h_1)^{dw[\xi]+d-1+\alpha}\, e^{-h_1} \mathrm{d}h_1 + \int_{0}^{\infty} (1+h_1)^{dw[\xi]}\, \frac{2\, c_1^2\, k^2}{e^{2h_1}}\, e^{h_1} \mathrm{d}h_1
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty (1+h_1)^{dw[\xi]+d-1+\alpha}\, e^{-h_1} \mathrm{d}h_1 + c_1\, k^2 \int_0^\infty (1+h_1)^{dw[\xi]}\, e^{-h_1} \mathrm{d}h_1 \\
&\leq c_1\, (dw[\xi] + d + \lceil \alpha \rceil)! + c_2\, k^2\, (dw[\xi]+1)! \\
&= c_1 + c_2\, k^2 \leq c_3\, k^2.
\end{aligned}
$$

Now, we have $p-1$ further height coordinates $h_2, \ldots, h_p$. The last computation shows that, up to constants, we get an additional factor k^2 for each of these integrals. Thus, the integration with respect to all height coordinates is bounded by a constant times k^{2p}. In view of the definition of T_5 and by (3.54), this leads to

$$
\begin{aligned}
|T_5| &\leq c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, k\, k!\, (dk)!\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{k-2}\, k^{2p} \sum_{L_1,\ldots,L_p \preceq [\![k]\!]} 1 \\
&\leq c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (dk)!\, (k!)^2\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{k-1}\, k^{2k} \\
&\leq c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (dk)!\, (k!)^2\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{3k}.
\end{aligned}
$$

This completes the proof. □

Lemma 3.3.6 *For T_6, we have that*

$$
|T_6| \leq c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (dk)!\, (k!)^2\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{3k}.
$$

Proof. As in the proof of Lemma 3.3.5, we start with the integral with respect to the variable h_1. Putting $a := \max\{|h_2|, \ldots, |h_p|\}$, we can rewrite this integral as

$$
\begin{aligned}
&\int_{-\infty}^{R_\lambda^\beta} (1+|h_1|)^{dw[\xi]} \exp\left(-\frac{e^{h_1 \vee 0}}{c_1\, k}\right) e^{h_1} \left(1 - \frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha} (\max\{|h_1|, \ldots, |h_p|\})^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&= \int_{-\infty}^{0} (1+|h_1|)^{dw[\xi]} \underbrace{\exp\left(-\frac{e^{h_1 \vee 0}}{c_1\, k}\right)}_{\leq 1} e^{h_1} \left(1 - \frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha} (\max\{|h_1|, a\})^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&\quad + \int_{0}^{R_\lambda^\beta} (1+|h_1|)^{dw[\xi]} \exp\left(-\frac{e^{h_1 \vee 0}}{c_2\, k}\right) e^{h_1} \underbrace{\left(1 - \frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha}}_{\leq 1} (\max\{|h_1|, a\})^{(d-1)(p-1)}\, \mathrm{d}h_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{0} (1+|h_1|)^{dw[\xi]}\, e^{h_1} \left(1 - \frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha} (\max\{|h_1|\,, a\})^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&\quad + \int_{0}^{R_\lambda^\beta} (1+h_1)^{dw[\xi]} \exp\left(-\frac{e^{h_1}}{c_1\, k}\right) e^{h_1} (\max\{|h_1|\,, a\})^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&= \int_{0}^{\infty} (1+h_1)^{dw[\xi]+d-1+\alpha}\, e^{-h_1} (\max\{|h_1|\,, a\})^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&\quad + \int_{0}^{\infty} (1+h_1)^{dw[\xi]} \exp\left(-\frac{e^{h_1}}{c_1\, k}\right) e^{h_1} (\max\{|h_1|\,, a\})^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&=: T_7 + T_8.
\end{aligned}
$$

Both T_7 and T_8 will be treated separately. Since

$$
dw[\xi] + (d-1) + \alpha + (d-1)(p-1) = dw[\xi] + (d-1)p + \alpha \leq dw[\xi] + dk + \alpha,
$$

we obtain, together with (2.7), that

$$
\begin{aligned}
T_7 &= \int_{0}^{\infty} (1+h_1)^{dw[\xi]+d-1+\alpha}\, e^{-h_1} (\max\{|h_1|\,, a\})^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&= \int_{0}^{a} (1+h_1)^{dw[\xi]+d-1+\alpha}\, e^{-h_1} a^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&\quad + \int_{a}^{\infty} (1+h_1)^{dw[\xi]+d-1+\alpha}\, e^{-h_1} h_1^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&\leq a^{(d-1)(p-1)} \int_{0}^{a} (1+h_1)^{dw[\xi]+d-1+\alpha}\, e^{-h_1} \mathrm{d}h_1 \\
&\quad + \int_{a}^{\infty} (1+h_1)^{dw[\xi]+d-1+\alpha}\, e^{-h_1} (1+h_1)^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&\leq a^{(d-1)(p-1)} \int_{0}^{\infty} (1+h_1)^{dw[\xi]+d-1+\alpha}\, e^{-h_1} \mathrm{d}h_1 + \int_{0}^{\infty} (1+h_1)^{dw[\xi]+dk+\alpha}\, e^{-h_1} \mathrm{d}h_1 \\
&\leq c_1\, (dw[\xi] + d + \lceil\alpha\rceil)!\, a^{(d-1)(p-1)} + c_2\, (dw[\xi] + dk + \lceil\alpha\rceil)! \\
&\leq c_1\, a^{(d-1)(p-1)} + c_2\, c_3^k\, (dk)!.
\end{aligned}
$$

Moreover, applying (2.5) with $n = 2$ in the second, the estimate

$$dw[\xi] + (d-1)(p-1) \leq dw[\xi] + dk$$

in the fourth and (2.7) in the last step, we conclude that

$$\begin{aligned}
T_8 &= \int_0^\infty (1+h_1)^{dw[\xi]} \exp\left(-\frac{e^{h_1}}{c_1\,k}\right) e^{h_1} \left(\max\{|h_1|\,, a\}\right)^{(d-1)(p-1)} \mathrm{d}h_1 \\
&\leq c_1\,k^2 \int_0^\infty (1+h_1)^{dw[\xi]}\, e^{-h_1} \left(\max\{|h_1|\,, a\}\right)^{(d-1)(p-1)} \mathrm{d}h_1 \\
&= c_1\,k^2 \int_0^a (1+h_1)^{dw[\xi]}\, e^{-h_1} a^{(d-1)(p-1)}\, \mathrm{d}h_1 + c_2\,k^2 \int_a^\infty (1+h_1)^{dw[\xi]}\, e^{-h_1} h_1^{(d-1)(p-1)}\, \mathrm{d}h_1 \\
&\leq c_1\,k^2\, a^{(d-1)(p-1)} \int_0^\infty (1+h_1)^{dw[\xi]}\, e^{-h_1} \mathrm{d}h_1 + c_2\,k^2 \int_0^\infty (1+h_1)^{dw[\xi]+dk}\, e^{-h_1} \mathrm{d}h_1 \\
&\leq c_1\,k^2\,(dw[\xi])!\, a^{(d-1)(p-1)} + c_2\,k^2\,(dw[\xi]+dk)! \\
&\leq c_1\,k^2\,a^{(d-1)(p-1)} + c_2\,c_3^k\,k^2\,(dk)!.
\end{aligned}$$

Combining the two bounds for T_7 and T_8, we arrive at

$$\begin{aligned}
&\int_{-\infty}^{R_\lambda^\beta} (1+|h_1|)^{dw[\xi]} \exp\left(-\frac{e^{h_1 \vee 0}}{c_1\,k}\right) e^{h_1} \left(1 - \frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha} \left(\max\{|h_1|\,, \ldots, |h_p|\}\right)^{(d-1)(p-1)} \mathrm{d}h_1 \\
&\qquad \leq c_1\,a^{(d-1)(p-1)} + c_2\,c_3^k\,(dk)! + c_4\,k^2\,a^{(d-1)(p-1)} + c_5\,c_6^k\,k^2\,(dk)! \\
&\qquad \leq c_1\,k^2\,\left(\max\{|h_2|, \ldots, |h_p|\}\right)^{(d-1)(p-1)} + c_2\,c_3^k\,k^2\,(dk)!.
\end{aligned}$$

For the term T_6, this implies that

$$\begin{aligned}
|T_6| &\leq c_1\,c_2^k\,\|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, k!\,(dk)!\,(k!)^{u[\xi]}\,(kdv[\xi])!\,k^{k-1}\,k^2 \sum_{L_1,\ldots,L_p \preceq [\![k]\!]} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} \cdots \int_{-\infty}^{R_\lambda^\beta} \\
&\quad \times \prod_{i=2}^p \left[(1+|h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_3\,k}\right) e^{h_i} \left(1 - \frac{h_i}{R_\lambda^\beta}\right)^{d-1+\alpha}\right] \mathrm{d}h_2 \ldots \mathrm{d}h_p\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&\quad + c_4\,c_5^k\,\|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, k!\,(k!)^{u[\xi]}\,(kdv[\xi])!\,k^{k-1}\,k^2
\end{aligned}$$

$$\times \sum_{L_1,\ldots,L_p \preceq [\![k]\!]} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} \cdots \int_{-\infty}^{R_\lambda^\beta} \left(\max_{i=2,\ldots,p} \{|h_i|\} \right)^{(d-1)(p-1)}$$
$$\times \prod_{i=2}^{p} \left[(1+|h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_6\, k} \right) e^{h_i} \left(1 - \frac{h_i}{R_\lambda^\beta} \right)^{d-1+\alpha} \right] \mathrm{d}h_2 \ldots \mathrm{d}h_p \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).$$

The first summand is almost the same as in Lemma 3.3.5. The only difference is that there are $p-1$ further integrals concerning the height coordinates $h_2, \ldots, h_p$ left. Carrying out the integrations as above, this yields another factor $k^{2(p-1)}$, up to constants. Furthermore, by computations similarly to those performed above, we have that

$$\int_{-\infty}^{R_\lambda^\beta} (1+|h_2|)^{dw[\xi]} \exp\left(-\frac{e^{h_2 \vee 0}}{c_1\, k} \right) e^{h_2} \left(1 - \frac{h_2}{R_\lambda^\beta} \right)^{d-1+\alpha} (\max\{|h_2|, \ldots, |h_p|\})^{(d-1)(p-1)} \, \mathrm{d}h_2$$
$$\leq c_1\, k^2 \, (\max\{|h_3|, \ldots, |h_p|\})^{(d-1)(p-1)} + c_2\, c_3^k\, k^2\, (dk)!,$$

giving that $|T_6|$ can be upper bounded by

$$c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (dk)!\, (k!)^2\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{3k}$$
$$+ c_3\, c_4^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, k!\, (dk)!\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{k-1}\, k^4 \sum_{L_1,\ldots,L_p \preceq [\![k]\!]} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} \cdots \int_{-\infty}^{R_\lambda^\beta}$$
$$\times \prod_{i=3}^{p} \left[(1+|h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_5\, k} \right) e^{h_i} \left(1 - \frac{h_i}{R_\lambda^\beta} \right)^{d-1+\alpha} \right] \mathrm{d}h_3 \ldots \mathrm{d}h_p \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$
$$+ c_6\, c_7^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, k!\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{k-1}\, k^4$$
$$\times \sum_{L_1,\ldots,L_p \preceq [\![k]\!]} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} \cdots \int_{-\infty}^{R_\lambda^\beta} \left(\max_{i=3,\ldots,p} \{|h_i|\} \right)^{(d-1)(p-1)}$$
$$\times \prod_{i=3}^{p} \left[(1+|h_i|)^{dw[\xi]} \exp\left(-\frac{e^{h_i \vee 0}}{c_8\, k} \right) e^{h_i} \left(1 - \frac{h_i}{R_\lambda^\beta} \right)^{d-1+\alpha} \right] \mathrm{d}h_3 \ldots \mathrm{d}h_p \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).$$

Repeating this procedure $p-2$ further times yields that $|T_6|$ is bounded by

$$
\begin{aligned}
& c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (dk)!\, (k!)^2\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{3k} \\
&\quad + c_3\, c_4^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, k!\, (dk)!\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{k-1}\, k^{2(p-1)} \sum_{L_1,\dots,L_p \preceq [\![k]\!]} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} \\
&\qquad \times (1+|h_p|)^{dw[\xi]} \exp\left(-\frac{e^{h_p \vee 0}}{c_5\, k}\right) e^{h_p} \left(1-\frac{h_p}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h_p\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&\quad + c_6\, c_7^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, k!\, (k!)^{u[\xi]}\, (kdv[\xi])!\, k^{k-1}\, k^{2(p-1)} \sum_{L_1,\dots,L_p \preceq [\![k]\!]} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} |h_p|^{(d-1)(p-1)} \\
&\qquad \times (1+|h_p|)^{dw[\xi]} \exp\left(-\frac{e^{h_p \vee 0}}{c_8\, k}\right) e^{h_p} \left(1-\frac{h_p}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h_p\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).
\end{aligned}
$$

The integral concerning h_p in the second summand is bounded by a constant times k^2, as we have already seen in Lemma 3.3.5. For the third summand, it follows by similar reasoning as above that

$$
\begin{aligned}
& \int_{-\infty}^{R_\lambda^\beta} (1+|h_p|)^{dw[\xi]} \exp\left(-\frac{e^{h_p \vee 0}}{c_1\, k}\right) e^{h_p} \left(1-\frac{h_p}{R_\lambda^\beta}\right)^{d-1+\alpha} |h_p|^{(d-1)(p-1)}\, \mathrm{d}h_p \\
&\quad = \int_{-\infty}^{0} (1+|h_p|)^{dw[\xi]} \underbrace{\exp\left(-\frac{e^{h_p \vee 0}}{c_1\, k}\right)}_{\le 1} e^{h_p} \left(1-\frac{h_p}{R_\lambda^\beta}\right)^{d-1+\alpha} |h_p|^{(d-1)(p-1)}\, \mathrm{d}h_p \\
&\qquad + \int_{0}^{R_\lambda^2} (1+|h_p|)^{dw[\xi]} \exp\left(-\frac{e^{h_p \vee 0}}{c_2\, k}\right) e^{h_p} \underbrace{\left(1-\frac{h_p}{R_\lambda^\beta}\right)^{d-1+\alpha}}_{\le 1} |h_p|^{(d-1)(p-1)}\, \mathrm{d}h_p \\
&\quad \le \int_{0}^{\infty} (1+h_p)^{dw[\xi]} e^{-h_p}\, (1+h_p)^{d-1+\alpha}\, (1+h_p)^{(d-1)(p-1)} \mathrm{d}h_p \\
&\qquad + \int_{0}^{\infty} (1+h_p)^{dw[\xi]} \exp\left(-\frac{e^{h_p}}{c\, k}\right) e^{h_p} (1+h_p)^{(d-1)(p-1)} \mathrm{d}h_p \\
&\quad \le c_1\, c_2^k\, k^2\, (dk)!.
\end{aligned}
$$

This completes the proof. □

We can now combine the two previous lemmas into a bound for the off-diagonal term, complementing the diagonal bound in Lemma 3.3.2.

Lemma 3.3.7 *Let $\xi \in \Xi$ and $f \in \mathcal{B}(\mathbb{R}^d)$. Then, for sufficiently large λ, it holds that*

$$\left| \sum_{S,T \preceq [\![k]\!]} \int_{\sigma(\{S,T\})} f_{R_\lambda}^k \, \mathrm{d}c_\lambda^k \right| \le c_1 \, c_2^k \, \|f\|_\infty^k \, R_\lambda^{\frac{\beta(d-1)}{2}} \, (dk)! \, (k!)^2 \, (k!)^{u[\xi]} \, (kdv[\xi])! \, k^{3k}.$$

Proof. Recalling (3.66), we have that

$$\left| \sum_{S,T \preceq [\![k]\!]} \int_{\sigma(\{S,T\})} f_{R_\lambda}^k \, \mathrm{d}c_\lambda^k \right| \le |T_5| + |T_6|.$$

Applying now Lemma 3.3.5 to T_5 and Lemma 3.3.6 to T_6 yields the desired upper bound. □

What is left is to combine the two estimates for the on- and the off-diagonal term to complete the proof of the cumulant bound.

Proof of Theorem 3.3.1. From Lemma 3.3.2 and Lemma 3.3.7, we deduce the bound

$$\begin{aligned} |\langle f_{R_\lambda}^k, c_\lambda^k \rangle| &\le \left| \int_\Delta f_{R_\lambda}^k \, \mathrm{d}c_\lambda^k \right| + \left| \sum_{S,T \preceq [\![k]\!]} \int_{\delta(\{S,T\})} f_{R_\lambda}^k \, \mathrm{d}c_\lambda^k \right| \\ &\le c_1 \, c_2^k \, \|f\|_\infty^k \, R_\lambda^{\frac{\beta(d-1)}{2}} \, (k!)^{u[\xi]} \, ((kdv[\xi])!)^2 \\ &\qquad + c_3 \, c_4^k \, \|f\|_\infty^k \, R_\lambda^{\frac{\beta(d-1)}{2}} \, (dk)! \, (k!)^2 \, (k!)^{u[\xi]} \, (kdv[\xi])! \, k^{3k} \\ &\le c_1 \, c_2^k \, \|f\|_\infty^k \, R_\lambda^{\frac{\beta(d-1)}{2}} \, (k!)^{u[\xi]+2dv[\xi]} + c_3 \, c_4^k \, \|f\|_\infty^k \, R_\lambda^{\frac{\beta(d-1)}{2}} \, (k!)^{u[\xi]+d+5+dv[\xi]} \\ &\le c_1 \, c_2^k \, \|f\|_\infty^k \, R_\lambda^{\frac{\beta(d-1)}{2}} \, (k!)^{u[\xi]+2dv[\xi]+z[\xi]}, \end{aligned}$$

for all sufficiently large λ, where we recall the definition of $z[\xi]$ in (3.62). This completes the proof of the cumulant bound. □

3.4 Main results

3.4.1 Intrinsic volumes and face numbers

Let us start with the expectation and variance asymptotics for the intrinsic volumes and face numbers of the generalized Gamma polytope K_λ, given by $V_i(K_\lambda)$, $i \in \{1, \dots, d\}$, and $f_j(K_\lambda)$, $j \in \{0, \dots, d-1\}$, respectively.
To streamline our presentation, following (3.62), we define the individual weights $z[f_0] := d+5$, $z[f_1] := 5$, $z[f_2] := 3$, $z[f_3] := 1$ and $z[f_j] := 0$, if $j \in \{4, \dots, d-1\}$.

Theorem 3.4.1 *(Expectation and variance asymptotics)*

(i) *Let $i \in \{1, \dots, d\}$. Then, it holds that*

$$\mathbb{E}[V_i(K_\lambda)] \sim \binom{d}{i} \frac{\kappa_d}{\kappa_{d-i}} (\beta \log \lambda)^{\frac{i}{\beta}} \qquad \text{and} \qquad \mathrm{var}[V_i(K_\lambda)] \sim c_1 \, (\beta \log \lambda)^{\frac{4i-\beta(d+3)}{2\beta}},$$

as $\lambda \to \infty$, where $c_1 \in (0, \infty)$ is a constant only depending on d, i, α and β.

(ii) *Let $j \in \{0, \dots, d-1\}$. Then, it holds that*

$$\mathbb{E}[f_j(K_\lambda)] \sim c_2 \, (\beta \log \lambda)^{\frac{d-1}{2}} \qquad \text{and} \qquad \mathrm{var}[f_j(K_\lambda)] \sim c_3 \, (\beta \log \lambda)^{\frac{d-1}{2}},$$

as $\lambda \to \infty$, where $c_2, c_3 \in (0, \infty)$ are constants only depending on d, j, α and β.

Remark 3.4.2 (a) Let us draw attention to an interesting phenomenon, appearing in the case that $i = d$, $\beta = 4$ and arbitrary $\alpha > -1$. Then, the order of magnitude of the variance of the volume of K_λ is independent of the dimension d. More formally, it holds that $\mathrm{var}[V_d(K_\lambda)] \sim c_1 \, (4 \log \lambda)^{-\frac{3}{2}}$, as $\lambda \to \infty$.

(b) Similarly to what has been done in the Gaussian model in [23], one can shift the results in Theorem 3.4.1 via a 'coupling' from our Poissonized model to the one where the number of underlying random points is deterministic. The same holds true for the results presented in Theorem 3.4.19 concerning the scaling limit properties. On the other hand, this is not possible in all other results stated in Section 3.4.1, since their proofs rely on the cumulant bound established above, which, unfortunately, cannot be shifted to the deterministic setting by using any kind of coupling. Therefore, we have decided to refrain from stating these results separately and to focus on the Poissonized model in this chapter.

Next, we state a central limit theorem with corresponding Berry-Esseen bound for the intrinsic volumes and face numbers of K_λ.

Theorem 3.4.3 *(Central limit theorems with Berry-Esseen bound)*

(i) *Let $i \in \{1, \dots, d\}$. Then, we have that for sufficiently large λ,*

$$\sup_{y\in\mathbb{R}} \left| \mathbb{P}\left(\frac{V_i(K_\lambda) - \mathbb{E}[V_i(K_\lambda)]}{\sqrt{\mathrm{var}[V_i(K_\lambda)]}} \le y \right) - \Phi(y) \right| \le c_1 (\log \lambda)^{-\frac{d-1}{4(4d+2i+9)}},$$

where $c_1 \in (0, \infty)$ is a constant only depending on d, i, α and β.

(ii) *Let $j \in \{0, \dots, d-1\}$. Then, we have that for sufficiently large λ,*

$$\sup_{y\in\mathbb{R}} \left| \mathbb{P}\left(\frac{f_j(K_\lambda) - \mathbb{E}[f_j(K_\lambda)]}{\sqrt{\mathrm{var}[f_j(K_\lambda)]}} \le y \right) - \Phi(y) \right| \le c_2 (\log \lambda)^{-\frac{d-1}{4(4j(d+1)+2z[f_j]-1)}},$$

where $c_2 \in (0, \infty)$ is a constant only depending on d, j, α and β.

Our technique also delivers an estimate for the relative error in the central limit theorem that was not available before.

Theorem 3.4.4 *(Bounds on the relative error in the central limit theorems)*

(i) *Let $i \in \{1, \dots, d\}$. Then, we have that for sufficiently large λ,*

$$\left| \log \frac{\mathbb{P}\big(V_i(K_\lambda) - \mathbb{E}[V_i(K_\lambda)] \ge y \sqrt{\mathrm{var}[V_i(K_\lambda)]}\,\big)}{1 - \Phi(y)} \right| \le c_1 (1 + y^3) (\log \lambda)^{-\frac{d-1}{4(4d+2i+9)}},$$

and

$$\left| \log \frac{\mathbb{P}\big(\,\mathrm{vol}_i(K_\lambda) - \mathbb{E}[\mathrm{vol}_i(K_\lambda)] \le -y \sqrt{\mathrm{var}[\mathrm{vol}_i(K_\lambda)]}\,\big)}{\Phi(-y)} \right| \le c_2 (1 + y^3) (\log \lambda)^{-\frac{d-1}{4(4d+2i+9)}},$$

for all

$$0 \le y \le c_3 (\log \lambda)^{\frac{d-1}{4(4d+2i+9)}},$$

where $c_1, c_2, c_3 \in (0, \infty)$ are constants only depending on d, i, α and β.

(ii) *Let $j \in \{0, \dots, d-1\}$. Then, we have that for sufficiently large λ,*

$$\left| \log \frac{\mathbb{P}\big(f_j(K_\lambda) - \mathbb{E}[f_j(K_\lambda)] \geq y \sqrt{\mathrm{var}[f_j(K_\lambda)]}\,\big)}{1 - \Phi(y)} \right| \leq c_4 \, (1 + y^3) \, (\log \lambda)^{-\frac{d-1}{4(4j(d+1)+2z[f_j]-1)}},$$

and

$$\left| \log \frac{\mathbb{P}\big(f_j(K_\lambda) - \mathbb{E}[f_j(K_\lambda)] \leq -y \sqrt{\mathrm{var}[f_j(K_\lambda)]}\,\big)}{\Phi(-y)} \right| \leq c_5 \, (1 + y^3) \, (\log \lambda)^{-\frac{d-1}{4(4j(d+1)+2z[f_j]-1)}},$$

for all

$$0 \leq y \leq c_6 (\log \lambda)^{\frac{d-1}{4(4j(d+1)+2z[f_j]-1)}},$$

where $c_4, c_5, c_6 \in (0, \infty)$ are constants only depending on d, j, α and β.

The next result contains new and powerful concentration inequalities.

Theorem 3.4.5 *(Concentration inequalities)*

(i) *Let $y \geq 0$ and $i \in \{1, \dots, d\}$. Then, we have that for sufficiently large λ,*

$$\begin{aligned} &\mathbb{P}\big(|V_i(K_\lambda) - \mathbb{E}[V_i(K_\lambda)]| \geq y \sqrt{\mathrm{var}[V_i(K_\lambda)]}\,\big) \\ &\qquad \leq 2 \exp\Big(-\frac{1}{4} \min\Big\{ \frac{y^2}{2^{2d+i+5}}, c_1 \, (\log \lambda)^{\frac{d-1}{4(2d+i+5)}} \, y^{\frac{1}{2d+i+5}} \Big\}\Big), \end{aligned}$$

where $c_1 \in (0, \infty)$ is a constant only depending on d, i, α and β.

(ii) *Let $y \geq 0$ and $j \in \{0, \dots, d-1\}$. Then, we have that for sufficiently large λ,*

$$\begin{aligned} &\mathbb{P}\big(|f_j(K_\lambda) - \mathbb{E}[f_j(K_\lambda)]| \geq y \sqrt{\mathrm{var}[f_j(K_\lambda)]}\,\big) \\ &\qquad \leq 2 \exp\Big(-\frac{1}{4} \min\Big\{ \frac{y^2}{2^{2j(d+1)+z[f_j]}}, c_2 \, (\log \lambda)^{\frac{d-1}{4(2j(d+1)+z[f_j])}} \, y^{\frac{1}{2j(d+1)+z[f_j]}} \Big\}\Big), \end{aligned}$$

where $c_2 \in (0, \infty)$ is a constant only depending on d, j, α and β.

Remark 3.4.6 For small arguments y, the Gaussian exponent $-y^2$ is already optimal. To improve the (presumably non-optimal) exponent for larger values of y by our method, which is based on sharp bounds for cumulants, one would need to improve the cumulant bound in Theorem 3.3.1. This point will further be discussed in Remark 3.4.18 below.

Consider once more the Gaussian setup. A strong law of large numbers dealing with the intrinsic volumes of Gaussian polytopes, arising from a deterministic number of Gaussian points, has been derived by Hug and Reitzner [73] by using the Chebychev inequality together with an upper bound on the variance obtained in the same paper. By using our concentration inequality from Theorem 3.4.5 (i), we prove a stronger result for our Poisson point process based model, which has the form of a so-called Marcinkiewicz-Zygmund-type strong law. While, in the Gaussian case, the classical strong law of large numbers for the random variables $V_i(K_\lambda)$, $i \in \{1, \dots, d\}$, says that

$$\frac{V_i(K_{\lambda_k}) - \mathbb{E}[V_i(K_{\lambda_k})]}{(\log \lambda_k)^{\frac{i}{2}}} \longrightarrow 0,$$

with probability one, along subsequences λ_k of the form $\lambda_k := a^k$, $a > 1$, as $k \to \infty$, our Marcinkiewicz-Zygmund-type strong law makes a statement about the almost sure convergence to 0, as $k \to \infty$, of the random variables

$$\frac{V_i(K_{\lambda_k}) - \mathbb{E}[V_i(K_{\lambda_k})]}{(\log \lambda_k)^{p\frac{i}{2}}},$$

for all $p > \frac{2i-(d+3)}{2i}$, again along all subsequences of the form $\lambda_k = a^k$, $a > 1$.
While for $p \geq 1$ such a result is a consequence of a classical strong law, the situation for $p < 1$ is not covered by such a result. We notice that for $p = \frac{2i-(d+3)}{2i}$, the denominator in the above expression equals $(\log \lambda_k)^{\frac{2i-(d+3)}{4}}$, which is precisely the rescaling that is necessary in the central limit theorem for the intrinsic volumes. Indeed, it holds that $\mathrm{var}[V_i(K_{\lambda_k})] \sim c(\log \lambda_k)^{\frac{2i-(d+3)}{2}}$, as $k \to \infty$, where $c \in (0, \infty)$ is a constant only depending on d and i (see Theorem 3.4.1 in the case that $\beta = 2$). This implies that our condition on p is in fact optimal and covers the whole possible range of parameter p.
In contrast to the intrinsic volume functionals and even in the case of an underlying binomial point process, a strong law of large numbers for the face numbers of Gaussian polytopes does not exist so far. In part (ii) of the next theorem, we present the first such result. Again, our condition on the parameter p is the best possible.
Moreover, we are able to state the Marcinkiewicz-Zygmund-type strong law of large numbers for all underlying densities presented in (3.1), not just the Gaussian case. For all choices of parameter α and β, the result stays optimal in the sense that it covers the whole possible range of scalings (see Theorem 3.4.3 and Theorem 3.4.1 for a comparison with the respective variances in the corresponding central limit theorems).

Theorem 3.4.7 *(Marcinkiewicz-Zygmund-type strong laws of large numbers)*

(i) *Let* $i \in \{1, \ldots, d\}$, $p > \frac{4i - \beta(d+3)}{4i}$, *and let* $(\lambda_k)_{k\in\mathbb{N}}$ *be a sequence of real numbers defined by* $\lambda_k := a^k$, $a > 1$. *Then, as* $k \to \infty$, *it holds that*

$$\frac{V_i(K_{\lambda_k}) - \mathbb{E}[V_i(K_{\lambda_k})]}{(\log \lambda_k)^{p\frac{i}{\beta}}} \longrightarrow 0,$$

with probability one.

(ii) *Let* $j \in \{0, \ldots, d-1\}$, $p > \frac{1}{2}$, *and let* $(\lambda_k)_{k\in\mathbb{N}}$ *be a sequence of real numbers defined by* $\lambda_k := a^k$, $a > 1$. *Then, as* $k \to \infty$, *it holds that*

$$\frac{f_j(K_{\lambda_k}) - \mathbb{E}[f_j(K_{\lambda_k})]}{(\log \lambda_k)^{p\frac{d-1}{2}}} \longrightarrow 0,$$

with probability one.

Remark 3.4.8 In [73], the strong law of large numbers for the intrinsic volumes was proved for the deterministic counterpart of the Gaussian random polytopes K_{λ_k} along the subsequence $\lambda_k = 2^k$ and, then, extended by monotonicity arguments to $\lambda_k = k$ (see [73, Corollary 1.4]). In our setup, such an extension is not possible, since the random variables $V_i(K_{\lambda_k}) - \mathbb{E}[V_i(K_{\lambda_k})]$, and also $f_j(K_{\lambda_k}) - \mathbb{E}[f_j(K_{\lambda_k})]$, are not monotone in k.

As a consequence of the cumulant bound presented in Theorem 3.3.1, we obtain upper and lower bounds for the k-th moment of the intrinsic volumes and the face numbers.

Theorem 3.4.9 *(Moment bounds)*

(i) *Let* $i \in \{1, \ldots, d\}$ *and* $k \in \mathbb{N}$. *Then, for sufficiently large* λ, *it holds that*

$$c_1\, c_2^k\, (\log \lambda)^{k\frac{i}{\beta}} \le \mathbb{E}[V_i(K_\lambda)^k] \le c_3\, c_4^k\, k!\, (\log \lambda)^{k\frac{i}{\beta}},$$

where the upper bound holds for all $\beta \ge \frac{2i}{d+1}$, *and* $c_1, c_2, c_3, c_4 \in (0, \infty)$ *are constants only depending on* d, i, α *and* β.

(ii) *Let* $j \in \{0, \ldots, d-1\}$ *and* $k \in \mathbb{N}$. *Then, for sufficiently large* λ, *it holds that*

$$c_5\, c_6^k\, (\log \lambda)^{k\frac{d-1}{2}} \le \mathbb{E}[f_j(K_\lambda)^k] \le c_7\, c_8^k\, k!\, (\log \lambda)^{k\frac{d-1}{2}},$$

where $c_5, c_6, c_7, c_8 \in (0, \infty)$ *are constants only depending on* d, j, α *and* β.

Remark 3.4.10 Let us emphasize that the upper bound for the k-th moment of the intrinsic volumes of K_λ is the only main result where we need a restriction to the parameter β. The reason for this phenomenon lies in the proof of Theorem 3.4.9 (see Section 3.5.3). On the other hand, if we put $\alpha = 0$ and $\beta = 2$ to obtain the most popular example, that is, the Gaussian case, the condition on β and, therefore, the upper bound, holds for all dimensions d.

After having investigated expectation and variance asymptotics, (the relative error in) the central limit theorems, concentration inequalities and strong laws of large numbers, we turn now to moderate deviation principles for the intrinsic volumes and face numbers of K_λ (recall the definition in Section 2.5). Although moderate (or large) deviations belong to the class of classical limit theorems in probability theory, to the best of our knowledge, they have not been investigated in the context of our class of generalized Gamma polytopes so far.

Theorem 3.4.11 *(Moderate deviation principles)*

(i) *Let $i \in \{1, \dots, d\}$, and let $(a_\lambda)_{\lambda>0}$ be a sequence of real numbers, satisfying*

$$\lim_{\lambda\to\infty} a_\lambda = \infty \qquad \text{and} \qquad \lim_{\lambda\to\infty} a_\lambda (\log\lambda)^{-\frac{d-1}{4(4d+2i+9)}} = 0.$$

Then, the family

$$\left(\frac{1}{a_\lambda}\frac{V_i(K_\lambda) - \mathbb{E}[V_i(K_\lambda)]}{\sqrt{\operatorname{var}[V_i(K_\lambda)]}}\right)_{\lambda>0}$$

satisfies a moderate deviation principle on $\mathbb{R}$ with speed a_λ^2 and rate function $I(x) = \frac{x^2}{2}$.

(ii) *Let $j \in \{0, \dots, d-1\}$, and let $(a_\lambda)_{\lambda>0}$ be a sequence of real numbers, satisfying*

$$\lim_{\lambda\to\infty} a_\lambda = \infty \qquad \text{and} \qquad \lim_{\lambda\to\infty} a_\lambda (\log\lambda)^{-\frac{d-1}{4(4j(d+1)+2z[f_j]-1)}} = 0.$$

Then, the family

$$\left(\frac{1}{a_\lambda}\frac{f_j(K_\lambda) - \mathbb{E}[f_j(K_\lambda)]}{\sqrt{\operatorname{var}[f_j(K_\lambda)]}}\right)_{\lambda>0}$$

satisfies a moderate deviation principle on $\mathbb{R}$ with speed a_λ^2 and rate function $I(x) = \frac{x^2}{2}$.

Remark 3.4.12 The results that we have presented in this section have immediate consequences for the model of randomly rotated and projected simplices, briefly discussed on page 10 in Chapter 1, if we randomize the model further. Namely, we let the space dimension $n = N(\lambda)$ be an independent random integer that is Poisson distributed with parameter λ and think of $\mathrm{pr}_d^{N(\lambda)}(\varrho(\Delta_{N(\lambda)}))$ as already being embedded in $\mathbb{R}^d$ in the case that $N(\lambda) \leq d$ (the probability of this event tends to 0, as $\lambda \to \infty$). Then, we conclude for the face numbers $f_j(\mathrm{pr}_d^{N(\lambda)}(\varrho(\Delta_{N(\lambda)})))$ of $\mathrm{pr}_d^{N(\lambda)}(\varrho(\Delta_{N(\lambda)}))$, for all $j \in \{0, \dots, d-1\}$, from Theorem 3.4.5 a concentration inequality, from Theorem 3.4.7 a Marcinkiewicz-Zygmund-type strong law of large numbers, from Theorem 3.4.9 bounds for the moments of all orders, from Theorem 3.4.4 a bound on the relative error in the central limit theorem that we have from Theorem 3.4.3, as well as a moderate deviation principle from Theorem 3.4.11. Moreover, Theorem 3.3.1 delivers a bound on the cumulants of these random variables. We refrain from presenting all these results formally since their statements are literally the same as in the theorems mentioned above in the case that $\alpha = 0$ and $\beta = 2$, with the random polytope K_λ replaced by the randomly rotated and projected simplex $\mathrm{pr}_d^{N(\lambda)}(\varrho(\Delta_{N(\lambda)}))$.

3.4.2 Empirical measures

In this section, we present a series of results for the empirical measures introduced in (3.26). The advantage of working with empirical measures instead of just their total masses is that they allow to capture also the spatial profile of the geometric functionals. Let us briefly recall the setup. We denote by $\mathcal{P}_\lambda$ a Poisson point process in $\mathbb{R}^d$, whose intensity measure is a multiple $\lambda > 0$ of the measure $\gamma_{d,\alpha,\beta}$. The generalized Gamma polytope K_λ is the random convex hull generated by $\mathcal{P}_\lambda$, while the class of key geometric functionals associated with K_λ is abbreviated by the symbol Ξ. Moreover, for $\xi \in \Xi$, we let μ_λ^ξ be the corresponding empirical measure (see (3.26)). To present our results in a unified way, recall the weights $u[\xi]$, $v[\xi]$ and $w[\xi]$ from the beginning of Section 3.2.2 and $z[\xi]$ from (3.62). Furthermore, define

$$\sigma_\lambda^\xi(f_{R_\lambda}) := (\mathrm{var}[\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle])^{\frac{1}{2}},$$

and recall the definition of $\sigma^2(\xi^{(\infty)})$ from (3.25).

We start with the following theorem, summarizing the expectation and variance asymptotics for the empirical measures. Recall that in the Gaussian setup, i.e., $\alpha = 0$ and $\beta = 2$, this has been obtained in [23, Theorem 2.1] for the volume and the face numbers.

Theorem 3.4.13 *Let $\xi \in \Xi$ and $f \in \mathcal{C}(\mathbb{R}^d;\mathbb{S}^{d-1})$. Then, it holds that*

$$\lim_{\lambda\to\infty} (\beta \log \lambda)^{-\frac{(d-1)}{2}}\, \mathbb{E}[\langle f_{R_\lambda}, \mu_\lambda^\xi\rangle] = \int_{-\infty}^{\infty} \mathbb{E}[\xi^{(\infty)}((\boldsymbol{o},h_0),\mathcal{P})]\, e^{h_0}\, \mathrm{d}h_0 \int_{\mathbb{S}^{d-1}} f(u)\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),$$

and, if $\langle f^2, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}\rangle > 0$,

$$\lim_{\lambda\to\infty} (\beta \log \lambda)^{-\frac{(d-1)}{2}}\, \mathrm{var}[\langle f_{R_\lambda}, \mu_\lambda^\xi\rangle] = \sigma^2(\xi^{(\infty)}) \int_{\mathbb{S}^{d-1}} f(u)^2\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \in (0,\infty).$$

All the upcoming results are again new and have no counterparts in the literature, not even in the Gaussian case. Our next result is a concentration inequality for the empirical measures, related to the intrinsic volumes and face numbers of K_λ.

Theorem 3.4.14 (Concentration inequality) *Let $\xi \in \Xi$ and $f \in \mathcal{C}(\mathbb{R}^d;\mathbb{S}^{d-1})$ with $\langle f^2, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}\rangle > 0$. Then, for all $y \geq 0$ and sufficiently large λ, it holds that*

$$\mathbb{P}\big(|\langle f_{R_\lambda}, \bar{\mu}_\lambda^\xi\rangle| \geq y\, \sigma_\lambda^\xi(f_{R_\lambda})\big) \leq 2\exp\Bigg(-\frac{1}{4}\min\Bigg\{\frac{y^2}{2^{u[\xi]+2dv[\xi]+z[\xi]}},\ c\,(\log\lambda)^{\frac{d-1}{4(u[\xi]+2dv[\xi]+z[\xi])}}\, y^{\frac{1}{u[\xi]+2dv[\xi]+z[\xi]}}\Bigg\}\Bigg),$$

where $c \in (0,\infty)$ is a constant only depending on d, α, β, ξ and f.

The next result is a generalization of Theorem 3.4.4 and assesses the relative error in the central limit theorem on a logarithmic scale, in a version taken from [43, Corollary 3.2].

Theorem 3.4.15 (Bounds on the relative error in the central limit theorem) *Let $\xi \in \Xi$ and $f \in \mathcal{C}(\mathbb{R}^d;\mathbb{S}^{d-1})$ with $\langle f^2, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}\rangle > 0$. Then, for all y with*

$$0 \leq y \leq c_1\,(\log\lambda)^{\frac{d-1}{4(2(u[\xi]+2dv[\xi]+z[\xi])-1)}},$$

and sufficiently large λ, we have that

$$\left|\log \frac{\mathbb{P}(\langle f_{R_\lambda}, \bar{\mu}_\lambda^\xi\rangle \geq y\,\sigma_\lambda^\xi(f_{R_\lambda}))}{1-\Phi(y)}\right| \leq c_2\,(1+y^3)\,(\log\lambda)^{-\frac{d-1}{4(2(u[\xi]+2dv[\xi]+z[\xi])-1)}},$$

and

$$\left|\log \frac{\mathbb{P}(\langle f_{R_\lambda}, \bar{\mu}_\lambda^\xi \rangle \leq -y\, \sigma_\lambda^\xi(f_{R_\lambda}))}{\Phi(-y)}\right| \leq c_2\,(1+y^3)\,(\log \lambda)^{-\frac{d-1}{4(2(u[\xi]+2dv[\xi]+z[\xi])-1)}},$$

with constants $c_1, c_2 \in (0,\infty)$ *only depending on* d, α, β, ξ *and* f.

We also get the following central limit theorem that is available from our technique. However, we point out that in the Gaussian case, the rate of convergence we obtain is weaker than the one derived in [9]. On the other hand, our result is more general since we consider integrals with respect to the empirical measures of general functions $f \in \mathcal{C}(\mathbb{R}^d; \mathbb{S}^{d-1})$, while even in the Gaussian case in [9] only constant functions were investigated.

Theorem 3.4.16 (Central limit theorem with Berry-Esseen bound) *Let* $\xi \in \Xi$ *and* $f \in \mathcal{C}(\mathbb{R}^d; \mathbb{S}^{d-1})$ *with* $\langle f^2, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}} \rangle > 0$. *Then, we have that for sufficiently large* λ,

$$\sup_{y \in \mathbb{R}} \left| \mathbb{P}\left(\frac{\langle f_{R_\lambda}, \bar{\mu}_\lambda^\xi \rangle}{\sigma_\lambda^\xi(f_{R_\lambda})} \leq y \right) - \Phi(y) \right| \leq c\,(\log \lambda)^{-\frac{d-1}{4(2(u[\xi]+2dv[\xi]+z[\xi])-1)}}, \tag{3.67}$$

where $c \in (0,\infty)$ *is a constant only depending on* d, α, β, ξ *and* f. *In particular, as* $\lambda \to \infty$, *the sequence*

$$\left(\frac{\langle f_{R_\lambda}, \bar{\mu}_\lambda^\xi \rangle}{\sigma_\lambda^\xi(f_{R_\lambda})} \right)_{\lambda > 0}$$

converges in distribution to a standard Gaussian random variable.

Now, we turn to moderate deviation principles for integrals with respect to the empirical measures.

Theorem 3.4.17 (Moderate deviation principle) *Let* $\xi \in \Xi$, $f \in \mathcal{C}(\mathbb{R}^d; \mathbb{S}^{d-1})$ *with* $\langle f^2, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}} \rangle > 0$, *and let* $(a_\lambda)_{\lambda > 0}$ *be a sequence of real numbers that satisfies the growth condition*

$$\lim_{\lambda \to \infty} a_\lambda = \infty \qquad \text{and} \qquad \lim_{\lambda \to \infty} a_\lambda\,(\log \lambda)^{-\frac{d-1}{4(2(u[\xi]+2dv[\xi]+z[\xi])-1)}} = 0.$$

Then, the family

$$\left(\frac{1}{a_\lambda}\frac{\langle f_{R_\lambda}, \bar{\mu}_\lambda^\xi\rangle}{\sigma_\lambda^\xi(f_{R_\lambda})}\right)_{\lambda>0}$$

fulfills a moderate deviation principle on $\mathbb{R}$ *with speed* a_λ^2 *and rate function* $I(x) = \frac{x^2}{2}$.

Remark 3.4.18 Except for Theorem 3.4.7, we do not claim that our findings are optimal. However, in order to improve them by our methods, one would have to decrease the exponent at $k!$ in the cumulant bound in Theorem 3.3.1 from $u[\xi] + 2dv[\xi] + z[\xi]$ to (optimally) 1. This would imply that $\gamma = 0$ in the application of Theorem 2.4.3, which is optimal. It is unclear to us and seems unlikely that such an improvement is possible in the framework of our class of generalized Gamma polytopes. We even doubt that the exponent can be chosen independently of d.

3.4.3 Germ-grain processes

The following theorem shows the connection between the generalized Gamma polytope K_λ and the four introduced germ-grain processes $\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$, $\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$, $\Phi(\mathcal{P})$ and $\Psi(\mathcal{P})$, respectively.

Theorem 3.4.19 *Fix* $L \in (0, \infty)$. *As* $\lambda \to \infty$, *the following assertions are true:*

(a) Under the scaling transformation T_λ, *the rescaled vertices of* K_λ *converge in distribution to the set of extreme points of* $\mathcal{P}$.

(b) Under the scaling transformation T_λ, *the rescaled boundaries* $T_\lambda(\partial K_\lambda) = \partial\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ *and* $\partial\Psi^{(\lambda)}(\mathcal{P}^{(\lambda)})$ *converge in probability to* $\partial(\Phi(\mathcal{P}))$ *and* $\partial(\Psi(\mathcal{P}))$, *respectively, on the space* $\mathcal{C}(\mathbb{B}_{d-1}(\boldsymbol{o}, L))$.

In particular and as already anticipated in Chapter 1, the latter theorem states that the scaling limit of the rescaled boundary of our generalized Gamma polytopes arises as a unique festoon of parabolic surfaces, not depending on the parameter α and β in the underlying distribution of the Poisson point process $\mathcal{P}_\lambda$.

3.5 Proof of the main results

3.5.1 Empirical measures: expectation and variance asymptotics

We start with the proof of Theorem 3.4.13. First, we need the following lemma, stating that the rescaled functionals $\xi^{(\lambda)} \in \Xi^{(\lambda)}$, as well as the corresponding second order correlation functions $c^{\xi^{(\lambda)}}((\boldsymbol{o}, h_0), (v_1, h_1), \mathcal{P}^{(\lambda)})$, recall the definition in (3.23), converge to their respective scaling limit counterparts.

Lemma 3.5.1 *Let $\xi \in \Xi$. Then, for all $h_0 \in (-\infty, R_\lambda^\beta]$ and $(v_1, h_1) \in W_\lambda$, it holds that*

$$\lim_{\lambda\to\infty} \mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o}, h_0), \mathcal{P}^{(\lambda)})] = \mathbb{E}[\xi^{(\infty)}((\boldsymbol{o}, h_0), \mathcal{P})],$$

and

$$\lim_{\lambda\to\infty} c^{\xi^{(\lambda)}}((\boldsymbol{o}, h_0), (v_1, h_1), \mathcal{P}^{(\lambda)}) = c^{\xi^{(\infty)}}((\boldsymbol{o}, h_0), (v_1, h_1), \mathcal{P}).$$

Proof. The proof of the first assertion follows step by step the proof of [23, Lemma 4.5 and 4.6] in the Gaussian setting. First, it is shown that the desired convergence holds restricted to the cylinder $C_{d-1}(\boldsymbol{o}, r) = \mathbb{B}^{d-1}(\boldsymbol{o}, r) \times \mathbb{R}$, $r > 0$, i.e.,

$$\lim_{\lambda\to\infty} \mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o}, h_0), \mathcal{P}^{(\lambda)} \cap C_{d-1}(\boldsymbol{o}, r))] = \mathbb{E}[\xi^{(\infty)}((\boldsymbol{o}, h_0), \mathcal{P} \cap C_{d-1}(\boldsymbol{o}, r))].$$

Then, the result is extended to hold also without this restriction. We remark that both related proofs just use the results obtained in Section 3.2, regarding to the properties of the functionals of interest and the germ-grain processes and, therefore, are completely independent of the parameter α and β in the underlying distribution of the Poisson point process $\mathcal{P}_\lambda$. Thus, we have decided to omit stating a word by word repetition of the proofs given in [23]. The same holds true for the second assertion in Lemma 3.5.1, which can be proven in the same spirit as [23, Lemma 4.7]. □

Proof of Theorem 3.4.13, the expectation. To analyze the expectation, we first use the Mecke equation (see Theorem 2.6.1). Secondly, putting $||x|| = R_\lambda(1 - h/R_\lambda^\beta)$ in view of the scaling transformation T_λ, the rotational invariance of the underlying Poisson point process $\mathcal{P}_\lambda$ leads to

$$\mathbb{E}[\xi(x, \mathcal{P}_\lambda)] = \mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o}, h), \mathcal{P}^{(\lambda)})].$$

Then, writing

$$u = \frac{x}{||x||} \quad \Leftrightarrow \quad x = u||x|| = uR_\lambda \left(1 - \frac{h}{R_\lambda^\beta}\right),$$

implies that

$$\mathrm{d}x = \left[R_\lambda \left(1 - \frac{h}{R_\lambda^\beta}\right)\right]^{d-1} R_\lambda^{-(\beta-1)} \, \mathrm{d}h \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),$$

see once more also (3.12) for further details. Moreover, we use (3.13) to get

$$\begin{aligned} &\phi_{\alpha,\beta}\left(u\left(1 - \frac{h}{R_\lambda^\beta}\right)\right) \\ &\quad = (\beta \log \lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}} \, R_\lambda^\alpha \left(1 - \frac{h}{R_\lambda^\beta}\right)^\alpha \exp\left(h - \frac{h^2}{2R_\lambda^\beta}(\beta - 1)(1 - C)^{\beta-2}\right), \end{aligned}$$

where $C \in (-\infty, 1)$. Lastly, combining all these explanations with

$$\phi_\lambda(u, h) = \frac{(\beta \log \lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}} \exp\left(h - \frac{h^2}{2R_\lambda^\beta}(\beta - 1)(1 - C)^{\beta-2}\right), \tag{3.68}$$

and the calculation

$$d + \alpha - \beta + \frac{\beta(d+1) - 2d - 2\alpha}{2} = \frac{2d + 2\alpha - 2\beta + \beta d + \beta - 2d - 2\alpha}{2} = \frac{\beta(d-1)}{2},$$

yields that

$$\begin{aligned} &\mathbb{E}[\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle] \\ &\quad = \lambda \int\limits_{\mathbb{R}^d} f\left(\frac{x}{R_\lambda}\right) \mathbb{E}[\xi(x, \mathcal{P}_\lambda)] \, \phi_{\alpha,\beta}(x) \, \mathrm{d}x \\ &\quad = \lambda \int\limits_{\mathbb{R}^d} f\left(\frac{x}{R_\lambda}\right) \mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o}, h), \mathcal{P}^{(\lambda)})] \, \phi_{\alpha,\beta}(x) \, \mathrm{d}x \end{aligned}$$

$$= \lambda \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} f\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right) \left[R_\lambda\left(1-\frac{h}{R_\lambda^\beta}\right)\right]^{d-1} R_\lambda^{-(\beta-1)}$$
$$\times \mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})]\,\phi_{\alpha,\beta}\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right) \mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$
$$= \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} f\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right) (\beta\log\lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}} R_\lambda^{d+\alpha-\beta}\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}$$
$$\times \exp\left(h-\frac{h^2}{2R_\lambda^\beta}(\beta-1)(1-C)^{\beta-2}\right)\mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})]\,\mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$
$$= R_\lambda^{\frac{\beta(d-1)}{2}} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} f\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right) \phi_\lambda(u,h)\,\mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})]$$
$$\times\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).$$

Thus,

$$R_\lambda^{-\frac{\beta(d-1)}{2}}\,\mathbb{E}[\langle f_{R_\lambda},\mu_\lambda^\xi\rangle] = \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} f\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right) \phi_\lambda(u,h)$$
$$\times\,\mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})]\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u). \tag{3.69}$$

Now, for fixed $h \in (-\infty, R_\lambda^\beta]$, we have by the continuity of f, the definition of $\phi_\lambda(u,h)$ and Lemma 3.5.1 that

$$\lim_{\lambda\to\infty}\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} = 1, \quad \lim_{\lambda\to\infty} f\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right) = f(u), \quad \lim_{\lambda\to\infty}\phi_\lambda(u,h) = e^h,$$

and

$$\lim_{\lambda\to\infty}\mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})] = \mathbb{E}[\xi^{(\infty)}((\boldsymbol{o},h),\mathcal{P})],$$

respectively. Moreover, since $C \in (-\infty, 1)$, we have that $\phi_\lambda(u,h) \le C_1\,e^h$, for all $u \in \mathbb{S}^{d-1}$ and all $h \in \mathbb{R}$, where $C_1 \in (0,\infty)$ is an absolute constant. This, combined with the moment bound derived in Theorem 3.2.12 (i), shows that for sufficiently large

λ, the integrand in (3.69) is bounded from above by

$$c_1 \, ||f||_\infty \, (1+|h|)^{(d-1)v[\xi]+2d+\alpha-1} \, e^h \, \exp\left(-\frac{e^{h\vee 0}}{c_2}\right).$$

Since this expression is integrable, the dominated convergence theorem implies the result. □

In order to deal with the variance, we first use again the Mecke equation to achieve that

$$\begin{aligned}
&\mathbb{E}[\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle^2] \\
&\quad = \lambda^2 \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} f\left(\frac{x}{R_\lambda}\right) f\left(\frac{y}{R_\lambda}\right) \mathbb{E}[\xi(x, \mathcal{P}_\lambda \cup \{y\}) \, \xi(y, \mathcal{P}_\lambda \cup \{x\})] \, \phi_{\alpha,\beta}(x) \, \phi_{\alpha,\beta}(y) \, \mathrm{d}x \, \mathrm{d}y \\
&\qquad + \lambda \int\limits_{\mathbb{R}^d} f\left(\frac{x}{R_\lambda}\right)^2 \mathbb{E}[\xi(x, \mathcal{P}_\lambda)^2] \, \phi_{\alpha,\beta}(x) \, \mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\mathbb{E}[\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle]\Big)^2 \\
&\qquad = \lambda^2 \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} f\left(\frac{x}{R_\lambda}\right) f\left(\frac{y}{R_\lambda}\right) \mathbb{E}[\xi(x, \mathcal{P}_\lambda)] \, \mathbb{E}[\xi(y, \mathcal{P}_\lambda)] \, \phi_{\alpha,\beta}(x) \, \phi_{\alpha,\beta}(y) \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

This yields that

$$\begin{aligned}
&R_\lambda^{-\frac{\beta(d-1)}{2}} \operatorname{var}[\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle] \\
&\quad = R_\lambda^{-\frac{\beta(d-1)}{2}} \left[\mathbb{E}[\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle^2] - \Big(\mathbb{E}[\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle]\Big)^2\right] \\
&\quad = R_\lambda^{-\frac{\beta(d-1)}{2}} \lambda^2 \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} f\left(\frac{x}{R_\lambda}\right) f\left(\frac{y}{R_\lambda}\right) \Big[\mathbb{E}[\xi(x, \mathcal{P}_\lambda \cup \{y\}) \, \xi(y, \mathcal{P}_\lambda \cup \{x\})] \\
&\qquad\qquad - \mathbb{E}[\xi(x, \mathcal{P}_\lambda)] \, \mathbb{E}[\xi(y, \mathcal{P}_\lambda)]\Big] \, \phi_{\alpha,\beta}(x) \, \phi_{\alpha,\beta}(y) \, \mathrm{d}x \, \mathrm{d}y \\
&\qquad + R_\lambda^{-\frac{\beta(d-1)}{2}} \lambda \int\limits_{\mathbb{R}^d} f\left(\frac{x}{R_\lambda}\right)^2 \mathbb{E}[\xi(x, \mathcal{P}_\lambda)^2] \, \phi_{\alpha,\beta}(x) \, \mathrm{d}x \\
&\quad =: I + II.
\end{aligned}$$

In the following, we analyze both terms separately.

Lemma 3.5.2 *As $\lambda \to \infty$, part II of the above sum converges to*

$$\int_{-\infty}^{\infty} \mathbb{E}[\xi^{(\infty)}((\boldsymbol{o},h),\mathcal{P})^2]\, e^h \,\mathrm{d}h \int_{\mathbb{S}^{d-1}} f(u)^2 \,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).$$

Proof of Lemma 3.5.2. As in the proof of the expectation asymptotic, we achieve

$$\begin{aligned} II &= R_\lambda^{-\frac{\beta(d-1)}{2}} \lambda \int_{\mathbb{R}^d} f\left(\frac{x}{R_\lambda}\right)^2 \mathbb{E}[\xi(x,\mathcal{P}_\lambda)^2]\, \phi_{\alpha,\beta}(x)\,\mathrm{d}x \\ &= \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{R_\lambda^\beta} f\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right)^2 \phi_\lambda(u,h)\,\mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})^2] \\ &\qquad \times \left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u). \end{aligned}$$

Similarly as above, for fixed $h \in (-\infty, R_\lambda^\beta]$, we find that

$$\lim_{\lambda\to\infty}\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} = 1, \quad \lim_{\lambda\to\infty} f\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right)^2 = f(u)^2, \quad \lim_{\lambda\to\infty}\phi_\lambda(u,h) = e^h,$$

and

$$\lim_{\lambda\to\infty}\mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o},h),\mathcal{P}^{(\lambda)})^2] = \mathbb{E}[\xi^{(\infty)}((\boldsymbol{o},h),\mathcal{P})^2].$$

Here, in view of the moment bounds in Theorem 3.2.12 (i), for sufficiently large λ, the integrand is bounded by

$$c_1\,||f||_\infty^2\,(1+|h|)^{2(d-1)v[\xi]+2d+\alpha-1}\, e^h \,\exp\left(-\frac{e^{h\vee 0}}{c_2}\right).$$

The limit result follows again by the dominated convergence theorem. □

Lemma 3.5.3 *As $\lambda \to \infty$, part I of the above sum converges to*

$$\int_{\mathbb{R}^{d-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^{\xi^{(\infty)}}((\boldsymbol{o},h),(v_1,h_1),\mathcal{P})\, e^{h+h_1}\,\mathrm{d}h\,\mathrm{d}h_1\,\mathrm{d}v_1 \int_{\mathbb{S}^{d-1}} f(u)^2\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),$$

where $c^{\xi^{(\infty)}}((\boldsymbol{o},h),(v_1,h_1),\mathcal{P})$ is the second order correlation function, given in (3.24).

Proof of Lemma 3.5.3. Due to the rotational invariance of the underlying Poisson point process $\mathcal{P}_\lambda$, we assume without loss of generality that $x = (\boldsymbol{o}, h_x) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and $y = (v_y, h_y) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Then, by putting $T_\lambda(x) := (\boldsymbol{o}, h)$ and $T_\lambda(y) := (v_1, h_1)$ in view of the scaling transformation, where h, h_1 and v_1 are defined in terms of

$$||x|| = R_\lambda \left(1 - \frac{h}{R_\lambda^\beta}\right), \quad ||y|| = R_\lambda \left(1 - \frac{h_1}{R_\lambda^\beta}\right) \quad \text{and} \quad v_1 = R_\lambda^{\frac{\beta}{2}} \exp\left(\frac{v_1}{||v_1||}\right),$$

respectively, we obtain

$$\begin{aligned}
&\mathbb{E}[\xi(x, \mathcal{P}_\lambda \cup \{y\})\, \xi(y, \mathcal{P}_\lambda \cup \{x\})] - \mathbb{E}[\xi(x, \mathcal{P}_\lambda)]\, \mathbb{E}[\xi(y, \mathcal{P}_\lambda)] \\
&\quad = \mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o}, h), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\})\, \xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(\boldsymbol{o}, h)\})] \\
&\qquad - \mathbb{E}[\xi^{(\lambda)}((\boldsymbol{o}, h), \mathcal{P}^{(\lambda)})]\, \mathbb{E}[\xi^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)})] \\
&\quad = c^{\xi^{(\lambda)}}((\boldsymbol{o}, h), (v_1, h_1), \mathcal{P}^{(\lambda)}).
\end{aligned}$$

Moreover, we get similarly as in the proof for the expectation that there is a constant $C \in (-\infty, 1)$ such that

$$\begin{aligned}
&R_\lambda^{-\frac{\beta(d-1)}{2}}\, \lambda\, \phi_{\alpha,\beta}(x)\, \mathrm{d}x \\
&\quad = \frac{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}}{(\beta \log \lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}} \left(1 - \frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \\
&\qquad \times \exp\left(h - \frac{h^2}{2R_\lambda^\beta}(\beta - 1)(1 - C)^{\beta-2}\right) \mathrm{d}h\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&\quad = \phi_\lambda(u, h) \left(1 - \frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}h\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),
\end{aligned}$$

where we recall the definition of $\phi_\lambda(u, h)$ in (3.68). Furthermore, as in (3.10), there is a $C_1 \in (-\infty, 1)$ with

$$\begin{aligned}
&\lambda\, \phi_{\alpha,\beta}(y)\, \mathrm{d}y \\
&\quad = \frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}} \|v_1\|)}{\|R_\lambda^{-\frac{\beta}{2}} v_1\|^{d-2}} \frac{(\beta \log \lambda)^{\frac{\beta(d+1)-2d-2\alpha}{2\beta}}}{R_\lambda^{\frac{\beta(d+1)-2d-2\alpha}{2}}} \\
&\qquad \times \exp\left(h_1 - \frac{h_1^2}{2R_\lambda^\beta}(\beta - 1)(1 - C_1)^{\beta-2}\right) \left(1 - \frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha} \mathrm{d}v_1\, \mathrm{d}h_1
\end{aligned}$$

$$= \phi_\lambda(u,h_1)\,\frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}}\|v_1\|)}{\|R_\lambda^{-\frac{\beta}{2}}v_1\|^{d-2}}\left(1-\frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}v_1\,\mathrm{d}h_1.$$

Note that the double integral over $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$ transforms to a quadruple integral over the set

$$(u,h,v_1,h_1)\in\mathbb{S}^{d-1}\times(-\infty,R_\lambda^\beta]\times T_\lambda(\mathbb{S}^{d-1})\times(-\infty,R_\lambda^\beta].$$

Combining these statements yields that

$$\begin{aligned}
I &= R_\lambda^{-\frac{\beta(d-1)}{2}}\lambda^2\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f\left(\frac{x}{R_\lambda}\right)f\left(\frac{y}{R_\lambda}\right)\big[\mathbb{E}[\xi(x,\mathcal{P}_\lambda\cup\{y\})\,\xi(y,\mathcal{P}_\lambda\cup\{x\})]\\
&\qquad -\mathbb{E}[\xi(x,\mathcal{P}_\lambda)]\,\mathbb{E}[\xi(y,\mathcal{P}_\lambda)]\big]\,\phi_{\alpha,\beta}(x)\,\phi_{\alpha,\beta}(y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \int\limits_{\mathbb{S}^{d-1}}\int\limits_{T_\lambda(\mathbb{S}^{d-1})}\int\limits_{-\infty}^{R_\lambda^\beta}\int\limits_{-\infty}^{R_\lambda^\beta} f\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right)f(R_\lambda^{-1}\,T_\lambda^{-1}(v_1,h_1))\,c^{\xi^{(\lambda)}}((\boldsymbol{o},h),(v_1,h_1),\mathcal{P}^{(\lambda)})\\
&\qquad\times\frac{\sin^{d-2}(R_\lambda^{-\frac{\beta}{2}}\|v_1\|)}{\|R_\lambda^{-\frac{\beta}{2}}v_1\|^{d-2}}\,\phi_\lambda(u,h)\,\phi_\lambda(u,h_1)\\
&\qquad\times\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}\left(1-\frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha}\mathrm{d}h\,\mathrm{d}h_1\,\mathrm{d}v_1\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).
\end{aligned}$$

Now, using

$$T_\lambda^{-1}(v_1,h_1)=u\,R_\lambda\left(1-\frac{h_1}{R_\lambda^\beta}\right)\exp(R_\lambda^{-\frac{\beta}{2}}v_1),$$

and the continuity of f, leads to

$$\lim_{\lambda\to\infty} f\left(u\left(1-\frac{h}{R_\lambda^\beta}\right)\right)f(R_\lambda^{-1}\,T_\lambda^{-1}(v_1,h_1))=f(u)^2.$$

Analogously as above, it holds that

$$\lim_{\lambda\to\infty}\phi_\lambda(u,h)\,\phi_\lambda(u,h_1)=e^{h+h_1}\quad\text{and}\quad\lim_{\lambda\to\infty}\left(1-\frac{h}{R_\lambda^\beta}\right)^{d-1+\alpha}\left(1-\frac{h_1}{R_\lambda^\beta}\right)^{d-1+\alpha}=1.$$

Furthermore, Lemma 3.5.1 implies that

$$\lim_{\lambda\to\infty} c^{\xi^{(\lambda)}}((\boldsymbol{o},h),(v_1,h_1),\mathcal{P}^{(\lambda)}) = c^{\xi^{(\infty)}}((\boldsymbol{o},h_0),(v_1,h_1),\mathcal{P}).$$

In order to use also the dominated convergence at this point, we have to bound the complete integrand from above. Corollary 3.2.16 and the fact that $\phi_\lambda(u,h)$ and $\phi_\lambda(u,h_1)$ are smaller than constants times e^h and e^{h_1}, respectively, imply that for sufficiently large λ, the integrand is indeed bounded by

$$\begin{aligned} c_1\,||f||_\infty^2\,(1+|h_0|)^{dw[\xi]}\,(1+|h_1|)^{dw[\xi]}\,\exp\left(-\frac{1}{c_2}(e^{h\vee 0}+e^{h_1\vee 0})+h+h_1\right) \\ \times\left(\exp(-c_3\,|v_1|)+\mathbf{1}(|v_1|\le 2\max\{|h|,|h_1|\})\right). \end{aligned}$$

In particular, this bound is exponentially decaying in all arguments. Thus, the result follows from the dominated convergence in combination with the four limits stated above, as $\lambda\to\infty$. □

Proof of Theorem 3.4.13, the variance. Combining the investigations of Lemma 3.5.2 and Lemma 3.5.3 implies that

$$\begin{aligned} &\lim_{\lambda\to\infty} R_\lambda^{-\frac{\beta(d-1)}{2}}\operatorname{var}[\langle f_{R_\lambda},\mu_\lambda^\xi\rangle] \\ &= \left[\int_{-\infty}^{\infty}\mathbb{E}[\xi^{(\infty)}((\boldsymbol{o},h),\mathcal{P})^2]\,e^h\,\mathrm{d}h + \int_{\mathbb{R}^{d-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^{\xi^{(\infty)}}((\boldsymbol{o},h),(v_1,h_1),\mathcal{P})\,e^{h+h_1}\,\mathrm{d}h\,\mathrm{d}h_1\,\mathrm{d}v_1\right] \\ &\qquad\times\int_{\mathbb{S}^{d-1}} f(u)^2\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\ &= \sigma^2(\xi^{(\infty)})\int_{\mathbb{S}^{d-1}} f(u)^2\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u), \end{aligned}$$

where we recall the definition of $\sigma^2(\xi^{(\infty)})$ in (3.25). The strict positivity of the limit follows from [8, 9] since $\sigma^2(\xi^{(\infty)})$ does not depend on α and β, and, therefore, stays exactly the same constant as in the Gaussian case. This completes the proof of the variance asymptotic. □

3.5.2 Empirical measures: cumulant bound

Fix $\xi \in \Xi$ and $f \in \mathcal{C}(\mathbb{R}^d, \mathbb{S}^{d-1})$ with $\langle f^2, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}} \rangle > 0$. If $f \in \mathcal{C}(\mathbb{R}^d; \mathbb{S}^{d-1})$, we recall from Theorem 3.4.13 that

$$\sigma_\lambda^\xi(f_{R_\lambda}) := (\mathrm{var}[\langle f_{R_\lambda}, \mu_\lambda^\xi \rangle])^{\frac{1}{2}}$$

fulfills, for sufficiently large λ, that

$$\sigma_\lambda^\xi(f_{R_\lambda}) \geq c\, \langle f^2, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}} \rangle^{\frac{1}{2}} (\log \lambda)^{\frac{d-1}{4}}. \tag{3.70}$$

The cumulant bound in Theorem 3.3.1 and the variance estimate (3.70) imply that, for all $k \in \{3, 4, \ldots\}$ and sufficiently large λ,

$$\frac{|\langle f_{R_\lambda}^k, c_\lambda^k \rangle|}{(\sigma_\lambda^\xi(f_{R_\lambda}))^k} \leq c_1\, c_2^k\, \|f\|_\infty^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\, (k!)^{u[\xi]+2dv[\xi]+z[\xi]} \left(c_3\, \langle f^2, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}} \rangle^{\frac{1}{2}} (\log \lambda)^{\frac{d-1}{4}} \right)^{-k}.$$

By the definition of R_λ, we see that $R_\lambda \leq C\, (\beta \log \lambda)^{\frac{1}{\beta}}$ and, hence,

$$\begin{aligned} \frac{|\langle f_{R_\lambda}^k, c_\lambda^k \rangle|}{(\sigma_\lambda^\xi(f_{R_\lambda}))^k} &\leq c_1\, c_2^k\, \|f\|_\infty^k\, (\log \lambda)^{\frac{d-1}{2}}\, (k!)^{u[\xi]+2dv[\xi]+z[\xi]} \left(\langle f^2, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}} \rangle^{\frac{1}{2}} (\log \lambda)^{\frac{d-1}{4}} \right)^{-k} \\ &\leq c_1\, c_2^k \left((\log \lambda)^{\frac{d-1}{4}} \right)^{-(k-2)} (k!)^{u[\xi]+2dv[\xi]+z[\xi]}. \end{aligned}$$

Proof of Theorem 3.4.14, 3.4.15, 3.4.16, and 3.4.17. Put

$$\gamma := u[\xi] + 2dv[\xi] + z[\xi] - 1 \qquad \text{and} \qquad \Delta_\lambda := (\log \lambda)^{\frac{d-1}{4}}. \tag{3.71}$$

Then, our computations performed above imply that the random variables

$$X_\lambda := \frac{\langle f_{R_\lambda}, \bar{\mu}_\lambda^\xi \rangle}{\sigma_\lambda^\xi(f_{R_\lambda})}$$

fulfill the conditions of Theorem 2.4.3 with the constants γ and Δ_λ given by (3.71). This completes the proof of the four theorems. □

3.5.3 Intrinsic volumes and face numbers

Proof of Theorem 3.4.1, 3.4.3, 3.4.4, 3.4.5, and 3.4.11. We start with the face numbers of the random polytope K_λ. In this case, all five theorems follow directly from the corresponding results in Section 3.4.2 by putting $f_{R_\lambda} \equiv 1$ since

$$\langle 1, \bar{\mu}_\lambda^{\xi_{f_j}} \rangle = \sum_{x \in \mathcal{P}_\lambda} \xi_{f_j}(x, \mathcal{P}_\lambda) - \mathbb{E}\left[\sum_{x \in \mathcal{P}_\lambda} \xi_{f_j}(x, \mathcal{P}_\lambda)\right] = f_j(K_\lambda) - \mathbb{E}[f_j(K_\lambda)],$$

for all $j \in \{0, \ldots, d-1\}$.
Now, we turn to the intrinsic volumes $V_i(K_\lambda)$, $i \in \{1, \ldots, d\}$, and start with the proof of Theorem 3.4.1. Likewise, setting $f_{R_\lambda} \equiv 1$ in Theorem 3.4.13 and by using

$$R_\lambda^i \sim (\beta \log \lambda)^{\frac{i}{\beta}},$$

as $\lambda \to \infty$, as well as (2.1), we get that

$$\begin{aligned}
& \mathbb{E}[(\beta \log \lambda)^{\frac{\beta(d+1)-2i}{2\beta}} (V_i(\mathbb{B}^d(\boldsymbol{o}, R_\lambda)) - V_i(K_\lambda))] && \sim c\,(\log \lambda)^{\frac{d-1}{2}} \\
\Leftrightarrow\quad & (\beta \log \lambda)^{-\frac{i}{\beta}} \binom{d}{i} \frac{\kappa_d}{\kappa_{d-i}} R_\lambda^i - (\beta \log \lambda)^{-\frac{i}{\beta}} \mathbb{E}[V_i(K_\lambda)] && \sim c\,(\log \lambda)^{-1} \\
\Leftrightarrow\quad & (\beta \log \lambda)^{-\frac{i}{\beta}} \mathbb{E}[V_i(K_\lambda)] && \sim (\beta \log \lambda)^{-\frac{i}{\beta}} \binom{d}{i} \frac{\kappa_d}{\kappa_{d-i}} R_\lambda^i \\
\Leftrightarrow\quad & (\beta \log \lambda)^{-\frac{i}{\beta}} \mathbb{E}[V_i(K_\lambda)] && \sim \binom{d}{i} \frac{\kappa_d}{\kappa_{d-i}} \\
\Leftrightarrow\quad & \mathbb{E}[V_i(K_\lambda)] && \sim \binom{d}{i} \frac{\kappa_d}{\kappa_{d-i}} (\beta \log \lambda)^{\frac{i}{\beta}},
\end{aligned}$$

proving the expectation asymptotics for the intrinsic volumes of K_λ.
In order to deal with the variance of the intrinsic volumes, we notice that part two of Theorem 3.4.13 yields that, as $\lambda \to \infty$,

$$\begin{aligned}
& \mathrm{var}[(\beta \log \lambda)^{\frac{\beta(d+1)-2i}{2\beta}} (V_i(\mathbb{B}^d(\boldsymbol{o}, R_\lambda)) - V_i(K_\lambda))] && \sim c\,(\beta \log \lambda)^{\frac{d-1}{2}} \\
\Leftrightarrow\quad & \mathrm{var}[V_i(\mathbb{B}^d(\boldsymbol{o}, R_\lambda)) - V_i(K_\lambda)] && \sim c\,(\beta \log \lambda)^{\frac{\beta(d-1)-2(\beta(d+1)-2i)}{2\beta}} \\
\Leftrightarrow\quad & \mathrm{var}[V_i(K_\lambda)] && \sim c\,(\beta \log \lambda)^{\frac{4i-\beta(d+3)}{2\beta}}.
\end{aligned}$$

This finishes the proof of the variance asymptotics for the intrinsic volumes of K_λ.

Regarding to the proof of the other four theorems, we first deduce from Theorem 3.3.1 and (3.22) that, for sufficiently large λ,

$$|c^k[R_\lambda^{\frac{\beta(d+1)-2i}{2}}(V_i(\mathbb{B}^d(\boldsymbol{o},R_\lambda))-V_i(K_\lambda))]| \le c_1\, c_2^k\, R_\lambda^{\frac{\beta(d-1)}{2}}\,(k!)^{2d+i+5}.$$

In view of the properties of cumulants, summarized in Lemma 2.4.1, we have that

$$\begin{aligned}|c^k[R_\lambda^{\frac{\beta(d+1)-2i}{2}}(V_i(\mathbb{B}^d(\boldsymbol{o},R_\lambda))-V_i(K_\lambda))]| &= |(-1)^k R_\lambda^{k\frac{\beta(d+1)-2i}{2}}\, c^k[V_i(K_\lambda)-V_i(\mathbb{B}^d(\boldsymbol{o},R_\lambda))]| \\ &= R_\lambda^{k\frac{\beta(d+1)-2i}{2}}\,|c^k[V_i(K_\lambda)]|.\end{aligned}$$

Therefore, for sufficiently large λ and $k \in \{3,4,\dots\}$,

$$|c^k[V_i(K_\lambda)]| \le c_1\, c_2^k\,(\log\lambda)^{\frac{\beta(d-kd-k-1)+2ki}{2\beta}}\,(k!)^{2d+i+5}. \tag{3.72}$$

In combination with the lower variance bound $\mathrm{var}[V_i(K_\lambda)] \ge c\,(\log\lambda)^{\frac{4i-\beta(d+3)}{2\beta}}$, following from Theorem 3.4.1, we get, similarly as above,

$$\begin{aligned}\frac{|c^k[V_i(K_\lambda)]|}{(\sqrt{\mathrm{var}[V_i(K_\lambda)]})^k} &\le c_1\, c_2^k\,(\log\lambda)^{\frac{\beta(d-kd-k-1)+2ki}{2\beta}}\,(k!)^{2d+i+5}\big(c_3\,(\log\lambda)^{\frac{4i-\beta(d+3)}{2\beta}}\big)^{-\frac{k}{2}} \\ &\le c_1\, c_2^k\,(\log\lambda)^{\frac{2\beta(d-kd-k-1)+4ki-4ki+k\beta(d+3)}{4\beta}}\,(k!)^{2d+i+5} \\ &\le c_1\, c_2^k\,(\log\lambda)^{\frac{2(\beta d-\beta)-k(\beta d-\beta)}{4\beta}}\,(k!)^{2d+i+5} \\ &\le c_1\, c_2^k\,\left((\log\lambda)^{\frac{d-1}{4}}\right)^{-(k-2)}\,(k!)^{2d+i+5}.\end{aligned}$$

Thus, the random variables

$$X_\lambda := \frac{V_i(K_\lambda)-\mathbb{E}[V_i(K_\lambda)]}{\sqrt{\mathrm{var}[V_i(K_\lambda)]}}$$

fulfill the conditions of Theorem 2.4.3 with

$$\gamma := 2d+i+4 \qquad \text{and} \qquad \Delta_\lambda := c\,(\log\lambda)^{\frac{d-1}{4}}.$$

This completes the proof for the intrinsic volume functionals. □

Proof of Theorem 3.4.7 (i). Let $i \in \{1, \dots, d\}$. Define the sequence $(\lambda_k)_{k\in\mathbb{N}}$ by putting $\lambda_k := a^k$, $a > 1$, for all $k \in \mathbb{N}$. The concentration inequality stated in Theorem 3.4.5 (i) and the lower variance bound $\operatorname{var}[V_i(K_{\lambda_k})] \geq c\,(\log \lambda_k)^{\frac{4i-\beta(d+3)}{2\beta}}$ from Theorem 3.4.1 (i) imply, together with the elementary inequality $\exp(-\min\{a,b\}) \leq \exp(-a)+\exp(-b)$, $a, b \geq 0$, for sufficiently large k, $p \in \mathbb{R}$ and $\varepsilon > 0$, that

$$\begin{aligned}
&\mathbb{P}\big(|V_i(K_{\lambda_k}) - \mathbb{E}[V_i(K_{\lambda_k})]| \geq \varepsilon\,(\log \lambda_k)^{p\frac{i}{\beta}}\big)\\
&\qquad \leq 2\exp\left(-c_1\left(\frac{\varepsilon\,(\log\lambda_k)^{p\frac{i}{\beta}}}{\sqrt{\operatorname{var}[V_i(K_{\lambda_k})]}}\right)^2\right)\\
&\qquad\qquad + 2\exp\left(-c_2\,(\log\lambda_k)^{\frac{d-1}{4(2d+i+5)}}\left(\frac{\varepsilon\,(\log\lambda_k)^{p\frac{i}{\beta}}}{\sqrt{\operatorname{var}[V_i(K_{\lambda_k})]}}\right)^{\frac{1}{2d+i+5}}\right)\\
&\qquad \leq 2\exp\left(-c_1\left(\frac{\varepsilon\,(\log\lambda_k)^{p\frac{i}{\beta}}}{(\log\lambda_k)^{\frac{4i-\beta(d+3)}{2\beta}}}\right)^2\right)\\
&\qquad\qquad + 2\exp\left(-c_2\,(\log\lambda_k)^{\frac{d-1}{4(2d+i+5)}}\left(\frac{\varepsilon\,(\log\lambda_k)^{p\frac{i}{\beta}}}{(\log\lambda_k)^{\frac{4i-\beta(d+3)}{2\beta}}}\right)^{\frac{1}{2d+i+5}}\right)\\
&\qquad \leq 2\exp\Big(-c_1\,\varepsilon^2\,(\log\lambda_k)^{\frac{i(4p-4)+\beta(d+3)}{2\beta}}\Big) + 2\exp\Big(-c_2\,\varepsilon^{\frac{1}{2d+i+5}}\,(\log\lambda_k)^{\frac{i(4p-4)+\beta(2d+2)}{4\beta(2d+i+5)}}\Big).
\end{aligned}$$

Next, we notice that

$$\sum_{k=1}^{\infty}\exp\Big(-(\log\lambda_k)^{\frac{i(4p-4)+\beta(2d+2)}{4\beta(2d+i+5)}}\Big) = \sum_{k=1}^{\infty}\exp\big(-(\log a)^{\frac{i(4p-4)+\beta(2d+2)}{4\beta(2d+i+5)}}\,k^{\frac{i(4p-4)+\beta(2d+2)}{4\beta(2d+i+5)}}\big) < \infty,$$

since $\frac{i(4p-4)+\beta(2d+2)}{4\beta(2d+i+5)} > 0$ and, thus, that the series converges. Indeed, the latter condition is equivalent to $p > \frac{2i-\beta(d+1)}{2i}$ and we observe that

$$\frac{4i-\beta(d+3)}{4i} \geq \frac{2i-\beta(d+1)}{2i}$$

holds true for all $d \geq 2$. Therefore, the exponent at k is indeed larger than 0 for all p in the corresponding range. Similarly, one has that

$$\sum_{k=1}^{\infty}\exp\big(-(\log\lambda_k)^{\frac{i(4p-4)+\beta(d+3)}{2\beta}}\big) = \sum_{k=1}^{\infty}\exp\big(-(\log a)^{\frac{i(4p-4)+\beta(d+3)}{2\beta}}\,k^{\frac{i(4p-4)+\beta(d+3)}{2\beta}}\big)$$

is finite, as long as $i(4p-4)+\beta(d+3) > 0$, which is equivalent to $p > \frac{4i-\beta(d+3)}{4i}$.

Thus, the series

$$\sum_{k=1}^{\infty} \mathbb{P}\big(|V_i(K_{\lambda_k}) - \mathbb{E}[V_i(K_{\lambda_k})]| \geq \varepsilon\,(\log \lambda_k)^{p\frac{i}{2}}\big)$$

converges for all $p > \frac{4i-\beta(d+3)}{4i}$ and the Borel-Cantelli lemma implies that

$$\frac{V_i(K_{\lambda_k}) - \mathbb{E}[V_i(K_{\lambda_k})]}{(\log \lambda_k)^{p\frac{i}{\beta}}} \longrightarrow 0, \tag{3.73}$$

with probability 1, as $k \to \infty$, for all $p > \frac{4i-\beta(d+3)}{4i}$. □

Proof of Theorem 3.4.7 (ii). Let $j \in \{0, \ldots, d-1\}$, and let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence defined by $\lambda_k := a^k$, for all $k \in \mathbb{N}$ and some $a > 1$. Using the concentration inequality for the number of j-dimensional faces in Theorem 3.4.5 (*ii*) in combination with the lower variance bound $\operatorname{var}[f_j(K_{\lambda_k})] \geq c\,(\log \lambda_k)^{\frac{d-1}{2}}$ (see Theorem 3.4.1 (*ii*)) yields, for sufficiently large k, $p \in \mathbb{R}$ and $\varepsilon > 0$, similarly as in the foregoing proof, that

$$\begin{aligned}
&\mathbb{P}\big(|f_j(K_{\lambda_k}) - \mathbb{E}[f_j(K_{\lambda_k})]| \geq \varepsilon\,(\log \lambda_k)^{p\frac{d-1}{2}}\big) \\
&\quad \leq 2\exp\left(-c_1\,\varepsilon^2\,(\log \lambda_k)^{\frac{2p(d-1)}{2} - \frac{d-1}{2}}\right) + 2\exp\left(-c_2\,\varepsilon^{\frac{1}{2j(d+1)+z[f_j]}}\,(\log \lambda_k)^{\frac{2p(d-1)}{4(2j(d+1)+z[f_j])}}\right).
\end{aligned}$$

Now, the series

$$\sum_{k=1}^{\infty} \exp\left(-\,(\log \lambda_k)^{\frac{p(d-1)}{2(2j(d+1)+z[f_j])}}\right) = \sum_{k=1}^{\infty} \exp\left(-\,(\log a)^{\frac{p(d-1)}{2(2j(d+1)+z[f_j])}}\;k^{\frac{p(d-1)}{2(2j(d+1)+z[f_j])}}\right)$$

converges, as long as the exponent is bigger than 0. This holds, since by assumption p is bigger than 0. On the other hand, the series

$$\sum_{k=1}^{\infty} \exp\left(-\,(\log \lambda_k)^{\frac{(d-1)(2p-1)}{2}}\right) = \sum_{k=1}^{\infty} \exp\left(-\,(\log a)^{\frac{(d-1)(2p-1)}{2}}\;k^{\frac{(d-1)(2p-1)}{2}}\right)$$

converges if $\frac{(d-1)(2p-1)}{2} > 0$, which is equivalent to $p > \frac{1}{2}$. Thus, the series

$$\sum_{k=1}^{\infty} \mathbb{P}\big(|f_j(K_{\lambda_k}) - \mathbb{E}[f_j(K_{\lambda_k})]| \geq \varepsilon\,(\log \lambda_k)^{p\frac{d-1}{2}}\big)$$

converges for all $p > \frac{1}{2}$ and the Borel-Cantelli lemma implies the desired almost sure convergence for all such p. □

Proof of Theorem 3.4.9. The k-th moment of a random variable X equals the k-th complete Bell polynomial evaluated in $c^1[X], \dots, c^k[X]$ (see [89]). Specifically,

$$\mathbb{E}[X^k] = B_k(c^1[X], \dots, c^k[X]),$$

with

$$B_k(x_1, \dots, x_k) := \sum_{i=1}^{k} B_{k,i}(x_1, \dots, x_{k-i+1}).$$

Here, the Bell polynomials $B_{k,i}$ are given by

$$\begin{aligned} &B_{k,i}(x_1, \dots, x_{k-i+1}) \\ &\quad := \sum \frac{k!}{j_1! \cdots j_{k-i+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{k-i+1}}{(k-i+1)!}\right)^{j_{k-i+1}}, \end{aligned} \tag{3.74}$$

where the sum runs over all $k-i+1$ tuples $j_1, \dots, j_{k-i+1} \in \mathbb{N}_0$, satisfying

$$\sum_{\ell=1}^{k-i+1} j_\ell = i \qquad \text{and} \qquad \sum_{\ell=1}^{k-i+1} \ell\, j_\ell = k.$$

Thus, the k-th moment can be written as a polynomial of the cumulants up to order k. In particular, for $i = k$ in the above sum, it contains the term $(c^1[X])^k$.
Let us first consider the intrinsic volumes of K_λ, and let $i \in \{1, \dots, d\}$. We know from Theorem 3.4.1 (i) that, for sufficiently large λ, $c^1[V_i(K_\lambda)] = \mathbb{E}[V_i(K_\lambda)] \le c_1 (\log \lambda)^{\frac{i}{\beta}}$ and $c^2[V_i(K_\lambda)] = \operatorname{var}[V_i(K_\lambda)] \le c_2 (\log \lambda)^{\frac{4i-\beta(d+3)}{2\beta}}$. The exponent appearing in the expectation is bigger than the one in the variance, as long as $\beta \ge \frac{2i}{d+3}$. Indeed, it holds that

$$\frac{i}{\beta} \ge \frac{4i - \beta(d+3)}{2\beta} \quad \Leftrightarrow \quad 2i \ge 4i - \beta(d+3) \quad \Leftrightarrow \quad \beta(d+3) \ge 2i \quad \Leftrightarrow \quad \beta \ge \frac{2i}{d+3}.$$

In view of the estimate provided in (3.72), for sufficiently large λ, all higher-order cumulants are bounded by a constant times

$$(\log \lambda)^{\frac{\beta(d-kd-k-1)+2ki}{2\beta}} = (\log \lambda)^{\frac{\beta(d-1)-k(\beta d+\beta-2i)}{2\beta}}.$$

Now, we have that $\beta d + \beta - 2i \ge 0$, for all $\beta \ge \frac{2i}{d+1}$.

Thus, if $\beta \geq \frac{2i}{d+1}$, for sufficiently large λ, it holds that

$$c^k[V_i(K_\lambda)] \leq c_1\, c_2^k\, (\log \lambda)^{\frac{\beta(d-1)-3(\beta d+\beta-2i)}{2\beta}} = c_1\, c_2^k\, (\log \lambda)^{\frac{3i-\beta(d+2)}{\beta}}, \tag{3.75}$$

for all $k \geq 3$. Similarly as above, we obtain that the exponent in the expectation is bigger than those appearing in the bound (3.75) for all $\beta \geq \frac{2i}{d+2}$. Hence, $(c^1[V_i(K_\lambda)])^k$ is the dominating term in the above sum, as long as the condition $\beta \geq \frac{2i}{d+1}$ is fulfilled. Moreover, it is known that the total number of terms appearing in the k-th complete Bell polynomial is the same as the total number of integer partitions of k, which in turn is bounded by $\exp(\pi\sqrt{2k/3})$ (see [106]). Furthermore, (3.74) shows that each coefficient in the sum is bounded from above by $k!$. Combining these facts, we get, for sufficiently large λ, $k \in \mathbb{N}$ and all $\beta \geq \frac{2i}{d+1}$, that

$$\mathbb{E}[V_i(K_\lambda)^k] \leq c_1^k\, k!\, (c^1[V_i(K_\lambda)])^k \leq c_2\, c_3^k\, k!\, (\log \lambda)^{k\frac{i}{\beta}}.$$

In a second step, we prove the upper moment bound for the number of j-dimensional faces of K_λ, $j \in \{0, \dots, d-1\}$. Each cumulant of $f_j(K_\lambda)$ of order bigger than 2 is bounded by a constant multiple of $(\log \lambda)^{\frac{d-1}{2}}$ in view of Theorem 3.3.1, for sufficiently large λ. The expectation and variance growth is of the same order (see Theorem 3.4.1 (*ii*)). Thus, again it follows that $(c^1[f_j(K_\lambda)])^k$ is the term of leading order, this time *for all* values of the parameter β. Hence, we achieve similarly as above, for sufficiently large λ and $k \in \mathbb{N}$, that

$$\mathbb{E}[f_j(K_\lambda)^k] \leq c_1^k\, k!\, (c^1[f_j(K_\lambda)])^k \leq c_2\, c_3^k\, k!\, (\log \lambda)^{k\frac{d-1}{2}}.$$

Finally, Jensen's inequality implies that the k-th moment of $V_i(K_\lambda)$, $i \in \{1, \dots, d\}$, and the k-th moment of $f_j(K_\lambda)$, $j \in \{0, \dots, d-1\}$, is bounded from below by $(\mathbb{E}[V_i(K_\lambda)])^k$ and $(\mathbb{E}[f_j(K_\lambda)])^k$, respectively. Using the estimates for the respective expectations completes the proof. □

3.5.4 Germ-grain processes

Proof of Theorem 3.4.19. We start with the following observation, a modification of [23, Lemma 3.1]. It shows that for fixed $w \in W_\lambda$, the quasi-paraboloids $[\Pi^\uparrow(w)]^{(\lambda)}$ and $[\Pi^\downarrow(w)]^{(\lambda)}$ locally approximate the paraboloids $[\Pi^\uparrow(w)]^{(\infty)}$ and $[\Pi^\downarrow(w)]^{(\infty)}$, respectively.

Let $w := (v_1, h_1) \in W_\lambda$, $L \in (0, \infty)$ and λ be sufficiently large. Then, it holds that

$$\begin{aligned}
&||\partial([\Pi^\uparrow(w)]^{(\lambda)} \cap C_{d-1}(v_1, L)) - \partial([\Pi^\uparrow(w)]^{(\infty)} \cap C_{d-1}(v_1, L))||_\infty \\
&\le C_1\, R_\lambda^{-\frac{1}{2}\beta}\, L^3 + C_2\, h_1\, R_\lambda^{-\beta}\, L^2,
\end{aligned}$$

and

$$\begin{aligned}
&||\partial([\Pi^\downarrow(w)]^{(\lambda)} \cap C_{d-1}(v_1, L)) - \partial([\Pi^\downarrow(w)]^{(\infty)} \cap C_{d-1}(v_1, L))||_\infty \\
&\le C_3\, R_\lambda^{-\frac{1}{2}\beta}\, L^3 + C_4\, h_1\, R_\lambda^{-\beta}\, L^2,
\end{aligned} \tag{3.76}$$

where $C_1, C_2, C_3, C_4 \in (0, \infty)$ are absolute constants. In particular, both right hand sides tend to 0, as $\lambda \to \infty$.
The rest of the proof is exactly the same as the proof of [23, Proposition 5.1] in the Gaussian case. It just uses the two bounds stated above, in combination with Theorem 3.2.4 from Section 3.2.1. As in the proof of Lemma 3.5.1, we omit stating a word by word repetition from [23, Page 34].
What is left is to prove the two assertions stated above. We start with the first one and recall from (3.18) that we have

$$\partial([\Pi^\uparrow(w)]^{(\lambda)}) = \Big\{(v, h) \in W_\lambda : h = R_\lambda^\beta(1 - \cos(d_\lambda(v_1, v))) + h_1\, \cos(d_\lambda(v_1, v))\Big\}.$$

Let $v \in \mathbb{B}^{d-1}(v_1, L)$. The Taylor expansion of the cosine function, together with

$$\begin{aligned}
d_\lambda(v_1, v) &= ||R_\lambda^{-\frac{\beta}{2}} v - R_\lambda^{-\frac{\beta}{2}} v_1|| + O(||R_\lambda^{-\frac{\beta}{2}} v - R_\lambda^{-\frac{\beta}{2}} v_1||^2) \\
&= R_\lambda^{-\frac{\beta}{2}}\, ||v - v_1|| + O(R_\lambda^{-\beta}\, ||v - v_1||^2) \\
&= R_\lambda^{-\frac{\beta}{2}}\, ||v - v_1|| + O(R_\lambda^{-\beta}\, L^2),
\end{aligned} \tag{3.77}$$

gives

$$\begin{aligned}
1 - \cos(d_\lambda(v_1, v)) &= \frac{d_\lambda(v_1, v)^2}{2} + O(d_\lambda(v_1, v)^3) \\
&= R_\lambda^{-\beta}\, \frac{||v - v_1||^2}{2} + O(R_\lambda^{-\frac{3}{2}\beta}\, L^3),
\end{aligned}$$

as $\lambda \to \infty$. Thus,

$$R_\lambda^\beta(1 - \cos(d_\lambda(v_1, v))) = \frac{||v - v_1||^2}{2} + O(R_\lambda^{-\frac{1}{2}\beta}\, L^3),$$

and

$$|h_1(1-\cos(d_\lambda(v_1,v)))| = O(h_1 R_\lambda^{-\beta} L^2),$$

as $\lambda \to \infty$. The two last equations prove that $\partial([\Pi^\uparrow(w)]^{(\lambda)} \cap C_{d-1}(v_1,L))$ and the boundary of $([\Pi^\uparrow(w)]^{(\infty)} \cap C_{d-1}(v_1,L))$, which is given by the graph

$$v \mapsto h_1 + \frac{||v-v_1||^2}{2},$$

(see the discussion before (3.20)), differ by at most $C_1 R_\lambda^{-\frac{1}{2}\beta} L^3 + C_2 h_1 R_\lambda^{-\beta} L^2$. This finishes the proof of the first assertion.
Moreover, we have from (3.19) that

$$\partial([\Pi^\downarrow(w)]^{(\lambda)}) = \left\{ (v,h) \in W_\lambda : h = R_\lambda^\beta - \frac{R_\lambda^\beta - h_1}{\cos(d_\lambda(v_1,v))} \right\}.$$

By using again the Taylor expansion up to second order, the fact that

$$\frac{1}{1-x} = 1 + x + x^2 + \dots,$$

and the preparation (3.77), we obtain for all $(v,h) \in \partial([\Pi^\downarrow(w)]^{(\lambda)}) \cap C_{d-1}(v_1,L))$ that

$$\begin{aligned}
h &= R_\lambda^\beta - \frac{R_\lambda^\beta - h_1}{\cos(d_\lambda(v_1,v))} \\
&= R_\lambda^\beta - \frac{R_\lambda^\beta - h_1}{\left(1 - \frac{d_\lambda(v_1,v)^2}{2}\right)} \\
&= R_\lambda^\beta - (R_\lambda^\beta - h_1)\left(1 + \frac{d_\lambda(v_1,v)^2}{2} + O(d_\lambda(v_1,v)^4)\right) \\
&= R_\lambda^\beta - (R_\lambda^\beta - h_1)\left(1 + R_\lambda^{-\beta}\frac{||v-v_1||^2}{2} + O(R_\lambda^{-\frac{3}{2}\beta} L^3)\right) \\
&= R_\lambda^\beta - R_\lambda^\beta - \frac{||v-v_1||^2}{2} + O(R_\lambda^{-\frac{1}{2}\beta} L^3) + h_1 + h_1 R_\lambda^{-\beta}\frac{||v-v_1||^2}{2} + O(h_1 R_\lambda^{-\frac{3}{2}\beta} L^3) \\
&= h_1 - \frac{||v-v_1||^2}{2} + O(R_\lambda^{-\frac{1}{2}\beta} L^3) + O(h_1 R_\lambda^{-\beta} L^2),
\end{aligned}$$

as $\lambda \to \infty$. Then, the result follows in the same way as in the first case. □

Chapter 4

Random simplices in high dimensions

Let $r = r(d)$ be a sequence of integers such that $r \leq d$, and let $X_1, \dots, X_{r+1}$ be independent random points, distributed according to the Gaussian, the Beta or the spherical distribution on $\mathbb{R}^d$, $d \geq 2$. In this chapter, limit theorems for the log-volume and the volume of the random convex hull of $X_1, \dots, X_{r+1}$ are established in high dimensions, that is, as r and d tend to infinity simultaneously.
This includes Berry-Esseen-type central limit theorems, log-normal limit theorems, moderate and large deviations, as well as concentration inequalities. Moreover, different types of mod-ϕ convergence are derived.
The results heavily depend on the asymptotic growth of r, relative to d. For example, we prove that the fluctuations of the volume of the simplex are normal (respectively, log-normal) if $r = o(d)$ (respectively, $r \sim \alpha d$, for some $\alpha \in (0, 1)$), as $d \to \infty$.

4.1 Models, volumes and probabilistic representations

4.1.1 The four models

In this chapter, we consider convex hulls of random points $X_1, X_2, \ldots$. More in detail, we consider the following four models which allow for explicit computations. These models were identified by Miles [101] and Ruben and Miles [115], respectively.

(a) The *Gaussian model*: $X_1, X_2, \ldots$ are independent and identically distributed with standard normal density

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{1}{2}\|x\|^2}, \qquad x \in \mathbb{R}^d.$$

(b) The *Beta model* with parameter $\nu > 0$: $X_1, X_2, \ldots$ are independent and identically distributed with density

$$f(x) = \frac{1}{\pi^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(1 - \|x\|^2\right)^{\frac{\nu-2}{2}}, \qquad x \in \mathbb{R}^d, \ \|x\| < 1.$$

(c) The *Beta prime model* with parameter $\nu > 0$: $X_1, X_2, \ldots$ are independent and identically distributed with density

$$f(x) = \frac{1}{\pi^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \|x\|^2\right)^{-\frac{d+\nu}{2}}, \qquad x \in \mathbb{R}^d.$$

(d) The *spherical model*: $X_1, X_2, \ldots$ are independent and uniformly distributed on the sphere $\mathbb{S}^{d-1}$.

Remark 4.1.1 Observe that in the Beta prime model, the power is $(d+\nu)/2$ (which depends on d) rather than just $(\nu - 2)/2$.

4.1.2 Moments

Let $1 \le r \le d$ be an integer and $X_1, \ldots, X_{r+1}$ be independent random points in $\mathbb{R}^d$, distributed according to one of the distributions introduced in Section 4.1.1.

Theorem 4.1.2 (Moments for simplices) *Let $\mathcal{V}_{d,r}$ be the volume of the r-dimensional simplex with vertices $X_1, \ldots, X_{r+1}$.*

(a) In the Gaussian model, for all $k \geq 0$, it holds that

$$\mathbb{E}\left[(r!\mathcal{V}_{d,r})^{2k}\right] = (r+1)^k \prod_{j=1}^{r} \left[2^k \frac{\Gamma\left(\frac{d-r+j}{2}+k\right)}{\Gamma\left(\frac{d-r+j}{2}\right)}\right].$$

(b) In the Beta model with parameter $\nu > 0$, for all $k \geq 0$, it holds that

$$\begin{aligned}&\mathbb{E}\left[(r!\mathcal{V}_{d,r})^{2k}\right]\\&= \prod_{j=1}^{r}\left[\frac{\Gamma\left(\frac{d-r+j}{2}+k\right)}{\Gamma\left(\frac{d-r+j}{2}\right)}\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d+\nu}{2}+k\right)}\right]\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d+\nu}{2}+k\right)}\frac{\Gamma\left(\frac{r(d+\nu-2)+(d+\nu)}{2}+(r+1)k\right)}{\Gamma\left(\frac{r(d+\nu-2)+(d+\nu)}{2}+rk\right)}.\end{aligned}$$

(c) In the Beta prime model with parameter $\nu > 0$, for all $0 \leq k < \frac{\nu}{2}$, it holds that

$$\begin{aligned}&\mathbb{E}\left[(r!\mathcal{V}_{d,r})^{2k}\right]\\&= \prod_{j=1}^{r}\left[\frac{\Gamma\left(\frac{d-r+j}{2}+k\right)}{\Gamma\left(\frac{d-r+j}{2}\right)}\frac{\Gamma\left(\frac{\nu}{2}-k\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right]\frac{\Gamma\left(\frac{\nu}{2}-k\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{\Gamma\left(\frac{(r+1)\nu}{2}-rk\right)}{\Gamma\left(\frac{(r+1)\nu}{2}-(r+1)k\right)}.\end{aligned}$$

(d) In the spherical model, for all $k \geq 0$, it holds that

$$\begin{aligned}&\mathbb{E}\left[(r!\mathcal{V}_{d,r})^{2k}\right]\\&= \prod_{j=1}^{r}\left[\frac{\Gamma\left(\frac{d-r+j}{2}+k\right)}{\Gamma\left(\frac{d-r+j}{2}\right)}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}+k\right)}\right]\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}+k\right)}\frac{\Gamma\left(\frac{r(d-2)+d}{2}+(r+1)k\right)}{\Gamma\left(\frac{r(d-2)+d}{2}+rk\right)}.\end{aligned}$$

Proof. Moments of integer orders can be computed by using the linear Blaschke-Petkantschin formula [119, Theorem 7.2.1] from integral geometry, together with an induction argument. Using this technique, as an example, [119, Theorem 8.2.3] provides a detailed proof in the spherical model. In particular, formula (a) is [101, Equation (70)], formula (b) is [101, Equation (74)] and formula (c) is [101, Equation (72)]. Finally, formula (d) is obtained from (b) by letting $\nu \to 0$. Note that the formula in [101] contains a typo, which is corrected, for example, in [28]. On the other hand, Miles [101] considers only integer moments. Extensions to non-integer moments for all four models can be found in [80]. □

Remark 4.1.3 Observe that the moments in the spherical case can be obtained from the ones in the Beta model by taking $\nu = 0$ there. Since the proofs of our limit theorems are based on the formulas for the moments, we may and will consider the spherical and the Beta model together, the former being the special case of the latter with $\nu = 0$.

4.1.3 Distributions

A random variable has a Gamma distribution with shape $\alpha > 0$ and scale $\lambda > 0$, if its density is

$$g(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\, t^{\alpha-1}\, e^{-\lambda t}, \qquad t \geq 0.$$

Especially, if $\alpha = \frac{a}{2}$, for some $a \in \mathbb{N}$, and $\lambda = \frac{1}{2}$, we speak about a χ^2 distribution with a degrees of freedom. Moreover, a random variable has a Beta distribution or Beta prime distribution with parameter $\alpha_1, \alpha_2 > 0$, if its density is

$$g(t) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\, t^{\alpha_1 - 1}\,(1-t)^{\alpha_2 - 1}, \qquad t \in (0,1),$$

or

$$g(t) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\, t^{\alpha_1 - 1}\,(1+t)^{-\alpha_1 - \alpha_2}, \qquad t > 0,$$

respectively. We denote by χ_a^2, respectively $\Gamma_{\alpha,\lambda}, \beta_{\alpha_1,\alpha_2}, \beta'_{\alpha_1,\alpha_2}$, a random variable with χ^2 distribution with $a \in \mathbb{N}$ degrees of freedom and the Gamma, Beta or Beta prime distribution with corresponding parameter, respectively. We use the notation $X \overset{D}{\sim} \mathrm{Beta}(\alpha_1, \alpha_2)$ or $X \overset{D}{\sim} \mathrm{Beta}'(\alpha_1, \alpha_2)$ to indicate that a random variable X has a Beta or a Beta prime distribution with parameter α_1 and α_2, respectively. Furthermore, the moments of order $k \geq 0$ of these distributions are given by

$$\mathbb{E}[\chi_a^{2k}] = 2^k \frac{\Gamma\left(\frac{a}{2} + k\right)}{\Gamma\left(\frac{a}{2}\right)}, \qquad \mathbb{E}[(\beta_{\alpha_1,\alpha_2})^k] = \frac{\Gamma(\alpha_1 + \alpha_2)\Gamma(\alpha_1 + k)}{\Gamma(\alpha_1)\Gamma(\alpha_1 + \alpha_2 + k)},$$

and

$$\mathbb{E}[(\beta'_{\alpha_1,\alpha_2})^k] = \frac{\Gamma(\alpha_1 + k)\Gamma(\alpha_2 - k)}{\Gamma(\alpha_1)\Gamma(\alpha_2)},$$

(see [79, Page 168], [78, Page 40] and [78, Page 87]), respectively.

The distribution of the volume of a random simplex generated by one of the four models under consideration can be derived from Theorem 4.1.2.

Theorem 4.1.4 (Distributions for simplices) *Let $\mathcal{V}_{d,r}$ denote the volume of the r-dimensional simplex with vertices $X_1, \ldots, X_{r+1}$, chosen according to the one of the above four models.*

(a) In the Gaussian model, we have

$$(r!\mathcal{V}_{d,r})^2 \overset{D}{\sim} (r+1) \prod_{j=1}^{r} \chi^2_{d-r+j}.$$

(b) In the Beta model, we have

$$\xi\,(1-\xi)^r\,(r!\mathcal{V}_{d,r})^2 \overset{D}{\sim} (1-\eta)^r \prod_{j=1}^{r} \beta_{\frac{d-r+j}{2},\frac{\nu+r-j}{2}},$$

where $\xi, \eta \overset{D}{\sim} \mathrm{Beta}(\frac{d+\nu}{2}, \frac{r(d+\nu-2)}{2})$ are independent random variables such that ξ is independent of $\mathcal{V}_{d,r}$, while η is independent of $\beta_{\frac{d-r+j}{2},\frac{\nu+r-j}{2}}$, $j = 1, \ldots, r$.

(c) In the Beta prime model, we have

$$(1+\eta)^r\,(r!\mathcal{V}_{d,r})^2 \overset{D}{\sim} \xi^{-1}(1+\xi)^{r+1} \prod_{k=1}^{r} \beta'_{\frac{d-r+j}{2},\frac{\nu}{2}},$$

where $\xi, \eta \overset{D}{\sim} \mathrm{Beta}'(\frac{\nu}{2}, \frac{r\nu}{2})$ are independent random variables such that η is independent of $\mathcal{V}_{d,r}$, while ξ is independent of $\beta'_{\frac{d-r+j}{2},\frac{\nu}{2}}$, $j = 1, \ldots, r$.

(d) In the spherical model, we have

$$\xi\,(1-\xi)^r\,(r!\mathcal{V}_{d,r})^2 \overset{D}{\sim} (1-\eta)^r \prod_{j=1}^{r} \beta_{\frac{d-r+j}{2},\frac{r-j}{2}},$$

where $\xi, \eta \sim \mathrm{Beta}(\frac{d}{2}, \frac{r(d-2)}{2})$ are independent random variables such that ξ is independent of $\mathcal{V}_{d,r}$, while η is independent of $\beta_{\frac{d-r+j}{2},\frac{r-j}{2}}$, $j = 1, \ldots, r$.

Proof. The assertion in (a) follows directly from Theorem 4.1.2 (a), combined with the fact that the k-th moment of a χ^2_{d-r+j} random variable is given by

$$2^{\frac{k}{2}}\frac{\Gamma\left(\frac{d-r+j}{2}+\frac{k}{2}\right)}{\Gamma(\frac{d-r+j}{2})}.$$

To prove (b), we define $\alpha_1 := \frac{d+\nu}{2}$ and $\alpha_2 := \frac{r(d+\nu-2)}{2}$. Since $\xi, \eta \overset{D}{\sim} \mathrm{Beta}(\alpha_1, \alpha_2)$,

$$\mathbb{E}[(1-\eta)^{rk}] = \frac{1}{B(\alpha_1,\alpha_2)}\int_0^1 x^{\alpha_1-1}(1-x)^{\alpha_2+rk-1}\,\mathrm{d}x = \frac{B(\alpha_1,\alpha_2+rk)}{B(\alpha_1,\alpha_2)},$$

and

$$\mathbb{E}[\xi^k(1-\xi)^{rk}] = \frac{1}{B(\alpha_1,\alpha_2)}\int_0^1 x^{\alpha_1+k-1}(1-x)^{\alpha_2+rk-1}\,\mathrm{d}x = \frac{B(\alpha_1+k,\alpha_2+rk)}{B(\alpha_1,\alpha_2)},$$

where we recall from Section 2.3 that $B(x,y)$ is the Beta function. This implies that

$$\begin{aligned}\frac{\mathbb{E}[(1-\eta)^{rk}]}{\mathbb{E}[\xi^k(1-\xi)^{rk}]} &= \frac{B(\alpha_1,\alpha_2+rk)}{B(\alpha_1+k,\alpha_2+rk)} = \frac{\Gamma(\alpha_1+\alpha_2+(r+1)k)\Gamma(\alpha_1)}{\Gamma(\alpha_1+k)\Gamma(\alpha_1+\alpha_2+rk)}\\ &= \frac{\Gamma\Big(\frac{r(d+\nu-2)+(d+\nu)}{2}+(r+1)k\Big)\Gamma\Big(\frac{d+\nu}{2}\Big)}{\Gamma\Big(\frac{d+\nu}{2}+k\Big)\Gamma\Big(\frac{r(d+\nu-2)+(d+\nu)}{2}+rk\Big)}.\end{aligned}$$

This is precisely the last factor in the formula for the moments (see Theorem 4.1.2 (b)). Next, we consider (c). Since $\xi, \eta \overset{D}{\sim} \mathrm{Beta}'(\alpha_1, \alpha_2)$ with $\alpha_1 = \frac{\nu}{2}$ and $\alpha_2 = \frac{r\nu}{2}$, we apply (2.11) to obtain

$$\mathbb{E}[(1+\eta)^{rk}] = \frac{1}{B(\alpha_1,\alpha_2)}\int_0^\infty x^{\alpha_1-1}\,(1+x)^{-\alpha_1-(\alpha_2-rk)}\,\mathrm{d}x = \frac{B(\alpha_1,\alpha_2-rk)}{B(\alpha_1,\alpha_2)},$$

and

$$\begin{aligned}\mathbb{E}\Big[\xi^{-k}(1+\xi)^{(r+1)k}\Big] &= \frac{1}{B(\alpha_1,\alpha_2)}\int_0^\infty x^{\alpha_1-k-1}\,(1+x)^{-\alpha_1-(\alpha_2-(r+1)k)}\,\mathrm{d}x\\ &= \frac{B(\alpha_1-k,\alpha_2-rk)}{B(\alpha_1,\alpha_2)}.\end{aligned}$$

It follows that

$$\frac{\mathbb{E}\Big[\xi^{-k}(1+\xi)^{(r+1)k}\Big]}{\mathbb{E}[(1+\eta)^{rk}]} = \frac{B(\alpha_1-k,\alpha_2-rk)}{B(\alpha_1,\alpha_2-rk)} = \frac{\Gamma(\alpha_1-k)\Gamma(\alpha_1+\alpha_2-rk)}{\Gamma(\alpha_1+\alpha_2-(r+1)k)\Gamma(\alpha_1)}$$
$$= \frac{\Gamma\Big(\frac{\nu}{2}-k\Big)}{\Gamma\Big(\frac{\nu}{2}\Big)} \frac{\Gamma\Big(\frac{(r+1)\nu}{2}-rk\Big)}{\Gamma\Big(\frac{(r+1)\nu}{2}-(r+1)k\Big)},$$

which is exactly the last factor in the formula for the moments given by Theorem 4.1.2 (c). The assertion in (d) follows as a limit case from that in (b), as $\nu \downarrow 0$. □

Remark 4.1.5 The distributional equality in Theorem 4.1.4 (a) has already been noted by Miles [101, Section 13]. The other probabilistic representations in (b)–(d) seem to be new.

4.2 Cumulants

Here, we concentrate on the Gaussian, the Beta and the spherical model, for which the random variables $\mathcal{V}_{d,r}$ admit finite moments of all orders for any $d \in \mathbb{N}$ and $r \leq d$.

4.2.1 Cumulant estimates

Let us define the random variables

$$\mathcal{L}_{d,r} := \log(r!\mathcal{V}_{d,r}),$$

and

$$\widetilde{\mathcal{L}}_{d,r} := \frac{\mathcal{L}_{d,r} - \mathbb{E}[\mathcal{L}_{d,r}]}{\sqrt{\operatorname{var}[\mathcal{L}_{d,r}]}}.$$

Theorem 4.2.1 *Let $r \leq d$ be an integer and $X_1, \ldots, X_{r+1}$ be chosen according to one of the four models presented above, and let $\alpha \in (0,1)$. Then, the following assertions are true:*

(a) *For the Gaussian model, we have*

$$\mathbb{E}[\mathcal{L}_{d,r}] \sim \frac{r}{2}\log d \qquad \text{and} \qquad \operatorname{var}[\mathcal{L}_{d,r}] \sim \begin{cases} \frac{1}{2}\,\frac{r}{d} & : r = o(d) \\ \frac{1}{2}\,\log\frac{1}{1-\alpha} & : r \sim \alpha d \\ \frac{1}{2}\,\log\frac{d}{d-r+1} & : d-r = o(d), \end{cases}$$

as $d \to \infty$. *Moreover, for all* $k \geq 3$,

$$|c^k[\mathcal{L}_{d,r}]| \leq \begin{cases} c_1^k\,(k-1)!\,r\,d^{1-k} & : r = o(d) \text{ or } r \sim \alpha d \\ 2\,(k-1)! & : \text{for arbitrary } r(d), \end{cases}$$

where $c_1 \in (0,\infty)$ *is a constant not depending on* d *and* k. *Thus, for all* $k \geq 3$ *and sufficiently large* d,

$$|c^k[\widetilde{\mathcal{L}}_{d,r}]| \leq \begin{cases} c_1^k\,(k-1)!\,\left(\frac{1}{\sqrt{rd}}\right)^{k-2} & : r = o(d) \text{ or } r \sim \alpha d \\ c_2^k\,(k-1)!\,\left(\frac{1}{\sqrt{\log\frac{d}{d-r+1}}}\right)^{k} & : d-r = o(d), \end{cases}$$

where $c_1, c_2 \in (0,\infty)$ *are constants not depending on* d *and* k.

(b) *For the Beta model and the spherical model, we have*

$$\operatorname{var}[\mathcal{L}_{d,r}] \sim \frac{1}{2}\log\frac{d}{d-r} - \frac{r^2}{2d(r+1)} \sim \begin{cases} \frac{1}{2}\left(\log\frac{1}{1-\alpha} - \alpha\right) & : r \sim \alpha d \\ \frac{1}{2}\log\frac{d}{d-r+1} & : d-r = o(d), \end{cases}$$

as $d \to \infty$. *Furthermore, for all* $k \geq 3$ *and* $d \geq 4$,

$$|c^k[\mathcal{L}_{d,r}]| \leq \begin{cases} c_1^k\,k!\,r\,d^{1-k} & : r = o(d) \text{ or } r \sim \alpha d \\ 2\,4^k\,k! & : \text{for arbitrary } r(d), \end{cases}$$

where $c_1 \in (0,\infty)$ *is a constant not depending on* d *and* k. *Thus, for all* $k \geq 3$ *and sufficiently large* d,

$$|c^k[\widetilde{\mathcal{L}}_{d,r}]| \leq \begin{cases} c_1^k\,k!\,\left(\frac{1}{d}\right)^{k-2} & : r \sim \alpha d \\ c_2^k\,k!\,\left(\frac{1}{\sqrt{\log\frac{d}{d-r+1}}}\right)^{k-2} & : d-r = o(d), \end{cases}$$

where $c_1, c_2 \in (0,\infty)$ *are constants not depending on* d *and* k.

The proof of Theorem 4.2.1 is to some extent canonical and roughly follows the method used in [38]. In particular, it is based on an asymptotic analysis of the digamma function

$$\psi(z) = \psi^{(0)}(z) := \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z),$$

and the polygamma functions

$$\psi^{(k)}(z) := \frac{\mathrm{d}^k}{\mathrm{d}z^k}\psi(z) = \frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}} \log \Gamma(z), \qquad k \in \mathbb{N},$$

whose properties have been analyzed in detail in Section 2.3.

Since the moments of $\mathcal{V}_{d,r}$ involve the same product of fractions of Gamma functions, we prepare the proof of Theorem 4.2.1 with the following lemma. We define

$$S_{d,r}(z) := \sum_{j=1}^{r} \left[\log \Gamma\left(\frac{d-r+j+z}{2}\right) - \log \Gamma\left(\frac{d-r+j}{2}\right) \right],$$

where $z > 0$.

Lemma 4.2.2 *(a) If $r = o(d)$, as $d \to \infty$, we have*

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} S_{d,r}(z)\Big|_{z=0} \sim \begin{cases} \frac{r}{2} \log d & : k = 1 \\ \frac{(-1)^k}{2}\,(k-2)!\, r\, d^{-(k-1)} & : k \geq 2. \end{cases}$$

(b) If $r \sim \alpha d$, for some $\alpha \in (0,1)$, as $d \to \infty$, we have

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} S_{d,r}(z)\Big|_{z=0} \sim \begin{cases} \frac{\alpha d}{2} \log d & : k = 1 \\ \frac{1}{2} \log \frac{1}{1-\alpha} & : k = 2 \\ \frac{(-1)^k}{2} \left(\frac{1}{(1-\alpha)^{k-2}} - 1\right) (k-3)!\, d^{-(k-2)} & : k \geq 3. \end{cases}$$

(c) If $d - r = o(d)$, as $d \to \infty$, we have

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} S_{d,r}(z)\Big|_{z=0} \sim \begin{cases} \frac{d}{2} \log d & : k = 1 \\ \frac{1}{2} \log \frac{d}{d-r+1} & : k = 2. \end{cases}$$

(d) *For $k \geq 2$, and if $r = o(d)$ or $r \sim \alpha d$, $\alpha \in (0,1)$, there is a constant $c \in (0,\infty)$ which may depends on α (but does not depend on d and k) such that*

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}z^k} S_{d,r}(z)\Big|_{z=0} \right| \leq c^k\,(k-1)!\,r\,d^{-(k-1)}.$$

(e) *Finally, for $k \geq 3$ and without any conditions on r, we have*

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}z^k} S_{d,r}(z)\Big|_{z=0} \right| \leq 2\,(k-1)!.$$

Proof. Let us prove (a), (b) and (c) for $k=1$. We have

$$\frac{\mathrm{d}}{\mathrm{d}z} S_{d,r}(z)\Big|_{z=0} = \frac{1}{2}\sum_{j=1}^{r} \psi\left(\frac{d-r+j}{2}\right) = \frac{1}{2}\sum_{j=1}^{d} \psi\left(\frac{j}{2}\right) - \frac{1}{2}\sum_{j=1}^{d-r} \psi\left(\frac{j}{2}\right),$$

and all three statements follow from (2.21). Next, we prove (a), (b) and (c) for $k \geq 2$. We have

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} S_{d,r}(z)\Big|_{z=0} = \frac{1}{2^k}\sum_{j=1}^{r} \psi^{(k-1)}\left(\frac{d-r+j}{2}\right).$$

We can conclude (a) by using (2.19). To prove (b) for $k=2$, apply (2.22) to get

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}z^2} S_{d,r}(z)\Big|_{z=0} &= \frac{1}{4}\sum_{j=1}^{r} \psi^{(1)}\left(\frac{d-r+j}{2}\right) = \frac{1}{2}\log d + c_1 - \frac{1}{2}\log(d-r) - c_1 + o(1)\\ &= \frac{1}{2}\log\frac{d}{d-r} + o(1) \sim \frac{1}{2}\log\frac{1}{1-\alpha},\end{aligned}$$

as $d \to \infty$. To prove (b) for $k \geq 3$, note that for $r \sim \alpha d$, using (2.19),

$$\begin{aligned}\frac{1}{2^k}\sum_{j=1}^{r} \psi^{(k-1)}\left(\frac{d-r+j}{2}\right) &= \frac{1}{2^k}\sum_{j=d-r+1}^{d} \psi^{(k-1)}\left(\frac{j}{2}\right)\\ &\sim \frac{1}{2^k}\sum_{j=d-r+1}^{d} \frac{(-1)^{k-2}(k-2)!}{\left(\frac{j}{2}\right)^{k-1}} = \frac{(-1)^k\,(k-2)!}{2}\left[\sum_{j=1}^{d}\frac{1}{j^{k-1}} - \sum_{j=1}^{d-r}\frac{1}{j^{k-1}}\right]\\ &\sim \frac{(-1)^k\,(k-2)!}{2}\left[\zeta(k-1) - \frac{1}{(k-2)d^{k-2}} - \zeta(k-1) + \frac{1}{(k-2)(d-r)^{k-2}}\right]\\ &\sim \frac{(-1)^k(k-3)!}{2\cdot d^{k-2}}\left(\frac{1}{(1-\alpha)^{k-2}} - 1\right),\end{aligned}$$

as $d \to \infty$, where we used (2.15) to deal with the ζ-function.

Finally, to prove (c) for $k = 2$, use the formula

$$\frac{1}{4}\sum_{j=1}^{r}\psi^{(1)}\left(\frac{d-r+j}{2}\right) = \frac{1}{2}\log d + O(1),$$

as $d \to \infty$, following from (2.22), to get

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}z^2}S_{d,r}(z)\Big|_{z=0} &= \frac{1}{4}\sum_{j=1}^{r}\psi^{(1)}\left(\frac{d-r+j}{2}\right) = \frac{1}{2}\log d + O(1) - \frac{1}{2}\log(d-r+1) - O(1)\\ &= \frac{1}{2}\log\frac{d}{d-r+1} + O(1) \sim \frac{1}{2}\log\frac{d}{d-r+1},\end{aligned}$$

as $d \to \infty$, since $\frac{d}{d-r+1} \to \infty$. We added the term $+1$ to make the formula work in the case $r = d$, too. Let us prove (d). Observe that the function

$$|\psi^{(k-1)}(z)| = \sum_{j=0}^{\infty}\frac{(k-1)!}{(z+j)^k},$$

(see (2.17)), is decreasing. Thus, we can write

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}S_{d,r}(z)\Big|_{z=0}\right| = \frac{1}{2^k}\sum_{j=1}^{r}\left|\psi^{(k-1)}\left(\frac{d-r+j}{2}\right)\right| \le \frac{r}{2^k}\left|\psi^{(k-1)}\left(\frac{d-r+1}{2}\right)\right|,$$

and the claim follows from the estimate

$$|\psi^{(k-1)}(z)| \le 2\,(k-1)!\,z^{1-k}, \tag{4.1}$$

$z \ge 1$, which is a consequence of (2.20). Let us prove (e). If $k \ge 3$ and r is arbitrary, we see from (2.17) that the function $\psi^{(k-1)}(z)$, $z > 0$, has the same sign as $(-1)^k$. Hence,

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}S_{d,r}(z)\Big|_{z=0}\right| = \frac{1}{2^k}\sum_{j=1}^{r}\left|\psi^{(k-1)}\left(\frac{d-r+j}{2}\right)\right| \le \frac{1}{2^k}\sum_{j=1}^{d}\left|\psi^{(k-1)}\left(\frac{j}{2}\right)\right|.$$

Then, the result follows in view of inequality (2.23). Thus, the proof is complete. □

Proof of Theorem 4.2.1. Denote the moment generating function of $\mathcal{L}_{d,r} = \log(r!\mathcal{V}_{d,r})$ by

$$M_{d,r}(z) := \mathbb{E}[\exp(z\mathcal{L}_{d,r})] = \mathbb{E}[(r!\mathcal{V}_{d,r})^z], \qquad z \ge 0.$$

We start with the Gaussian model. Recalling the moment formula from Theorem 4.1.2 (a), we see that

$$\log M_{d,r}(z) = S_{d,r}(z) + \frac{z}{2}\log(r+1) + \frac{zr}{2}\log 2.$$

Hence,

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\log M_{d,r}(z) = \frac{\mathrm{d}^k}{\mathrm{d}z^k}S_{d,r}(z) + \mathbf{1}(k=1)\,\frac{1}{2}\log(r+1) + \mathbf{1}(k=1)\,\frac{r}{2}\log 2,$$

for all $k \in \mathbb{N}$. Taking $z = 0$, it follows that

$$c^k[\mathcal{L}_{d,r}] = \frac{\mathrm{d}^k}{\mathrm{d}z^k}S_{d,r}(z)\Big|_{z=0} + \mathbf{1}(k=1)\,\frac{1}{2}\log(r+1) + \mathbf{1}(k=1)\,\frac{r}{2}\log 2.$$

By using Lemma 4.2.2, we immediately get the required asymptotic formula for $\mathbb{E}[\mathcal{L}_{d,r}] = c^1[\mathcal{L}_{d,r}]$ and $\mathrm{var}[\mathcal{L}_{d,r}] = c^2[\mathcal{L}_{d,r}]$. The estimates for the cumulants $c^k[\mathcal{L}_{d,r}]$, $k \geq 3$, follow from Lemma 4.2.2 (d),(e).

Next, we consider the Beta model and prove part (b) of the theorem. Recalling the moment formula from Theorem 4.1.2 (b) and denoting by $M_{d,r}(z)$ again the moment generating function of $\mathcal{L}_{d,r}$, we get that

$$\begin{aligned}
&\log M_{d,r}(z)\\
&\quad= S_{d,r}(z) + \log\Gamma\left(\frac{r(d+\nu-2)+(d+\nu)}{2} + \frac{(r+1)z}{2}\right) + (r+1)\log\Gamma\left(\frac{d+\nu}{2}\right)\\
&\qquad - \log\Gamma\left(\frac{r(d+\nu-2)+(d+\nu)}{2} + \frac{rz}{2}\right) - (r+1)\log\Gamma\left(\frac{d+\nu}{2} + \frac{z}{2}\right).
\end{aligned}$$

It follows that, for $k \in \mathbb{N}$,

$$\begin{aligned}
&\frac{\mathrm{d}^k}{\mathrm{d}z^k}\log M_{d,r}(z)\\
&= \frac{\mathrm{d}^k}{\mathrm{d}z^k}S_{d,r}(z) + \left(\frac{r+1}{2}\right)^k \psi^{(k-1)}\left(\frac{r(d+\nu-2)+(d+\nu)}{2} + \frac{(r+1)z}{2}\right)\\
&\quad - \left(\frac{r}{2}\right)^k \psi^{(k-1)}\left(\frac{r(d+\nu-2)+(d+\nu)}{2} + \frac{rz}{2}\right) - \frac{r+1}{2^k}\psi^{(k-1)}\left(\frac{d+\nu}{2} + \frac{z}{2}\right).
\end{aligned} \tag{4.2}$$

Taking $z = 0$, we obtain

$$\begin{aligned} c^k[\mathcal{L}_{d,r}] = \frac{\mathrm{d}^k}{\mathrm{d}z^k} S_{d,r}(z)\Big|_{z=0} + \left(\frac{r+1}{2}\right)^k \psi^{(k-1)}\left(\frac{r(d+\nu-2)+(d+\nu)}{2}\right) \\ - \left(\frac{r}{2}\right)^k \psi^{(k-1)}\left(\frac{r(d+\nu-2)+(d+\nu)}{2}\right) - \frac{r+1}{2^k}\psi^{(k-1)}\left(\frac{d+\nu}{2}\right). \end{aligned} \tag{4.3}$$

Let us compute the asymptotic of $\mathrm{var}[\mathcal{L}_{d,r}] = c^2[\mathcal{L}_{d,r}]$ in the case that $r \sim \alpha d$. First of all, by using (2.19) in the case that $k = 1$, we obtain

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2} S_{d,r}(z)\Big|_{z=0} &= \frac{1}{4}\sum_{j=1}^{r} \psi^{(1)}\left(\frac{d-r+j}{2}\right) \\ &= \frac{1}{4}\sum_{j=1}^{r} \frac{2}{d-r+j} + O\left(\frac{r}{d^2}\right) \\ &= \frac{H_d - H_{d-r}}{2} + O\left(\frac{r}{d^2}\right), \end{aligned}$$

as $d \to \infty$, where H_d is the d-th harmonic number. By using (2.16), we arrive at

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}z^2} S_{d,r}(z)\Big|_{z=0} &= \frac{1}{2}\log\frac{d}{d-r} + \frac{1}{4}\left(\frac{1}{d} - \frac{1}{d-r}\right) + O\left(\frac{1}{d^2}\right) + O\left(\frac{r}{d^2}\right) \\ &= \frac{1}{2}\log\frac{d}{d-r} + O\left(\frac{r}{d^2}\right), \end{aligned}$$

as $d \to \infty$. On the other hand, using (2.19) for $k = 1$, it follows that

$$\begin{aligned} \psi^{(1)}\left(\frac{r(d+\nu-2)+(d+\nu)}{2}\right) &= \frac{2}{d(r+1)+O(r)} + O\left(\frac{1}{d^2r^2}\right) \\ &= \frac{2}{d(r+1)} + O\left(\frac{1}{d^2r}\right), \end{aligned}$$

and

$$\psi^{(1)}\left(\frac{d+\nu}{2}\right) = \frac{2}{d} + O\left(\frac{1}{d^2}\right),$$

as $d \to \infty$. Recalling (4.3) and combining the previous estimates, as $d \to \infty$,

$$\begin{aligned} \mathrm{var}[\mathcal{L}_{d,r}] &= \frac{1}{2}\log\frac{d}{d-r} + \frac{2r+1}{4}\frac{2}{d(r+1)} - \frac{r+1}{4}\frac{2}{d} + O\left(\frac{r}{d^2}\right) \\ &= \frac{1}{2}\log\frac{d}{d-r} - \frac{r^2}{2d(r+1)} + O\left(\frac{r}{d^2}\right). \end{aligned} \tag{4.4}$$

In the case that $r \sim \alpha d$, we evidently have

$$\lim_{d\to\infty} \operatorname{var}[\mathcal{L}_{d,r}] = \frac{1}{2}\log\frac{1}{1-\alpha} - \frac{\alpha}{2}.$$

Let us now compute the asymptotic of $\operatorname{var}[\mathcal{L}_{d,r}] = c^2[\mathcal{L}_{d,r}]$ in the case that $d-r = o(d)$. By using once more (2.19) in the case that $k=1$, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} S_{d,r}(z)\Big|_{z=0} = \frac{1}{4}\sum_{j=1}^{r} \psi^{(1)}\left(\frac{d-r+j}{2}\right) = \frac{H_d - H_{d-r}}{2} + O\left(\frac{1}{d}\right),$$

as $d \to \infty$. Applying the formulas $H_d = \log d + O(1)$ and $H_{d-r} = \log(d-r+1) + O(1)$, following directly from (2.16) (where $+1$ is needed to make the expression well-defined in the case that $r = d$, too), we arrive at

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} S_{d,r}(z)\Big|_{z=0} = \frac{1}{2}\log\frac{d}{d-r+1} + O(1),$$

as $d \to \infty$. By the formula $\psi^{(1)}(z) = O(1/z)$, as $z \to \infty$, implied by (2.19), we have

$$\psi^{(1)}\left(\frac{r(d+\nu-2)+(d+\nu)}{2}\right) = O\left(\frac{1}{d^2}\right),$$

and

$$\psi^{(1)}\left(\frac{d+\nu}{2}\right) = O\left(\frac{1}{d}\right),$$

as $d \to \infty$. Plugging everything into (4.3) yields

$$\operatorname{var}[\mathcal{L}_{d,r}] = c^2[\mathcal{L}_{d,r}] = \frac{1}{2}\log\frac{d}{d-r+1} + O(1) \sim \frac{1}{2}\log\frac{d}{d-r+1},$$

as $d \to \infty$, because $\frac{d}{d-r+1} \to \infty$, proving the required asymptotic of the variance.

Next, we prove the bounds on the cumulants assuming that $r = o(d)$ or $r \sim \alpha d$. Recall from Lemma 4.2.2 (d) the estimate

$$\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k} S_{d,r}(z)\Big|_{z=0}\right| \le c^k\,(k-1)!\,r\,d^{-(k-1)}.$$

Further, since $\nu \ge 0$, we have

$$\frac{r(d+\nu-2)+(d+\nu)}{2} \ge \frac{r(d-2)}{2}.$$

Since the function $|\psi^{(k-1)}(z)|$ is non-increasing, we have, by using (4.1), that

$$\begin{aligned}\left|\psi^{(k-1)}\left(\frac{r(d+\nu-2)+(d+\nu)}{2}\right)\right| &\le \left|\psi^{(k-1)}\left(\frac{r(d-2)}{2}\right)\right| \\ &\le 2^k\,(k-1)!\,r^{1-k}\,(d-2)^{1-k}.\end{aligned}$$

By the mean value theorem, we also have

$$(r+1)^k - r^k \le k\,(r+1)^{k-1}.$$

Hence,

$$\begin{aligned}\frac{(r+1)^k - r^k}{2^k}\left|\psi^{(k-1)}\left(\frac{r(d+\nu-2)+(d+\nu)}{2}\right)\right| &\le k!\,\left(\frac{r+1}{r}\right)^{k-1}\,(d-2)^{1-k} \\ &\le 4^k\,k!\,d^{1-k},\end{aligned}$$

because $d-2 \ge d/2$, for $d \ge 4$. Similarly, by the non-increasing property of $|\psi^{(k-1)}(z)|$ and (4.1), we have

$$\frac{r+1}{2^k}\left|\psi^{(k-1)}\left(\frac{d+\nu}{2}\right)\right| \le \frac{r+1}{2^k}\left|\psi^{(k-1)}\left(\frac{d}{2}\right)\right| \le 2\,r\,(k-1)!\,d^{1-k},$$

since $r+1 \le 2r$. Recalling (4.3) and combining the above estimates, we arrive at the required bound

$$|c^k[\mathcal{L}_{d,r}]| \le c^k\,k!\,r\,d^{1-k}.$$

To prove the bound $|c^k[\mathcal{L}_{d,r}]| \le 2\,4^k\,k!$, without restrictions on $r(d)$, we argue as above, except for using Lemma 4.2.2 (e) to bound the derivative of $S_{d,r}$, that is,

$$|c^k[\mathcal{L}_{d,r}]| \le 2\,(k-1)! + 4^k\,k!\,d^{1-k} + 2\,r\,(k-1)!\,d^{1-k} \le 2\,4^k\,k!.$$

Finally, we consider the spherical model. Since the results for the Beta model are independent of the parameter ν, they carry over to the spherical model, appearing as a limiting case, as $\nu \downarrow 0$.

The cumulant bounds of the normalized random variables $\widetilde{\mathcal{L}}_{d,r}$ follow by applying the obtained cumulant bound in combination with the variance estimates. $\square$

4.2.2 Implications of the cumulant bound

By using the cumulant bounds stated above in conjunction with Theorem 2.4.3 from Section 2.4, we are able to derive a list of companion results. Recall that

$$\Phi(y) := (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}}\,\mathrm{d}t, \qquad y \in \mathbb{R},$$

is the distribution function of a standard Gaussian random variable. We start with the Gaussian model. The following theorem includes a central limit theorem with corresponding Berry-Esseen bound (part (a)), an estimate on the relative error in the central limit theorem (part (b)), a moderate deviation principle (part (c)), as well as a concentration inequality (part (d)) for the log-volume of the Gaussian simplex, that is,

$$\widetilde{\mathcal{L}}_{d,r} = \frac{\mathcal{L}_{d,r} - \mathbb{E}[\mathcal{L}_{d,r}]}{\sqrt{\mathrm{var}[\mathcal{L}_{d,r}]}}.$$

Theorem 4.2.3 *Let $r \leq d$ be an integer and $X_1, \ldots, X_{r+1}$ be chosen according to the Gaussian model. Define*

$$\varepsilon_d := \begin{cases} \frac{1}{\sqrt{rd}} & : r = o(d) \text{ or } r \sim \alpha d \\ \frac{1}{\sqrt{\log \frac{d}{d-r+1}}} & : d - r = o(d), \end{cases}$$

where $\alpha \in (0,1)$. Then, the following assertions are true:

(a) For sufficiently large d, we have the Berry-Esseen bound

$$\sup_{y \in \mathbb{R}} \left| \mathbb{P}\left(\widetilde{\mathcal{L}}_{d,r} \leq y\right) - \Phi(y) \right| \leq c_1\, \varepsilon_d,$$

where $c_1 \in (0,\infty)$ is a constant not depending on d.

(b) For sufficiently large d, there exist constants $c_1, c_2, c_3 \in (0,\infty)$ not depending on d such that

$$\left| \log \frac{\mathbb{P}\big(\widetilde{\mathcal{L}}_{d,r} \geq y\big)}{1 - \Phi(y)} \right| \leq c_1\,(1 + y^3)\,\varepsilon_d,$$

and

$$\left| \log \frac{\mathbb{P}\big(\widetilde{\mathcal{L}}_{d,r} \le -y\big)}{\Phi(-y)} \right| \le c_2\,(1+y^3)\,\varepsilon_d,$$

for all $0 \le y \le c_3\,\varepsilon_d^{-1}$.

(c) Let $(a_d)_{d\in\mathbb{N}}$ *be a sequence of real numbers such that*

$$\lim_{d\to\infty} a_d = \infty \qquad \text{and} \qquad \lim_{d\to\infty} a_d\,\varepsilon_d = 0.$$

Then, the family

$$\left(\frac{1}{a_d}\widetilde{\mathcal{L}}_{d,r}\right)_{d>0}$$

satisfies a moderate deviation principle on $\mathbb{R}$ *with speed* a_d^2 *and rate function* $I(x) = \frac{x^2}{2}$.

(d) For sufficiently large d, it holds that

$$\mathbb{P}\left(|\mathcal{L}_{d,r} - \mathbb{E}[\mathcal{L}_{d,r}]| \ge y\sqrt{\operatorname{var}[\mathcal{L}_{d,r}]}\right) \le 2\exp\Big(-\frac{1}{4}\min\Big\{\frac{y^2}{2}, c_1\,\varepsilon_d^{-1}\,y\Big\}\Big),$$

where $c_1 \in (0,\infty)$ *is a constant not depending on d.*

Proof. The proofs of the four statements follow directly by applying the cumulant bounds for the normalized random variables $\widetilde{\mathcal{L}}_{d,r}$ from Theorem 4.2.1 there to Theorem 2.4.3 in Section 2.4. □

In the Beta and the spherical case, we get a similar result. The difference is in the definition of the parameter ε_d in the regimes of r. Unfortunately, we do not get the results in the case that $r = o(d)$ (see also Remark 4.2.5).

Theorem 4.2.4 *Let* $r \le d$ *be an integer and* $X_1, \ldots, X_{r+1}$ *be chosen according to the Beta or the spherical model. Define*

$$\varepsilon_d := \begin{cases} \frac{1}{d} & : r \sim \alpha d \\ \frac{1}{\sqrt{\log \frac{d}{d-r+1}}} & : d-r = o(d), \end{cases}$$

where $\alpha \in (0,1)$. *Then, the following assertions are true:*

(a) *For sufficiently large d, we have the Berry-Esseen bound*

$$\sup_{y\in\mathbb{R}}\left|\mathbb{P}\left(\widetilde{\mathcal{L}}_{d,r}\le y\right)-\Phi(y)\right|\le c_1\,\varepsilon_d,$$

where $c_1\in(0,\infty)$ is a constant not depending on d.

(b) *For sufficiently large d, there exist constants $c_1,c_2,c_3\in(0,\infty)$ not depending on d such that*

$$\left|\log\frac{\mathbb{P}\big(\widetilde{\mathcal{L}}_{d,r}\ge y\big)}{1-\Phi(y)}\right|\le c_1\,(1+y^3)\,\varepsilon_d,$$

and

$$\left|\log\frac{\mathbb{P}\big(\widetilde{\mathcal{L}}_{d,r}\le -y\big)}{\Phi(-y)}\right|\le c_2\,(1+y^3)\,\varepsilon_d,$$

for all $0\le y\le c_3\,\varepsilon_d^{-1}$.

(c) *Let $(a_d)_{d\in\mathbb{N}}$ be a sequence of real numbers such that*

$$\lim_{d\to\infty}a_d=\infty\qquad\text{and}\qquad\lim_{d\to\infty}a_d\,\varepsilon_d=0.$$

Then, the family

$$\left(\frac{1}{a_d}\widetilde{\mathcal{L}}_{d,r}\right)_{d>0}$$

satisfies a moderate deviation principle on $\mathbb{R}$ with speed a_d^2 and rate function $I(x)=\frac{x^2}{2}$.

(d) *For sufficiently large d, it holds that*

$$\mathbb{P}\left(|\mathcal{L}_{d,r}-\mathbb{E}[\mathcal{L}_{d,r}]|\ge y\sqrt{\operatorname{var}[\mathcal{L}_{d,r}]}\right)\le 2\exp\Big(-\frac{1}{4}\min\Big\{\frac{y^2}{2},c_1\,\varepsilon_d^{-1}\,y\Big\}\Big),$$

where $c_1\in(0,\infty)$ is a constant not depending on d.

Remark 4.2.5 Theorem 4.2.4 (*a*) does not yield a central limit theorem for $\widetilde{\mathcal{L}}_{d,r}$, if $r=o(d)$. However, also in this case we still get the estimate (4.4). Now, since $r=o(d)$,

$$\log\frac{d}{d-r}\ge\frac{r}{d},$$

so that, for sufficiently large d,

$$\frac{1}{2}\log\frac{d}{d-r}-\frac{r^2}{2d(r+1)}\geq\frac{r}{2d}-\frac{r^2}{2d(r+1)}\geq\frac{r}{2d(r+1)}.$$

Thus, we have

$$\frac{r}{d^2}=o\left(\frac{1}{2}\log\frac{d}{d-r}-\frac{r^2}{2d(r+1)}\right),$$

as $d\to\infty$, and we can conclude that

$$\operatorname{var}[\mathcal{L}_{d,r}]\sim\frac{1}{2}\log\frac{d}{d-r}-\frac{r^2}{2d(r+1)}\geq\frac{r}{2d(r+1)},$$

as $d\to\infty$. This yields, for sufficiently large d,

$$|c^k[\widetilde{\mathcal{L}}_{d,r}]|\leq\frac{c_1^k\,k!\,r\,d^{1-k}}{\left(\frac{r}{2d(r+1)}\right)^{\frac{k}{2}}}\leq c_2^k\,k!\,r\,d^{-\frac{1}{2}(k-2)},$$

with constants $c_1,c_2\in(0,\infty)$ not depending on d. Thus, $|c^k[\widetilde{\mathcal{L}}_{d,r}]|\to 0$, as $d\to\infty$, for all $k\geq 4$, and Theorem 2.4.4 implies asymptotic normality for $\widetilde{\mathcal{L}}_{d,r}$ also in the case that $r=o(d)$.

While, in the three cases $r=o(d)$, $r\sim\alpha d$ and $d-r=o(d)$, we were able to derive precise Berry-Esseen bounds by using cumulant bounds, we can state a 'pure' central limit theorem for the log-volume in an even more general setup. The following result can directly be concluded by extracting subsequences and, then, by applying the results of Theorem 4.2.3, Theorem 4.2.4 and Remark 4.2.5, respectively.

Corollary 4.2.6 (Central limit theorem for the log-volume) *Let $r=r(d)$ be an arbitrary sequence of integers such that $r(d)\leq d$, for any $d\in\mathbb{N}$. Further, let for each $d\in\mathbb{N}$, $X_1,\ldots,X_{r+1}$ be independent random points, chosen according to the Gaussian, the Beta or the spherical model, and put $\mathcal{L}_{d,r}=\log(r!\mathcal{V}_{d,r})$. Then,*

$$\frac{\mathcal{L}_{d,r}-\mathbb{E}[\mathcal{L}_{d,r}]}{\sqrt{\operatorname{var}\left[\mathcal{L}_{d,r}\right]}}\xrightarrow{D}Z,$$

where $Z\overset{D}{\sim}N(0,1)$ is a standard Gaussian random variable.

4.2.3 Central and non-central limit theorem

After having investigated asymptotic normality for the log-volume of a random simplex, we turn now to its actual volume, that is, the random variable $\mathcal{V}_{d,r}$ itself.

Theorem 4.2.7 (Distributional limit theorem for the volume) *Let $X_1, \dots, X_{r+1}$ be chosen according to the Gaussian model, the Beta model or the spherical model, and let $\alpha \in (0,1)$. Let $Z \overset{D}{\sim} N(0,1)$ be a standard Gaussian random variable.*

1. *If $r = o(d)$, then, for suitable normalizing sequences $a_{d,r}$ and $b_{d,r}$, as $d \to \infty$,*

$$\frac{\mathcal{V}_{d,r} - a_{d,r}}{b_{d,r}} \overset{D}{\longrightarrow} Z.$$

2. *If $r \sim \alpha d$, for some $\alpha \in (0,1)$, then, for a suitable normalizing sequence $b_{d,r}$,*

$$\frac{\mathcal{V}_{d,r}}{b_{d,r}} \overset{D}{\longrightarrow} \begin{cases} e^{\sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}}\,Z} & : \textit{in the Gaussian model} \\ e^{\sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}-\frac{\alpha}{2}}\,Z} & : \textit{in the Beta or spherical model.} \end{cases}$$

Remark 4.2.8 In the third case, i.e., $d-r = o(d)$, there is no non-trivial distributional limit theorem for the random variable $\mathcal{V}_{d,r}$ under affine rescaling. The reason is that the variance of $\log \mathcal{V}_{d,r}$ tends to infinity in this case.

Proof of Theorem 4.2.7. From Corollary 4.2.6, we know that with the sequences $c_{d,r} := \mathbb{E}[\log \mathcal{V}_{d,r}]$ and $c'_{d,r} := \sqrt{\operatorname{var}[\log \mathcal{V}_{d,r}]}$, it holds that

$$\frac{\log \mathcal{V}_{d,r} - c_{d,r}}{c'_{d,r}} \overset{D}{\longrightarrow} Z,$$

as $d \to \infty$. By the Skorokhod–Dudley lemma [81, Theorem 4.30], we can construct random variables $\mathcal{V}^*_{d,r}$ and Z^* on a different probability space such that $\mathcal{V}^*_{d,r} \overset{D}{\sim} \mathcal{V}_{d,r}$, $Z^* \overset{D}{\sim} Z$ and

$$Z^*_d := \frac{\log \mathcal{V}^*_{d,r} - c_{d,r}}{c'_{d,r}} \longrightarrow Z^* \qquad \text{almost surely,}$$

as $d \to \infty$. So, we have

$$\mathcal{V}^*_{d,r} = e^{c'_{d,r} Z^*_d + c_{d,r}},$$

where $Z^*_d \to Z^*$ almost surely, as $d \to \infty$.

Consider first the Gaussian model in the case that $r \sim \alpha d$. Then, by Theorem 4.2.1 (a), we have

$$c'_{d,r} = \sqrt{\operatorname{var}\left[\log \mathcal{V}_{d,r}\right]} \sim \sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}},$$

as $d \to \infty$. With the aid of the Skorokhod–Dudley lemma, it follows that

$$\frac{\mathcal{V}^*_{d,r}}{e^{c_{d,r}}} = e^{c'_{d,r}Z^*_d} \longrightarrow e^{\sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}}\,Z^*} \qquad \text{almost surely,}$$

as $d \to \infty$. Passing back to the original probability space, we obtain the distributional convergence

$$\frac{\mathcal{V}_{d,r}}{e^{c_{d,r}}} \xrightarrow{D} e^{\sqrt{\frac{1}{2}\log\frac{1}{1-\alpha}}\,Z}.$$

The proof for the Beta or spherical model in the case that $r \sim \alpha d$ is similar, only the expression for the asymptotic variance being different.

Consider now the Gaussian model in the case that $r = o(d)$. Then, by Theorem 4.2.1 (a),

$$c'_{d,r} = \sqrt{\operatorname{var}\left[\log \mathcal{V}_{d,r}\right]} \longrightarrow 0,$$

as $d \to \infty$. Using the asymptotic $\lim_{x\to 0}(e^x - 1)/x = 1$ and the Skorokhod–Dudley lemma, we obtain

$$\frac{\frac{\mathcal{V}^*_{d,r}}{e^{c_{d,r}}} - 1}{c'_{d,r}} = \frac{e^{c'_{d,r}Z^*_d} - 1}{c'_{d,r}Z^*_d} \cdot Z^*_d \longrightarrow Z^* \qquad \text{almost surely,}$$

as $d \to \infty$. Passing back to the original probability space and taking $b_{d,r} = e^{c_{d,r}}c'_{d,r}$ and $a_{d,r} = e^{c_{d,r}}$, we obtain the required distributional convergence. Again, the proof for the Beta and the spherical model is in the same spirit. □

4.3 Mod-ϕ convergence

The aim of the present section is to establish mod-ϕ convergence for the log-volumes of the random simplices under consideration. Recall the definition of mod-ϕ convergence from Section 2.7.

4.3.1 The Gaussian model

In this section, we prove mod-ϕ convergence for the random variables $\mathcal{L}_{d,r} = \log(r!\mathcal{V}_{d,r})$ in the Gaussian model. The following theorem focuses on four different regimes for the parameter r. First, the case where r is fixed is treated. Then, we turn to the full-dimensional setting, that is, $r = d$. Thirdly, we investigate the case in which the co-dimension of the simplex, that is, $d - r$, stays fixed, as $d \to \infty$. Finally, we consider the case when the co-dimension of the random simplex goes to infinity. To streamline our presentation, recall that G denotes the Barnes G-function (see Section 2.3) and put

$$m_d := \frac{1}{2}\left(d \log d - d + \frac{1}{2}\log d + \log 2^{\frac{3}{2}}\pi\right). \tag{4.5}$$

Theorem 4.3.1 *(a) Fix some $r \in \mathbb{N}$. Then, as $d \to \infty$, the sequence*

$$d\,\mathcal{Y}_{d,r} := d\left(\mathcal{L}_{d,r} - \frac{r}{2}\log d - \frac{1}{2}\log(r+1)\right)$$

converges in the mod-ϕ sense with $\eta(t) = (t+1)\log(t+1) - t$ and parameter $w_d = \frac{rd}{2}$, namely,

$$\lim_{d\to\infty} \frac{\mathbb{E}\left[e^{td(\mathcal{L}_{d,r} - \frac{r}{2}\log d - \frac{1}{2}\log(r+1))}\right]}{e^{\frac{rd}{2}((t+1)\log(t+1)-t)}} = (t+1)^{-\frac{r(r+1)}{4}},$$

uniformly as long as t stays in any compact subset of $\mathbb{C} \setminus (-\infty, -1)$. Thus, we have mod-$\phi$ convergence modulo a tilted totally skewed 1-stable distribution (see Theorem 2.7.3).

(b) Let $r = d$. Then, as $d \to \infty$, the sequence

$$\mathcal{L}_{d,d} - m_d$$

converges in the mod-Gaussian sense, that is, $\eta(t) = \frac{1}{2}t^2$, and parameter $w_d = \frac{1}{2}\log\frac{d}{2}$, namely,

$$\lim_{d\to\infty} \frac{\mathbb{E}\left[e^{t(\mathcal{L}_{d,d} - m_d)}\right]}{e^{\frac{1}{4}t^2 \log\frac{d}{2}}} = \frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2} + \frac{t}{2}\right) G\left(1 + \frac{t}{2}\right)},$$

uniformly as long as t stays in any compact subset of $\mathbb{C}\setminus\{-1, -2, \ldots\}$.

(c) Let $a \in \mathbb{N}$ be fixed and take $r = d - a$. Then, as $d \to \infty$, the sequence

$$\mathcal{L}_{d,r} - m_d$$

converges in the mod-Gaussian sense, that is, $\eta(t) = \frac{1}{2}t^2$, and parameter $w_d = \frac{1}{2}\log\frac{d}{2}$, namely,

$$\lim_{d\to\infty} \frac{\mathbb{E}\left[e^{t(\mathcal{L}_{d,r}-m_d)}\right]}{e^{\frac{1}{4}t^2\log\frac{d}{2}}} = \frac{G\left(\frac{a+1}{2}\right)\, G\left(\frac{a+2}{2}\right)}{2^{\frac{ta}{2}}\, G\left(\frac{a+1}{2}+\frac{t}{2}\right)\, G\left(\frac{a+2}{2}+\frac{t}{2}\right)},$$

uniformly as long as t stays in any compact subset of $\mathbb{C}\backslash\{-a-1, -a-2, \ldots\}$.

(d) If $r = r(d)$ is such that $d - r = o(d)$, as $d \to \infty$, then, the sequence

$$\mathcal{Y}^*_{d,r} := \mathcal{L}_{d,r} - (m_d - m_{d-r}) - \frac{1}{2}\log\frac{(r+1)(d-r)}{d}$$

converges in the mod-Gaussian sense, that is, $\eta(t) = \frac{1}{2}t^2$, and parameter $w_d = \frac{1}{2}\log\frac{d}{d-r}$, namely,

$$\lim_{d\to\infty} \frac{\mathbb{E}\left[e^{t\left(\mathcal{L}_{d,r}-(m_d-m_{d-r})-\frac{1}{2}\log\frac{(r+1)(d-r)}{d}\right)}\right]}{e^{\frac{1}{4}t^2\log\frac{d}{d-r}}} = 1,$$

uniformly for all $t \in \mathbb{C}$.

Remark 4.3.2 We notice that in the full dimensional case $r = d$, our random variables are equivalent to those considered in [17] and one can follow our result also from their Theorem 5.1. Nevertheless, we decided to include our completely independent and much shorter proof. The paper [17] deals with the determinant of certain random matrix models and has a completely different focus. On the other hand, let us emphasize that even in this special case, the distributions appearing in [17] are in fact different from (but very close to) those we obtain.

Proof. We start by proving (a). From the moment formula in Theorem 4.1.2 (a), we obtain

$$\mathbb{E}\left[e^{td\mathcal{L}_{d,r}}\right] = (r+1)^{\frac{td}{2}}\, 2^{\frac{tdr}{2}} \prod_{j=1}^{r} \frac{\Gamma\left(\frac{(t+1)d-r+j}{2}\right)}{\Gamma\left(\frac{d-r+j}{2}\right)}.$$

By using the asymptotic behavior of the Gamma function, stated in (2.13), we deduce that

$$\prod_{j=1}^{r} \frac{\Gamma\left(\frac{(t+1)d-r+j}{2}\right)}{\Gamma\left(\frac{d-r+j}{2}\right)} \sim \prod_{j=1}^{r} e^{-\frac{td}{2}} \left(\frac{d}{2}\right)^{\frac{td}{2}} (t+1)^{\frac{(t+1)d}{2}+\frac{j-r-1}{2}}$$
$$= e^{-\frac{tdr}{2}} \left(\frac{d}{2}\right)^{\frac{rtd}{2}} (t+1)^{\left(\frac{(t+1)d}{2}-\frac{1}{2}\right)r-\frac{r(r-1)}{4}}, \tag{4.6}$$

as $d \to \infty$, since

$$\sum_{j=1}^{r} j = \frac{r^2+r}{2}.$$

Thus, as $d \to \infty$,

$$\mathbb{E}\left[e^{td\mathcal{L}_{d,r}}\right] \sim (r+1)^{\frac{td}{2}} e^{-\frac{tdr}{2}} d^{\frac{tdr}{2}} (t+1)^{\left(\frac{(t+1)d}{2}-\frac{1}{2}\right)r-\frac{r(r-1)}{4}}.$$

Taking the logarithm and subtracting $\frac{r}{2}\log d$ and $\frac{1}{2}\log(r+1)$, we conclude that

$$\log \mathbb{E}\left[e^{td(\mathcal{L}_{d,r}-\frac{r}{2}\log d-\frac{1}{2}\log(r+1))}\right]$$
$$= \frac{dr}{2}\left((t+1)\log(t+1)-t\right) - \frac{r(r+1)}{4}\log(t+1) + o(1), \tag{4.7}$$

and part (a) follows. Let us prove (b). In view of Theorem 4.1.2 (a) and (2.26), we can express the moment generating function of $\mathcal{L}_{d,d}$ in terms of the Barnes G-function as

$$\mathbb{E}\left[e^{t\mathcal{L}_{d,d}}\right] = (d+1)^{\frac{t}{2}} 2^{\frac{td}{2}} \frac{G\left(\frac{1}{2}\right)}{G\left(\frac{d+1}{2}\right)} \frac{G(1)}{G\left(\frac{d+2}{2}\right)} \frac{G\left(\frac{d+1}{2}+\frac{t}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)} \frac{G\left(\frac{d+2}{2}+\frac{t}{2}\right)}{G\left(1+\frac{t}{2}\right)}, \tag{4.8}$$

where $G(1) = 1$. For the function

$$\psi(t) := \frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right) G\left(1+\frac{t}{2}\right)}, \tag{4.9}$$

we have

$$\log \mathbb{E}\left[e^{t\mathcal{L}_{d,d}}\right] = \frac{t}{2}\log(d+1) + \frac{td}{2}\log 2 + \log\psi(t) + \log G\left(\frac{d-1}{2}+\frac{t}{2}+1\right)$$
$$-\log G\left(\frac{d-1}{2}+1\right) + \log G\left(\frac{d}{2}+\frac{t}{2}+1\right) - \log G\left(\frac{d}{2}+1\right).$$

Applying Lemma 2.3.5 two times and using

$$((d+b)\log(d+b)-(d+b))-(d\log d-d)=b\log d+o(1),$$

as $d\to\infty$, where $b\in(0,\infty)$ is any constant, leads to

$$\begin{aligned}
&\log\mathbb{E}\left[e^{t\mathcal{L}_{d,d}}\right]\\
&\quad=\frac{t}{2}\log(d+1)+\frac{td}{2}\log 2+\log\psi(t)\\
&\qquad+\frac{t}{2}\left(\frac{d-1}{2}\log\frac{d-1}{2}-\frac{d-1}{2}+\log\sqrt{2\pi}\right)+\frac{t^2}{8}\log\frac{d-1}{2}\\
&\qquad+\frac{t}{2}\left(\frac{d}{2}\log\frac{d}{2}-\frac{d}{2}+\log\sqrt{2\pi}\right)+\frac{t^2}{8}\log\frac{d}{2}+o(1)\\
&\quad=\log\psi(t)+\frac{t}{2}\left(\frac{d-1}{2}\log\frac{d-1}{2}-\frac{d-1}{2}-\left(\frac{d}{2}\log\frac{d}{2}-\frac{d}{2}\right)\right)\\
&\qquad+\frac{t}{2}\left(d\log\frac{d}{2}-d+\log 2\pi+\log(d+1)+d\log 2\right)+\frac{t^2}{4}\log\frac{d}{2}+o(1)\\
&\quad=\log\psi(t)+\frac{t}{2}\left(-\frac{1}{2}\log\frac{d}{2}+d\log d-d+\log d+\log 2\pi\right)+\frac{t^2}{4}\log\frac{d}{2}+o(1)\\
&\quad=\log\psi(t)+\frac{t}{2}\left(d\log d-d+\frac{1}{2}\log d+\log 2^{\frac{3}{2}}\pi\right)+\frac{t^2}{4}\log\frac{d}{2}+o(1),
\end{aligned}\tag{4.10}$$

as $d\to\infty$. This completes the argument and proves (b). We turn to part (c). First, we observe that Theorem 4.1.4 (a) implies the distributional representation

$$\mathcal{L}_{d,d}-\frac{1}{2}\log(d+1)\overset{D}{\sim}\left(\mathcal{L}_{d-r,d-r}-\frac{1}{2}\log(d-r+1)\right)+\left(\mathcal{L}'_{d,r}-\frac{1}{2}\log(r+1)\right),\tag{4.11}$$

where $\mathcal{L}'_{d,r}$ is a copy of $\mathcal{L}_{d,r}$, independent of $\mathcal{L}_{d-r,d-r}$. Since $d-r=a$, this implies that

$$\mathbb{E}\left[e^{t(\mathcal{L}_{d,r}-m_d)}\right]=\frac{\mathbb{E}\left[e^{t(\mathcal{L}_{d,d}-m_d)}\right]}{\mathbb{E}\left[e^{t\mathcal{L}_{a,a}}\right]}\,e^{\frac{t}{2}\log(a+1)}\,e^{\frac{t}{2}\log\frac{d-a+1}{d+1}}.$$

Applying part (b) of this theorem to the numerator and (4.8) to the denominator, we conclude that, as $d\to\infty$,

$$\mathbb{E}\left[e^{t(\mathcal{L}_{d,r}-m_d)}\right]\sim\frac{e^{\frac{1}{4}t^2\log\frac{d}{2}}\,\frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)G\left(1+\frac{t}{2}\right)}}{(a+1)^{\frac{t}{2}}\,2^{\frac{ta}{2}}\,\frac{G\left(\frac{1}{2}\right)}{G\left(\frac{a+1}{2}\right)}\,\frac{G(1)}{G\left(\frac{a+2}{2}\right)}\,\frac{G\left(\frac{a+1}{2}+\frac{t}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)}\,\frac{G\left(\frac{a+2}{2}+\frac{t}{2}\right)}{G\left(1+\frac{t}{2}\right)}}(a+1)^{\frac{t}{2}}.$$

Finally, we prove (d). From part (b), we know that

$$\lim_{d\to\infty} \frac{\mathbb{E}\left[e^{t(\mathcal{L}_{d,d}-m_d)}\right]}{e^{\frac{1}{4}t^2\log\frac{d}{2}}} = \lim_{d\to\infty} \frac{\mathbb{E}\left[e^{t(\mathcal{L}_{d-r,d-r}-m_{d-r})}\right]}{e^{\frac{1}{4}t^2\log\frac{d-r}{2}}} = \frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)\,G\left(1+\frac{t}{2}\right)}.$$

Using the distributional identity (4.11) yields that

$$\begin{aligned}\mathbb{E}\left[e^{t\left(\mathcal{L}_{d,r}-(m_d-m_{d-r})-\frac{1}{2}\log\frac{(r+1)(d-r+1)}{d+1}\right)}\right] &= \frac{\mathbb{E}\left[e^{t(\mathcal{L}_{d,d}-m_d)}\right]}{\mathbb{E}\left[e^{t(\mathcal{L}_{d-r,d-r}-m_{d-r})}\right]}\\ &\sim \frac{e^{\frac{1}{4}t^2\log\frac{d}{2}}}{e^{\frac{1}{4}t^2\log\frac{d-r}{2}}}\\ &= e^{\frac{1}{4}t^2\log\frac{d}{d-r}},\end{aligned}$$

as $d\to\infty$. This implies the claim after observing that, as $d\to\infty$, $\log(d+1) = \log d + o(1)$ and $\log(d-r+1) = \log(d-r)+o(1)$, respectively. Note also that if $d-r = o(d)$, then $w_d\to\infty$, as $d\to\infty$, which is otherwise not the case. □

By using Theorem 4.3.1 and the results presented in Section 2.7, we deduce the following companion results. Recall also the definition of a stable distributed random variable from Section 2.7.

Theorem 4.3.3 *(a) Fix some $r\in\mathbb{N}$. Then,*

$$\frac{2}{r}\mathcal{Y}_{d,r} + \log\frac{rd}{2} \xrightarrow{D} \phi_{\frac{\pi}{2},1,-1},$$

as $d\to\infty$, where $\phi_{\frac{\pi}{2},1,-1}$ indicates a stable distributed random variable with parameter $c=\frac{\pi}{2}$, $\alpha=1$ and $\beta=-1$. Moreover, for all $x_d = o(d^{\frac{1}{12}})$, it holds that

$$\mathbb{P}\left(\mathcal{Y}_{d,r}\le x_d\sqrt{\frac{r}{2d}}\right) = \Phi(x_d)\,(1+o(1)),$$

and, for all $x>0$,

$$\mathbb{P}\left(\mathcal{Y}_{d,r}\ge x\,\frac{r}{2}\right) = \frac{\exp(-\frac{rd}{2}\,(e^x-x-1))}{(e^x-1)\sqrt{\pi rd\,e^{-x}}}\,e^{-\frac{r(r+1)}{4}x}\,(1+o(1)),$$

as $d\to\infty$.

(b) Let $a \in \mathbb{N}_0$ *be fixed and take* $r = d - a$*. Then, for all* $x_d = o(\sqrt{\log d})$*,*

$$\mathbb{P}\left(\mathcal{L}_{d,r} - m_d \le x_d \sqrt{\frac{1}{2}\log\frac{d}{2}}\right) = \Phi(x_d)\,(1 + o(1)),$$

and, for all $x > 0$*,*

$$\mathbb{P}\left(\mathcal{L}_{d,r} - m_d \ge \frac{x}{2}\,\log\frac{d}{2}\right) = \frac{\exp(-\frac{x^2}{4}\log\frac{d}{2})}{x\sqrt{\pi\log\frac{d}{2}}}\;\frac{G\left(\frac{a+1}{2}\right)\,G\left(\frac{a+2}{2}\right)}{2^{\frac{xa}{2}}\,G\left(\frac{a+1}{2}+\frac{x}{2}\right)\,G\left(\frac{a+2}{2}+\frac{x}{2}\right)}\,(1 + o(1)),$$

as $d \to \infty$*.*

(c) If $r = r(d)$ *is such that* $d - r = o(d)$*, as* $d \to \infty$*, then, for all* $x_d = o(\sqrt{\log d})$*,*

$$\mathbb{P}\left(\mathcal{Y}^*_{d,r} \le x_d \sqrt{\frac{1}{2}\log\frac{d}{d-r}}\right) = \Phi(x_d)\,(1 + o(1)),$$

and, for all $x > 0$*,*

$$\mathbb{P}\left(\mathcal{Y}^*_{d,r} \ge \frac{x}{2}\,\log\frac{d}{d-r}\right) = \frac{\exp(-\frac{x^2}{4}\log\frac{d}{2})}{x\sqrt{\pi\log\frac{d}{d-r}}}\,(1 + o(1)),$$

as $d \to \infty$*.*

Remark 4.3.4 Theorem 2.7.2 also yields rates of convergences in the central limit theorems for the sequences $\mathcal{Y}_{d,r}$, $\mathcal{L}_{d,r} - m_d$ and $\mathcal{Y}^*_{d,r}$, analyzed in Theorem 4.3.3. We refrain from stating them separately since they show exactly the same rates as the ones in the corresponding parts of Theorem 4.2.3. Indeed, by using mod-ϕ convergence, these rates have been calculated in [42, Theorem 6.13] (see also Remark 4.3.7).

Proof. We start by proving (a). The first statement follows from (2.32), since $c = \frac{\pi}{2}$, $\alpha = 1$ and $\beta = -1$. The second and third one are direct consequences of Theorem 2.7.1, since $w_d = \frac{rd}{2}$,

$$\psi(t) = (t+1)^{-\frac{r(r+1)}{4}}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\eta(t) = \log(t+1), \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\eta(t) = \frac{1}{t+1} \quad \text{and} \quad h = e^x - 1,$$

as well as $F(x) = \sup_{t\in\mathbb{R}}\{tx - \eta(t)\} = e^x - x - 1$.

Indeed, since $\eta(t) = (t+1)\log(t+1) - t$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[tx - ((t+1)\log(t+1) - t)\right] = x - \log(t+1).$$

Thus, for all $x \in \mathbb{R}$, the supremum is attained at $t = e^x - 1$, which yields the claim. Now, let us prove (b) and (c). Again, the assertions follow from the respective conditions on w_d, $\psi(t)$ and $\eta(t)$ from Theorem 4.3.1 (b), (c), (d), applied to Theorem 2.7.1. □

4.3.2 The Beta and the spherical model

Now, we consider the Beta model with parameter $\nu > 0$ or the spherical model (in which case $\nu = 0$). We establish mod-ϕ convergence in the same four regimes as in the Gaussian setting in the foregoing section. Similarly as above, put

$$\widetilde{m}_d = \frac{1}{2}\left(\frac{1}{2}\log d - d + 1 - \nu + \log 2^{\frac{3}{2}}\pi\right). \tag{4.12}$$

Theorem 4.3.5 *(a) Fix some $r \in \mathbb{N}$. Then, as $d \to \infty$, the sequence*

$$d\,\mathcal{L}_{d,r}$$

converges in the mod-ϕ sense with

$$\eta(t) = \frac{(r+1)(t+1)}{2}\log((r+1)(t+1)) - \frac{r(t+1)+1}{2}\log(r(t+1)+1)$$
$$- \frac{t+1}{2}\log(t+1),$$

and parameter $w_d = d$, namely,

$$\lim_{d\to\infty}\frac{\mathbb{E}\left[e^{td\mathcal{L}_{d,r}}\right]}{e^{d\eta(t)}} = (1+t)^{\frac{1-\nu(r+1)}{2} - \frac{r(r-1)}{4}}\left(\frac{(r+1)(t+1)}{r(t+1)+1}\right)^{\frac{\nu(r+1)-2r-1}{2}},$$

uniformly as long as t stays in any compact subset of $\mathbb{C}\setminus(-\infty,-1)$.

(b) Let $r = d$. Then, as $d \to \infty$, the sequence

$$\mathcal{L}_{d,d} - \widetilde{m}_d$$

converges in the mod-Gaussian sense, that is, $\eta(t) = \frac{1}{2}t^2$, *and parameter* $w_d = \frac{1}{2}\log\frac{d}{2} - \frac{1}{2}$*, namely,*

$$\lim_{d\to\infty} \frac{\mathbb{E}\left[e^{t(\mathcal{L}_{d,d}-\widetilde{m}_d)}\right]}{e^{\frac{1}{4}t^2(\log\frac{d}{2}-1)}} = \frac{G\left(\frac{1}{2}\right)}{G\left(\frac{1}{2}+\frac{t}{2}\right)G\left(1+\frac{t}{2}\right)},$$

uniformly as long as t *stays in any compact subset of* $\mathbb{C}\backslash\{-1,-2,\dots\}$.

(c) Let $a\in\mathbb{N}$ *be fixed and take* $r = d-a$*, as* $d\to\infty$*. Then, as* $d\to\infty$*, the sequence*

$$\mathcal{L}_{d,r} - \tilde{m}_d - \frac{a}{2}\log\frac{d}{2}$$

converges in the mod-Gaussian sense, that is, $\eta(t) = \frac{1}{2}t^2$*, and parameter* $w_d = \frac{1}{2}\log\frac{d}{2} - \frac{1}{2}$*, namely,*

$$\lim_{d\to\infty} \frac{\mathbb{E}\left[e^{t(\mathcal{L}_{d,r}-\widetilde{m}_d-\frac{a}{2}\log\frac{d}{2})}\right]}{e^{\frac{1}{4}t^2(\log\frac{d}{2}-1)}} = \frac{G\left(\frac{a+1}{2}\right)G\left(\frac{a+2}{2}\right)}{2^{\frac{ta}{2}}G\left(\frac{a+1}{2}+\frac{t}{2}\right)G\left(\frac{a+2}{2}+\frac{t}{2}\right)},$$

uniformly as long as t *stays in any compact subset of* $\mathbb{C}\backslash\{-a-1,-a-2,\dots\}$.

(d) If $r = r(d)$ *is such that* $d - r = o(d)$*, as* $d\to\infty$*, then, the sequence*

$$\mathcal{L}_{d,r} - \left(m_d - m_{d-r} - \frac{r+1}{4d}(t-2+2\nu)\right) - \frac{1}{2}\log\frac{(d-r)(1+r)}{d^{1+r}}$$

converges in the mod-Gaussian sense, that is, $\eta(t) = \frac{1}{2}t^2$*, and parameter* $w_d = \frac{1}{2}\log\frac{d}{d-r}$*, namely,*

$$\lim_{d\to\infty} \frac{\mathbb{E}\left[e^{t\left(\mathcal{L}_{d,r}-(m_d-m_{d-r}-\frac{r+1}{4d}(t-2+2\nu))-\frac{1}{2}\log\frac{(d-r)(1+r)}{d^{1+r}}\right)}\right]}{e^{\frac{1}{4}t^2\log\frac{d}{d-r}}} = 1,$$

uniformly for all $t\in\mathbb{C}$.

Proof. We start by proving (a). From the moment formula in Theorem 4.1.2 (b),

$$\mathbb{E}\left[e^{td\mathcal{L}_{d,r}}\right] = \prod_{j=1}^{r}\left[\frac{\Gamma\left(\frac{d-r+j}{2}+\frac{td}{2}\right)}{\Gamma\left(\frac{d-r+j}{2}\right)}\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d+\nu}{2}+\frac{td}{2}\right)}\right]\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d+\nu}{2}+\frac{td}{2}\right)}\frac{\Gamma\left(\frac{r(d+\nu-2)+(d+\nu)}{2}+\frac{(r+1)td}{2}\right)}{\Gamma\left(\frac{r(d+\nu-2)+(d+\nu)}{2}+\frac{rtd}{2}\right)}.$$

First of all, by (2.13), it holds that

$$\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d+\nu}{2}+\frac{td}{2}\right)} \sim (1+t)^{\frac{1}{2}-\frac{\nu}{2}-\frac{(1+t)d}{2}} \left(\frac{d}{2}\right)^{-\frac{td}{2}} e^{\frac{td}{2}},$$

as $d \to \infty$. It follows from (4.6) that the first product in the moment formula asymptotically behaves like

$$\prod_{j=1}^{r} \left[\frac{\Gamma\left(\frac{d-r+j}{2}+\frac{td}{2}\right)}{\Gamma\left(\frac{d-r+j}{2}\right)} \frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d+\nu}{2}+\frac{td}{2}\right)} \right] \sim (1+t)^{-\frac{r\nu}{2}-\frac{r(r-1)}{4}},$$

as $d \to \infty$. Again, using (2.13), leads to, as $d \to \infty$,

$$\begin{aligned} \frac{\Gamma\left(\frac{r(d+\nu-2)+(d+\nu)}{2}+\frac{(r+1)td}{2}\right)}{\Gamma\left(\frac{r(d+\nu-2)+(d+\nu)}{2}+\frac{rtd}{2}\right)} &\sim ((r+1)(t+1))^{\frac{d(r+1)(t+1)}{2}+\frac{\nu(r+1)-2r-1}{2}} \left(\frac{d}{2}\right)^{\frac{td}{2}} e^{-\frac{td}{2}} \\ &\quad \times (r(t+1)+1)^{-\frac{d(r(t+1)+1)}{2}-\frac{\nu(r+1)-2r-1}{2}}. \end{aligned}$$

Thus, as $d \to \infty$, we get

$$\begin{aligned} &\log \mathbb{E}\left[e^{td\mathcal{L}_{d,r}}\right] \\ &\quad = \left(\frac{1-\nu(r+1)}{2} - \frac{r(r-1)}{4} - \frac{(1+t)d}{2}\right) \log(1+t) \\ &\qquad + \left(\frac{d(r+1)(t+1)}{2} + \frac{\nu(r+1)-2r-1}{2}\right) \log((r+1)(t+1)) \\ &\qquad - \left(\frac{d(r(t+1)+1)}{2} + \frac{\nu(r+1)-2r-1}{2}\right) \log(r(t+1)+1) + o(1), \end{aligned} \tag{4.13}$$

and the result of part (a) follows. Let us prove part (b). For the purpose of this proof, let $\mathcal{L}^{\mathrm{G}}_{d,d}$ denote the Gaussian analogue of $\mathcal{L}_{d,d}$. In view of the connection between the Gaussian and the Beta model (see Theorem 4.1.2 (a),(b)), the moment generating function of $\mathcal{L}_{d,d}$ is given by

$$\begin{aligned} \log \mathbb{E}\left[e^{t\mathcal{L}_{d,d}}\right] &= \log \mathbb{E}\left[e^{t\mathcal{L}^{\mathrm{G}}_{d,d}}\right] - \frac{t}{2}\log(d+1) - \frac{td}{2}\log 2 \\ &\quad + (d+1)\log\left(\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d}{2}+\frac{\nu+t}{2}\right)}\right) + \log\left(\frac{\Gamma\left(\frac{d(d+\nu-1)+dt+t+\nu}{2}\right)}{\Gamma\left(\frac{d(d+\nu-1)+nt+\nu}{2}\right)}\right). \end{aligned}$$

Using the asymptotic relation (2.14) implies that

$$(d+1)\log\left(\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d}{2}+\frac{\nu+t}{2}\right)}\right)=\frac{(d+1)t}{2}\log\frac{2}{d}-\frac{t}{4}(t-2+2\nu)+o(1),$$

and, similarly,

$$\log\left(\frac{\Gamma\left(\frac{d(d+\nu-1)+dt+t+\nu}{2}\right)}{\Gamma\left(\frac{d(d+\nu-1)+dt+\nu}{2}\right)}\right)=t\log d-\frac{t}{2}\log 2+o(1),$$

as $d\to\infty$. Denoting by $\psi(t)$ the function defined in (4.9), and using (4.10), yields that

$$\begin{aligned}
&\log\mathbb{E}\left[e^{t\mathcal{L}_{d,d}}\right]\\
&=\log\psi(t)+\frac{t}{2}\left(d\log d-d+\frac{1}{2}\log d+\log 2^{\frac{3}{2}}\pi\right)+\frac{t^2}{4}\log\frac{d}{2}-\frac{t}{2}\log(d+1)\\
&\quad-\frac{td}{2}\log 2+\frac{(d+1)t}{2}\log\frac{2}{d}-\frac{t}{4}(t-2+2\nu)+t\log d-\frac{t}{2}\log 2+o(1)\\
&=\log\psi(t)+\frac{t}{2}\left(\log d\left(d+\frac{1}{2}-1-d-1+2\right)-d+\log 2^{\frac{3}{2}}\pi+1-\nu\right)\\
&\quad+\frac{t}{2}\log 2\,(d+1-d-1)+\frac{t^2}{4}\Big(\log\frac{d}{2}-1\Big)+o(1)\\
&=\log\psi(t)+\frac{t}{2}\left(\frac{1}{2}\log d-d+\log 2^{\frac{3}{2}}\pi+1-\nu\right)+\frac{t^2}{4}\Big(\log\frac{d}{2}-1\Big)+o(1)\\
&=\log\psi(t)+t\widetilde{m}_d+\frac{t^2}{4}\Big(\log\frac{d}{2}-1\Big)+o(1),
\end{aligned}$$

as $d\to\infty$. Thus, the proof of (b) is complete and we progress with (c). The computations are similar to those in the proof of (b), but slightly more involved. Again, we let $\mathcal{L}^{\mathrm{G}}_{d,r}$ be the Gaussian analogue of $\mathcal{L}_{d,r}$. By Theorem 4.1.2 (a),(b), the moment generating function of $\mathcal{L}_{d,r}$ is given by

$$\begin{aligned}
&\log\mathbb{E}\left[e^{t\mathcal{L}_{d,r}}\right]\\
&=\log\mathbb{E}\left[e^{t\mathcal{L}^{\mathrm{G}}_{d,r}}\right]-\frac{t}{2}\log(r+1)-\frac{tr}{2}\log 2\\
&\quad+(r+1)\log\left(\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d}{2}+\frac{\nu+t}{2}\right)}\right)+\log\left(\frac{\Gamma\left(\frac{r(d+\nu-2)+d+\nu}{2}+\frac{(r+1)t}{2}\right)}{\Gamma\left(\frac{r(d+\nu-2)+d+\nu}{2}+\frac{rt}{2}\right)}\right).
\end{aligned}\tag{4.14}$$

Using again (2.14) and $r = d - a$ implies that

$$(r+1)\log\left(\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d}{2}+\frac{\nu+t}{2}\right)}\right) = \frac{(d+1)t}{2}\log\frac{2}{d} - \frac{t}{4}(t-2+2\nu) + \frac{at}{2}\log\frac{d}{2} + o(1),$$

and

$$\log\left(\frac{\Gamma\left(\frac{r(d+\nu-2)+d+\nu}{2}+\frac{(r+1)t}{2}\right)}{\Gamma\left(\frac{r(d+\nu-2)+d+\nu}{2}+\frac{rt}{2}\right)}\right) = t\log d - \frac{t}{2}\log 2 + o(1),$$

as $d \to \infty$. By using the behavior of $\mathcal{L}^{\mathrm{G}}_{d,r}$, stated in Theorem 4.3.1 (c), we obtain, as $d \to \infty$,

$$\begin{aligned}
&\log\mathbb{E}\left[e^{t\mathcal{L}_{d,r}}\right] \\
&= \log\left(\frac{G\left(\frac{a+1}{2}\right)G\left(\frac{a+2}{2}\right)}{2^{\frac{ta}{2}}G\left(\frac{a+1}{2}+\frac{t}{2}\right)G\left(\frac{a+2}{2}+\frac{t}{2}\right)}\right) + tm_d + \frac{t^2}{4}\log\frac{d}{2} - \frac{t}{2}\log(r+1) \\
&\quad - \frac{tr}{2}\log 2 + \frac{(d+1)t}{2}\log\frac{2}{d} - \frac{t}{4}(t-2+2\nu) + \frac{at}{2}\log\frac{d}{2} \\
&\qquad + t\log d - \frac{t}{2}\log 2 + o(1) \\
&= \log\left(\frac{G\left(\frac{a+1}{2}\right)G\left(\frac{a+2}{2}\right)}{2^{\frac{ta}{2}}G\left(\frac{a+1}{2}+\frac{t}{2}\right)G\left(\frac{a+2}{2}+\frac{t}{2}\right)}\right) + \frac{t^2}{4}\Big(\log\frac{d}{2}-1\Big) + \frac{at}{2}\log\frac{d}{2} \\
&\quad + \frac{t}{2}(d\log d - d + \frac{1}{2}\log d + \log 2^{\frac{3}{2}}\pi - \log d - r\log 2 \\
&\qquad + (d+1)\log\frac{2}{d} + 1 - \nu + 2\log d - \log 2) + o(1) \\
&= \log\left(\frac{G\left(\frac{a+1}{2}\right)G\left(\frac{a+2}{2}\right)}{2^{\frac{ta}{2}}G\left(\frac{a+1}{2}+\frac{t}{2}\right)G\left(\frac{a+2}{2}+\frac{t}{2}\right)}\right) + t\,\widetilde{m}_d + \frac{t^2}{4}\Big(\log\frac{d}{2}-1\Big) \\
&\quad + \frac{at}{2}\log\frac{d}{2} + o(1),
\end{aligned}$$

which yields the claim of part (c) in view of the definition of $\widetilde{m}_d$. Finally, we arrive at part (d). Observe that relation (4.14) still holds. Regarding to the first term in this relation, we know from Theorem 4.3.1 (d) that, as $d \to \infty$,

$$\log\mathbb{E}\left[e^{t\mathcal{L}^{\mathrm{G}}_{d,r}}\right] = t(m_d - m_{d-r}) + \frac{t}{2}\log\frac{(r+1)(d-r)}{d} + \frac{1}{4}t^2\log\frac{d}{d-r} + o(1).$$

Once more, (2.14) yields

$$(r+1)\log\left(\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{d}{2}+\frac{\nu+t}{2}\right)}\right)=(r+1)\frac{t}{2}\log\frac{2}{d}-\frac{r+1}{d}\frac{t}{4}(t-2+2\nu)+o\left(\frac{r}{d^2}\right),$$

and

$$\begin{aligned}\log\left(\frac{\Gamma\left(\frac{r(d+\nu-2)+d+\nu}{2}+\frac{(r+1)t}{2}\right)}{\Gamma\left(\frac{r(d+\nu-2)+d+\nu}{2}+\frac{rt}{2}\right)}\right)&=\frac{t}{2}\log\left(\frac{r(d+\nu-2)+d+\nu}{2}+\frac{rt}{2}\right)+o(1)\\&=\frac{t}{2}\log\frac{(r+1)d}{2}+o(1),\end{aligned}$$

as $d\to\infty$. Combining these estimates implies that, as $d\to\infty$,

$$\begin{aligned}&\log\mathbb{E}\left[e^{t\mathcal{L}_{d,r}}\right]\\&=t(m_d-m_{d-r})+\frac{t}{2}\log\frac{(r+1)(d-r)}{d}+\frac{1}{4}t^2\log\frac{d}{d-r}-\frac{t}{2}\log(r+1)-\frac{tr}{2}\log 2\\&\quad+(r+1)\frac{t}{2}\log\frac{2}{d}-\frac{r+1}{d}\frac{t}{4}(t-2+2\nu)+\frac{t}{2}\log\frac{(r+1)d}{2}+o(1)\\&=t\left(m_d-m_{d-r}-\frac{r+1}{4d}(t-2+2\nu)\right)+\frac{t}{2}\log\frac{(d-r)(r+1)}{d^{1+r}}\\&\quad+\frac{1}{4}t^2\log\frac{d}{d-r}+o(1).\end{aligned}$$

This yields the claim, since $w_d=\frac{1}{2}\log\frac{d}{d-r}\to\infty$, as $d\to\infty$, by the assumption that $d-r=o(d)$. □

Remark 4.3.6 We do not formulate the corresponding extended central limit theorems, precise deviations and Berry-Esseen bounds in the Beta and the spherical model, because these results can be stated similarly as in Theorem 4.3.3.

Remark 4.3.7 In their paper, Eichelsbacher and Knichel [42] analyze positive random variables X that fulfill, for all k in some interval,

$$\mathbb{E}[X^k] = c_1\, c_2^k \prod_{j=1}^{r} \frac{\Gamma(\beta k + \alpha(j+l))}{\Gamma(\alpha(j+l))}, \tag{4.15}$$

where $r \le d$ is allowed to depend on d, while $c_1, c_2 \in (0,\infty)$ are allowed to depend on r and $\alpha \in (0,\infty)$ is an absolute constant. Moreover, $\beta \in (0,\infty)$ is allowed to depend on j and $l \in (0,\infty)$ may depends on d and r.
For this huge class of random variables, the authors use methods similar to those applied in this section to derive mod-ϕ convergence in the regimes where

- r is fixed,
- $r = d$,
- $d - r \to 0$,
- $d - r = a$, $a \in \mathbb{N}$, and,
- $d - r = o(d)$,

as $d \to \infty$. Now, let $\mathcal{V}_{d,r}$ be the volume of the r-dimensional simplex with vertices $X_1, \ldots, X_{r+1}$, $r \le d$, chosen according to the Gaussian model (see Section 4.1.1). Then, by using (4.15) with $X = r!\mathcal{V}_{d,r}$, $c_1 = 1$, $c_2 = 2^r(r+1)$, $\alpha = \beta = \frac{1}{2}$ and $l = d - r$, we obtain that, for all $k \ge 0$,

$$\mathbb{E}\left[(r!\mathcal{V}_{d,r})^k\right] = (r+1)^k \prod_{j=1}^{r} \left[2^k \frac{\Gamma\left(\frac{d-r+j}{2} + \frac{k}{2}\right)}{\Gamma\left(\frac{d-r+j}{2}\right)} \right],$$

that is, the moment formula presented in Theorem 4.1.2 (a). Similar choices of the involved parameter recover also the moments in the Beta-, the Beta-prime and the spherical model, respectively. In particular, the results presented in [42, Theorem 7.3 and Theorem 7.7] include our Theorem 4.3.1, as well as Theorem 4.3.5, as special cases. Further, Eichelsbacher and Knichel extend our results by proving mod-ϕ convergence also in the regime where $d - r \to 0$, as $d \to \infty$. On the other hand, while we are able to state a central limit theorem with Berry-Esseen bound in the regimes where $r = o(d)$ and $r \sim \alpha d$, $\alpha \in (0,1)$, by using the method of cumulants, such results are not included in [42]. This due to the fact that there is no mod-ϕ convergence in the latter mentioned regimes for the parameter r.

4.4 Large deviations

The purpose of this section is to derive large deviation principles (recall the definition in Section 2.5). Again, we restrict to the Gaussian, the Beta and the spherical model, which admit finite moments of all orders.

4.4.1 The Gaussian model

We start with the Gaussian model and recall the notation $\mathcal{L}_{d,r} = \log(r!\mathcal{V}_{d,r})$. By using the Gärtner–Ellis theorem (see Theorem 2.5.2), we derive large deviation principles from the following assertions.

Theorem 4.4.1 *(a) Let $r \in \mathbb{N}$ be fixed. Then, we have*

$$\begin{aligned} j_1(t) :&= \lim_{d\to\infty} \frac{1}{rd} \log \mathbb{E}\left[e^{td(\mathcal{L}_{d,r} - \frac{r}{2}\log d - \frac{1}{2}\log(r+1))}\right] \\ &= \begin{cases} \frac{1}{2}((t+1)\log(t+1) - t) & : t \geq -1 \\ +\infty & : \textit{otherwise.} \end{cases} \end{aligned}$$

(b) Let $r \sim \alpha d$, $\alpha \in (0,1)$, as $d \to \infty$. Then, we have

$$\begin{aligned} j_2(t) :&= \lim_{d\to\infty} \frac{1}{\alpha d^2} \log \mathbb{E}\left[e^{td(\mathcal{L}_{d,r} - \frac{\alpha d}{2}(\log d + \log(1-\alpha)))}\right] \\ &= \begin{cases} \frac{2+2t-\alpha}{4} \log \frac{1+t-\alpha}{1-\alpha} - \frac{t}{2} & : t \geq \alpha - 1 \\ +\infty & : \textit{otherwise.} \end{cases} \end{aligned}$$

(c) Let $a \in \mathbb{N}_0$ and assume that $r = d - a$, as $d \to \infty$, and m_d as in (4.5). Then, we have

$$\lim_{d\to\infty} \frac{1}{\frac{1}{2}\log\frac{d}{2}} \log \mathbb{E}\left[e^{t(\mathcal{L}_{d,r} - m_d)}\right] = \frac{1}{2}t^2, \qquad t \in \mathbb{R}.$$

(d) Let $r = r(d)$ be such that $d - r = o(d)$, as $d \to \infty$. Then, we have

$$\lim_{d\to\infty} \frac{1}{\frac{1}{2}\log\frac{d}{d-r}} \log \mathbb{E}\left[e^{t\left(\mathcal{L}_{d,r} - (m_d - m_{d-r}) - \frac{1}{2}\log\frac{(r+1)(d-r+1)}{d+1}\right)}\right] = \frac{1}{2}t^2, \qquad t \in \mathbb{R}.$$

Proof. For $t \geq -1$, part (a) is a consequence of Theorem 4.3.1 (a). Now, recall from Theorem 4.1.4 (a) that the distribution of $\mathcal{L}_{d,r}$ involves Gamma distributed random variables $Z \overset{D}{\sim} \Gamma_{\frac{d-r+j}{2},\frac{1}{2}}$ with $j \leq r$. Writing

$$\mathbb{E}\left[e^{\frac{td}{2}\log Z}\right] = \mathbb{E}\left[Z^{\frac{td}{2}}\right] = c\int_0^\infty z^{\frac{d-r+j}{2}+\frac{td}{2}-1}\, e^{-\frac{x}{2}}\,\mathrm{d}z,$$

where $c \in (0,\infty)$ is some absolute constant, we get that the exponent at z is less than -1 for sufficiently large d, if $t < -1$. This implies that in this regime the expectation tends to infinity and, thus, completes the proof of part (a).

The proofs of (c) and (d) directly follow from the proofs of Theorem 4.3.1 (b), (c), (d) in the previous section, respectively.

We turn now to the case that $r \sim \alpha d$. Due to the asymptotic formula (2.13), we obtain for all $\alpha \in (0,1)$, $t \geq \alpha - 1$ and $j \in \mathbb{N}$ that

$$\begin{aligned}
&\log\left(\frac{\Gamma\left(\frac{(1+t-\alpha)d+j}{2}\right)}{\Gamma\left(\frac{(1-\alpha)d+j}{2}\right)}\right)\\
&\sim \log\left(\frac{\exp(-\frac{(1+t-\alpha)d}{2})\left(\frac{(1+t-\alpha)d}{2}\right)^{\frac{(1+t-\alpha)d}{2}+\frac{j-1}{2}}}{\exp(-\frac{(1-\alpha)d}{2})\left(\frac{(1-\alpha)d}{2}\right)^{\frac{(1-\alpha)d}{2}+\frac{j-1}{2}}}\right)\\
&= \log\left(\exp\left(-\frac{td}{2}\right)\left(\frac{d}{2}\right)^{\frac{td}{2}}\frac{(1+t-\alpha)^{\frac{(1+t-\alpha)d}{2}}}{(1-\alpha)^{\frac{(1-\alpha)d}{2}}}\left(\frac{1+t-\alpha}{1-\alpha}\right)^{\frac{j-1}{2}}\right)\\
&= -\frac{td}{2} + \frac{td}{2}\log\frac{d}{2} + \frac{(1+t-\alpha)d}{2}\log(1+t-\alpha)\\
&\qquad - \frac{(1-\alpha)d}{2}\log(1-\alpha) + \frac{j-1}{2}\log\frac{1+t-\alpha}{1-\alpha},
\end{aligned}$$

as $d \to \infty$, and, thus,

$$\begin{aligned}
&\frac{1}{\alpha d^2}\log\mathbb{E}\left[e^{td\mathcal{L}_{d,r}}\right] = \frac{1}{\alpha d^2}\left[\frac{td}{2}\log(\alpha d+1) + \frac{t\alpha d^2}{2}\log 2 + \sum_{j=1}^{\alpha d}\log\left(\frac{\Gamma\left(\frac{(1+t-\alpha)d+j}{2}\right)}{\Gamma\left(\frac{(1-\alpha)d+j}{2}\right)}\right)\right]\\
&\sim -\frac{t}{2} + \frac{t}{2}\log d + \frac{1+t-\alpha}{2}\log(1+t-\alpha) - \frac{1-\alpha}{2}\log(1-\alpha) + \frac{\alpha}{4}\log\frac{1+t-\alpha}{1-\alpha}\\
&= -\frac{t}{2} + \frac{t}{2}\log d + \frac{2+2t-\alpha}{4}\log(1+t-\alpha) - \frac{2-\alpha}{4}\log(1-\alpha).
\end{aligned}$$

This yields the result in the case that $r \sim \alpha d$ in view of the moment formula for Gaussian simplices stated in Section 4.1.2. For $t < \alpha - 1$, we obtain the result completely similar as in the case where r is fixed, presented above. $\square$

We turn now to the large deviation principles for the log-volume of Gaussian simplices.

Theorem 4.4.2 (LDP for Gaussian simplices) *(a) Let $r \in \mathbb{N}$ be fixed. Then, the sequence*

$$\frac{1}{r}\left(\mathcal{L}_{d,r} - \frac{r}{2}\log d - \frac{1}{2}\log(r+1)\right)$$

satisfies a large deviation principle on $\mathbb{R}$ with speed rd and rate function

$$I(x) = \frac{1}{2}(e^{2x} - 1) - x, \qquad x \in \mathbb{R}.$$

(b) If $r \sim \alpha d$, $\alpha \in (0,1)$, then, the sequence

$$\frac{1}{\alpha d}\left(\mathcal{L}_{d,r} - \frac{\alpha d}{2}(\log d + \log(1-\alpha))\right)$$

satisfies a large deviation principle on $\mathbb{R}$ with speed αd^2 and rate function

$$I(x) = \sup_{t \geq \alpha - 1} \{tx - j_2(t)\}, \qquad x \in \mathbb{R},$$

where $j_2(t)$ is the function from Theorem 4.4.1 (b).

(c) Let $a \in \mathbb{N}_0$ and assume that $r = d - a$, as $d \to \infty$, and m_d as in (4.5). Then, the sequence

$$\frac{1}{\frac{1}{2}\log\frac{d}{2}}(\mathcal{L}_{d,r} - m_d)$$

satisfies a large deviation principle on $\mathbb{R}$ with speed $\frac{1}{2}\log\frac{d}{2}$ and rate function

$$I(x) = \frac{1}{2}x^2, \qquad x \in \mathbb{R}.$$

(d) Let $r = r(d)$ be such that $d - r = o(d)$, as $d \to \infty$. Then, the sequence

$$\frac{1}{\frac{1}{2}\log\frac{d}{d-r}}\left(\mathcal{L}_{d,r} - (m_d - m_{d-r}) - \frac{1}{2}\log\frac{(r+1)(d-r+1)}{d+1}\right)$$

satisfies a large deviation principle on $\mathbb{R}$ with speed $\frac{1}{2}\log\frac{d}{d-r}$ and rate function

$$I(x) = \frac{1}{2}x^2, \qquad x \in \mathbb{R}.$$

Proof. Let $r \in \mathbb{N}$ be fixed. Then, by the Gärtner–Ellis theorem (see Section 2.5) and Theorem 4.4.1 (a), the random variables $\frac{1}{r}(\mathcal{L}_{d,r} - \frac{r}{2}\log d - \frac{1}{2}\log(r+1))$ satisfy a large deviation principle with speed rd and rate function

$$I(x) = \sup_{t\in\mathbb{R}}\left[tx - \frac{1}{2}((t+1)\log(t+1) - t)\right], \qquad x \in \mathbb{R},$$

i.e., the Legendre-Fenchel transformation of the function $f(t) := \frac{1}{2}((t+1)\log(t+1)-t)$. For each $x \in \mathbb{R}$, the supremum is attained at $t = e^{2x} - 1$, which yields the result of (a) since the function $\frac{1}{2}((t+1)\log(t+1) - t)$ is lower-semicontinuous, differentiable on $(-1, \infty)$ and satisfies

$$\lim_{t\downarrow -1}\left|\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right| = \lim_{t\downarrow -1}\left|\frac{1}{2}\log(t+1)\right| = \infty. \tag{4.16}$$

Similar arguments imply the large deviation principles for the other regimes of r as well. □

4.4.2 The Beta and the spherical model

Now, we turn to the Beta model with parameter $\nu > 0$ and the spherical model, i.e., $\nu = 0$, and recall that $\mathcal{L}_{d,r} = \log(r!\mathcal{V}_{d,r})$, where $\mathcal{V}_{d,r}$ is the volume of the r-dimensional simplex with vertices $X_1, \ldots, X_{r+1}$, chosen according to the Beta or the spherical distribution, respectively. Similar to the Gaussian case, we start with the following theorem that implies the large deviation principles.

Theorem 4.4.3 *(a) Let $r \in \mathbb{N}$ be fixed. Then, we have*

$$j_3(t) := \lim_{d\to\infty} \frac{1}{d} \log \mathbb{E}\left[e^{td\mathcal{L}_{d,r}}\right] = \begin{cases} \eta(t) & : t \geq -1 \\ +\infty & : \text{otherwise}, \end{cases}$$

where $\eta(t)$ is the function from Theorem 4.3.5 (a).

(b) If $r \sim \alpha d$, $\alpha \in (0,1)$, we have

$$j_4(t) := \lim_{d\to\infty} \frac{1}{\alpha d^2} \log \mathbb{E}\left[e^{td\mathcal{L}_{d,r}}\right] = \begin{cases} \eta(t) & : t \geq \alpha - 1 \\ +\infty & : \text{otherwise}, \end{cases}$$

where $\eta(t)$ is the function given by

$$\eta(t) := \frac{2+2t-\alpha}{4} \log(1+t-\alpha) - \frac{2-\alpha}{4} \log(1-\alpha) - \frac{1+t}{2} \log(1+t).$$

(c) Let $a \in \mathbb{N}_0$ and assume that $r = d - a$, as $d \to \infty$, and let $\widetilde{m}_d$ as in (4.12). Then,

$$\lim_{d\to\infty} \frac{1}{\frac{1}{2}(\log\frac{d}{2} - 1)} \log \mathbb{E}\left[e^{t(\mathcal{L}_{d,r} - \widetilde{m}_d - \frac{a}{2}\log\frac{d}{2})}\right] = \frac{1}{2}t^2, \qquad t \in \mathbb{R}.$$

(d) Let $r = r(d)$ be such that $d - r = o(d)$, and let m_d as in (4.5). Then,

$$\lim_{d\to\infty} \frac{1}{\frac{1}{2}\log\frac{d}{d-r}} \log \mathbb{E}\left[e^{t\left(\mathcal{L}_{d,r} - (m_d - m_{d-r} - \frac{r+1}{4d}(t-2+2\nu)) - \frac{1}{2}\log\frac{(d-r)(1+r)}{d^{1+r}}\right)}\right] = \frac{1}{2}t^2, \quad t \in \mathbb{R}.$$

Proof. For $t \geq -1$, the assertion in (a) follows from Theorem 4.3.5 (a). Recall from Theorem 4.1.4 (b), that the distribution of $\mathcal{V}_{d,r}$ involves Beta random variables $Z \overset{D}{\sim} \beta_{\frac{\nu+r-j}{2}, \frac{d-r+j}{2}}$ with $j \leq r$. Writing once more

$$\mathbb{E}\left[e^{\frac{td}{2}\log Z}\right] = \mathbb{E}\left[Z^{\frac{td}{2}}\right] = c \int_0^1 z^{\frac{d-r+j}{2} + \frac{td}{2} - 1}(1-z)^{\frac{\nu+r-j}{2} - 1}\,\mathrm{d}z,$$

we see that the exponent at z is less than -1 for sufficiently large d, if $t < -1$. This shows that the expected value tends to infinity and, thus, completes the proof of (a).

Now, let us turn towards the case that $r \sim \alpha d$, $\alpha \in (0,1)$. Similar to the Gaussian setting, we obtain, by using the asymptotic formula (2.13), for all $\nu > 0$, as $d \to \infty$,

$$
\begin{aligned}
&(\alpha d+1)\log\left(\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{(1+t)d+\nu}{2}\right)}\right)\\
&\qquad\sim \frac{t\alpha d^2}{2}-\frac{t\alpha d^2}{2}\log\frac{d}{2}-\frac{(1+t)\alpha d^2+\alpha d(\nu-1)}{2}\log(1+t),
\end{aligned}
$$

and, for all $t \geq 0$,

$$
\begin{aligned}
\log&\left(\frac{\Gamma\left(\frac{\alpha d(d+\nu-2)+d+td(\alpha d+1)+\nu}{2}\right)}{\Gamma\left(\frac{\alpha d(d+\nu-2)+d+td\alpha d+\nu}{2}\right)}\right)\\
&\sim -\frac{td}{2}+\frac{td}{2}\log\frac{d}{2}\\
&\qquad+\frac{\alpha d(d+\nu-2)+d+td(\alpha d+1)+\nu}{2}\log\left(\alpha(d+\nu-2)+1+t(\alpha d+1)\right)\\
&\qquad-\frac{\alpha d(d+\nu-2)+d+td\alpha d+\nu}{2}\log\left(\alpha(d+\nu-2)+1+t\alpha d\right).
\end{aligned}
$$

Thus, by using the calculations made in the Gaussian case above, we conclude that, for $t \geq \alpha - 1$, as $d \to \infty$,

$$
\begin{aligned}
&\frac{1}{\alpha d^2}\log\mathbb{E}\left[e^{td\mathcal{L}_{d,r}}\right]\\
&=\frac{1}{\alpha d^2}\left[(\alpha d+1)\log\left(\frac{\Gamma\left(\frac{d+\nu}{2}\right)}{\Gamma\left(\frac{(1+t)d+\nu}{2}\right)}\right)\right.\\
&\qquad\left.+\log\left(\frac{\Gamma\left(\frac{\alpha d(d+\nu-2)+d+td(\alpha d+1)+\nu}{2}\right)}{\Gamma\left(\frac{\alpha d(d+\nu-2)+d+td\alpha d+\nu}{2}\right)}\right)+\sum_{j=1}^{\alpha d}\log\left(\frac{\Gamma\left(\frac{(1+t-\alpha)d+j}{2}\right)}{\Gamma\left(\frac{(1-\alpha)d+j}{2}\right)}\right)\right]\\
&\sim\frac{t}{2}-\frac{t}{2}\log\frac{d}{2}-\frac{1+t}{2}\log(1+t)+\frac{1+t}{2}\log\left(\alpha(d+\nu-2)+1+t(\alpha d+1)\right)\\
&\qquad-\frac{1+t}{2}\log\left(\alpha(d+\nu-2)+1+t\alpha d\right)-\frac{t}{2}+\frac{t}{2}\log\frac{d}{2}\\
&\qquad+\frac{2+2t-\alpha}{4}\log\left(1+t-\alpha\right)-\frac{2-\alpha}{4}\log\left(1-\alpha\right)\\
&\sim-\frac{1+t}{2}\log(1+t)+\frac{2+2t-\alpha}{4}\log\left(1+t-\alpha\right)-\frac{2-\alpha}{4}\log\left(1-\alpha\right).
\end{aligned}
$$

This directly yields the result in the case where $r \sim \alpha d$, taking into account the moment representation in the Beta model stated in Section 4.1.2.

For $t < \alpha - 1$, we conclude the result similarly as above in the case that $r = o(d)$. Since there is no dependence on the parameter ν in the result concerning the Beta model, the one regarding to the spherical model is implied by considering the limiting case $\nu \downarrow 0$, as seen several times before.

The proofs of (c) and (d) directly follow from the proofs of Theorem 4.4.3 (b), (c), (d) in the previous section, respectively. □

Now, we are able to state the large deviation principles for the Beta and the spherical model. Their proofs follow the same lines as the ones in the Gaussian case, using again the Gärtner–Ellis theorem. Therefore, we have decided to omit them.

Theorem 4.4.4 (LDP for Beta-type and spherical simplices) *(a) Let $r \in \mathbb{N}$ be fixed. Then, the sequence $\mathcal{L}_{d,r}$ satisfies a large deviation principle on $\mathbb{R}$ with speed d and rate function*

$$I(x) = \sup_{t\in\mathbb{R}} \{tx - j_3(t)\},$$

where $j_3(t)$ is the function from Theorem 4.4.3 (a).

(b) If $r \sim \alpha d$, $\alpha \in (0,1)$, then, the sequence

$$\frac{1}{\alpha d}\,\mathcal{L}_{d,r}$$

satisfies a large deviation principle on $\mathbb{R}$ with speed αd^2 and rate function

$$I(x) = \sup_{t\in\mathbb{R}} \{tx - j_4(t)\},$$

where $j_4(t)$ is the function from Theorem 4.4.3 (b).

(c) Let $a \in \mathbb{N}_0$ and assume that $r = d - a$, as $d \to \infty$, and $\widetilde{m}_d$ as in (4.12). Then,

$$\frac{1}{\frac{1}{2}(\log\frac{d}{2} - 1)}\left(\mathcal{L}_{d,r} - \widetilde{m}_d - \frac{a}{2}\log\frac{d}{2}\right)$$

satisfies a large deviation principle on $\mathbb{R}$ with speed $\frac{1}{2}(\log\frac{d}{2} - 1)$ and rate function

$$I(x) = \frac{1}{2}x^2, \qquad x \in \mathbb{R}.$$

(d) Let $r = r(d)$ be such that $d - r = o(d)$, and let m_d be as in (4.5). Then,

$$\frac{1}{\frac{1}{2}\log\frac{d}{d-r}}\left(\mathcal{L}_{d,r} - (m_d - m_{d-r} - \frac{r+1}{4d}(t-2+2\nu)) - \frac{1}{2}\log\frac{(d-r)(1+r)}{d^{1+r}}\right)$$

satisfies a large deviation principle on $\mathbb{R}$ with speed $\frac{1}{2}\log\frac{d}{d-r}$ and rate function

$$I(x) = \frac{1}{2}x^2, \qquad x \in \mathbb{R}.$$

Finally, we combine Theorem 4.4.4 with the contraction principle (see Theorem 2.5.3) to obtain a large deviation principle for $\mathcal{V}_{d,r}$, that is, for the volume of the random simplex itself, in the case that $r \in \mathbb{N}$ is fixed or $r \sim \alpha d$, for some $\alpha \in (0,1)$.

Corollary 4.4.5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.*

(a) Let $r \in \mathbb{N}$ be fixed. Then, the sequence $r!\mathcal{V}_{d,r}$ satisfies a large deviation principle on $\mathbb{R}$ with speed d and rate function

$$I^*(y) := \inf\left\{I(x) : x \in \mathbb{R}, e^x = y\right\}, \qquad y \in \mathbb{R},$$

where $I(x)$ is the rate function from Theorem 4.4.4 (a).

(b) Let $r \sim \alpha d$, $\alpha \in (0,1)$. Then, the sequence $\frac{r!}{\alpha d}\mathcal{L}_{d,r}$ satisfies a large deviation principle on $\mathbb{R}$ with speed αd^2 and rate function

$$I^*(y) := \inf\left\{I(x) : x \in \mathbb{R}, e^x = y\right\}, \qquad y \in \mathbb{R},$$

where $I(x)$ is the rate function from Theorem 4.4.4 (b).

Chapter 5

Approximation of smooth convex bodies

Let K be a convex body in $\mathbb{R}^d$, $d \geq 2$, with twice continuously differentiable boundary ∂K and strictly positive Gaussian curvature $\kappa_K(x)$, $x \in \partial K$. Further, let $f : \partial K \to \mathbb{R}_+$ be a continuous and strictly positive function, satisfying

$$\int_{\partial K} f(x)\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) = 1.$$

In this chapter, we give an upper bound for the approximation of K in the symmetric difference metric by an arbitrarily positioned polytope P_f, having a fixed number of vertices. This generalizes a result by Ludwig, Schütt and Werner [91]. The polytope P_f is obtained by a random construction via a probability measure with density f. In our result, the dependence on the number of vertices is optimal. Moreover, with the optimal density f, the dependence on K is also optimal.

5.1 Main result

Recall the symmetric difference metric of two convex bodies K and L in $\mathbb{R}^d$, defined as

$$\mathrm{vol}_d(K\Delta L) := \mathrm{vol}_d\,(K \cup L) - \mathrm{vol}_d\,(K \cap L)\,.$$

In this chapter, $C, C_1, C_2 \in (0,\infty)$ will always denote absolute constants that may change from line to line.

Theorem 5.1.1 *Let K be a convex body in $\mathbb{R}^d$, $d \geq 2$, with twice continuously differentiable boundary ∂K and strictly positive Gaussian curvature $\kappa_K(x)$, $x \in \partial K$. Further, let $f : \partial K \to \mathbb{R}_+$ be a continuous and strictly positive function, satisfying*

$$\int\limits_{\partial K} f(x)\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) = 1.$$

Then, there exists a polytope P_f in $\mathbb{R}^d$, having n vertices, such that for sufficiently large n, it holds that

$$\mathrm{vol}_d(K\Delta P_f) \leq C\,n^{-\frac{2}{d-1}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x).$$

We discuss this bound for different densities. First, it was shown in [121, Page 8] that the minimum of the right hand side is attained for the normalized affine surface area measure, having density

$$f_{\mathrm{as}}(x) := \frac{\kappa_K(x)^{\frac{1}{d+1}}}{\int\limits_{\partial K} \kappa_K(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x)}.$$

In this case, the theorem yields that

$$\mathrm{vol}_d(K\Delta P_f) \leq C\,n^{-\frac{2}{d-1}}\,\mathrm{as}(K)^{\frac{d+1}{d-1}},$$

where $\mathrm{as}(K)$ is the affine surface area of K (see Section 2.2). In particular, choosing the vertices of the approximating polytope according to the Gaussian curvature yields the optimal bound.

Now, let K be centered, that is, its centroid is positioned at the origin. Recall that for $x \in \partial K$, we denote by $N_K(x)$ the corresponding outer unit normal. Put

$$f_{\beta,\alpha}(x) := \frac{\langle x, N_K(x)\rangle^{\alpha}\, \kappa_K(x)^{\beta}}{\int_{\partial K} \langle x, N_K(x)\rangle^{\alpha}\, \kappa_K(x)^{\beta}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x)},$$

where $\alpha, \beta \in \mathbb{R}$. Then, the theorem yields that

$$\begin{aligned} \mathrm{vol}_d(K\Delta P_f) \leq C\, n^{-\frac{2}{d-1}} & \int_{\partial K} \frac{\kappa_K(x)^{\frac{1-2\beta}{d-1}}}{\langle x, N_K(x)\rangle^{\frac{2\alpha}{d-1}}}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \\ & \times \left(\int_{\partial K} \langle x, N_K(x)\rangle^{\alpha}\, \kappa_K(x)^{\beta}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \right)^{\frac{2}{d-1}}, \end{aligned}$$

(see also [121, Page 10]). The second integral is a p-affine surface area $\mathrm{as}_p(K)$ of K, if and only if

$$\alpha = -\frac{d(p-1)}{d+p} \qquad \text{and} \qquad \beta = \frac{p}{d+p}.$$

In this case, it holds that

$$\mathrm{vol}_d(K\Delta P_f) \leq C\, n^{-\frac{2}{d-1}}\, \mathrm{as}_q(K)\, \mathrm{as}_p(K)^{\frac{2}{d-1}},$$

where

$$q = \frac{d-p}{d+p-2}.$$

Finally, we discuss the surface measure, given by the constant density

$$f_{\mathrm{sm}}(x) := \frac{1}{\mathrm{vol}_{d-1}(\partial K)}.$$

Then, the theorem implies that

$$\mathrm{vol}_d(K\Delta P_f) \leq C\, n^{-\frac{2}{d-1}}\, \mathrm{vol}_{d-1}(\partial K)^{\frac{2}{d-1}} \int_{\partial K} \kappa_K(x)^{\frac{1}{d-1}}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x),$$

(see also [121, Page 9]).

5.2 Preliminaries

Recall that for fixed $u \in \mathbb{S}^{d-1}$ and $h \geq 0$, we denote by $H := H(u,h)$ the unique hyperplane orthogonal to u at distance h from the origin. Let $\mathbb{P}_f$ be the probability measure on ∂K given by

$$\mathrm{d}\mathbb{P}_f := f(x)\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x).$$

Now, let $H \cap K \neq \emptyset$. Then, $\mathbb{P}_{f_{\partial K\cap H}}$ is the probability measure on $\partial K \cap H$ given by

$$\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}} := \frac{f(x)\,\mathcal{H}^{d-2}_{\partial K\cap H}(\mathrm{d}x)}{\int\limits_{\partial K\cap H} f(x)\,\mathcal{H}^{d-2}_{\partial K\cap H}(\mathrm{d}x)}.$$

The following results are crucial to prove the main theorem. The first two are stated in [121, Theorem 1.1 and Lemma 4.3].

Theorem 5.2.1 *Denote by $\mathbb{E}[\mathrm{vol}_d(P_n)]$ the expected volume of the convex hull of n points, chosen independently on ∂K with respect to $\mathbb{P}_f$. Then, it holds that*

$$\frac{\mathrm{vol}_d(K) - \mathbb{E}[\mathrm{vol}_d(P_n)]}{n^{-\frac{2}{d-1}}} \sim \frac{(d-1)^{\frac{d+1}{d-1}}\,\Gamma\left(d+1+\frac{2}{d-1}\right)}{2\,(d+1)!\,\omega_{d-1}^{\frac{2}{d-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x),$$

as $n \to \infty$.

Lemma 5.2.2 *Let $\sigma = (\sigma_i)_{1\leq i\leq d}$ be a sequence of signs, that is, $\sigma_i \in \{-1,1\}$, $i \in \{1,\dots,d\}$. We define*

$$K^\sigma := \{x = (x_1,\dots,x_d) \in K : \mathrm{sign}(x_i) = \sigma_i,\, i \in \{1,\dots,d\}\}.$$

Then, it holds that

$$\mathbb{P}_f^n(\{\boldsymbol{o} \notin \mathrm{conv}(x_1,\dots,x_n)\}) \leq 2^d \left(1 - \min_\sigma \int\limits_{\partial K^\sigma} f(x)\,\mathcal{H}^{d-1}_{\partial K^\sigma}(\mathrm{d}x)\right)^n,$$

where $\mathbb{P}_f^n$ indicates that the n points are chosen independently on ∂K with respect to $\mathbb{P}_f$.

Moreover, we need the following Blaschke-Petkantschin-type formula. It arises as a special case of a result derived in [137]. An alternative and simpler proof of this version is also provided in [107, Page 2247].

Theorem 5.2.3 *Let $g(x_1, \dots, x_d)$ be a continuous and non-negative function. Then, it holds that*

$$\int_{\partial K} \cdots \int_{\partial K} g(x_1, \dots, x_d) \, \mathrm{d}\mathbb{P}_f(x_1) \dots \mathrm{d}\mathbb{P}_f(x_d)$$
$$= (d-1)! \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{\partial K \cap H} \cdots \int_{\partial K \cap H} g(x_1, \dots, x_d) \, \mathrm{vol}_{d-1}(\mathrm{conv}(x_1, \dots, x_d))$$
$$\times \prod_{j=1}^d l_H(x_j) \, \mathrm{d}\mathbb{P}_{f_{\partial K \cap H}}(x_1) \dots \mathrm{d}\mathbb{P}_{f_{\partial K \cap H}}(x_d) \, \mathrm{d}h \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),$$

where, for $j \in \{1, \dots, d\}$,

$$l_H(x_j) := \| N_K(x_j) | H \|^{-1},$$

and $N_K(x_j)|H$ is the orthogonal projection of $N_K(x_j)$ onto the hyperplane $H := H(u, h)$.

Finally, from Theorem 4.1.2 (d), (2.9) and (2.10), we deduce that

$$\int_{\mathbb{S}^{d-2}} \cdots \int_{\mathbb{S}^{d-2}} \mathrm{vol}_{d-1}(\mathrm{conv}(x_1, \dots, x_d))^2 \, \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}x_1) \dots \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}x_d)$$
$$= \frac{\omega_{d-1}^d}{((d-1)!)^2} \underbrace{\prod_{j=1}^{d-1} \frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{j}{2}\right)}}_{=\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}}} \underbrace{\left(\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}+1\right)}\right)^d}_{=\left(\frac{2}{d-1}\right)^d} \underbrace{\frac{\Gamma\left(\frac{(d-1)(d-3)+(d-1)}{2}+d\right)}{\Gamma\left(\frac{(d-1)(d-3)+(d-1)}{2}+d-1\right)}}_{=\frac{(d-1)d}{2}} \tag{5.1}$$
$$= \frac{\omega_{d-1}^d \, 2^d \, (d-1) \, d}{((d-1)!)^2 \sqrt{\pi} \, (d-1)^d \, 2} \underbrace{\Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d+1}{2}\right)}_{=\frac{\sqrt{\pi}(d-1)!}{2^{d-1}}} = \frac{d \, \omega_{d-1}^d}{(d-1)! \, (d-1)^{d-1}}.$$

5.3 Proof of the main result

Without loss of generality, we assume that the origin is in the interior of K. As already explained detailed in Section 1.2, we obtain the approximating polytope in a probabilistic way. More precisely, we choose n random points $X_1, \dots, X_n$ on the boundary of K according to $\mathbb{P}_f$, and let $P_n := \operatorname{conv}(X_1, \dots, X_n)$. Then, we approximate a slightly smaller body, namely, $(1-\gamma)K$, where $\gamma := \gamma_{n,d}$ depends on the dimension d and the number of points n. In fact, we choose γ such that

$$\mathbb{E}[\operatorname{vol}_d(P_n)] = \operatorname{vol}_d((1-\gamma)K) = (1-\gamma)^d \operatorname{vol}_d(K). \tag{5.2}$$

By Theorem 5.2.1, we have that

$$\operatorname{vol}_d(K) - \mathbb{E}[\operatorname{vol}_d(P_n)] \sim n^{-\frac{2}{d-1}} \frac{(d-1)^{\frac{d+1}{d-1}}\, \Gamma\left(d+1+\frac{2}{d-1}\right)}{2\,(d+1)!\,\omega_{d-1}^{\frac{2}{d-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}_{\partial K}^{d-1}(\mathrm{d}x),$$

as $n \to \infty$. Hence, with the choice (5.2) of γ,

$$\begin{aligned} &\operatorname{vol}_d(K) - (1-\gamma)^d \operatorname{vol}_d(K) \\ &\quad \sim n^{-\frac{2}{d-1}} \frac{(d-1)^{\frac{d+1}{d-1}}\, \Gamma\left(d+1+\frac{2}{d-1}\right)}{2\,(d+1)!\,\omega_{d-1}^{\frac{2}{d-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}_{\partial K}^{d-1}(\mathrm{d}x), \end{aligned}$$

as $n \to \infty$. Since

$$\left(1-(1-\gamma)^d\right) \sim d\,\gamma,$$

as $d \to \infty$, this leads to

$$\gamma \sim n^{-\frac{2}{d-1}} \frac{(d-1)^{\frac{d+1}{d-1}}\, \Gamma\left(d+1+\frac{2}{d-1}\right)}{2\,(d+1)!\,\omega_{d-1}^{\frac{2}{d-1}}} \frac{1}{d\operatorname{vol}_d(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}_{\partial K}^{d-1}(\mathrm{d}x), \tag{5.3}$$

as $n \to \infty$. In particular, for sufficiently large n, γ can be bounded from below by

$$\left(1-\frac{1}{d}\right) n^{-\frac{2}{d-1}} \frac{(d-1)^{\frac{d+1}{d-1}}\, \Gamma\left(d+1+\frac{2}{d-1}\right)}{2\,(d+1)!\,\omega_{d-1}^{\frac{2}{d-1}}} \frac{1}{d\operatorname{vol}_d(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}_{\partial K}^{d-1}(\mathrm{d}x). \tag{5.4}$$

We split the proof of the main theorem into several lemmas. Recall, if H is some hyperplane, we denote by H^+ the corresponding half-space containing the origin and by H^- the opposite one. Now, define

$$\mathbb{P}_f(\partial K \cap H^+) := \int_{\partial K \cap H^+} f(x)\, \mathcal{H}^{d-1}_{\partial K \cap H^+}(\mathrm{d}x). \tag{5.5}$$

Furthermore, for fixed $u \in \mathbb{S}^{d-1}$ and sufficiently large n, let $\varepsilon > 0$ be such that

$$\gamma\, h_K(u) \leq \varepsilon \leq \frac{h_K(u)}{d},$$

where $h_K(u)$ is the support function of K in direction u.

Lemma 5.3.1 *For sufficiently large n, for all $\varepsilon \geq \gamma h_K(u)$ sufficiently small, it holds that*

$$\begin{aligned}
&\mathbb{E}[\mathrm{vol}_d((1-\gamma)K \Delta P_n)] \\
&\quad \leq C \binom{n}{d} (d-1)! \int_{\mathbb{S}^{d-1}} \int_{h_K(u)-\varepsilon}^{h_K(u)} \mathbb{P}_f(\partial K \cap H^+)^{n-d} \max\{0, ((1-\gamma)h_K(u) - h)\} \\
&\quad \times \int_{\partial K \cap H} \cdots \int_{\partial K \cap H} \mathrm{vol}_{d-1}(\mathrm{conv}(x_1, \dots, x_d))^2 \prod_{j=1}^{d} l_H(x_j) \\
&\quad \times \mathrm{d}\mathbb{P}_{f_{\partial K \cap H}}(x_1) \dots \mathrm{d}\mathbb{P}_{f_{\partial K \cap H}}(x_d)\, \mathrm{d}h\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).
\end{aligned}$$

Proof of Lemma 5.3.1. The choice of the parameter γ in (5.2) yields for sufficiently large n,

$$\mathrm{vol}_d(K \setminus (1-\gamma)K) = \int_{\partial K} \cdots \int_{\partial K} \mathrm{vol}_d(K \setminus P_n)\, \mathrm{d}\mathbb{P}_f(x_1) \dots \mathrm{d}\mathbb{P}_f(x_n).$$

We combine this observation with the relation

$$\mathrm{vol}_d((1-\gamma)K \Delta P_n) = \mathrm{vol}_d(K \setminus (1-\gamma)K) - \mathrm{vol}_d(K \setminus P_n) + 2\, \mathrm{vol}_d((1-\gamma)K \cap P_n^c).$$

Figure 1.13 in the guideline corresponding to this chapter illustrates an example for this equality in the planar case.

Thus, Lemma 5.2.2 implies for sufficiently large n,

$$
\begin{aligned}
&\mathbb{E}[\mathrm{vol}_d((1-\gamma)K\Delta P_n)] \\
&\quad= \int\limits_{\partial K}\cdots\int\limits_{\partial K} \mathrm{vol}_d((1-\gamma)K\Delta P_n)\,\mathrm{d}\mathbb{P}_f(x_1)\ldots\mathrm{d}\mathbb{P}_f(x_n) \\
&\quad= \mathrm{vol}_d(K\setminus(1-\gamma)K) - \int\limits_{\partial K}\cdots\int\limits_{\partial K} \mathrm{vol}_d(K\setminus P_n)\,\mathrm{d}\mathbb{P}_f(x_1)\ldots\mathrm{d}\mathbb{P}_f(x_n) \\
&\qquad+ 2\int\limits_{\partial K}\cdots\int\limits_{\partial K} \mathrm{vol}_d((1-\gamma)K\cap P_n^c)\,\mathrm{d}\mathbb{P}_f(x_1)\ldots\mathrm{d}\mathbb{P}_f(x_n) \\
&\quad= 2\int\limits_{\partial K}\cdots\int\limits_{\partial K} \mathrm{vol}_d((1-\gamma)K\cap P_n^c)\,\mathrm{d}\mathbb{P}_f(x_1)\ldots\mathrm{d}\mathbb{P}_f(x_n) \\
&\quad= 2\int\limits_{\partial K}\cdots\int\limits_{\partial K} \mathrm{vol}_d((1-\gamma)K\cap P_n^c)\,\mathbf{1}(\boldsymbol{o}\in P_n)\,\mathrm{d}\mathbb{P}_f(x_1)\ldots\mathrm{d}\mathbb{P}_f(x_n) \\
&\qquad+ 2\int\limits_{\partial K}\cdots\int\limits_{\partial K} \mathrm{vol}_d((1-\gamma)K\cap P_n^c)\,\mathbf{1}(\boldsymbol{o}\notin P_n)\,\mathrm{d}\mathbb{P}_f(x_1)\ldots\mathrm{d}\mathbb{P}_f(x_n) \\
&\quad\le 2\int\limits_{\partial K}\cdots\int\limits_{\partial K} \mathrm{vol}_d((1-\gamma)K\cap P_n^c)\,\mathbf{1}(\boldsymbol{o}\in P_n)\,\mathrm{d}\mathbb{P}_f(x_1)\ldots\mathrm{d}\mathbb{P}_f(x_n) \\
&\qquad+ 2\,\mathrm{vol}_d(K)\,\mathbb{P}_f^n(\{\boldsymbol{o}\notin \mathrm{conv}(x_1,\ldots,x_n)\}) \\
&\quad\le 2\int\limits_{\partial K}\cdots\int\limits_{\partial K} \mathrm{vol}_d((1-\gamma)K\cap P_n^c)\,\mathbf{1}(\boldsymbol{o}\in P_n)\,\mathrm{d}\mathbb{P}_f(x_1)\ldots\mathrm{d}\mathbb{P}_f(x_n) \\
&\qquad+ 2\,\mathrm{vol}_d(K)\,2^d\left(1-\min_\sigma \int\limits_{\partial K^\sigma} f(x)\,\mathcal{H}^{d-1}_{\partial K^\sigma}(\mathrm{d}x)\right)^n.
\end{aligned}
$$

The density f is strictly positive everywhere and since the origin is in the interior of K, the second summand is essentially of order C^{-n}, where $C>1$. Later, we derive that the first summand is of order $n^{-\frac{2}{d-1}}$. Thus, it is enough to consider the first one in what follows.
Next, we introduce the function $\Phi_{j_1,\ldots,j_d}:\partial K\times\cdots\times\partial K\to\mathbb{R}$ by

$$
\begin{aligned}
&\Phi_{j_1,\ldots,j_d}(x_1,\ldots,x_n) \\
&:= \begin{cases} 0 & : \mathrm{conv}(x_{j_1},\ldots,x_{j_d})\notin\mathcal{F}_{d-1}(P_n)\text{ or }\boldsymbol{o}\notin P_n \\ \mathrm{vol}_d((1-\gamma)K\cap P_n^c\cap\mathrm{cone}(x_{j_1},\ldots,x_{j_d})) & : \mathrm{conv}(x_{j_1},\ldots,x_{j_d})\in\mathcal{F}_{d-1}(P_n)\text{ and }\boldsymbol{o}\in P_n, \end{cases}
\end{aligned}
$$

where $\mathcal{F}_{d-1}(P_n)$ denotes the set of facets of P_n and

$$\operatorname{cone}(x_1,\dots,x_d) := \left\{\sum_{i=1}^{d} a_i\, x_i : a_i \geq 0, 1 \leq i \leq d\right\}.$$

For all random polytopes P_n containing the origin as an interior point, it holds that

$$\mathbb{R}^d = \bigcup_{\operatorname{conv}(x_{j_1},\dots,x_{j_d}) \in \mathcal{F}_{d-1}(P_n)} \operatorname{cone}(x_{j_1},\dots,x_{j_d}).$$

Moreover,

$$\begin{aligned}
&\mathbb{P}_f^{n-d}(\{(x_{d+1},\dots,x_n) : \operatorname{conv}(x_1,\dots,x_d) \in \mathcal{F}_{d-1}(P_n) \,\text{and}\, \boldsymbol{o} \in P_n\}) \\
&= \left(\int_{\partial K \cap H^+} f(x)\, \mathcal{H}^{d-1}_{\partial K \cap H^+}(\mathrm{d}x) \right)^{n-d} = \mathbb{P}_f(\partial K \cap H^+)^{n-d},
\end{aligned}$$

where H is the hyperplane spanned by the points $x_1,\dots,x_d$ and we recall the definition of $\mathbb{P}_f(\partial K \cap H^+)$, given in (5.5).
Since the n points are independent and identically distributed, we arrive at

$$\begin{aligned}
&\int_{\partial K} \cdots \int_{\partial K} \operatorname{vol}_d((1-\gamma)K \cap P_n^c)\, \mathbf{1}(\boldsymbol{o} \in P_n)\, \mathrm{d}\mathbb{P}_f(x_1) \dots \mathrm{d}\mathbb{P}_f(x_n) \\
&= \int_{\partial K} \cdots \int_{\partial K} \sum_{\{j_1,\dots,j_d\} \subseteq \{1,\dots,n\}} \Phi_{j_1,\dots,j_d}(x_1,\dots,x_n)\, \mathrm{d}\mathbb{P}_f(x_1) \dots \mathrm{d}\mathbb{P}_f(x_n) \\
&= \binom{n}{d} \int_{\partial K} \cdots \int_{\partial K} \Phi_{1,\dots,d}(x_1,\dots,x_n)\, \mathrm{d}\mathbb{P}_f(x_1) \dots \mathrm{d}\mathbb{P}_f(x_n) \\
&= \binom{n}{d} \int_{\partial K} \cdots \int_{\partial K} \mathbb{P}_f(\partial K \cap H^+)^{n-d} \\
&\qquad \times \operatorname{vol}_d((1-\gamma)K \cap H^- \cap \operatorname{cone}(x_1,\dots,x_d))\, \mathbf{1}(\boldsymbol{o} \in P_n)\, \mathrm{d}\mathbb{P}_f(x_1) \dots \mathrm{d}\mathbb{P}_f(x_d) \\
&\leq \binom{n}{d} \int_{\partial K} \cdots \int_{\partial K} \mathbb{P}_f(\partial K \cap H^+)^{n-d} \\
&\qquad \times \operatorname{vol}_d((1-\gamma)K \cap H^- \cap \operatorname{cone}(x_1,\dots,x_d))\, \mathrm{d}\mathbb{P}_f(x_1) \dots \mathrm{d}\mathbb{P}_f(x_d),
\end{aligned}$$

where the sum runs over all unordered partitions of $\{1,\dots,n\}$.

Now, Theorem 5.2.3 yields for sufficiently large n,

$$
\begin{aligned}
&\mathbb{E}[\mathrm{vol}_d((1-\gamma)K\Delta P_n)] \\
&\quad \le C\binom{n}{d}(d-1)! \int_{\mathbb{S}^{d-1}}\int_0^\infty \int_{\partial K\cap H}\cdots\int_{\partial K\cap H} \mathbb{P}_f(\partial K\cap H^+)^{n-d}\,\mathrm{vol}_{d-1}(\mathrm{conv}(x_1,\dots,x_d)) \\
&\qquad \times \mathrm{vol}_d((1-\gamma)K\cap H^-\cap \mathrm{cone}(x_1,\dots,x_d))\prod_{j=1}^d l_H(x_j) \\
&\qquad \times \mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\cdots \mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_d)\,\mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).
\end{aligned}
$$

Notice that $h\in[0,h_K(u)]$. On the other hand, due to the same arguments as in [91, Page 9] and [107, Page 2255], it is possible to bound the range of integration for h from below by $h_K(u)-\varepsilon$, where $\varepsilon>0$ is sufficiently small. Indeed, for all $h\le h_K(u)-\varepsilon$, it holds that $\mathbb{P}_f(\partial K\cap H^+)<1$. Thus, in the latter mentioned regime, the whole integral decays exponentially fast in n.
In particular, for sufficiently large n, we can choose ε such that

$$\gamma\, h_K(u)\le\varepsilon\le\frac{h_K(u)}{d}.$$

Furthermore, it holds that

$$
\begin{aligned}
&\mathrm{vol}_d((1-\gamma)K\cap H^-\cap\mathrm{cone}(x_1,\dots,x_d)) \\
&\qquad \le \frac{h}{d}\,\mathrm{vol}_{d-1}(\mathrm{conv}(x_1,\dots,x_d))\cdot\max\left\{0,\left(\frac{(1-\gamma)h_K(u)}{h}\right)^d-1\right\}.
\end{aligned}
$$

Indeed, let H_* be the hyperplane orthogonal to u at distance $(1-\gamma)h_K(u)$ from the origin, i.e., the tangent at $(1-\gamma)K$ in direction u. Then,

$$
\begin{aligned}
&\mathrm{vol}_d((1-\gamma)K\cap H^-\cap\mathrm{cone}(x_1,\dots,x_d)) \\
&\quad \le \mathrm{vol}_d(H_*^-\cap\mathrm{cone}(x_1,\dots,x_d))-\mathrm{vol}_d(H^-\cap\mathrm{cone}(x_1,\dots,x_d)) \\
&\quad = \left(\frac{(1-\gamma)h_K(u)}{h}\right)^d\mathrm{vol}_d(H^-\cap\mathrm{cone}(x_1,\dots,x_d))-\mathrm{vol}_d(H^-\cap\mathrm{cone}(x_1,\dots,x_d)) \\
&\quad = \left[\left(\frac{(1-\gamma)h_K(u)}{h}\right)^d-1\right]\mathrm{vol}_d(H^-\cap\mathrm{cone}(x_1,\dots,x_d)) \\
&\quad = \frac{h}{d}\left[\left(\frac{(1-\gamma)h_K(u)}{h}\right)^d-1\right]\mathrm{vol}_{d-1}(\mathrm{conv}(x_1,\dots,x_d)).
\end{aligned}
$$

Besides, since γ is of order $n^{-\frac{2}{d-1}}$ and $\varepsilon \leq \frac{h_K(u)}{d}$, for sufficiently large n,

$$\frac{(1-\gamma)h_K(u)-h}{h} \leq \frac{(1-\gamma)h_K(u)-h_K(u)+\varepsilon}{h_K(u)-\varepsilon} \leq \frac{\frac{1}{d}-\gamma}{1-\frac{1}{d}} \leq \frac{1}{d-1}.$$

Thus, by using the latter estimate,

$$(1+x)^d = \sum_{k=0}^{d} x^k \binom{d}{k},$$

and

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

we achieve that, for sufficiently large n,

$$\begin{aligned}
&\frac{1}{d}\left[\left(\frac{(1-\gamma)h_K(u)}{h}\right)^d - 1\right] \\
&\quad = \frac{1}{d}\left[\left(\frac{h+(1-\gamma)h_K(u)-h}{h}\right)^d - 1\right] \\
&\quad = \frac{1}{d}\left[\left(1+\frac{(1-\gamma)h_K(u)-h}{h}\right)^d - 1\right] \\
&\quad = \frac{1}{d}\left[d\,\frac{(1-\gamma)h_K(u)-h}{h} + \frac{d\,(d-1)}{2}\left(\frac{(1-\gamma)h_K(u)-h}{h}\right)^2 + \cdots\right] \\
&\quad \leq \frac{(1-\gamma)h_K(u)-h}{h} \cdot \sum_{k=0}^{\infty} \frac{d^k}{k!}\left(\frac{(1-\gamma)h_K(u)-h}{h}\right)^k \\
&\quad \leq \frac{(1-\gamma)h_K(u)-h}{h} \cdot \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{d}{d-1}\right)^k \\
&\quad = \exp\left(\frac{d}{d-1}\right)\frac{(1-\gamma)h_K(u)-h}{h} \leq C\,\frac{(1-\gamma)h_K(u)-h}{h}.
\end{aligned}$$

Therefore, for sufficiently large n,

$$\begin{aligned}
&\mathrm{vol}_d((1-\gamma)K \cap H^- \cap \mathrm{cone}(x_1,\ldots,x_d)) \\
&\qquad \leq C\,\mathrm{vol}_{d-1}(\mathrm{conv}(x_1,\ldots,x_d)) \cdot \max\{0,((1-\gamma)h_K(u)-h)\}.
\end{aligned}$$

This proves the lemma. □

To evaluate the innermost integral in the expression of the foregoing lemma, we first recall geometric results derived by Reitzner [107]. Let $x(u)$ be the point on ∂K with fixed outer unit normal vector $u \in \mathbb{S}^{d-1}$. Since K has a twice differentiable boundary, there is a paraboloid $Q_2^{(x(u))}$, given by a quadratic form $b_2 := b_2^{(x(u),x(u))}$, that osculates ∂K at $x(u)$. In order to keep our presentation reasonably self contained, we provide the reader with an explicit construction (see, for example, [107, Page 2265]).
We identify the support hyperplane of ∂K at $x(u)$ with $\mathbb{R}^{d-1}$ and $x(u)$ with the origin of $\mathbb{R}^{d-1}$. Then, there exists a twice differentiable convex function $g(y) := g^{x(u)}(y)$, $y = (y_1, \ldots, y_{d-1}) \in \mathbb{R}^{d-1}$, such that, in some neighborhood of $x(u)$, ∂K can be represented by $(y, g^{x(u)}(y))$. To formalize this further, we denote by

$$\frac{\mathrm{d}^2}{\mathrm{d}y_i \mathrm{d}y_j}\, g(y)\big|_{y=\boldsymbol{o}}$$

the second partial derivative of the function g, evaluated at the origin. Then,

$$b_2(y) := \frac{1}{2} \sum_{i,j} \frac{\mathrm{d}^2}{\mathrm{d}y_i \mathrm{d}y_j}\, g(y)\big|_{y=\boldsymbol{o}}\, y_i\, y_j,$$

and

$$Q_2^{x(u)} := \{(y, z) : z \geq b_2(y)\}.$$

Thus, K is contained in the half space corresponding to $z \geq 0$. Now, let $\mathbb{R}^d = (\mathbb{R}_+ \times \mathbb{S}^{d-2}) \times \mathbb{R}$, and denote by (rv, z) a point in $\mathbb{R}^d$, where $v \in \mathbb{S}^{d-2}$, $r \in \mathbb{R}_+$ and $z \in \mathbb{R}$. The following lemma summarizes results from [107, Page 2265 and 2271]. In particular, it states that for each boundary point $x(u) \in \partial K$, the distance between $\partial Q_2^{x(u)}$ and ∂K is uniformly bounded, in some specific neighborhood of $x(u)$.

Lemma 5.3.2 *Let $u \in \mathbb{S}^{d-1}$ and $\delta > 0$ be sufficiently small. Then, there exists some $\lambda > 0$ only depending on δ and K such that for $x(u) \in \partial K$, the λ-neighborhood U^λ of $x(u)$ in ∂K, defined by*

$$U^\lambda | \mathbb{R}^{d-1} = \mathbb{B}^{d-1}(\boldsymbol{o}, \lambda),$$

can be represented by a twice differentiable convex function $z := g(rv) := g^{(x(u))}(rv)$. In particular, in this neighborhood, it satisfies

$$(1+\delta)^{-\frac{1}{2}}\, b_2(v)^{-\frac{1}{2}}\, z^{\frac{1}{2}} \leq r \leq (1+\delta)^{\frac{1}{2}}\, b_2(v)^{-\frac{1}{2}}\, z^{\frac{1}{2}}, \tag{5.6}$$

and

$$(1+\delta)^{-\frac{3}{2}}\,2^{-1}\,b_2(v)^{-\frac{1}{2}}\,z^{-\frac{1}{2}} \leq \frac{l_H(rv)}{\langle v, N_{K\cap H}(rv)\rangle} \leq (1+\delta)^{\frac{3}{2}}\,2^{-1}\,b_2(v)^{-\frac{1}{2}}\,z^{-\frac{1}{2}}. \tag{5.7}$$

Here, for fixed rv*,* H *is the hyperplane that contains* $(rv, g(rv))$ *and is parallel to* $\mathbb{R}^{d-1}$*, and* $N_{K\cap H}(rv)$ *is the outer unit normal vector to* $\partial K \cap H$ *at this point. Furthermore, for the density* f *and all* $p \in U^\lambda$*, it holds that*

$$(1+\delta)^{-1}\,f(x(u)) \leq f(p) \leq (1+\delta)f(x(u)). \tag{5.8}$$

We next estimate the innermost integral in Lemma 5.3.1.

Lemma 5.3.3 *Let* $x(u)$ *be the point on* ∂K *with fixed outer unit normal vector* $u \in \mathbb{S}^{d-1}$*. Denote by* z *the distance from* H *to the support hyperplane of* ∂K *at* $x(u)$*, and note that* $h = h_K(u) - z$ *by construction. Then, for all sufficiently small* $\delta > 0$*,*

$$\begin{aligned}&\int\limits_{\partial K\cap H} \cdots \int\limits_{\partial K\cap H} \mathrm{vol}_{d-1}(\mathrm{conv}(x_1,\ldots,x_d))^2 \prod_{j=1}^{d} l_H(x_j)\,\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\ldots\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_d)\\ &\quad\leq (1+\delta)^{\frac{d(d+3)}{2}}\,2^{\frac{d^2-d-2}{2}}\,z^{\frac{d^2-d-2}{2}}\,\frac{d\,\omega_{d-1}^d}{(d-1)!\,(d-1)^{d-1}}\,f(x(u))^d\,\kappa_K(x(u))^{-\frac{d}{2}-1}\\ &\qquad+\delta O(z^{\frac{d^2-d-2}{2}}),\end{aligned}$$

where the constant in $O(\cdot)$ *can be chosen independently of* $x(u)$ *and* δ*.*

Proof of Lemma 5.3.3. The proof follows closely the arguments given in [107]. First, we replace the random points x_i, $i \in \{1,\ldots,d\}$, chosen on $\partial K \cap H$, by random points chosen on the intersection of H with the approximating paraboloid $Q_2^{(x(u))}$. Hence, we write x_i, $i \in \{1,\ldots,d\}$, as $x_i = r(v_i)v_i$, where $r(v_i)$ is the radial function of $K \cap H$, estimated in (5.6). Now, [107, Equation (68)] yields that

$$|\mathrm{vol}_{d-1}(\mathrm{conv}(x_1,\ldots,x_d)) - \mathrm{vol}_{d-1}(\mathrm{conv}(r_2(v_1)v_1,\ldots,r_2(v_d)v_d))| \leq \delta O(z^{\frac{d-1}{2}}),$$

where

$$r_2(v) := b_2(v)^{-\frac{1}{2}} z^{\frac{1}{2}},$$

$\delta > 0$ is arbitrarily small and the constant in $O(\cdot)$ is independent of $x(u)$ and δ.

Therefore,

$$\begin{aligned}&\int\limits_{\partial K\cap H}\cdots\int\limits_{\partial K\cap H}\operatorname{vol}_{d-1}(\operatorname{conv}(x_1,\ldots,x_d))^2\prod_{j=1}^{d}l_H(x_j)\,\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\ldots\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_d)\\ &\quad=\int\limits_{\partial K\cap H}\cdots\int\limits_{\partial K\cap H}\left[\operatorname{vol}_{d-1}(\operatorname{conv}(r_2(v_1)v_1,\ldots,r_2(v_d)v_d))^2+\delta O(z^{d-1})\right]\\ &\qquad\times\prod_{j=1}^{d}l_H(x_j)\,\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\ldots\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_d),\end{aligned}$$

where the constant in $O(\cdot)$ can be chosen independently of $x(u)$ and δ.
First, we evaluate the integral involving the $O(\cdot)$ term. The density f is uniformly bounded and by (5.7), the integration concerning each

$$l_H(x_j)\,\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_j),$$

$j\in\{1,\ldots,d\}$, results in terms of order

$$O(z^{-\frac{1}{2}})\operatorname{vol}_{d-2}(\partial K\cap H).$$

Since, in view of (5.6),

$$\operatorname{vol}_{d-2}(\partial K\cap H)=O(z^{\frac{d-2}{2}}),$$

we achieve that

$$\begin{aligned}&\int\limits_{\partial K\cap H}\cdots\int\limits_{\partial K\cap H}\delta O(z^{d-1})\prod_{j=1}^{d}l_H(x_j)\,\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\ldots\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_d)\\ &\quad=\delta O(z^{d-1})O(z^{-\frac{d}{2}})\operatorname{vol}_{d-2}(\partial K\cap H)^d\\ &\quad=\delta O(z^{d-1-\frac{d}{2}+d\frac{d-2}{2}})\\ &\quad=\delta O(z^{\frac{d^2-d-2}{2}}),\end{aligned}$$

where the constant in $O(\cdot)$ can be chosen independently of $x(u)$ and δ.

Secondly, we turn to the first summand. Rewriting the integral over $\mathbb{S}^{d-2}$ and using (5.6), (5.7) and (5.8) yields similarly as in [107, Page 2274] that

$$
\begin{aligned}
&\int\limits_{\partial K\cap H} \cdots \int\limits_{\partial K\cap H} \operatorname{vol}_{d-1}(\operatorname{conv}(r_2(v_1)v_1,\dots,r_2(v_d)v_d))^2 \\
&\qquad \times \prod_{j=1}^{d} l_H(x_j)\,\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\dots\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_d) \\
&\quad = \int\limits_{\mathbb{S}^{d-2}} \cdots \int\limits_{\mathbb{S}^{d-2}} \operatorname{vol}_{d-1}(\operatorname{conv}(r_2(v_1)v_1,\dots,r_2(v_d)v_d))^2 \\
&\qquad \times \prod_{j=1}^{d} f(r(v_j)v_j)\,\frac{l_H(r(v_j)v_j)\,r(v_j)^{d-2}}{\langle v_j, N_{K\cap H}(r(v_j)v_j)\rangle}\,\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_1)\dots\,\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_d) \\
&\quad \le (1+\delta)^{\frac{d(d+3)}{2}}\,2^{-d}\,z^{-d}\,f(x(u))^d \int\limits_{\mathbb{S}^{d-2}} \cdots \int\limits_{\mathbb{S}^{d-2}} \operatorname{vol}_{d-1}(\operatorname{conv}(r_2(v_1)v_1,\dots,r_2(v_d)v_d))^2 \\
&\qquad \times \prod_{j=1}^{d} r_2(v_j)^{d-1}\,\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_1)\dots\,\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_d),
\end{aligned}
$$

where again $r_2(v) = b_2(v)^{-\frac{1}{2}}z^{\frac{1}{2}}$. Now, define an ellipsoid E as the $(d-1)$-dimensional convex body having radial function $b_2(v)^{-\frac{1}{2}}$, i.e., as the intersection of $Q_2^{(x(u))}$ with the hyperplane corresponding to $z=1$. Since the Lebesgue measure is homogeneous, the integral appearing in the latter expression can be rewritten as an integral where the random points are chosen in the interior of E, according to the uniform distribution (see [107, Page 2275]). That is,

$$
\begin{aligned}
&\int\limits_{\mathbb{S}^{d-2}} \cdots \int\limits_{\mathbb{S}^{d-2}} \operatorname{vol}_{d-1}(\operatorname{conv}(r_2(v_1)v_1,\dots,r_2(v_d)v_d))^2 \\
&\qquad \times \prod_{j=1}^{d} r_2(v_j)^{d-1}\,\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_1)\dots\,\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_d) \\
&= z^{\frac{d(d-1)}{2}} \int\limits_{\mathbb{S}^{d-2}} \cdots \int\limits_{\mathbb{S}^{d-2}} \int\limits_{0}^{b_2(v_1)^{-\frac{1}{2}}} \cdots \int\limits_{0}^{b_2(v_d)^{-\frac{1}{2}}} \operatorname{vol}_{d-1}(\operatorname{conv}(b_2(v_1)^{-\frac{1}{2}}z^{\frac{1}{2}}v_1,\dots,b_2(v_d)^{-\frac{1}{2}}z^{\frac{1}{2}}v_d))^2 \\
&\qquad \times \prod_{j=1}^{d}((d-1)t_j^{d-2})\,\mathrm{d}t_1\dots\mathrm{d}t_d\,\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_1)\dots\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_d)
\end{aligned}
$$

$$= z^{\frac{d(d-1)}{2}+d-1} \int\limits_{\mathbb{S}^{d-2}} \cdots \int\limits_{\mathbb{S}^{d-2}} \int\limits_0^{b_2(v_1)^{-\frac{1}{2}}} \cdots \int\limits_0^{b_2(v_d)^{-\frac{1}{2}}} \mathrm{vol}_{d-1}(\mathrm{conv}(b_2(v_1)^{-\frac{1}{2}}v_1, \dots, b_2(v_d)^{-\frac{1}{2}}v_d))^2$$

$$\times \prod_{j=1}^{d}((d-1)t_j^{d-2})\, \mathrm{d}t_1 \cdots \mathrm{d}t_d\, \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_1) \dots \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}v_d)$$

$$= z^{\frac{d^2+d-2}{2}} (d-1)^d \int\limits_E \cdots \int\limits_E \mathrm{vol}_{d-1}(\mathrm{conv}(\tilde{x}_1, \dots, \tilde{x}_d))^2 \mathrm{d}x_1 \dots \mathrm{d}x_d,$$

where $\tilde{x}_i$ arises as the orthogonal projection of the point x_i onto the boundary of E, i.e.,

$$\tilde{x}_i = \frac{x_i}{\|x_i\|}\, r_E \left(\frac{x_i}{\|x_i\|} \right).$$

Here, r_E is the radial function of E, and $\| \cdot \|$ is the Euclidean norm with the origin placed at the center of E. The random elements $\mathrm{d}x_i$, $i \in \{1 \dots, d\}$, as well as vol_{d-1}, are homogeneous and invariant with respect to volume preserving affine transforms acting in the affine subspace $\{z = 1\}$.
Moreover, the volume of E equals

$$2^{\frac{d-1}{2}} \kappa_K(x(u))^{-\frac{1}{2}} \kappa_{d-1},$$

(see [107, Page 2275]). Thus, by first transforming the ellipsoid E into the Euclidean ball $\mathbb{B}^{d-1}$ (using a suitable affinity), then, rewriting the integral as an integral over the sphere $\mathbb{S}^{d-2}$ and, finally, using (5.1), it follows that

$$z^{\frac{d^2+d-2}{2}} (d-1)^d \int\limits_E \cdots \int\limits_E \mathrm{vol}_{d-1}(\mathrm{conv}(\tilde{x}_1, \dots, \tilde{x}_d))^2\, \mathrm{d}x_1 \dots \mathrm{d}x_d$$

$$= z^{\frac{d^2+d-2}{2}} \left(2^{\frac{d-1}{2}} \kappa_K(x(u))^{-\frac{1}{2}}\right)^2 \left(2^{\frac{d-1}{2}} \kappa_K(x(u))^{-\frac{1}{2}}\right)^d$$

$$\times \int\limits_{\mathbb{S}^{d-2}} \cdots \int\limits_{\mathbb{S}^{d-2}} \mathrm{vol}_{d-1}(\mathrm{conv}(x_1, \dots, x_d))^2\, \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}u_1) \dots \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}u_d)$$

$$= z^{\frac{d^2+d-2}{2}}\, 2^{\frac{d^2+d-2}{2}}\, \kappa_K(x(u))^{-\frac{d}{2}-1} \frac{d\, \omega_{d-1}^d}{(d-1)!\,(d-1)^{d-1}}.$$

Combining the above calculations yields for all sufficiently small $\delta > 0$,

$$\begin{aligned}\int\limits_{\partial K\cap H} \cdots \int\limits_{\partial K\cap H} &\mathrm{vol}_{d-1}(\mathrm{conv}(x_1,\dots,x_d))^2 \prod_{j=1}^{d} l_H(x_j)\,\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_1)\cdots\mathrm{d}\mathbb{P}_{f_{\partial K\cap H}}(x_d)\\ &\leq (1+\delta)^{\frac{d(d+3)}{2}}\, 2^{\frac{d^2-d-2}{2}}\, z^{\frac{d^2-d-2}{2}}\, \frac{d\,\omega_{d-1}^{d}}{(d-1)!\,(d-1)^{d-1}}\, f(x(u))^d\,\kappa_K(x(u))^{-\frac{d}{2}-1}\\ &\quad + \delta O(z^{\frac{d^2-d-2}{2}}),\end{aligned}$$

where the constant in $O(\cdot)$ can be chosen independently of $x(u)$ and δ. This proves the lemma. □

Now, we further analyze the expression appearing in Lemma 5.3.1. We put

$$s := \mathbb{P}_f(\partial K \cap H^-).$$

Consequently,

$$\mathbb{P}_f(\partial K \cap H^+) = 1 - s.$$

Moreover, the result stated in [107, Equation (71)] implies the following estimates.

Lemma 5.3.4 *Let $x(u)$ be the point on ∂K with fixed outer unit normal vector $u \in \mathbb{S}^{d-1}$. Denote by z the distance from H to the support plane of ∂K at $x(u)$, i.e., $z = h_K(u) - h$. Then, for all sufficiently small $\delta > 0$, it holds that*

$$\begin{aligned}&(1+\delta)^{-d}\, 2^{\frac{d-1}{2}}\, f(x(u))\,\kappa_K(x(u))^{-\frac{1}{2}}\,\kappa_{d-1}\, z^{\frac{d-1}{2}}\\ &\quad\leq s \leq (1+\delta)^{d+1}\, 2^{\frac{d-1}{2}}\, f(x(u))\,\kappa_K(x(u))^{-\frac{1}{2}}\,\kappa_{d-1}\, z^{\frac{d-1}{2}}.\end{aligned} \tag{5.9}$$

Therefore, we achieve that

$$z \leq (1+\delta)^{\frac{2d}{d-1}}\, \frac{\kappa_K(x(u))^{\frac{1}{d-1}}\,(d-1)^{\frac{2}{d-1}}}{2\,f(x(u))^{\frac{2}{d-1}}\,\omega_{d-1}^{\frac{2}{d-1}}}\, s^{\frac{2}{d-1}}, \tag{5.10}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}s} \leq (1+\delta)^d\, \frac{\kappa_K(x(u))^{\frac{1}{2}}\, 2^{-\frac{d-3}{2}}}{f(x(u))\,\omega_{d-1}}\, z^{-\frac{d-3}{2}}. \tag{5.11}$$

Using the two latter estimates, we continue the proof of the main theorem as follows.

Lemma 5.3.5 *For sufficiently large n and sufficiently small $\delta > 0$, it holds that*

$$\mathbb{E}[\mathrm{vol}_d((1-\gamma)K\Delta P_n)] \leq I + II,$$

where

$$I := (1+\delta)^{\frac{3d^2+3d}{2}}\, C \binom{n}{d}\, d \int_{\mathbb{S}^{d-1}} \kappa_K(x(u))^{-1} \times \int_0^1 (1-s)^{n-d}\, s^{d-1}\, (z - \gamma h_K(u))\, \mathrm{d}s\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u),$$

and

$$II := (1+\delta)^{\frac{3d^2+3d}{2}}\, C \binom{n}{d}\, d \int_{\mathbb{S}^{d-1}} \kappa_K(x(u))^{-1} \times \int_0^{s(\gamma h_K(u))} (1-s)^{n-d}\, s^{d-1}\, (\gamma h_K(u) - z)\, \mathrm{d}s\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).$$

Here, we have that $z := z(s)$ and

$$s(\gamma h_K(u)) := \int_{\partial K \cap H^-} f(x)\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x),$$

where H is the unique hyperplane orthogonal to $u \in \mathbb{S}^{d-1}$ at distance $(1-\gamma)h_K(u)$ to the origin and H^- the corresponding half-space not containing the origin.

Proof of Lemma 5.3.5. First, observe that

$$\max\{0, ((1-\gamma)h_K(u) - h)\} = 0, \qquad \text{if} \quad h > (1-\gamma)h_K(u).$$

This, Lemma 5.3.1, Lemma 5.3.3 and the substitution $z = h_K(u) - h$ yield that

$$\mathbb{E}[\mathrm{vol}_d((1-\gamma)K\Delta P_n)] \leq (1+\delta)^{\frac{d(d+3)}{2}}\, C\, 2^{\frac{d^2-d-2}{2}} \binom{n}{d} \frac{d\,\omega_{d-1}^d}{(d-1)^{d-1}} \int_{\mathbb{S}^{d-1}} f(x(u))^d\, \kappa_K(x(u))^{-\frac{d}{2}-1}$$

$$\times \int_{h_K(u)-\epsilon}^{(1-\gamma)h_K(u)} \mathbb{P}_f(\partial K \cap H^+)^{n-d} z^{\frac{d^2-d-2}{2}} ((1-\gamma)h_K(u) - h) \, \mathrm{d}h \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$

$$+ \delta \binom{n}{d} (d-1)! \int_{\mathbb{S}^{d-1}} \int_{h_K(u)-\epsilon}^{(1-\gamma)h_K(u)} \mathbb{P}_f(\partial K \cap H^+)^{n-d} O(z^{\frac{d^2-d-2}{2}})$$

$$\times ((1-\gamma)h_K(u) - h) \, \mathrm{d}h \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$

$$= (1+\delta)^{\frac{d(d+3)}{2}} C \, 2^{\frac{d^2-d-2}{2}} \binom{n}{d} \frac{d\,\omega_{d-1}^d}{(d-1)^{d-1}} \int_{\mathbb{S}^{d-1}} f(x(u))^d \, \kappa_K(x(u))^{-\frac{d}{2}-1}$$

$$\times \int_{\gamma h_K(u)}^{\epsilon} \mathbb{P}_f(\partial K \cap H^+)^{n-d} z^{\frac{d^2-d-2}{2}} (z - \gamma h_K(u)) \, \mathrm{d}z \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$

$$+ \delta \binom{n}{d} (d-1)! \int_{\mathbb{S}^{d-1}} \int_{\gamma h_K(u)}^{\epsilon} \mathbb{P}_f(\partial K \cap H^+)^{n-d} O(z^{\frac{d^2-d-2}{2}})$$

$$\times (z - \gamma h_K(u)) \, \mathrm{d}z \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).$$

As the upcoming calculations show, the order of both summands is $n^{-\frac{2}{d-1}}$. Since δ is arbitrarily small, it is enough to consider the first one in what follows.

We use (5.11) and (5.10) to change from $z^{\frac{(d-1)^2}{2}}$ to s^{d-1} and obtain that, for sufficiently large n,

$$\mathbb{E}[\mathrm{vol}_d((1-\gamma)K \Delta P_n)]$$

$$\leq (1+\delta)^{\frac{d(d+3)}{2}+d} C \, 2^{\frac{d^2-d-2}{2}} \, 2^{-\frac{d-3}{2}} \binom{n}{d} \frac{d\,\omega_{d-1}^d}{(d-1)^{d-1}} \int_{\mathbb{S}^{d-1}} f(x(u))^{d-1} \, \kappa_K(x(u))^{-\frac{d}{2}-\frac{1}{2}}$$

$$\times \int_{s(\gamma h_K(u))}^{1} (1-s)^{n-d} z^{\frac{d^2-d-2-d+3}{2}} (z - \gamma h_K(u)) \, \mathrm{d}s \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$

$$\leq (1+\delta)^{\frac{d^2+5d}{2}} C \, 2^{\frac{d^2-2d+1}{2}} \binom{n}{d} \frac{d\,\omega_{d-1}^{d-1}}{(d-1)^{d-1}} \int_{\mathbb{S}^{d-1}} f(x(u))^{d-1} \, \kappa_K(x(u))^{-\frac{d}{2}-\frac{1}{2}}$$

$$\times \int_{s(\gamma h_K(u))}^{1} (1-s)^{n-d} z^{\frac{(d-1)^2}{2}} (z - \gamma h_K(u)) \, \mathrm{d}s \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$

$$\leq (1+\delta)^{\frac{d^2+5d}{2}+d(d-1)} C \, 2^{\frac{(d-1)^2}{2}} \, 2^{-\frac{(d-1)^2}{2}} \binom{n}{d} d \int_{\mathbb{S}^{d-1}} \kappa_K(x(u))^{-1}$$

$$
\begin{aligned}
&\times \int_{s(\gamma h_K(u))}^{1} (1-s)^{n-d}\, s^{d-1}\, (z-\gamma h_K(u))\, \mathrm{d}s\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&\le (1+\delta)^{\frac{3d^2+3d}{2}}\, C \binom{n}{d}\, d \int_{\mathbb{S}^{d-1}} \kappa_K(x(u))^{-1} \\
&\quad\times \int_{s(\gamma h_K(u))}^{1} (1-s)^{n-d}\, s^{d-1}\, (z-\gamma h_K(u))\, \mathrm{d}s\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&= (1+\delta)^{\frac{3d^2+3d}{2}}\, C \binom{n}{d}\, d \int_{\mathbb{S}^{d-1}} \kappa_K(x(u))^{-1} \\
&\quad\times \int_{0}^{1} (1-s)^{n-d}\, s^{d-1}\, (z-\gamma h_K(u))\, \mathrm{d}s\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&\quad + (1+\delta)^{\frac{3d^2+3d}{2}}\, C \binom{n}{d}\, d \int_{\mathbb{S}^{d-1}} \kappa_K(x(u))^{-1} \\
&\quad\times \int_{0}^{s(\gamma h_K(u))} (1-s)^{n-d}\, s^{d-1}\, (\gamma h_K(u)-z)\, \mathrm{d}s\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).
\end{aligned}
$$

This proves the lemma in view of the definitions of I and II. □

We start with the first term.

Lemma 5.3.6 *For sufficiently large n and sufficiently small $\delta > 0$, it holds that*

$$
I \le (1+\delta)^{\frac{3d^2+3d}{2}}\, C\, n^{-\frac{2}{d-1}} \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x).
$$

Proof of Lemma 5.3.6. We apply (5.10) and (5.4) to get for all sufficiently small $\delta > 0$ and sufficiently large n,

$$
\begin{aligned}
I &\le (1+\delta)^{\frac{3d^2+3d}{2}}\, C \binom{n}{d} \frac{d}{2} \frac{(d-1)^{\frac{2}{d-1}}}{\omega_{d-1}^{\frac{2}{d-1}}} \\
&\quad\times \Bigg[(1+\delta)^{\frac{2d}{d-1}} \int_{\mathbb{S}^{d-1}} \frac{\kappa_K(x(u))^{-1+\frac{1}{d-1}}}{f(x(u))^{\frac{2}{d-1}}}\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \int_0^1 (1-s)^{n-d}\, s^{d-1+\frac{2}{d-1}}\, \mathrm{d}s
\end{aligned}
$$

$$- \left(1 - \frac{1}{d}\right) n^{-\frac{2}{d-1}} \frac{(d-1)\,\Gamma\left(d+1+\frac{2}{nd-1}\right)}{(d+1)!} \frac{1}{d\,\mathrm{vol}_d(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}} \,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x)$$
$$\times \int\limits_{\mathbb{S}^{d-1}} h_K(u)\,\kappa_K(x(u))^{-1}\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \int\limits_0^1 (1-s)^{n-d}\,s^{d-1}\,\mathrm{d}s \Bigg].$$

For $u \in \mathbb{S}^{d-1}$, let $x = x(u) \in \partial K$ be such that $N_K(x) = u$. Then, relation (2.4) implies that

$$\begin{aligned} d\,\mathrm{vol}_d(K) &= \int\limits_{\partial K} \langle x, N_K(x)\rangle\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \\ &= \int\limits_{\partial K} h_K(u(x))\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \\ &= \int\limits_{\mathbb{S}^{d-1}} \frac{h_K(u)}{\kappa_K(x(u))}\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u), \end{aligned} \tag{5.12}$$

and

$$\int\limits_{\mathbb{S}^{d-1}} \frac{\kappa_K(x(u))^{-1+\frac{1}{d-1}}}{f(x(u))^{\frac{2}{d-1}}}\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) = \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x).$$

We use those, together with the definition and properties of the Beta function, to arrive at

$$\begin{aligned} I &\le (1+\delta)^{\frac{3d^2+3d}{2}}\, C \binom{n}{d} \frac{d}{2} \frac{(d-1)^{\frac{2}{d-1}}}{\omega_{d-1}^{\frac{2}{d-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \\ &\quad \times \Bigg[(1+\delta)^{\frac{2d}{d-1}} \frac{\Gamma(n-d+1)\Gamma\left(d+\frac{2}{d-1}\right)}{\Gamma\left(n+1+\frac{2}{d-1}\right)} \\ &\quad - \left(1-\frac{1}{d}\right) n^{-\frac{2}{d-1}} \frac{(d-1)\Gamma\left(d+1+\frac{2}{d-1}\right)}{(d+1)!} \frac{\Gamma(n-d+1)\Gamma(d)}{\Gamma(n+1)} \Bigg] \\ &= (1+\delta)^{\frac{3d^2+3d}{2}}\, C \frac{d}{2} \binom{n}{d} \frac{(d-1)^{\frac{2}{d-1}}}{\omega_{d-1}^{\frac{2}{d-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x)\, \frac{\Gamma(n-d+1)\Gamma\left(d+\frac{2}{d-1}\right)}{\Gamma\left(n+1+\frac{2}{d-1}\right)} \\ &\quad \times \Bigg[(1+\delta)^{\frac{2d}{d-1}} - \left(1-\frac{1}{d}\right) n^{-\frac{2}{d-1}} \frac{(d-1)\left(d+\frac{2}{d-1}\right)}{d(d+1)} \frac{\Gamma\left(n+1+\frac{2}{d-1}\right)}{\Gamma(n+1)} \Bigg]. \end{aligned}$$

Here, in the last equality we have also used that

$$\Gamma\left(d+1+\frac{2}{d-1}\right)=\Gamma\left(d+\frac{2}{d-1}\right)\left(d+\frac{2}{d-1}\right),$$

in view of (2.9). Now, observe that, due to (2.13),

$$\Gamma\left(n+1+\frac{2}{d-1}\right)\sim\sqrt{2\pi}\,e^{-n}\,n^{n+1-\frac{1}{2}+\frac{2}{d-1}}\sim n^{\frac{2}{d-1}}\,\Gamma\left(n+1\right), \tag{5.13}$$

as $n\to\infty$. Thus,

$$\frac{\Gamma(n-d+1)\Gamma\left(d+\frac{2}{d-1}\right)}{\Gamma\left(n+1+\frac{2}{d-1}\right)}\sim\frac{(n-d)!\,(d-1)!}{n!\,n^{\frac{2}{d-1}}}=\frac{1}{\binom{n}{d}\,d\,n^{\frac{2}{d-1}}},$$

as $n,d\to\infty$. By using the latter estimates, we obtain for sufficiently large n and sufficiently small $\delta>0$,

$$\begin{aligned}I&\leq(1+\delta)^{\frac{3d^2+3d}{2}}\,C\,n^{-\frac{2}{d-1}}\,\frac{(d-1)^{\frac{2}{d-1}}}{\omega_{d-1}^{\frac{2}{d-1}}}\int\limits_{\partial K}\frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x)\\&\quad\times\left[\,(1+\delta)^{\frac{2d}{d-1}}-\left(1-\frac{1}{d}\right)\frac{(d-1)\left(d+\frac{2}{d-1}\right)}{d(d+1)}\,\right]\\&\leq(1+\delta)^{\frac{3d^2+3d}{2}}\,C\,n^{-\frac{2}{d-1}}\int\limits_{\partial K}\frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\,\mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x),\end{aligned}$$

where in the last inequality we have again used (2.13) to get that

$$\omega_{d-1}^{\frac{2}{d-1}}=\left(\frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\right)^{\frac{2}{d-1}}\sim\left(\frac{2\pi^{\frac{d-1}{2}}}{\sqrt{2\pi}e^{-\frac{d}{2}}\left(\frac{d}{2}\right)^{\frac{d-2}{2}}}\right)^{\frac{2}{d-1}}=\frac{2\,\pi^{\frac{d-2}{d-1}}\,e^{\frac{d}{d-1}}}{d^{\frac{d-2}{d-1}}}\sim\frac{2e\pi}{d}, \tag{5.14}$$

as $d\to\infty$, and $(d-1)^{\frac{2}{d-1}}\leq 2$. This proves the assertion. □

Now, we deal with the second summand in Lemma 5.3.5.

Lemma 5.3.7 *For sufficiently large n and sufficiently small $\delta > 0$, it holds that*

$$II \leq (1+\delta)^{\frac{3d^2+3d}{2}} C\, n^{-\frac{2}{d-1}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}} \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x).$$

Proof of Lemma 5.3.7. Recall that

$$II = (1+\delta)^{\frac{3d^2+3d}{2}} C \binom{n}{d} d \int\limits_{\mathbb{S}^{d-1}} \kappa_K(x(u))^{-1}$$
$$\times \int\limits_0^{s(\gamma h_K(u))} (1-s)^{n-d} s^{d-1} (\gamma h_K(u) - z)\, \mathrm{d}s\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u).$$

First of all, by (5.3), for sufficiently large d and n,

$$\gamma \leq C \frac{1}{\mathrm{vol}_d(K)\, n^{\frac{2}{d-1}}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}} \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x). \tag{5.15}$$

Indeed, in view of (5.14) and (5.13), it holds that

$$\frac{(d-1)^{\frac{d+1}{d-1}} \Gamma\left(d+1+\frac{2}{d-1}\right)}{2\, d\, (d+1)!\, \omega_{d-1}^{\frac{2}{d-1}}} \sim \frac{1}{4\pi e} \frac{(d-1)^{\frac{d+1}{d-1}} d^{\frac{2}{d-1}} \Gamma(d+1)}{(d+1)!} \leq \frac{d^{\frac{4}{d-1}}}{2\pi e} \leq C,$$

as $d \to \infty$. Moreover, it holds that

$$s(\gamma h_K(u))$$
$$\leq \frac{(1+\delta)^{d+1} d}{e\, n} \frac{f(x(u))\, h_K(u)^{\frac{d-1}{2}}}{\kappa_K(x(u))^{\frac{1}{2}}} \left(\frac{1}{d\ \mathrm{vol}_d(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}} \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \right)^{\frac{d-1}{2}}, \tag{5.16}$$

for sufficiently large d and n.

Truly, by Lemma 5.3.4, (5.3) and (5.14), as $n, d \to \infty$,

$$\begin{aligned}
&s(\gamma h_K(u)) \\
&\leq (1+\delta)^{d+1}\, 2^{\frac{d-1}{2}}\, \kappa_{d-1}\, \frac{f(x(u))\, h_K(u)^{\frac{d-1}{2}}}{\kappa_K(x(u))^{\frac{1}{2}}}\, \gamma^{\frac{d-1}{2}} \\
&\leq (1+\delta)^{d+1}\, 2^{\frac{d-1}{2}}\, \kappa_{d-1}\, \frac{f(x(u))\, h_K(u)^{\frac{d-1}{2}}}{\kappa_K(x(u))^{\frac{1}{2}}} \\
&\quad \times \left(n^{-\frac{2}{d-1}}\, \frac{(d-1)^{\frac{d+1}{d-1}}\, \Gamma\left(d+1+\frac{2}{d-1}\right)}{2\,(d+1)!\, \omega_{d-1}^{\frac{2}{d-1}}}\, \frac{1}{d \operatorname{vol}_d(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \right)^{\frac{d-1}{2}} \\
&\sim (1+\delta)^{d+1}\, \frac{f(x(u))\, h_K(u)^{\frac{d-1}{2}}}{n\, \kappa_K(x(u))^{\frac{1}{2}}} \\
&\quad \times \left(\frac{(d-1)\, d^{\frac{2}{d-1}}}{d+1}\, \frac{1}{d \operatorname{vol}_d(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \right)^{\frac{d-1}{2}} \\
&\sim \frac{(1+\delta)^{d+1}\, d}{e\, n}\, \frac{f(x(u))\, h_K(u)^{\frac{d-1}{2}}}{\kappa_K(x(u))^{\frac{1}{2}}} \left(\frac{1}{d\, \operatorname{vol}_d(K)} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \right)^{\frac{d-1}{2}},
\end{aligned}$$

since

$$\left(\frac{d-1}{d+1}\right)^{\frac{d-1}{2}} \sim \frac{1}{e},$$

as $d \to \infty$. Now, we distinguish two cases.

<u>Case 1:</u>

$$s(\gamma h_K(u)) \leq \frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n}.$$

The function

$$f(s) := (1-s)^{n-d}\, s^{d-1},$$

$s \in [0,1]$, attains its maximum at

$$s^* := \frac{d-1}{n-1}.$$

Indeed, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}s} f(s) = -(n-d)(1-s)^{n-d-1} s^{d-1} + (d-1)(1-s)^{n-d} s^{d-2},$$

and

$$\begin{aligned}
& -(n-d)(1-s)^{n-d-1} s^{d-1} + (d-1)(1-s)^{n-d} s^{d-2} &&= 0 \\
\Leftrightarrow \quad & (d-1)(1-s)^{n-d} s^{d-2} &&= (n-d)(1-s)^{n-d-1} s^{d-1} \\
\Leftrightarrow \quad & (d-1)(1-s) &&= (n-d)s \\
\Leftrightarrow \quad & s(n-1) &&= d-1.
\end{aligned}$$

Therefore,

$$s^* = \frac{d-1}{n-1}.$$

Since

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2} f(s)\Big|_{s=s^*} &= (n-d)(n-d-1)\left(1-\frac{d-1}{n-1}\right)^{n-d-2}\left(\frac{d-1}{n-1}\right)^{d-1} \\
&\quad - 2(d-1)(n-d)\left(1-\frac{d-1}{n-1}\right)^{n-d-1}\left(\frac{d-1}{n-1}\right)^{d-2} \\
&\quad + (d-1)(d-2)\left(1-\frac{d-1}{n-1}\right)^{n-d}\left(\frac{d-1}{n-1}\right)^{d-3} \\
&= (n-d)^{n-d-1}(n-d-1)(d-1)^{d-1}(n-1)^{-(n-3)} \\
&\quad - 2(n-d)^{n-d}(d-1)^{d-1}(n-1)^{-(n-3)} \\
&\quad + (n-d)^{n-d}(d-1)^{d-2}(d-2)(n-d)^{-(n-3)} \\
&< (n-d)^{n-d}(d-1)^{d-1}(n-d)^{-(n-3)} \\
&\quad - 2(n-d)^{n-d}(d-1)^{d-1}(n-1)^{-(n-3)} \\
&\quad + (n-d)^{n-d}(d-1)^{d-1}(n-d)^{-(n-3)} \\
&= 0,
\end{aligned}$$

the function $f(s)$ has its maximum at s^*. Now, because $f(0) = 0$ and

$$\frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n} \leq \frac{(d-1)^{\frac{2d^2-5d+2}{2d^2-2d}}}{n} \leq \frac{d-1}{n-1},$$

the function $f(s)$ is increasing on

$$\left[0, \frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n}\right].$$

Observe that, in view of (2.13),

$$\binom{n}{d} d = \frac{n(n-1)\cdots(n-d+1)}{d!} d \sim \frac{n^d\, e^d}{\sqrt{2\,\pi}\sqrt{d}\, d^{d-1}},$$

as $n, d \to \infty$. Furthermore, for all $x \geq 0$, it holds that

$$(1-x)^{n-d} \leq \exp(-(n-d)x),$$

(see [1, Equation (4.2.29)]). Thus, combining the above estimates, for sufficiently large d and n, it holds that

$$\begin{aligned}
&\binom{n}{d} d \int_0^{s(\gamma h_K(u))} (1-s)^{n-d}\, s^{d-1}\, (\gamma h_K(u) - z)\, \mathrm{d}s \\
&\leq \gamma\, h_K(u) \binom{n}{d} d \int_0^{s(\gamma h_K(u))} (1-s)^{n-d}\, s^{d-1}\, \mathrm{d}s \\
&\leq \gamma\, h_K(u) \binom{n}{d} d\, s(\gamma h_K(u)) \left(1 - \frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n}\right)^{n-d} \left(\frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n}\right)^{d-1} \\
&\leq C\, \gamma\, h_K(u) \frac{n^d\, e^d}{\sqrt{2\pi}\, \sqrt{d}\, d^{d-1}} \frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n} \left(1 - \frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n}\right)^{n-d} \left(\frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n}\right)^{d-1} \\
&\leq C\, \gamma\, h_K(u)\, e^d\, d^{\frac{d-1}{d} + \frac{(d-1)^2}{d} - d + \frac{1}{2} - \frac{1}{2(d-1)} - \frac{1}{2}} \exp\left(-\frac{(n-d)(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n}\right) \\
&\leq C\, \gamma\, h_K(u)\, e^d\, d^{\frac{2(d-1)^2 + 2(d-1)^3 - 2d^2(d-1) - 2d}{2d(d-1)}} \exp\left(-\frac{(n-d)(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n}\right) \\
&\leq C\, \gamma\, h_K(u)\, e^d\, d^{-\frac{d}{d-1}} \exp\left(-\frac{(n-d)(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}\, n}\right) \\
&\leq C\, \frac{\gamma\, h_K(u)}{d},
\end{aligned}$$

where we also used in the last step that

$$\frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}} \sim d,$$

as $d \to \infty$. Hence, with (5.15) and (5.12), it follows that

$$\begin{aligned} II &\leq (1+\delta)^{\frac{3d^2+3d}{2}} \, C \, \frac{\gamma}{d} \int_{\mathbb{S}^{d-1}} \kappa_K(x(u))^{-1} \, h_K(u) \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\ &\leq (1+\delta)^{\frac{3d^2+3d}{2}} \, C \, n^{-\frac{2}{d-1}} \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}} \, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x), \end{aligned}$$

finishing the proof of the lemma in Case 1.

<u>Case 2:</u>

$$s(\gamma h_K(u)) > \frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}} \, n}.$$

This inequality is in view of (5.16) equivalent to

$$\begin{aligned} &(1+\delta)^{d+1} \, \frac{d \, f(x(u)) \, h_K(u)^{\frac{d-1}{2}}}{e \, \kappa_K(x(u))^{\frac{1}{2}}} \left(\frac{1}{d \, \mathrm{vol}_d(K)} \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}} \, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \right)^{\frac{d-1}{2}} \\ &\quad > \frac{(d-1)^{\frac{d-1}{d}}}{d^{\frac{1}{2(d-1)}}}, \end{aligned}$$

which itself is equivalent to

$$h_K(u) \, \frac{1}{d \, \mathrm{vol}_d(K)} \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}} \, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x) \; > \; \frac{e^{\frac{2}{d-1}} \, (d-1)^{\frac{2}{d}}}{(1+\delta)^{\frac{2(d+1)}{d-1}} \, d^{\frac{2d-1}{(d-1)^2}}} \, \frac{\kappa_K(x(u))^{\frac{1}{d-1}}}{f(x(u))^{\frac{2}{d-1}}}.$$

We integrate both sides over ∂K according to the $(d-1)$-dimensional Hausdorff measure to achieve that

$$(1+\delta)^{\frac{2(d+1)}{d-1}} > \frac{e^{\frac{2}{d-1}} \, (d-1)^{\frac{2}{d}}}{d^{\frac{2d-1}{(d-1)^2}}}.$$

Thus, we arrive at a contradiction. Indeed, the right hand side is strictly bigger than 1. On the other hand, $\delta > 0$ can be chosen arbitrarily small. This shows that Case 2 never arises and, therefore, finishes the proof of the lemma. $\square$

Proof of Theorem 5.1.1. Lemma 5.3.6 and Lemma 5.3.7 imply that for sufficiently large n and sufficiently small $\delta > 0$,

$$\mathbb{E}[\mathrm{vol}_d((1-\gamma)K\Delta P_n)] \le (1+\delta)^{\frac{3d^2+3d}{2}}\, C\, n^{-\frac{2}{d-1}} \int\limits_{\partial K} \frac{\kappa_K(x)^{\frac{1}{d-1}}}{f(x)^{\frac{2}{d-1}}}\, \mathcal{H}^{d-1}_{\partial K}(\mathrm{d}x). \tag{5.17}$$

Taking into account that we were approximating the body $(1-\gamma)K$ instead of K, we need to multiply the bound (5.17) by $(1-\gamma)^{-d}$. Since

$$(1-\gamma)^d \ge 1 - d\gamma,$$

for sufficiently large n, we have that $(1-\gamma)^{-d} \le C$. Finally, since the bound (5.17) holds for all $\delta > 0$, the theorem follows. □

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