Discrete stochastic geometry: Beta-polytopes, random cones and empty simplices

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Meiner Familie in Dankbarkeit gewidmet

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My parents Ilona and Stefan.

Contents

Chapter 1

Introduction

In this thesis we are concerned with problems arising in the fields of *stochastic geometry* and *random polytopes*. We provide the reader with a brief review of the historical development of these fields as well as an overview of their applications. We will, however, mainly focus on topics of relevance for the contents of this work. We start by recalling Buffon's needle problem and the famous four-point problem of Sylvester. This serves as a starting point for the somewhat related problem presented in Chapter 5, which, just as Sylvester's problem, arises from a very simple geometric question. Although the nature of the problem seems very trivial, it is all but that when one tries to tackle it. We go on by introducing the historical pathway which the theory of random polytopes has taken. As we will see, a lot of great results were achieved in this field of study and it's practical applications are manifold. A large literature can be found on topics from random polytope theory, which we readily mention and point out to the reader. However, as already mentioned, we focus on aspects of relevance for this work. We hope to be able to provide the reader with a useful entry point into the tightly linked Chapters 3 and 4. Lastly, we give an outline of the contents of the endeavor awaiting the reader. We briefly introduce the problems of the respective chapters and provide sources of the research papers in which their solutions where first presented.

1.1 General

Stochastic geometry and *random polytope* theory are on the crossroads of a number of mathematical fields. Immediately, *probability theory* and *convex geometry* come to mind. However, in fact, a whole lot more mathematical fields are needed to be able to appropriately approach the problems posed by these fields. For instance, methods from *integral geometry*, *differential geometry* and *manifold theory*, *asymptotic analysis* and many more are employed. In turn, the results obtained in these disciplines find a wide variety of uses in other mathematical and scientific fields, for instance, *asymptotic geometric analysis*, *computational geometry*, *multivariate statistics*, *optimization*, *compressed sensing* and many more.

The onset of the field of stochastic geometry can be traced back to the year 1733 to Georges-Louis Leclerc, Comte de Buffon [70, 71] who posed a problem that is nowadays known as *Buffon's needle problem*:

> "Suppose we have a floor made of parallel strips of wood, each the same width a, and we drop a needle of length $2r$ onto the floor. The length of the needle should be at most the width of the strips, i.e., $2r < a$. What is the probability that the needle will lie across a line between two strips?"

Leclerk de Buffon himself gave in [70] the answer: The probability is $1 - \frac{4r}{\pi a}$. In 1812 Laplace [67] reconsidered Buffon's needle problem and realized that one may approximate π with the procedure prescribed by Leclerc de Buffon. Nowadays such an approximation is known as Monte Carlo method. Later Lazzarini [69] used this method to reproduce the, at the time already well known, approximation 355/113 for π by conducting 3408 throws. However, there is strong doubt about the sincerity of Lazzarini, since the rate of convergence of this method is very slow and it is believed that he may have faked his results, see for instance [11, 74].

Much later, namely in 1864, another problem of great relevance to stochastic geometry was posed by James Joseph Sylvester [103] known as *Sylvester's four-point problem.* It also marked the start of random polytope theory. In its naïve formulation it reads as:

> "Show, that the probability that the convex hull of four points taken at random in an indefinite plane is a triangle, is 1/4."

One immediately notices that this question leaves a lot to be desired. Neither is it clear with respect to which probability measure the points should be taken, nor whether they are independent or even identically distributed. Hence, no concise answer can be given. This was mirrored by the first results obtained by Sylvester [103], DeMorgan and Wilson, published by Ingleby [57], and Woolhouse [112] who obtained all different results, due to their respective methods of approach. Sylvester thus reformulated his question into its modern and precise version:

> "What is the probability that the convex hull of four points chosen independently and uniformly from a convex body K form a triangle and which classes of convex bodies minimize, respectively maximize, this probability?"

This question quickly sparked fruitful results. Namely, Woolhouse's former results implied that this probability is $\frac{35}{12\pi^2}$ for K being a circle (respectively, ellipse) [112], Sylvester showed that it is 1/3 for a triangle [104], and Woolhouse showed that it is 11/36 for a square (respectively, parallelogram) and 289/972 for a hexagon [111]. Later, it was shown by Crofton [37] that circles (respectively, ellipses) are the minimizers of this probability, while Blaschke [24] proved that triangles are the maximizers. Note that these probabilities can also be interpreted as the portion of area of K *covered* by the convex hull of three independent uniform random points from K. Later Alagar [5] gave the distribution function of this area if K is a triangle, while Henze [52] established those for the parallelogram and the circle. Another natural generalization of the question, namely, the probability that n such random points in a convex body K form an n-gon, was treated. Valtr [106, 107] and Peyerimhoff [83] gave answers for special classes of convex bodies.

But let us consider the most obvious generalization: higher dimensions. So, the question of the probability that the convex hull of $d+2$ independent uniform random points from a d-dimensional convex body K form a d-simplex. An explicit formula for this probability in the case of K being a d-dimensional ball (respectively, ellipsoid) was given by Kingman [65], while, shortly after, Groemer [46] showed that the class of ellipsoids is indeed the minimizer for this probability. Naturally, the candidates of convex bodies K for maximizers of this probability are d-simplices. However, to this day it is not clear whether this is in fact true, not even in the seemingly simple case of three-dimensional space. It's correctness would carry a large number of implications, not least the affirmation of the famous and long open *hyperplane* or *slicing conjecture*. It states that for any convex body K of fixed volume there exists a hyperplane H such that the volume of the intersection $K \cap H$ is lower bounded by a universal constant independent of the dimension or the convex body. See Bourgain [27] and Klartag [66].

Let us now turn our attention to the broader historic development of random polytope theory. Very good survey articles on random polytopes were written by Bárány [13, 14], Hug [54], Majumdar, Comtet and Randon-Furling [75], Reitzner [89] and Schneider [96], which we suggest the reader to consult for an in-depth overview.

We first establish the general construction principle of the random polytopes under investigation. Given a probability distribution in $\mathbb{R}^d$, which may or may not have compact support, we sample n independent random points $X_1, \dots, X_n$ according to that distribution. The convex hull $\mathrm{conv}(X_1, \dots, X_n)$ of these points is then a random polytope. Efron's [42] and Rényi and Sulanke's [91, 92] papers could be considered as starting points to investigations of topics related to the work presented in this thesis. One may also find similar problems and question in [110]. While Efron established formulas for expected areas, perimeters and edge numbers of random polytopes generated for a number of different planar distributions, already hinting at the importance of the normal distribution or the uniform distribution in a ball or a sphere, Rényi and Sulanke turned their attention to questions concerning approximation of two-dimensional convex bodies by random polytopes. In particular, Rényi and Sulanke's gave asymptotic formulas for the expected number of edges, for the area difference and length difference of the boundaries of a random polytope to the convex body from which its point are uniformly and independently sampled, in terms of the number of points n. Rényi and Sulanke's work was carried forward to arbitrary dimension by works from McClure and Vitale [78], Gruber [49], Bárány [12], Ludwig [72], Ludwig, Schütt and Werner [73], Reitzner [88], Schütt [100] and Schütt and Werner [101], to name just a few.

The line of work of Efron was resumed by Kingman [65] giving explicit formulas for all the integer moments of the volume of a simplex formed by uniformly and independently chosen random points form the unit ball, and vastly generalized by Miles [79] who gave formulas for these quantities for Gaussian simplices, the whole class of beta- and beta′-type simplices and simplices formed from uniform and independent random points from the unit sphere. The importance of these classes of distributions as well as the simplicity of the formulas obtained in these cases became quite apparent through a work of Ruben and Miles [95]. They showed that there are only very few distributions that satisfy the property that their marginal distributions are from the same class of distributions as the original distribution. Later, we will make heavy use of this fact. For the moment, let us just briefly introduce these distributions. We introduce the Gaussian, the beta- and beta′-type distributions via their densities with respect to the Lebesgue measure on $\mathbb{R}^d$.

(1) Centered Gaussian distribution:

$$\phi_\sigma(x) := (2\pi\sigma^2)^{-\frac{d}{2}} \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right), \quad x \in \mathbb{R}^d, \quad \sigma > 0.$$

(2) Beta-type distribution:

$$f_{d,\beta,\sigma}(x) := \sigma^d \frac{\Gamma\left(\frac{d}{2}+\beta+1\right)}{\pi^{\frac{d}{2}}\Gamma(\beta+1)}\left(1-\frac{\|x\|^2}{\sigma^2}\right)^{\beta}, \quad \|x\| \leq 1, \quad \beta > -1, \quad \sigma > 0.$$

(3) Beta′-type distribution:

$$\tilde{f}_{d,\beta,\sigma}(x) := \sigma^d \frac{\Gamma(\beta)}{\pi^{\frac{d}{2}}\Gamma\left(\beta-\frac{d}{2}\right)}\left(1+\frac{\|x\|^2}{\sigma^2}\right)^{-\beta}, \quad x \in \mathbb{R}^d, \quad \beta > \frac{d}{2}, \quad \sigma > 0.$$

(4) Uniform distribution on the sphere:

$$\sigma_{d-1}(A) := \frac{\mathrm{Vol}_d(\{tx : x \in A, t \in [0,1]\})}{\mathrm{Vol}_d(\mathbb{B}^d)}, \quad A \subset \mathbb{S}^{d-1} \text{ measurable.}$$

Note that the beta-type distribution for $\beta = 0$ and $\sigma = 1$ is just the uniform distribution in the unit ball, the uniform distribution on the unit sphere is the weak limit of beta-type distributions with parameter $\sigma = 1$ for $\beta(n) \to -1$, as $n \to \infty$, and, appropriately rescaled with $\sigma(n) = \sqrt{2\beta(n)}$, both the beta- and beta′-type densities converge weakly to the standard Gaussian distribution for $\beta(n) \to \infty$, as $n \to \infty$.

In the wake of this discovery a lot of results regarding expectations of geometric quantities, like the Lebesgue volume, surface area, mean width or facet numbers of random polytopes coming from these classes where derived, for instance by Buchta [29], Buchta and Müller [31], Buchta, Müller and Tichy [32], Affentranger [2, 3], Affentranger and Schneider [4], Mathai [76, 77], Hug, Munsonius and Reitzner [55], Grote and Thäle [48], Kabluchko and Zaporozhets [62], Bonnet, Grote, Temesvari, Thäle, Turchi and Wespi [26], Bonnet, Chasapis, Grote, Temesvari and Turchi [25], Kabluchko, Temesvari and Thäle [59], Grote, Kabluchko and Thäle [47], or Kabluchko, Thäle and Zaporozhets [61].

More recently, also central limit theorems for various geometric quantities of random polytopes have been investigated. We refer the reader to Bárány, Fodor and Vigh [15], Bárány and Vu [20], Calka, Schreiber and Yukich [33], Reitzner [88, 87] and Vu [109].

1.2 Guideline

Here we want to give a quick outline of the chapters to come. We briefly describe their contents, their interconnections to each other as well as to the general introduction.

Chapter 2:

Chapter 2 serves the purpose of introducing the notation used throughout this thesis as well as recalling important theorems and lemmas from different fields of mathematics employed in this work. These fields are *convex geometry*, *integral geometry*, *random measure* and *random set theory*, *Poisson point processes* and *analysis*, which all get their respective sections. We refrain from giving proofs of these theorems and lemmas and rather give references where the proofs can be found in the literature. Furthermore, we introduce original theorems and lemmas which most appropriately fit with the respective sections and were first introduced and proved in papers of the author. This is to avoid unnecessary overloading of the later chapters with auxiliary theorems and lemmas which are not directly connected to its contents. For these theorems and lemmas we state proofs and also point out where one may find them in the author's works.

Chapter 3:

This chapter focuses on the classes of distributions given by Ruben and Miles that where already mentioned in the general introduction. We investigate geometric properties of beta- and beta$'$-type random polytopes generated from n random points, as well as the limiting case of random polytopes coming from the uniform distribution on the unit sphere. Furthermore, we consider their symmetrized versions and their versions obtained by conditioning on the containment of the origin $\boldsymbol{o}$. Moreover, also the Poissonized analogues of these polytopes are investigated. The following illustrations give an idea how realizations of such random polytopes, respectively symmetrized polytopes, generated from the uniform distribution in the 2-dimensional Euclidean ball, respectively uniform distribution on the Euclidean 1-sphere, may look depending on the number of random points.

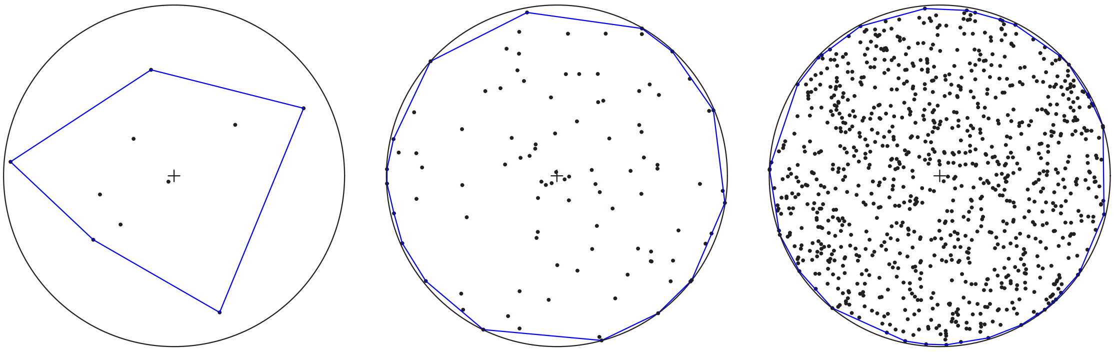

FIGURE 1.1: Random polytopes generated as convex hull of $n = 10, 100, 1000$ independent uniform random points in the unit ball.

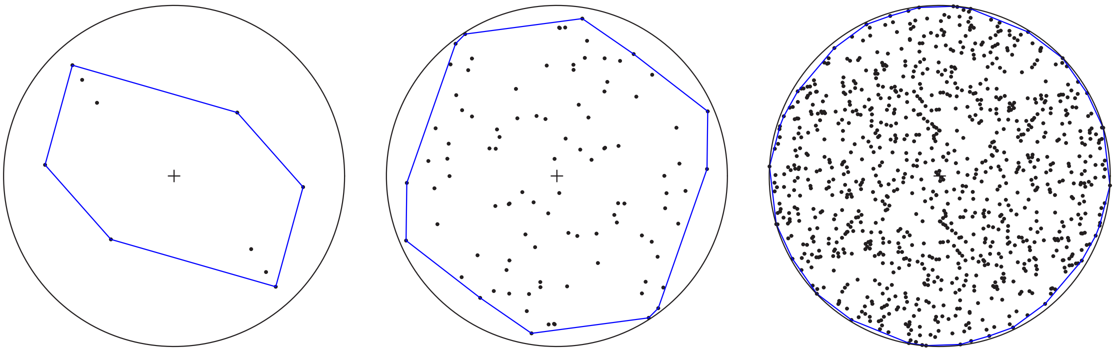

FIGURE 1.2: Random polytopes generated as symmetric convex hull of $n = 5, 50, 500$ independent uniform random points in the unit ball.

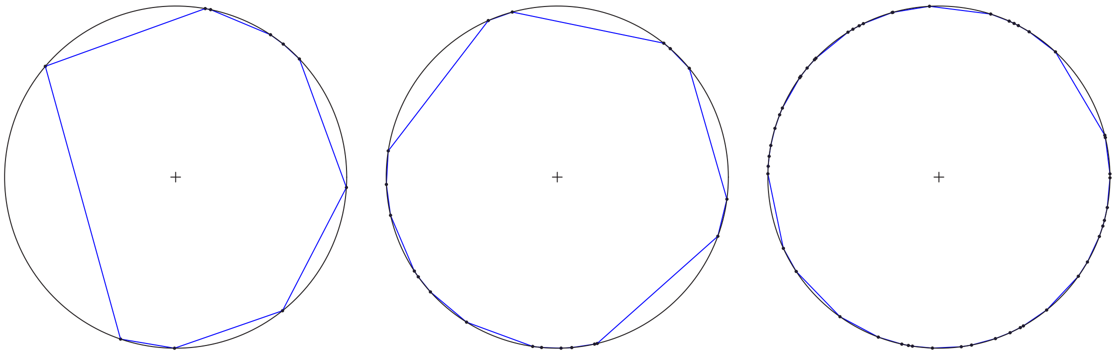

FIGURE 1.3: Random polytopes generated as convex hull of $n = 10, 20, 50$ independent uniform random points on the unit sphere.

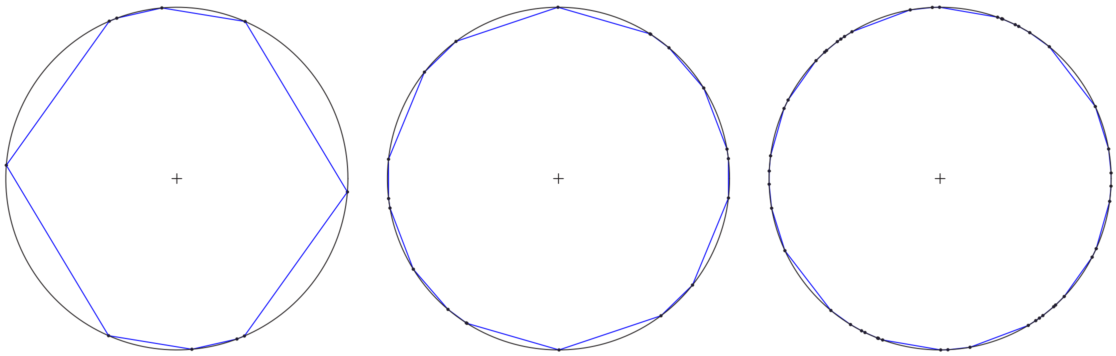

FIGURE 1.4: Random polytopes generated as symmetric convex hull of $n = 5, 10, 25$ independent uniform random points on the unit sphere.

The chapter is subdivided into two topics:

(1) We start off with the topic of deriving explicit integral formulae for the *expected volume*, *intrinsic volumes* and *facet numbers* of these types of polytopes from these classes of distributions for arbitrary dimensions. More generally, we will derive explicit formulas for the expectation of the so-called *T-functional*, introduced by Wieacker [110, Definition 3.1], which for a polytop $P \subset \mathbb{R}^d$, $a, b > 0$ and $k = 0, \ldots, d-1$ is defined as

$$T_{a,b}^{d,k}(P) = \sum_{F \in \mathcal{F}_k(P)} \eta^a(F) \operatorname{Vol}_k^b(F),$$

where $\mathcal{F}_k(P)$ is the set of k-dimensional faces of P, $\eta(F)$ is the distance of the affine hull of the face F to the origin $\boldsymbol{o}$ and $\operatorname{Vol}_k(F)$ is the k-dimensional Lebesgue volume of the face F. We will see that in fact we are only able to handle the case where $k = d - 1$. Meaning that explicit formulas for expected numbers of lower dimensional faces can not be obtained by our method.

Closely related to the contents of this section are the works of Efron [42], Aldous, Fristedt, Griffin and Pruit [6], Affentranger [2, 3], Buchta [29], Buchta and Müller [31], Buchta, Müller and Tichy [32], Carnal [34], Dwyer [40], Eddy and Gale [41] and Raynaud [85], of which we already mentioned a few in the general introduction. Each of these investigated random polytopes coming from spherically symmetric distributions in $\mathbb{R}^d$, in particular, special cases of beta- and beta′-type distributions. A similar analysis for the Gaussian random polytope is the content of the works of Affentranger and Schneider [4], Hug, Munsonius and Reitzner [55] and Kabluchko and Zaporozhets [62]. One of the goals of this section is to unify and to generalize these results to the whole class of polytopes coming from the distributions given by Ruben and Miles, as well as extend them to the bigger class of *symmetric* beta- and beta′-type polytopes and beta- and beta′-type polytopes *containing the origin* $\boldsymbol{o}$. Furthermore, we also treat the *Poissonized versions* of these three different types of random polytopes. We shall discuss the special cases $d = 2$ and $d = 3$ separately, where in some cases our formulae coincide with results known from the existing literature and in other cases lead to formulae that were not available before. In this context we especially refer to the early results in [29, 32, 42]. Furthermore, it is worth mentioning that postdating our work, Kabluchko, Thäle and Zaporozhets [61] answered some of the open cases of the T-functional by giving more or less explicit formulas for the expected number of lower dimensional faces in both the beta- and beta′-type case.

This section is build upon the paper [59]:

Kabluchko, Z., Temesvari, D., and Thäle, C. Expected intrinsic volumes and facet numbers of random beta-polytopes. *Mathematisch Nachrichten*, doi:10.1002/mana.201700255 (2018).

(2) The second part is about answering the question of *the monotonicity of the expected facet numbers* of beta and beta′-type polytopes in the number of points n generating the polytope. The question was put forward and answered positively by Devillers, Glisse, Gaoac, Moroz and Reitzner [39] for random uniformly and independently distributed points from a convex body $K \subset \mathbb{R}^d$ if $d = 2$, and also $d = 3$, if additionally the boundary of K is twice differentiable, has strictly positive Gaussian curvature and n is sufficiently large. In her Ph.D. thesis, Beermann [21] reconsidered this question for Gaussian polytopes as well as polytopes generated from the uniform distribution in the unit ball for arbitrary dimension $d \geq 2$ and answered it affirmatively, see also [22]. Quite recently Kabluchko and Thäle [60] proved that for Gaussian polytopes the monotonicity in expectation actually holds for the whole f-vector.

We would also like to remark that monotonicity questions related to the volume of random convex hulls have recently attracted some interest in convex geometry because of their connection to the *slicing conjecture*. Namely, if $K, L \subset \mathbb{R}^d$ are two convex bodies and S_K and S_L are two random simplices generated by independent uniform random points from K and L, respectively, one is interested in the question whether the set inclusion $K \subset L$ implies the inequality $\mathbb{E}\,\mathrm{Vol}_d(S_K) \leq \mathbb{E}\,\mathrm{Vol}_d(S_L)$, where Vol_d stands for the d-dimensional Lebesgue volume. In particular, the work of Rademacher [84] shows that this is false in general whenever $d \geq 4$. Higher moments were treated by Reichenwallner and Reitzner [86], and we refer to the discussion therein for further details and background material.

It should also be mentioned that in a work postdating the one this section is based on, Kabluchko, Thäle and Zaporozhets [61] showed that also for the class of beta-type and beta′-type polytopes the monotonicity in expectation holds for the *whole f-vector*.

We stick to the results first established in [26] and show the monotonicity in expectation for the facet numbers of beta- and beta′-type polytopes. However, we extend this result also to symmetric beta- and beta′-type polytopes as well as the Poissonized versions of these polytopes.

This section is build upon the paper [26]:

Bonnet, G., Grote, J., Temesvari, D., Thäle, C., Turchi, N., and Wespi, F.: Monotonicity of facet numbers of random convex hulls. *J. Math. Anal. Appl. 455* (2017), 1351–1364.

Chapter 4:

We focus on extending a recent work of Bárány, Hug, Reitzner and Schneider [16], where they investigated the f-vector, the *spherical volume* and some other quantities for the *spherical convex hull* of n uniformly and independently distributed random points on the d-dimensional *upper half-sphere*. Among other results, they showed that the expected number of facets and the expected number of vertices and edges of such spherical random polytopes tend to finite constants as $n \to \infty$. This surprising result is the starting point for our work in which we consider the $(d+1)$-*dimensional random convex cone* generated by random points on the half-sphere chosen according to a power law density with respect to the normalized Hausdorff measure thereon. Our first main result will be a weak limit theorem for the sections of these random cones with the tangent hyperplane of the half-sphere at its north pole. It turns out that, appropriately rescaled, these intersections are distributed like d-dimensional beta′-type polytopes, uncovering a tight link between these random cones and random polytopes.

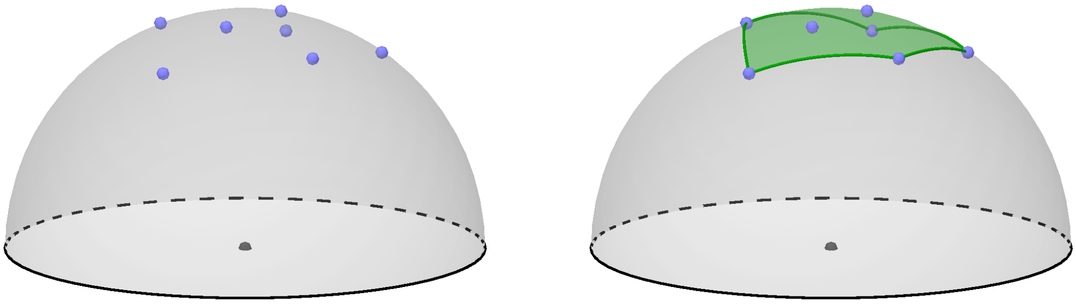

FIGURE 1.5: Left: Random points distributed according to a power law density with respect to the uniform probability measure on the half-sphere. Right: The spherical convex hull generated by these points.

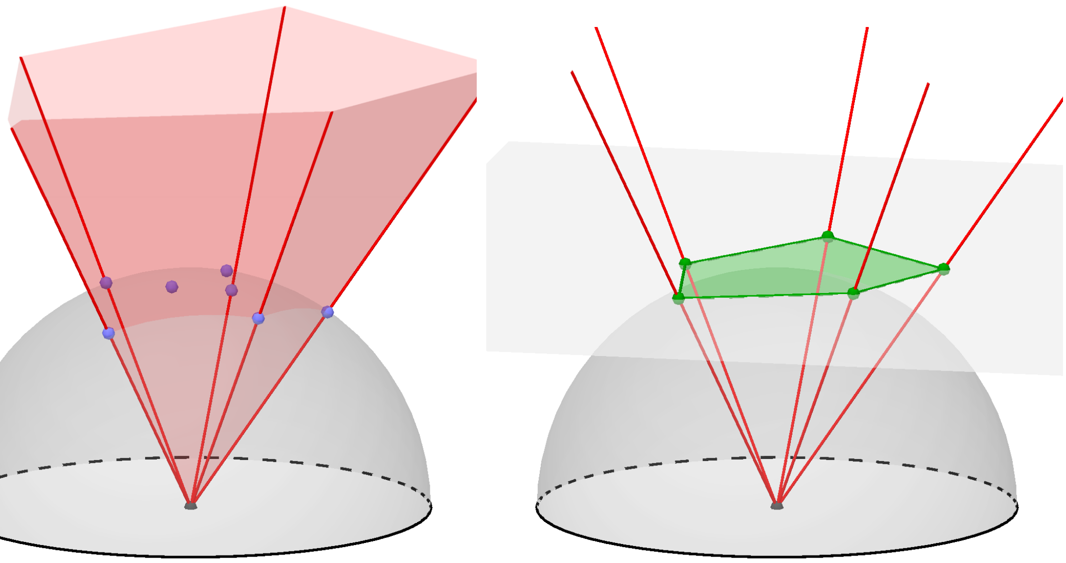

FIGURE 1.6: Left: Random convex cone generated by the above points. Right: Unrescaled intersection of the cone with the tangent plane at the north pole (generating points removed for better visibility).

For the number of uniform points on the half-sphere going to infinity, we shall identify the limiting random polytope of the sequence of such appropriately rescaled intersections as the convex hull of a Poisson point process in the tangent hyperplane with a power-law intensity function. This in turn leads to limit theorems for the whole f-vector and the volume of the corresponding spherical convex hull on the half-sphere, which complements the findings in [16]. In addition, the weak limit theorem allows us to describe the expectation asymptotics of the conic intrinsic volumes (in fact, all three versions of them) of the random cone. This solves in an extended form a conjecture posed by Bárány, Hug, Reitzner and Schneider [16, Section 9]. We also separately study the expected T-functional of the convex hull of a general class of Poisson point processes in $\mathbb{R}^d$ with a power-law intensity function $\|x\|^{-(d+\gamma)}$, where $\gamma > 0$ is a real parameter. In particular, we will compute explicitly the expected volume (and, more generally, expected intrinsic volumes) and the expected number of facets of this type of random polytope, thus generalizing a two-dimensional result of Davis, Mulrow and Resnick [38]. Furthermore, the tight link between random cones of this type and beta′-type polytopes will allow us to carry over the monotonicity results for beta′-type polytopes from Chapter 3.

This is build upon the paper [58]:

Kabluchko, Z., Marynych, A., Temesvari, D., and Thäle, C.: Cones generated by random points on half-spheres and convex hulls of Poisson point processes. *arXiv: 1801.08008* (2018).

Chapter 5:

In the last chapter we tackle a problem from elementary geometry. Consider an n-element point set $X = \{x_1, \dots, x_n\}$ from $\mathbb{R}^d$ in general position. We say that the d-simplex $\operatorname{conv}(x_{i_1}, \dots, x_{i_{d+1}})$ formed from the $(d+1)$-element subset $\{x_{i_1}, \dots, x_{i_{d+1}}\} \subset X$ is *empty*, if $\operatorname{int}(\operatorname{conv}(x_{i_1}, \dots, x_{i_d}, y)) \cap X = \emptyset$ holds. For a subset $\{x_{i_1}, \dots, x_{i_d}\} \subset X$ of d elements we define the *degree* $\deg(x_{i_1}, \dots, x_{i_d})$ as the number of empty simplices one can form with elements $y \in X \setminus \{x_{i_1}, \dots, x_{i_d}\}$. The degree $\deg(X)$ is defined as the maximum of the degrees of all d-element subsets of X. We are interested in the asymptotic behavior of $\deg(X)$ as $n \to \infty$. Note that so far X is a deterministic set.

This problem was introduced by Erdős in the planar case in the early nineties of the last century, asking the question whether the degree of the point set goes to infinity as the number of points go to infinity. Bárány conjectured that this is indeed true. This conjecture was later repeated in [17] and [28]. Although Bárány and Károlyi [17] showed that $\deg(X) \geq 10$ for sufficiently large n and Bárány and Valtr [19] constructed a set X in general position such that $\deg(X) = 4\sqrt{n}(1 + o(1))$, it is still unknown if the conjectures is true. However, in [18] Bárány, Marckert and Reitzner showed that in the planar case $\mathbb{E}(\deg(X)) \geq cn/\log n$, for some constant c, if the points of X are chosen uniformly and independently from a convex body $K \subset \mathbb{R}^2$. Furthermore, they also showed that the degree of X converges in probability to infinity as n goes to infinity.

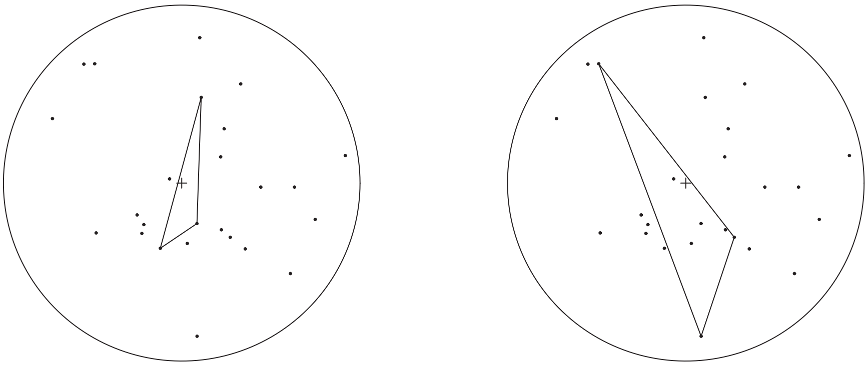

FIGURE 1.7: Point set with 25 points. Left: Instance of an empty triangle. Right: Instance of a nonempty triangle.

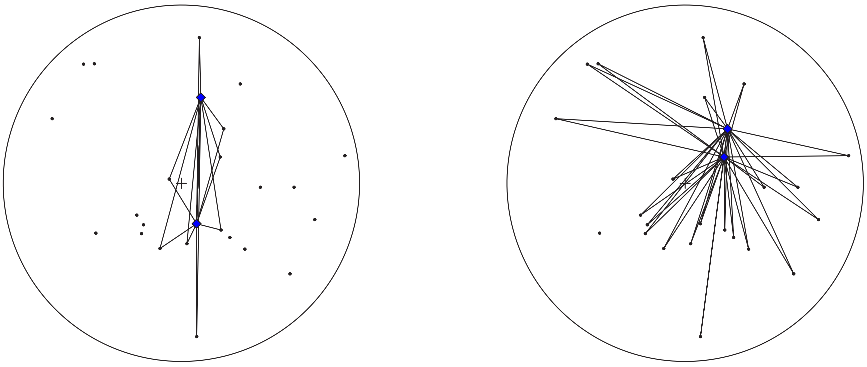

FIGURE 1.8: Point set with 25 point. Left: Degree of this basis is 8. Right: Degree of this basis is 22 and is also the degree of the point set.

We give two approaches to the probabilistic version of the problem. The first one vastly generalizes the method of Bárány, Marckert and Reitzner used in [18], to arbitrary dimension and all integer moments. The second approach still relies to some degree on this method, but yet significantly alters it. As it turns out, this approach gives the correct asymptotics for $\mathbb{E}(\deg(X))$ as well as for the integer moments of $\deg(X)$.

This is build upon the papers [90] and [105]:

TEMESVARI, D.: Moments of the maximal number of empty simplices of a random point set. *Discrete Comput. Geom. 60(3)* (2018), 646–664.

REITZNER, M. AND TEMESVARI, D.: Stars of empty simplices. *arXiv:1808.08734* (2018).

Chapter 2

Preliminaries

The purpose of this chapter is to introduce the reader to general notation used throughout the thesis as well as to specific knowledge from the fields of convex and integral geometry, Poisson point processes, random measures, random sets and analysis. We start by establishing our notation. We refrain from recalling probabilistic definitions and theorems which can be considered general knowledge and rather refer the reader to the book of Kallenberg [64] on the foundations of probability theory. We go on by presenting aspects of convex and integral geometry needed in this work. For a more in-depth treatment of these topics we recommend the books of Gruber [50] and Schneider [98] for convex geometry and the book of Schneider and Weil [99] regarding integral geometry. Furthermore, the papers of Ameluxen and Lotz [8] and Ameluxen, Lotz, McCoy and Tropp [9] are good treatises on spherical convex and spherical integral geometry, while a very general version of the particularly important integral geometric formulas of Blaschke and Petkantschin can be found in Vedel Jensen's book [108]. This will be followed by a section on random measures and random sets and, more specifically, one about Poisson point processes. Again Kallenberg [63] provides a good source for the theory of random measure, Molchanov [80] for the theory of random sets, while the book of Last and Penrose [68] treats Poisson point processes very neatly. Lastly, we give some general background on topics from analysis that we use. The parts of this section based on variation of functions are taken from the book of Ambrosio, Fusco and Pallara [7].

To keep this chapter reasonably compact, only proofs that were first presented in the works of the author will be given, while proofs of theorems and lemmas taken from other sources will only be referenced.

2.1 Notation

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of *natural numbers* and *natural numbers with zero*, respectively, and, for any $n \in \mathbb{N}$, denote by $[n] = \{1, 2, 3, \ldots, n\}$ the set of *natural numbers up to n*. For any $d \in \mathbb{N}$, we mean by $\mathbb{R}^d$ the d-dimensional *Euclidean space* with origin $\boldsymbol{o}$ equipped with the *Euclidean scalar product*, denoted by $\langle \cdot, \cdot \rangle$, and the induced *Euclidean norm*, denoted by $\|\cdot\|$. $\mathbb{R}_0^+$ and $\mathbb{R}^+$ are the *nonnegative*, respectively *positive, real numbers*. We set $e_1, \ldots, e_d$ to be the *standard basis* of $\mathbb{R}^d$ and put $x = (x_1, \ldots, x_d)$ for the representation of a vector $x \in \mathbb{R}^d$ with respect to the standard basis. For a set $A \subset \mathbb{R}^d$ we say that $\text{int}(A)$ and $\text{cl}(A)$ are its *interior* and *closure*, respectively, whereas by $\partial A = \text{cl}(A) \setminus \text{int}(A)$ we mean its *boundary*. We write $\mathbb{B}^d(x, r) \subset \mathbb{R}^d$ for the closed d-dimensional *Euclidean ball* of radius $r \geq 0$ centered at $x \in \mathbb{R}^d$, that is the set $\mathbb{B}^d(x, r) = \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$. In particular, we write $\mathbb{B}^d := \mathbb{B}^d(\boldsymbol{o}, 1)$ for the Euclidean unit ball and $\mathbb{S}^{d-1} := \partial \mathbb{B}^d$ for the *Euclidean unit sphere* in $\mathbb{R}^d$. Furthermore, $\mathbb{S}_+^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1, x_d \geq 0\}$ defines the *closed upper half-sphere* with north pole e_d.

Two special functions will find frequent use throughout this thesis, namely, the *gamma function*, defined as

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t \tag{2.1}$$

for any $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, and the *beta function*, defined as

$$\mathrm{B}(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t \tag{2.2}$$

for any $x, y \in \mathbb{C}$ with $Re(x), Re(y) > 0$. For details on these two functions see [1].

By λ_d we introduce the d-dimensional *Lebesgue measure* on $\mathbb{R}^d$ and by $\text{Vol}_d(\cdot)$ the corresponding d-dimensional *Lebesgue volume*. In particular, we denote by $\kappa_d = \text{Vol}_d(\mathbb{B}^d)$ the d-dimensional Lebesgue volume of the Euclidean unit ball for which we have that

$$\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2} + 1\right)}, \tag{2.3}$$

see [99, p.13]. We define the $(d-1)$-dimensional *surface area* S_{d-1} on the unit sphere $\mathbb{S}^{d-1}$ via $S_{d-1}(A) := \text{Vol}_d(\{tx : x \in A, t \in [0, 1]\})$ for any measurable $A \subset \mathbb{S}^{d-1}$. To obtain the uniform distribution σ_{d-1} on the unit sphere we normalize S_{d-1}. As before we abbreviate by $\omega_d = S_{d-1}(\mathbb{S}^{d-1})$ the $(d-1)$-dimensional surface area of the unit sphere for which we have that

$$\omega_d = d\kappa_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}; \tag{2.4}$$

see [99, p.13]. Similarly, we define the uniform measure $\bar{\sigma}_{d-1}$ on the closed upper half-sphere $\mathbb{S}^{d-1}_+$. In particular, we have $\bar{\sigma}_{d-1}(A) = 2\sigma_{d-1}(A)$ for any measurable $A \subset \mathbb{S}^{d-1}_+$.

Furthermore, we denote by $\mathcal{S}_k$ the *set of all k-dimensional great sub-spheres* of $\mathbb{S}^{d-1}$, i.e., the set of all intersections of $\mathbb{S}^{d-1}$ with k-dimensional linear subspaces, and equip it with the uniquely determined rotationally invariant probability measure τ_k thereon.

Let $A \subset \mathbb{R}^d$. The *linear hull* $\text{lin}(A)$ of A is defined as the smallest linear subset of $\mathbb{R}^d$ containing A, the *affine hull* $\text{aff}(A)$ of A is the smallest affine subset of $\mathbb{R}^d$ containing A and the *positive hull* $\text{pos}(A)$ of A is defined by

$$\text{pos}(A) := \left\{ \sum_{i=1}^{m} \lambda_i x_i : m \in \mathbb{N}, x_1, \ldots, x_m \in A, \lambda_1, \ldots, \lambda_m \in \mathbb{R}_0^+ \right\}. \tag{2.5}$$

Furthermore, for a set $A \in \mathbb{R}^d$ we mean by $\text{relint}(A)$ its *relative interior*, that is, the interior of A with $\text{aff}(A)$ as ambient space. The *polar set* A° of A in $\mathbb{R}^d$ is given by

$$A^\circ := \left\{ x \in \mathbb{R}^d : \sup_{y \in A} |\langle x, y \rangle| \leq 1 \right\}, \tag{2.6}$$

while, if $A \subset \mathbb{S}^{d-1}$, the *polar set A^* of A with respect to the sphere* $\mathbb{S}^{d-1}$, by

$$A^* := \left\{ x \in \mathbb{S}^{d-1} : \sup_{y \in A} |\langle x, y \rangle| \leq 0 \right\}. \tag{2.7}$$

We make use of three different projections. The first one being the *orthogonal projection* $P_L : \mathbb{R}^d \to \mathbb{R}^d$ onto a k-dimensional linear subspace $L \subset \mathbb{R}^d$, that is, the projection fulfilling $P_L(x) \in L$ for all $x \in \mathbb{R}^d$ and $\langle P_L(x) - x, y \rangle = 0$ for all $y \in L$. The second one is the *metric projection* $\Pi_C : \mathbb{R}^d \to \mathbb{R}^d$ onto a convex set $C \subset \mathbb{R}^d$, that is, the projection fulfilling $\|\Pi_C(x) - x\| = \inf\{\|y - x\| : y \in C\}$. Note, that the assumption on C being convex is not necessary. However, since we only apply $\Pi_C(x)$ with convex sets C, this assumption allows us to avoid technical inconveniences. The reason being that, if C is convex, then $\Pi_C(x)$ maps to a uniquely determined point, for any $x \in \mathbb{R}^d$.

And lastly, we introduce the *spherical projection* $A|S$ of a set $A \subset \mathbb{R}^d$ to a k-dimensional great sub-sphere $S \in \mathcal{S}_k$, defined as follows: Let $A \vee S := \mathbb{S}^{d-1} \cap \operatorname{pos}(A \cup S)$, then the spherical projection is given by

$$A|S := S \cap (A \vee S^*). \tag{2.8}$$

For more details regarding the spherical projection, see [99, p. 263].

Let $U \subset \mathbb{R}^d$. We denote by $\mathcal{C}_c^1(U, \mathbb{R})$ and $\mathcal{C}_c^1(U, \mathbb{R}^d)$ the *spaces of continuously differentiable functions* from U to $\mathbb{R}$ and U to $\mathbb{R}^d$, respectively, equipped with the *supremum norm* $\|\cdot\|_\infty$, defined as $\|f\|_\infty = \sup\{|f(x)| : x \in U\}$ for any $f \in \mathcal{C}_c^1(U, \mathbb{R})$ and $\|f\|_\infty = \sup\{\|f(x)\| : x \in U\}$ for any $f \in \mathcal{C}_c^1(U, \mathbb{R}^d)$, respectively. Whereas by $L^1_{loc}(U)$ and $L^1(U)$ we mean the *spaces of locally integrable functions* over U and *integrable functions* over U, respectively. For a function $f \in L^1_{loc}(U)$ the *variation* in U is defined as

$$V(f, U) := \sup \left\{ \int_U f(x) \operatorname{div} \varphi(x) \, \mathrm{d}x : \varphi \in \mathcal{C}_c^1(U, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}, \tag{2.9}$$

while the *directional variation* in U in the direction $u \in \mathbb{S}^{d-1}$ is defined as

$$V_u(f, U) := \sup \left\{ \int_U f(x) \frac{\partial \varphi}{\partial u}(x) \, \mathrm{d}x : \varphi \in \mathcal{C}_c^1(U, \mathbb{R}), \|\varphi\|_\infty \leq 1 \right\}. \tag{2.10}$$

Furthermore, we introduce the *perimeter* of a set $K \in \mathbb{R}^d$ in U as $\operatorname{Per}(K, U) := V(\mathbb{1}_K, U)$. If $U = \mathbb{R}^d$, we write $\operatorname{Per}(K) = \operatorname{Per}(K, \mathbb{R}^d)$. Note that, if K is convex, then $\operatorname{Per}(K) = S_{d-1}(\partial K)$. The directional variation of a set K in direction $u \in \mathbb{S}^{d-1}$ is defined as $V_u(K, U) := V_u(\mathbb{1}_K, U)$. Again, if $U = \mathbb{R}^d$, we write $V_u(K) := V_u(K, \mathbb{R}^d)$.

Lastly, we also make use of the *Landau notation*. Let $g, h : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. We say

(i) $g \in o(h)$ as $t \to a$, if $\lim_{t \to a} \left| \frac{g(t)}{h(t)} \right| = 0$,

(ii) $g \in \mathcal{O}(h)$ as $t \to a$, if $\limsup_{t \to a} \left| \frac{g(t)}{h(t)} \right| < \infty$,

(iii) $g \in \Omega(h)$ as $t \to a$, if $\liminf_{t \to a} \left| \frac{g(t)}{h(t)} \right| > 0$,

(iv) $g \sim h$ as $t \to a$, if $\lim_{t \to a} \left| \frac{g(t)}{h(t)} \right| = 1$, and

(v) $g \in \Theta(h)$ as $t \to a$, if $g \in \mathcal{O}(h)$ and $f \in \Omega(g)$.

In abuse of notation, we will write $g = o(h)$, $g = \mathcal{O}(h)$, $g = \Omega(h)$ and $g = \Theta(h)$.

2.2 Convex geometry

A set $K \subset \mathbb{R}^d$ is called *convex*, if for each pair of points $x, y \in K$ also the straight line segment $\{\lambda x + (a - \lambda)y : \lambda \in [0, 1]\}$ is contained in K. The *convex hull* $\mathrm{conv}(A)$ of a set $A \subset \mathbb{R}^d$ is defined as the smallest convex set in $\mathbb{R}^d$ containing A, i.e.,

$$\mathrm{conv}(A) := \bigcap_{\substack{A \subset K \\ K \subset \mathbb{R}^d \, convex}} K. \tag{2.11}$$

For a finite point set $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$ we also write $[x_1, \dots, x_n] := \mathrm{conv}(\{x_1, \dots, x_n\})$. *Caratheodory's theorem* gives us a very helpful constructive description of the convex hull of a set A, that is

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{d+1} \lambda_i x_i : x_1, \dots, x_{d+1} \in A, \lambda_1, \dots, \lambda_{d+1} \in [0, 1], \sum_{i=1}^{d+1} \lambda_i = 1 \right\}. \tag{2.12}$$

By a *convex body* $K \subset \mathbb{R}^d$ we mean a compact, convex set with nonempty interior. The *space of compact subsets of* $\mathbb{R}^d$ is denoted by $\mathcal{C}^d$, while the *space of convex bodies in* $\mathbb{R}^d$ by $\mathcal{K}^d$. We define the *Minkowski sum* on $\mathcal{C}^d$ by $X + Y := \{x + y : x \in X, y \in Y\}$, for any $X, Y \in \mathcal{C}^d$. We can now introduce the *symmetric convex hull*, sometimes also called the *absolute convex hull*, as

$$\mathrm{sconv}(A) := \mathrm{conv}(A \cup (-A)). \tag{2.13}$$

Similar to before we write $[\pm x_1, \dots, \pm x_n] := \mathrm{sconv}(\{x_1, \dots, x_n\})$ for a finite point set $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$. Furthermore, we equip $\mathcal{C}^d$ with the *Hausdorff distance*, that is, the metric defined by

$$\mathrm{d}_H(X, Y) := \max \left\{ \sup_{x \in X} \inf_{y \in Y} \|x - y\|, \sup_{y \in Y} \inf_{x \in X} \|x - y\| \right\} \tag{2.14}$$

for any two elements $X, Y \in \mathcal{C}^d$. We will use the notation $C_n \overset{\mathrm{d}_H}{\to} C_0$ to indicate that $d_H(C_n, C_0) \to 0$, as $n \to \infty$, for a sequence $(C_n)_{n \in \mathbb{N}_0} \subset \mathcal{C}^d$ and a fixed $C_0 \in \mathcal{C}^d$. Note that $\mathcal{K}^d$ is a closed subspace of $\mathcal{C}^d$ with respect to the Hausdorff metric.

Furthermore, we define the *surface area* S_{d-1} of a convex body $K \in \mathcal{K}^d$ as the $(d-1)$-dimensional Hausdorff measure of its boundary ∂K, i.e., $S_{d-1}(K) = \mathcal{H}^{d-1}(\partial K)$, see [99, p.607]. We refrain from going into detail regarding the k-dimensional Hausdorff measure of a set $A \subset \mathbb{R}^d$, $k = 0, \dots, d$, and instead refer the reader to [99, p.634] for a rigorous definition and treatment.

Affine $(d-1)$-dimensional subspaces of $\mathbb{R}^d$, so-called *hyperplanes*, and *half-spaces* generated by them play an outstanding role in convex geometry. They can be uniquely characterized very conveniently via two parameters. Namely a unit vector $u \in \mathbb{S}^{d-1}$, describing the unit normal vector of the hyperplane, and a distance $h \geq 0$. Thus, for any hyperplane $H \subset \mathbb{R}^d$ there exist $u \in \mathbb{S}^{d-1}$ and $h \geq 0$ such that

$$H = H(u,h) := \{x \in \mathbb{R}^d : \langle x, u \rangle = h\}. \tag{2.15}$$

Note, that this characterization of hyperplanes is unique if $h > 0$. If $h = 0$, then $u \in \mathbb{S}^{d-1}$ is not unique anymore and it holds that $H(u,0) = H(-u,0)$.

Any such hyperplane H bounds two open half-space, which we denote by

$$H^-(u,h) := \{x \in \mathbb{R}^d : \langle x, u \rangle < h\} \quad \text{and} \quad H^+(u,h) := \{x \in \mathbb{R}^d : \langle x, u \rangle > h\}. \tag{2.16}$$

By $\eta(H)$ we mean the *distance* of the affine hull $\mathrm{aff}(H)$ of H to the origin $\boldsymbol{o}$.

For a convex body $K \in \mathcal{K}^d$ the map $h_K : \mathbb{R}^d \to \mathbb{R}$, $x \mapsto \sup\{\langle x, y \rangle : y \in K\}$ is called *support function* of K. For $u \in \mathbb{S}^{d-1}$ the hyperplane $H(u, h_K(u))$ is called *supporting hyperplane* of K in direction u.

Theorem 2.2.1 (Supporting hyperplane theorem) *Let $K \subset \mathbb{R}^n$ be convex and closed. Then through each boundary point there exists a supporting hyperplane of K. If $K \neq \emptyset$ is bounded, then to each vector $u \in \mathbb{R}^n \setminus \{\boldsymbol{o}\}$ there exists a supporting hyperplane to K with outer normal vector u.*

Another very important theorem in convex geometry is the *separating hyperplane theorem.* Let $K, L \subset \mathbb{R}^n$ be two sets and let $u \in \mathbb{R}^d \setminus \{\boldsymbol{o}\}$ and $h \geq 0$. We say that the hyperplane $H(u,h)$ separates K and L if $K \subset H^-(u,h)$ and $L \subset H^+(u,h)$, or vice versa, while we say that $H(u,h)$ strongly separates them if there exists an $\varepsilon > 0$ such that $H(u, h-\varepsilon)$ and $H(u, h+\varepsilon)$ both separate K and L.

Theorem 2.2.2 (Separating hyperplane theorem) *Let $K, L \subset \mathbb{R}^n$ be nonempty convex sets with $K \cap L = \emptyset$. Then K and L can be separated. If K is compact and L is closed, then K and L can be strongly separated.*

For proofs of these two theorems the reader may turn to [98, Theorem 1.3.2] and [98, Theorem 1.3.7].

For a convex body $K \in \mathcal{K}^d$, we denote by $V_i(K)$, $i = 0, \dots, d$, the so-called *intrinsic volumes* of K. They arise from *Steiner's formula*, see [99, p.600], given by

$$\mathrm{Vol}_d(K + \mathbb{B}^d(\boldsymbol{o}, t)) = \sum_{i=0}^{d} \kappa_{d-i}\, V_i(K)\, t^{d-i}, \tag{2.17}$$

as the coefficients of this polynomial decomposition of the Lebesgue volume of the Minkowski sum $K + \mathbb{B}^d(\boldsymbol{o}, t)$ in t. This family of functionals is of great interest in convex geometry since they constitute a basis of the *space of motion invariant and continuous valuations on* $\mathcal{K}^d$, where valuations are maps $\varphi : \mathcal{K}^d \to \mathbb{R}$ fulfilling $\varphi(K) + \varphi(L) = \varphi(K \cup L) + \varphi(K \cap L)$ for any $K, L \in \mathcal{K}^d$. In particular, for $K \in \mathcal{K}^d$ it holds that $V_d(K)$ is its d-dimensional Lebesgue volume, $V_{d-1}(K)$ is half of its $(d-1)$-dimensional surface area, $V_1(K)$ is a constant multiple of its mean width, denoted by $\mathcal{W}_d(K)$, and $V_0(K)$ is its Euler characteristic, which in the case of a convex body always satisfies $V_0(K) = 1$.

The main class of convex bodies we are focusing our attention on, is the class of *polytopes*. To some extent we are also concerned with *polyhedrons*. A polytope is the convex hull of a finite point set in $\mathbb{R}^d$, while a polyhedron is the intersection of finitely many closed half-spaces in $\mathbb{R}^d$. Furthermore, we call a polyhedron a *(polyhedral) cone*, if its supporting hyperplanes all contain the origin $\boldsymbol{o}$. Note, that every polytope is also a polyhedron, the reverse is not true anymore.

Let $P \subset \mathbb{R}^d$ be a polyhedron and H be a supporting hyperplane of P, then the intersection $F = P \cap H$ is called a *face of the polyhedron* and its dimension is defined via $\dim(F) = \dim(\mathrm{aff}(P \cap H))$, i.e., as the dimension of its affine hull. We write $\mathcal{F}_k(P)$ for the *set of k-dimensional faces* and $\mathbf{f}_k(P)$ for the *number of k-dimensional faces of the polyhedron P*, for any $k = 0, \dots, d-1$. The vector $\mathbf{f}(P) = (\mathbf{f}_0(P), \dots, \mathbf{f}_{d-1}(P))$, the so-called **f**-*vector of the polyhedron P*, is an object of great interest in convex geometry and is also of great importance in this thesis.

We call a polytope P a *simplicial polytope* if all its $(d-1)$-dimensional faces are $(d-1)$-simplices. For simplicial polytopes the so-called *Dehn-Sommerville equations* provide tight relations between the components of the **f**-vector $\mathbf{f}(P)$, see for instance [50] and [102]. Set $\mathbf{f}_{-1}(P) = \mathbf{f}_d(P) = 1$.

Theorem 2.2.3 (Dehn-Sommerville equations) *Let $P \subset \mathbb{R}^d$ be a simplicial polytope. Then, for any $k = -1, \dots, d-2$, we have*

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} \mathbf{f}_j(P) = (-1)^{d-1} \mathbf{f}_k(P).$$

In particular, for $k = d-2$, we have $d\,\mathbf{f}_{d-1}(P) = 2\,\mathbf{f}_{d-2}(P)$.

Next we introduce the notion of *conic*, respectively *spherical*, *intrinsic volumes*. But first, recall that for two points $x, y \in \mathbb{S}^{d-1}$, with $x \neq y$ and $x \neq -y$, there exists a unique great circle $S(x,y) \in \mathcal{S}_{d-1}$ going through x and y. We say that a set $A \subset \mathbb{S}^{d-1}$ is *spherically convex* if for any two points $x, y \in A$ the shorter of the two arcs of $S(x,y)$ entirely lies in A. Analogously, we can define spherical convexity on the upper half-sphere $\mathbb{S}^{d-1}_+$. Note, that there exists a one to one correspondence between convex cones and spherically convex sets. Namely, if $C \subset \mathbb{R}^d$ is a convex cone with apex $\boldsymbol{o}$, then $C \cap \mathbb{S}^{d-1}$ is a spherically convex set, and vice versa, if $A \subset \mathbb{S}^{d-1}$ is spherically convex, then $\mathrm{pos}(A)$ is a convex cone in $\mathbb{R}^d$. Via this correspondence we can easily define what a spherical polytope is. A spherically convex set $P \subset \mathbb{S}^{d-1}$ is called a *spherical polytope* if and only if $C := \mathrm{pos}(P)$ is a polyhedral cone. Furthermore, we say that $F \subset P$ is a k-face of P if and only if $G := \mathrm{pos}(F)$ is a $(k+1)$-face of C, for any $k = 0, \dots, d-2$.

We go over to define conic intrinsic volumes and, via the aforementioned correspondence, also spherical intrinsic volumes. Note that there exist three different, tightly linked notions of this. Namely, the *conic intrinsic volumes*, the *Grassmann angles* and the *conic mean projection volumes*. For an in-depth treatment of conic intrinsic volumes the reader may consult [8, 9], while a treatment from the spherical point of view is provided in [45] and [99].

Let $C \subset \mathbb{R}^d$ be a polyhedral cone and $F \subset C$ be a face of C. Let g be a standard Gaussian random vector in $\mathbb{R}^d$ and set $v_F := \mathbb{P}(\Pi_C(g) \in \mathrm{relint}(F))$. For any $k = 0, \dots, d$, the *k-th conic intrinsic volume* v_k is defined as

$$v_k(C) := \sum_{F \in \mathcal{F}_k(C)} v_F. \tag{2.18}$$

For convenience we also set $v_k(C) = 0$ for any $k > d$. Consider for instance the upper half-space $H_{\mathrm{up}} := \{x \in \mathbb{R}^d : x_d \geq 0\}$. Then, $v_k(H_{\mathrm{up}}) = 0$ for $k \in \{0, \dots, d-2\}$ and $v_{d-1}(H_{\mathrm{up}}) = v_d(H_{\mathrm{up}}) = 1/2$. If C is a k-dimensional linear subspace, then $v_k(C) = 1$, while all the other conic intrinsic volumes vanish. Henceforth, we shall always exclude the case when C is a linear subspace (since formulae (2.19) and (2.20) below are not valid in this case). One important property of conic intrinsic volumes is the *Gauss-Bonnet formula* [9, Equation (5.3)]

$$v_0(C) + v_2(C) + \ldots = v_1(C) + v_3(C) + \ldots = \frac{1}{2}. \tag{2.19}$$

Before we go over to introduce Grassmann angles and conic mean projection volumes we briefly anticipates some of the notation and definitions from Section 2.3 needed here: For any $k \in \{0, 1, \ldots, d-1\}$, let $L \in G(d, d-k)$ be a $(d-k)$-dimensional linear subspace of $\mathbb{R}^d$ chosen at random according to the unique rotationally invariant Haar probability measure ν_{d-k} defined on the group $G(d, d-k)$ of all $(d-k)$-dimensional linear subspaces of $\mathbb{R}^d$.

The $(k+1)$-*th Grassmann angle* of C is defined as

$$h_{k+1}(C) := \frac{1}{2}\mathbb{P}(C \cap L \neq \{\boldsymbol{o}\}), \tag{2.20}$$

and was introduced by Grünbaum [51]. Note that all the Grassmann angles $h_1, \ldots, h_d$ of the upper half-space H_{up} are equal to $1/2$. The d-th Grassmann angle $h_d(C)$ of the cone C is also called the *solid angle* $\alpha(C)$ of C, and has an equivalent formulation as

$$h_d(C) = \alpha(C) = \bar{\sigma}_{d-1}(C \cap \mathbb{S}^{d-1}). \tag{2.21}$$

The connection between conic intrinsic volumes and Grassmann angles is given by the *conic Crofton formula* [8, Equation (2.10)] and reads, for any $k \in \{0, \ldots, d-1\}$, as

$$h_{k+1}(C) = \sum_{\substack{i \geq 1 \\ i \text{ odd}}} v_{k+i}(C). \tag{2.22}$$

In the terminology of [9], the above sums (which are in fact finite) are called the *half-tail functionals*.

Lastly, we consider the *conic mean projection volumes* w_{k+1} defined for any $k \in \{0, 1, \ldots, d-1\}$ by

$$w_{k+1}(C) := \frac{1}{\kappa_{k+1}} \int\limits_{G(d,k+1)} \text{Vol}_{k+1}\left(P_L(C) \cap \mathbb{B}^d\right) \nu_{k+1}(\mathrm{d}L). \tag{2.23}$$

The relation between the conic mean projection volumes and the conic intrinsic volumes, can be called *conic Kubota formula* and states that

Lemma 2.2.4 (Conic Kubota formula) *For $k \in \{0, 1, \dots, d-1\}$ and a cone $C \subset \mathbb{R}^d$ we have that*

$$w_{k+1}(C) = \sum_{i=k+1}^{d} v_i(C) = h_{k+1}(C) + h_{k+2}(C). \tag{2.24}$$

Thus, the conic mean projection volumes coincide with the *tail functionals* in the language of [9]. For the half-space H_{up} we have $w_{k+1}(H_{\text{up}}) = 1$, for $k \in \{0, 1, \dots, d-2\}$, and $w_d(H_{\text{up}}) = 1$. Let us briefly give a proof, which can also be found in [58].

Proof. Recall that $\mathcal{S}_k$ is the space of k-dimensional great sub-spheres of $\mathbb{S}^{d-1}$, supplied with the unique rotationally invariant probability measure τ_k. The spherical mean projection volume of a spherically convex set $K \subset \mathbb{S}^{d-1}$ is given by

$$W_k(K) := \frac{1}{\omega_{k+1}} \int\limits_{\mathcal{S}_k} \sigma_k(K|S)\, \tau_k(\mathrm{d}S),$$

where, in this particular instance, by σ_k we mean the normalized Hausdorff measure on $S \in \mathcal{S}_k$, see [99, p.263]. Putting $C = \mathrm{pos}(K)$ and using the fact that τ_k is the probability measure on $L \cap \mathbb{S}^{d-1}$, where $L \in G(d, k+1)$ is distributed according to the unique rotationally invariant Haar probability measure ν_{k+1} on $G(d, k+1)$, we obtain

$$W_k(K) = \frac{1}{\omega_{k+1}} \int\limits_{\mathcal{S}_k} \sigma_k(K|S)\, \tau_k(\mathrm{d}S) = \frac{1}{\kappa_{k+1}} \int\limits_{G(d,k+1)} \mathrm{Vol}_{k+1}\left(P_L(C) \cap \mathbb{B}^d\right)\, \nu_{k+1}(\mathrm{d}L).$$

This leads to the equality $W_k(K) = w_{k+1}(C)$. On the other hand, from [99, p.263] we have the relationship

$$W_k(K) = \sum_{i=k}^{d-1} v_i(K)$$

with the spherical intrinsic volumes $v_i(K) := v_{i+1}(C)$. This yields the required formula for $w_{k+1}(C)$. □

We close this section with two lemmas that will be needed in the course of the discussion in Chapter 4.

Lemma 2.2.5 *Suppose that for each* $(\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, +1\}^d$ *a point in* $\mathbb{R}^d$ *is given whose coordinates have the same signs as* $\varepsilon_1, \ldots, \varepsilon_d$. *Then, the convex hull of these* 2^d *points contains the origin* $\boldsymbol{o}$.

Proof. We argue by induction over the dimension d. The claim obviously holds for $d = 1$. Suppose it is true for dimension $d-1$. Then we can take 2^{d-1} points corresponding to $\varepsilon_1 = 1$ and construct a convex combination a_+ of these points such that all coordinates of a_+ vanish except the first one (which is positive). Similarly, taking 2^{d-1} points corresponding to $\varepsilon_1 = -1$ we construct a convex combination a_- with negative first coordinate and all other coordinates being 0. Clearly, the origin $\boldsymbol{o}$ can now be written as a convex combination of these two points a_+ and a_-. □

Lemma 2.2.6 *For* $r \geq 0$ *and* $\varepsilon_2, \ldots, \varepsilon_d \in \{-1, +1\}$ *define the set*

$$A_{\varepsilon_2,\ldots,\varepsilon_d}(r) := \{(z_1, \ldots, z_d) \in \mathbb{R}^d : z_1 > r,\ \varepsilon_2 z_2 > 0, \ldots, \varepsilon_d z_d > 0\}.$$

Suppose that for every choice of $(\varepsilon_2, \ldots, \varepsilon_d)$ *a point in* $A_{\varepsilon_2,\ldots,\varepsilon_d}(r)$ *and another point in* $-A_{\varepsilon_2,\ldots,\varepsilon_d}(0)$ *are given. Then* $(r, 0, \ldots, 0)$ *can be represented as a convex combination of these points.*

Proof. By Lemma 2.2.5 we can take all points in $A_{\varepsilon_2,\ldots,\varepsilon_d}(r)$ or all points in $-A_{\varepsilon_2,\ldots,\varepsilon_d}(0)$, respectively, corresponding to all choices of $\varepsilon_2, \ldots, \varepsilon_d$ and construct a convex combination of these points such that all coordinates are zero except the first one (which is larger than r or smaller than 0, respectively). Obviously, there exists a convex combination of these two points which is equal to $(r, 0, \ldots, 0)$. □

2.3 Integral geometry

In this section we collect all the tools of integral geometry needed for this work. We start with the basic objects: Let $G(d,k)$ and $A(d,k)$ be the set of all *k-dimensional linear*, respectively *affine*, *subspaces* of $\mathbb{R}^d$. For $L \in G(d,k)$ we define by $\|\cdot\|_L$, $\mathbb{B}^k_L$, $\mathbb{S}^{k-1}_L$, λ_L and σ_L the *norm*, the *unit ball*, the *unit sphere*, the *Lebesgue measure* and the *unique rotationally invariant probability measure on the sphere*, as we did in the case of $\mathbb{R}^d$. Analogously, we define these quantities for $H \in A(d,k)$ and denote them by indexing with H instead of L. Since elements from $G(d,k)$ and $A(d,k)$ are isometrically isomorphic to $\mathbb{R}^k$ we will often identify them and the above defined quantities thereon with the Euclidean space $\mathbb{R}^k$. Moreover, by $L^\perp$, respectively $H^\perp$, we mean the *orthogonal complement of* L, respectively H. We have $L^\perp, H^\perp \in G(d,d-k)$.

We equip $G(d,k)$ and $A(d,k)$ with the *unique rotationally invariant*, respectively *rigid motion invariant*, *Haar probability measure* ν_k and μ_k, respectively. These two probability measure are connected via the identity

$$\mu_k(\cdot) = \int\limits_{G(d,k)} \int\limits_{L^\perp} \mathbb{1}_{\{L+x\in\,\cdot\}}(x)\, \lambda_{L^\perp}(\mathrm{d}x)\, \nu_k(\mathrm{d}L), \tag{2.25}$$

making use of the fact that for any $H \in A(d,k)$ there exists a unique parallel linear subspace $L \in G(d,k)$ and a unique point $x \in L^\perp$, such that $H = L + x$, see [99, p.591].

Next we introduce a very neat integral identity between intrinsic volumes of a convex body $K \subset \mathbb{R}^d$ and the k-dimensional Lebesgue volumes of it's projections to k-dimensional subspaces $L \in G(d,k)$, known as *Kubota's formula*, see [99, p.222]:

$$V_k(K) = \binom{d}{k} \frac{\kappa_d}{\kappa_k \kappa_{d-k}} \int\limits_{G(d,k)} \mathrm{Vol}_k(P_L K)\, \nu_k(\mathrm{d}L), \tag{2.26}$$

for any $K \in \mathcal{K}^d$ and $k = 1, \dots, d-1$. The case of $k = d-1$ is known as *Cauchy's surface area formula* and can be reformulated as

$$V_{d-1}(K) = \frac{1}{\kappa_{d-1}} \int\limits_{\mathbb{S}^{d-1}} \mathrm{Vol}_{d-1}\left(P_{u^\perp}(K)\right)\, \sigma_{d-1}(\mathrm{d}u). \tag{2.27}$$

We will also very frequently use *Blaschke-Petkantschin type formulas*. The classical (linear) formula is an integral transformations which transform a k-fold integral over $(\mathbb{R}^d)^k$

into an integral where one first calculates a k-fold integral over a fixed k-dimensional linear subspace and then integrates over all k-dimensional linear subspaces. A similar version exists for affine subspace. But before we introduce these, let us fix some more notation. For any $k = 1, \dots, d$ we define the constant

$$b_{d,k} := \frac{\omega_{d-k+1} \cdots \omega_d}{\omega_1 \cdots \omega_k}. \tag{2.28}$$

Furthermore, we denote by $\Delta_k(x_0, \dots, x_k)$ and $\nabla_k(x_1, \dots, x_k)$ the *k-dimensional Lebesgue volume of the simplex*, respectively the *parallelepiped*, generated by the points $x_0, \dots, x_k \in \mathbb{R}^d$, for all $k = 1, \dots, d$. The two quantities are tightly connected via the formula

$$\Delta_k(x_0, \dots, x_k) = \frac{1}{k!} \nabla_k(x_1 - x_0, \dots, x_k - x_0); \tag{2.29}$$

see [99, p.271]. They play a fundamental role in the Blaschke-Petkantschin formulas, since, up to multiplicative constants depending only on k and d, they appear to be the *Jacobians* of the aforementioned transformations. Let us now formulate the classical *linear Blaschke-Petkantschin formula.*

Lemma 2.3.1 (Linear Blaschke-Petkantschin formula)
Let $k \in \{1, \dots, d\}$ and let $f : (\mathbb{R}^d)^k \to \mathbb{R}$ be a nonnegative measurable function. Then,

$$\begin{aligned} &\int\limits_{(\mathbb{R}^d)^k} f(x_1, \dots, x_k) \, \lambda_d^k(\mathrm{d}(x_1, \dots, x_k)) \\ &\quad = b_{d,k} \int\limits_{G(d,k)} \int\limits_{L^k} f(x_1, \dots, x_k) \nabla_k^{d-k}(x_1, \dots, x_k) \, \lambda_L^k(\mathrm{d}(x_1, \dots, x_k)) \, \nu_k(\mathrm{d}L). \end{aligned} \tag{2.30}$$

The *affine Blaschke-Petkantschin formula* reads as:

Lemma 2.3.2 (Affine Blaschke-Petkantschin formula)
Let $k \in \{1, \dots, d\}$ and let $f : (\mathbb{R}^d)^{k+1} \to \mathbb{R}$ be a nonnegative measurable function. Then,

$$\begin{aligned} &\int\limits_{(\mathbb{R}^d)^{k+1}} f(x_0, \dots, x_k) \, \lambda_d^{k+1}(\mathrm{d}(x_0, \dots, x_k)) = \\ &b_{d,k} (k!)^{d-k} \int\limits_{A(d,k)} \int\limits_{H^{k+1}} f(x_0, \dots, x_k) \Delta_k^{d-k}(x_0, \dots, x_k) \, \lambda_H^{k+1}(\mathrm{d}(x_0, \dots, x_k)) \, \mu_k(\mathrm{d}H). \end{aligned} \tag{2.31}$$

Both of these formulas and their proofs can be found in [99, p.271 and p.278].

There also exists a vastly generalized version of the Blaschke-Petkantschin formula, due to Zähle [113], that can also be found in [108]. Furthermore, we consider a specific type of the affine Blaschke-Petkantschin formula. It is the *spherical counterpart of the affine Blaschke-Petkantschin formula* in $\mathbb{R}^d$, and can be found in [79, Theorem 4] .

Lemma 2.3.3 (Spherical affine Blaschke-Petkantschin formula)
Let $f : (\mathbb{S}^{d-1})^d \to \mathbb{R}$ *be a nonnegative measurable function. Then,*

$$\int\limits_{(\mathbb{S}^{d-1})^d} f(x_1, \dots, x_d)\, \sigma_{d-1}^d(\mathrm{d}(x_1, \dots, x_d)) = (d-1)! \int\limits_{A(d,d-1)} \int\limits_{\left(\mathbb{S}_H^{k-1}\right)^d} f(x_1, \dots, x_d)$$

$$\times\, \Delta_{d-1}(x_1, \dots, x_d)(1-h^2)^{-\frac{d}{2}}\, \sigma_H^d(\mathrm{d}(x_1, \dots, x_d))\, \mu_{d-1}(\mathrm{d}H),$$

where h *denotes the distance from* H *to the origin* $\boldsymbol{o}$*.*

Lastly, we will also need the so-called *slice integration formula*, see [10, Theorem A4] where also its proof can be found. It is yet another geometric integral transformation, that transform the integration over the $(d-1)$-dimensional unit sphere to the integration over $(d-2)$-dimensional spheres perpendicular to a fixed 1-dimensional linear subspace and then integrating these slices up along the interval $[-1, 1]$. Hence, the name.

Lemma 2.3.4 (Slice integration formula) *Let* $f : \mathbb{S}^{d-1} \to \mathbb{R}$ *be a nonnegative measurable function. Then,*

$$\omega_d \int\limits_{\mathbb{S}^{d-1}} f(x)\, \sigma_{d-1}(\mathrm{d}x) = \omega_{d-1} \int\limits_{-1}^{1} \left(1-t^2\right)^{\frac{d-3}{2}} \int\limits_{\mathbb{S}^{d-2}} f\left(\sqrt{1-t^2}\, y, t\right) \sigma_{d-2}(\mathrm{d}y)\, \mathrm{d}t, \qquad (2.32)$$

where $f\left(\sqrt{1-t^2}\, y, t\right) := f\left((\sqrt{1-t^2}\, y_1, \dots, \sqrt{1-t^2}\, y_{d-1}, t)\right)$.

Note that this is a reformulated version with the uniform probability measure on $\mathbb{S}^{d-1}$ and $\mathbb{S}^{d-2}$, respectively, whereas in [10] the result is presented with respect to the unnormalized surface area measures.

The following lemma is an application of the slice integration formula. It describes how the projection of a uniformly distributed random point on the unit sphere onto a 1-dimensional linear subspace is distributed.

Lemma 2.3.5 *Let $L \in G(d,1)$. Then the image measure of the uniform distribution σ_{d-1} on $\mathbb{S}^{d-1}$ under the orthogonal projection P_L has density*

$$f(x) = \pi^{-\frac{1}{2}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} (1-x^2)^{\frac{d-3}{2}}, \qquad x \in [-1,1].$$

Proof. Due to rotational symmetry we can assume that $L = \operatorname{lin}(e_d)$. We denote by F the distribution function of the image measure of σ_{d-1} under the orthogonal transformation P_L and let $x_d \in [-1,1]$. Using the slice integration formula (2.32), we obtain

$$\begin{aligned} F(x_d) &= \sigma_{d-1}\left(\left\{u \in \mathbb{S}^{d-1} : P_L(u) \in [-1, x_d]\right\}\right) \\ &= \int_{\mathbb{S}^{d-1}} \mathbb{1}\left\{P_L(u) \in [-1, x_d]\right\} \, \sigma_{d-1}(\mathrm{d}u) \\ &= \frac{\omega_{d-1}}{\omega_d} \int_{-1}^{x_d} (1-t^2)^{\frac{d-3}{2}} \int_{\mathbb{S}^{d-2}} \sigma_{d-2}(\mathrm{d}y) \, \mathrm{d}t \\ &= \frac{\omega_{d-1}}{\omega_d} \int_{-1}^{x_d} (1-t^2)^{\frac{d-3}{2}} \, \mathrm{d}t. \end{aligned}$$

Differentiating with respect to x_d completes the proof. □

2.4 Random measures and random sets

The purpose of this section is to support us with the necessary tools from *random measure theory* and *random set theory* needed to keep this work self-contained. We follow along the lines of the books of Kallenberg [63] and Molchanov [80]. We will apply the content of this section to Chapter 4.

A *random compact set* is a random variable X, defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, which takes values in the measurable space $\mathcal{C}^d$ equipped with the Borel-σ-field generated by the *Fell*-Topology, for details see [80, Appendix B]. The functional $C_X : \mathcal{K}^d \to [0,1]$, $K \mapsto \mathbb{P}(X \subset K)$, is the so-called *containment functional.* They are valuable tools in describing the distribution of compact convex sets.

Theorem 2.4.1 (Distribution of convex compact sets) *The distribution of a random convex compact set X in $\mathbb{R}^d$ is uniquely determined by the values of the containment functional $C_X(K)$ for $K \in \mathcal{K}^d$. Moreover, it suffices to consider all K being convex polytopes.*

The proof of this statement can be found in [80, Theorem 7.8]. The containment functionals are also very useful, when it comes to formulating *distributional convergence of a sequence of random compact convex sets* $(X_n)_{n\in\mathbb{N}}$ to another random compact convex set X_0. In order to do so, we have to extend the notion of the containment function. For an open set A we define the containment functional by $C_X(A) := \sup\{C_X(K) : K \in \mathcal{K}^d, K \subset A\}$, see [80, Definition 1.32]. Then, we have

Theorem 2.4.2 *A sequence $(X_n)_{n\in\mathbb{N}}$ of random compact convex sets converges weakly to a random closed set X_0 if*

$$C_{X_n}(K) \to C_{X_0}(K), \quad \text{as } n \to \infty, \tag{2.33}$$

for every $K \in \mathcal{K}^d$ with $C_{X_0}(K) = C_{X_0}(\mathrm{int}K)$.

The proof can be found in [80, Theorem 7.12]. We will denote this convergence by $X_n \xrightarrow{\mathrm{w}} X_0$. Also the following lemma will be needed:

Lemma 2.4.3 *Let $(K_n)_{n\in\mathbb{N}_0} \subset \mathcal{K}^d$ be a sequence of deterministic compact convex sets such that $K_n \xrightarrow{\mathrm{d}_H} K_0$ as $n \to \infty$. Then, we have for every $x \in \mathbb{R}^d \setminus \partial K_0$ that $\lim_{n\to\infty} \mathbb{1}_{K_n}(x) = \mathbb{1}_{K_0}(x)$.*

Proof. Assume first $x \notin K_0$. Then there is a hyperplane H such that x and K_0 are contained in different open half-spaces H^+ and H^- defined by H. For sufficiently small $\varepsilon > 0$, the ε-neighborhood of K_0 is still contained in H^-. Hence, for sufficiently large n, we have $K_n \subset H^-$ and at the same time $x \in H^+$. It follows that $\mathbb{1}_{K_n}(x) = 0 = \mathbb{1}_{K_0}(x)$ for sufficiently large n, which proves the claim. Suppose now that x is in the interior of K_0 and without loss of generality that $x = \boldsymbol{o}$. We argue by contradiction and assume that $0 \notin K_n$ for infinitely many n. By the hyperplane separation theorem, there is a unit vector $\theta_n \in \mathbb{R}^d$ such that $\langle z, \theta_n \rangle < 0$ for all $z \in K_n$. By passing to a subsequence we may assume that $\theta_n \to \theta$, as $n \to \infty$, for some unit vector $\theta \in \mathbb{R}^d$. Since the origin $\boldsymbol{o}$ is in the interior of K_0, we can find $\varepsilon > 0$ such that $\varepsilon\theta \in K_0$. The distance between $\varepsilon\theta$ and K_n is bounded from below by the distance between $\varepsilon\theta$ and the half-space $\{z \in \mathbb{R}^d : \langle z, \theta \rangle < 0\}$ containing K_n. Thus, the distance between $\varepsilon\theta$ and K_n is at least $\langle \varepsilon\theta, \theta_n \rangle$ which is larger than $\varepsilon/2$, for sufficiently large n. Therefore, $\varepsilon\theta \in K_0$ but at the same time $\varepsilon\theta$ is not contained in the $\varepsilon/2$-neighborhood of K_n, a contradiction to the assumption $K_n \overset{\mathrm{d}_H}{\to} K_0$, as $n \to \infty$. □

Furthermore, when dealing with weak convergence, we shall frequently make use of the following *Skorokhod representation theorem*, which can be found in [64, Theorem 4.30]:

Theorem 2.4.4 (Skorokhod representation theorem) *Let $(X_n)_{n \in \mathbb{N}_0}$ be a sequence of random elements with values in a separable metric space and let X_n converge weakly to X_0 as $n \to \infty$. Then, there exist random elements $(X'_n)_{n \in \mathbb{N}_0}$ defined on a common probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ such that X'_n has the same distribution as X_n for all $n \in \mathbb{N}_0$, and X'_n converges to X'_0 $\mathbb{P}'$-almost surely.*

Now we go over to introduce random measures. Let S be a *locally compact metric space*. We denote by $\mathcal{M}_S$ and $\mathcal{N}_S$, respectively, the *space of locally finite measures* and *locally finite integer-valued measures on* S, respectively. We supply $\mathcal{M}_S$ and $\mathcal{N}_S$ with the *topology of vague convergence* and recall that a sequence $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}_S$ *vaguely converges to a measure* $\mu \in \mathcal{M}_S$ provided that

$$\lim_{n \to \infty} \int_S f(x)\, \mu_n(\mathrm{d}x) = \int_S f(x)\, \mu(\mathrm{d}x),$$

for all continuous functions $f : S \to [0, \infty)$ with compact support. We shall write $\mu_n \overset{\mathrm{v}}{\to} \mu$ in such a case. It is known from [63, Lemma 15.7.4] that $\mathcal{N}_S$ is a vaguely closed subset of $\mathcal{M}_S$. Furthermore, the vague topology turns $\mathcal{M}_S$ and $\mathcal{N}_S$ into Polish spaces, i.e., separable and completely metrizable topological spaces (see [63, Lemma 15.7.7]).

A *random measure* (or *point process*, respectively) is a random variable, defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and taking values in $\mathcal{M}_S$ (or $\mathcal{N}_S$, respectively). As in the case of random sets, we denote the *weak convergence of a sequence of random measures* $(\zeta_n)_{n\in\mathbb{N}}$ on S to another random measure ζ, as $n \to \infty$, by $\zeta_n \overset{\mathrm{w}}{\to} \zeta$.

The following lemma plays a crucial role in Chapter 4. We set $\mathcal{N} := \mathcal{N}_{\mathbb{R}^d \cup \{\infty\} \setminus \{\boldsymbol{o}\}}$ to be the *space of locally finite integer-valued measures on the one-point compactification* $\mathbb{R}^d \cup \{\infty\}$ of $\mathbb{R}^d$.

Lemma 2.4.5 *Assume that $(\zeta_n)_{n\in\mathbb{N}_0}$ is a sequence of deterministic measures in $\mathcal{N}$ and suppose that $\zeta_n \overset{\mathrm{v}}{\to} \zeta_0$, as $n \to \infty$. Suppose further that ζ_0 satisfies $\zeta_0(\{\infty\}) = 0$ and that the following two conditions are satisfied:*

(a) $\zeta_0(H) > 0$ for every open half-space $H \subset \mathbb{R}^d$ such that $\boldsymbol{o} \in \partial H$,

(b) the atoms of ζ_0 are in general position, that is, no $k+2$ atoms of ζ_0 lie in the same k-dimensional affine subspace for all $k = 1, \dots, d-1$.

Then, $\mathrm{conv}(\zeta_0)$ *is a convex polytope containing the origin $\boldsymbol{o}$ in its interior. Moreover, as $n \to \infty$, we have the convergence*

$$\mathrm{conv}(\zeta_n) \overset{\mathrm{d}_H}{\to} \mathrm{conv}(\zeta_0)$$

on the space $\mathcal{K}^d$ as well as the convergence of the f-vectors

$$\mathbf{f}\left(\mathrm{conv}(\zeta_n)\right) \to \mathbf{f}\left(\mathrm{conv}(\zeta_0)\right).$$

Proof. By the local finiteness of ζ_0 and since $\zeta_0(\{\infty\}) = 0$, the set of atoms of ζ_0 is bounded. Hence $\mathrm{conv}(\zeta_0)$ is a compact convex set. We show that it is in fact a polytope. By the supporting hyperplane theorem, i.e., Theorem 2.2.1, assumption (a) implies that the origin $\boldsymbol{o}$ is an interior point of $\mathrm{conv}(\zeta_0)$. Thus, there exists an open ball $\mathbb{B}^d(\boldsymbol{o}, 2r) \subset \mathrm{conv}(\zeta_0)$ with $r > 0$. Since $\mathbb{B}^d(\boldsymbol{o}, r)$ is open, the set $\mathbb{R}^d \cup \{\infty\} \setminus \mathbb{B}^d(\boldsymbol{o}, r)$ is compact and thus ζ_0 has only a finite number of atoms, say $A_1, \dots, A_k$ outside of $\mathbb{B}^d(\boldsymbol{o}, r)$. We claim that

$$\mathrm{conv}(\zeta_0) = \mathrm{conv}(\{A_1, \dots, A_k\}) \tag{2.34}$$

and, in particular, $\mathrm{conv}(\zeta_0)$ is a convex polytope. To prove (2.34), it suffices to show $\mathbb{B}^d(\boldsymbol{o}, r) \subset \mathrm{conv}(\{A_1, \ldots, A_k\})$. Assume that $x \in \mathbb{B}^d(\boldsymbol{o}, r)$ but $x \notin \mathrm{conv}(\{A_1, \ldots, A_k\})$. By the separating hyperplane theorem, i.e., Theorem 2.2.2, there exists an open half-space H such that $x \notin H$ and $\mathrm{conv}(\{A_1, \ldots, A_k\}) \subset H$. After applying an orthogonal transformation, we may assume that $H = \{y \in \mathbb{R}^d \colon y_1 < a\}$, where y_1 is the first coordinate of $y \in \mathbb{R}^d$. Since $x \notin H$, its first coordinate satisfies $x_1 \geq a$, hence $a < r$. Now,

$$\mathrm{conv}(\zeta_0) \subset \mathrm{conv}(\{A_1, \ldots, A_k\} \cup \mathbb{B}^d(\boldsymbol{o}, r)) \subset \mathrm{conv}(\{H \cup \mathbb{B}^d(\boldsymbol{o}, r)\}) \subset \{y \in \mathbb{R}^d \colon y_1 \leq r\},$$

which is in contradiction with $\mathbb{B}^d(\boldsymbol{o}, 2r) \subset \mathrm{conv}(\zeta_0)$. This proves (2.34).

By Proposition 3.13 in [93], the assumed vague convergence of ζ_n to ζ_0, as $n \to \infty$, implies that for sufficiently large n, each ζ_n has exactly k atoms, say $\{A_1^{(n)}, \ldots, A_k^{(n)}\}$, in $\mathbb{R}^d \setminus \mathrm{cl}(\mathbb{B}^d(\boldsymbol{o}, r))$ and

$$\left\{A_1^{(n)}, \ldots, A_k^{(n)}\right\} \xrightarrow{\mathrm{d}_H} \{A_1, \ldots, A_k\}, \tag{2.35}$$

as $n \to \infty$, on the space $\mathcal{C}^d$. Since the mapping $\mathrm{conv} : \mathcal{C}^d \to \mathcal{C}^d$ is continuous with respect to the Hausdorff distance (see [99, Theorem 12.3.5]), we also have that

$$\mathrm{conv}\left(\left\{A_1^{(n)}, \ldots, A_k^{(n)}\right\}\right) \xrightarrow{\mathrm{d}_H} \mathrm{conv}(\{A_1, \ldots, A_k\}),$$

as $n \to \infty$, on the space $\mathcal{C}^d$ as well as on the space $\mathcal{K}^d$. Now, since $\mathbb{B}^d(\boldsymbol{o}, 2r) \subset \mathrm{conv}(\zeta_0) = \mathrm{conv}(\{A_1, \ldots, A_k\})$, this yields that $\mathbb{B}(\boldsymbol{o}, r) \subset \mathrm{conv}(\{A_1^{(n)}, \ldots, A_k^{(n)}\})$ for large n and therefore,

$$\mathrm{conv}(\zeta_n) = \mathrm{conv}\left(\left\{A_1^{(n)}, \ldots, A_k^{(n)}\right\}\right), \tag{2.36}$$

for all sufficiently large n, which can be proved in the same way as (2.34).

Assumption (b) implies that the points of $\{A_1, \ldots, A_k\}$ are in general position, which in conjunction with (2.35) yields that also the points of $\{A_1^{(n)}, \ldots, A_k^{(n)}\}$ are in general position for sufficiently large n. Hence, (2.35) implies that for each $k \in \{0, 1, \ldots, d-1\}$ the number of k-dimensional faces of $\mathrm{conv}(\{A_1^{(n)}, \ldots, A_k^{(n)}\})$ is the same as the number of k-dimensional faces of $\mathrm{conv}(\{A_1, \ldots, A_k\})$ for all $k \in \{0, \ldots, d-1\}$ and large enough n. This completes the proof of the lemma. □

2.5 Poisson point processes

As in Section 2.4 we consider a locally compact metric space S. Let $\mathcal{S}$ be a *σ-field* thereon, making $(S, \mathcal{S})$ into a *measurable space*. We denoted by $\mathcal{N}_S$ the set of locally finite integer-valued measures on S and equip it with the *σ-field* $\mathfrak{N}_S$ *generated by all subset of the form*

$$\{\mu \in \mathcal{N}_S : \mu(B) = k\},$$

for any $B \in \mathcal{S}$ and $k \in \mathbb{N}_0$, i.e., the smallest σ-field on $\mathcal{N}_S$ such that the maps $\mu \mapsto \mu(B)$ are measurable for all $B \in \mathcal{S}$. Furthermore, let $(\Omega, \mathcal{A}, \mathbb{P})$ be an underlying probability space.

Definition 2.5.1 (Point process) A point process on S is a random variable ζ defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ taking values in the measurable space $(\mathcal{N}_S, \mathfrak{N}_S)$.

If ζ is a *point process*, we denote by $\zeta(B)$ the mapping $\omega \mapsto \zeta(\omega, B)$ for any $B \in \mathcal{S}$, that is, the *number of points of the point process* ζ in the set B. A very important characteristic of point processes is the mean number of points lying in a given measurable subset of S. This generates a measure on $(S, \mathcal{S})$, namely, the so-called *intensity measure*.

Definition 2.5.2 (Intensity measure) The intensity measure of a point process ζ on S is the measure ν defined by

$$\begin{aligned} \nu : \mathcal{S} &\to [0, \infty] \\ B &\mapsto \mathbb{E}(\zeta(B)). \end{aligned}$$

The intensity measure is an important tool in calculations regarding point processes. We have

Theorem 2.5.3 (Campbell's formula) *Let ζ be a point process on $(S, \mathcal{S})$ with intensity measure ν. Let $u : S \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a measurable function. Then $\int_S u(x)\, \zeta(\mathrm{d}x)$ is a random variable and*

$$\mathbb{E}\left(\int_S u(x)\, \zeta(\mathrm{d}x)\right) = \int_S u(x)\, \nu(\mathrm{d}x) \tag{2.37}$$

whenever u is nonnegative or $\int_S |u(x)|\, \nu(\mathrm{d}x) < \infty$.

Let us now introduce *Poisson point processes*. We say that a measure ν is s-finite if it is a countable sum of finite measures. Note, that σ-finiteness implies s-finiteness.

Definition 2.5.4 (Poisson point process) Let ν be an s-finite measure on S. A Poisson point process with intensity measure ν is a point process Π on S fulfilling that

(a) $\Pi(B)$ is Poisson distributed with parameter $\nu(B)$, for every $B \in \mathcal{S}$, that is

$$\mathbb{P}(\Pi(B) = k) = \frac{\nu(B)^k}{k!}\, e^{-\nu(B)}, \text{ for all } k \in \mathbb{N}_0.$$

(b) the random variables $\Pi(B_1), \ldots, \Pi(B_n)$ are independent for every $n \in \mathbb{N}$ and all pairwise disjoint sets $B_1, \ldots, B_n \in \mathcal{S}$.

We remark that a Poisson point process Π can almost surely be represented as $\Pi = \sum_{i=1}^{\kappa} \delta_{x_i}$ with random points $x_1, x_2, \ldots \in S$ and a Poisson random variable κ with mean $\nu(S)$ (which is interpreted as $+\infty$ if ν is not a finite measure). Here, and in the rest of this thesis, δ_x stands for the *Dirac measure centered at the point* $x \in S$.

For $k \in \mathbb{N}$, denote by $\Pi^k_{\neq}$ the k-tuples of distinct points charged by the Poisson point process Π. It is a crucial fact that the Poisson point process Π satisfies the *multivariate Mecke equation.*

Theorem 2.5.5 (Multivariate Mecke equation) *Let Π be a Poisson point process on S with s-finite intensity measure ν. Then, for every $k \in \mathbb{N}$ and every nonnegative measurable function $f : S^k \times \mathcal{N}_S \to [0, +\infty]$,*

$$\mathbb{E} \sum_{(x_1,\ldots,x_k)\in\Pi^k_{\neq}} f(x_1, \ldots, x_k; \Pi) = \int_{S^k} \mathbb{E} f\left(x_1, \ldots, x_k; \Pi + \sum_{i=1}^{k} \delta_{x_i}\right) \prod_{i=1}^{k} \nu(\mathrm{d}x_i). \tag{2.38}$$

2.6 Analysis

In this section we will gather a few additional lemmas that will be needed later one, but do not fit thematically with the other sections.

We introduce one more lemma that is needed in the discussions surrounding beta- and beta′-type polytopes, namely, in Section 3.4. It was first stated in [21] without proof. A proof was added in [26]. The lemma was later restated in a slightly corrected form in [61].

Lemma 2.6.1 *Let $h : (0,1) \to \mathbb{R}$ be a nonnegative measurable function such that $0 < \int_0^1 h(s)\,\mathrm{d}s < \infty$, let $g : (0,1) \to \mathbb{R}$ be a linear function with negative slope and root $s^* \in (0,1)$, and let $L : (0,1) \to \mathbb{R}$ be nonnegative and strictly concave on $(0,1)$. Then, for any $d > 1$,*

$$\int_0^1 h(s)g(s)L(s)^{d-1}\,\mathrm{d}s > \int_0^1 h(s)g(s)\ell(s)^{d-1}\,\mathrm{d}s, \tag{2.39}$$

where $\ell(s) = \frac{s}{s^*}\,L(s^*)$.

Proof. We start by exploiting the nonnegativity and strict concavity of L. For $s \in (0, s^*)$, it implies that

$$L(s) = L\left(\frac{s}{s^*}s^*\right) > \frac{s}{s^*}\,L(s^*) = \ell(s), \tag{2.40}$$

while for $s \in (s^*, 1)$, it gives

$$L(s) < \frac{s}{s^*}\,L(s^*) = \ell(s). \tag{2.41}$$

Since g has a negative slope, it is positive on $(0, s^*)$ and negative on $(s^*, 1)$. Splitting the integral on the left hand side of (2.39) at the point s^* and using (2.40) and (2.41), respectively, yields

$$\begin{aligned}\int_0^1 h(s)g(s)L(s)^{d-1}\,\mathrm{d}s &= \int_0^{s^*} h(s)g(s)L(s)^{d-1}\,\mathrm{d}s + \int_{s^*}^1 h(s)g(s)L(s)^{d-1}\,\mathrm{d}s \\ &> \int_0^{s^*} h(s)g(s)\ell(s)^{d-1}\,\mathrm{d}s + \int_{s^*}^1 h(s)g(s)\ell(s)^{d-1}\,\mathrm{d}s\end{aligned}$$

$$= \int_0^1 h(s)g(s)\ell(s)^{d-1}\,\mathrm{d}s.$$

This completes the argument. □

We go on by introducing definitions and lemmas regarding Chapter 5. We begin with the notion of the *covariogram* and *generalized covariogram* of a set. Recall, that the covariogram of a Lebesgue measurable set $K \subset \mathbb{R}^d$ is the map $g_K : \mathbb{R}^d \to \mathbb{R}_0^+$, $y \mapsto \mathrm{Vol}_d(K \cap (y + K))$. We extend this notion

Definition 2.6.2 Let $K \in \mathbb{R}^d$ be a Lebesgue measurable set of finite Lebesgue measure. The generalized covariogram of K is the map $g_K : (\mathbb{R}^d)^{d-1} \to \mathbb{R}_0^+$ defined by

$$g_K(y) = g_K(y_1, \ldots, y_{d-1}) = \mathrm{Vol}_d(K \cap (y_1 + K) \cap \ldots \cap (y_{d-1} + K)).$$

Note that the generalized covariogram can be written as an integral over indicator functions of the set K, i.e.,

$$g_K(y) = \mathrm{Vol}_d(K \cap (y_1 + K) \cap \ldots \cap (y_{d-1} + K)) = \int_{\mathbb{R}^d} \mathbb{1}_K(x) \prod_{i=1}^{d-1} \mathbb{1}_K(x - y_i)\,\mathrm{d}x, \quad (2.42)$$

and that it is symmetric with respect to permutations of the vectors $y_1, \ldots, y_{d-1}$, i.e.,

$$g_K(y_1, \ldots, y_{d-1}) = g_K(y_{\sigma(1)}, \ldots, y_{\sigma(d-1)}) \quad (2.43)$$

for any permutation $\sigma : [d-1] \to [d-1]$. Observe that $g_K(\boldsymbol{o}) = \mathrm{Vol}_d(K)$ and that $g_K(y) \leq \mathrm{Vol}_d(K)$.

Lemma 2.6.3 *Let $K \subset \mathbb{R}^d$ be Lebesgue measurable and let g_K be its generalized covariogram. Let $\tilde{y}, \tilde{z} \in \mathbb{R}^d$. Define $y, z \in \left(\mathbb{R}^d\right)^{d-1}$ by $y := (\tilde{y}, \boldsymbol{o}, \ldots, \boldsymbol{o})$ and $z := (\tilde{z}, \boldsymbol{o}, \ldots, \boldsymbol{o})$. Then,*

$$|g_K(y) - g_K(z)| \leq g_K(\boldsymbol{o}) - g_K(y - z).$$

Proof. Let $A_1, A_2, A_3 \subset \mathbb{R}^d$ be Lebesgue measurable sets. We have

$$\begin{aligned} \mathrm{Vol}_d(A_1 \cap A_2) - \mathrm{Vol}_d(A_1 \cap A_3) &\leq \mathrm{Vol}_d(A_1 \cap A_2) - \mathrm{Vol}_d(A_1 \cap A_2 \cap A_3) \\ &= \mathrm{Vol}_d((A_1 \cap A_2) \setminus (A_1 \cap A_2 \cap A_3)) \end{aligned}$$

$$\leq \mathrm{Vol}_d(A_2 \setminus (A_2 \cap A_3))$$
$$= \mathrm{Vol}_d(A_2) - \mathrm{Vol}_d(A_2 \cap A_3).$$

Set now $A_1 = K$, $A_2 = \tilde{y} + K$ and $A_3 = \tilde{z} + K$. Then

$$\begin{aligned} g_K(y) - g_K(z) &= \mathrm{Vol}_d(K \cap (\tilde{y} + K)) - \mathrm{Vol}_d(K \cap (\tilde{z} + K)) \\ &\leq \mathrm{Vol}_d(\tilde{y} + K) - \mathrm{Vol}_d((\tilde{y} + K) \cap (\tilde{z} + K)) \\ &= \mathrm{Vol}_d(K) - \mathrm{Vol}_d(K \cap (\tilde{y} - \tilde{z} + K)) \\ &= g_K(\boldsymbol{o}) - g_K(y - z). \end{aligned}$$

Due to $g_K(z - y) = g_K(y - z)$, the inequality follows for $|g_K(y) - g_K(z)|$. □

Lemma 2.6.4 *Let $K \subset \mathbb{R}^d$ be Lebesgue measurable and let g_K be its generalized covariogram. Let $\tilde{y} \in \mathbb{R}^d$ and define $y \in \left(\mathbb{R}^d\right)^{d-1}$ by $y := (\tilde{y}, \boldsymbol{o}, \ldots, \boldsymbol{o})$. Then,*

$$g_K(\boldsymbol{o}) - g_K(y) = \frac{1}{2} \int\limits_{\mathbb{R}^d} |\mathbb{1}_K(x + \tilde{y}) - \mathbb{1}_K(x)| \, \mathrm{d}x.$$

Proof. Using basic properties of the indicator function of a set, we have

$$\begin{aligned} &\int\limits_{\mathbb{R}^d} |\mathbb{1}_K(x + \tilde{y}) - \mathbb{1}_K(x)| \, \mathrm{d}x = \int\limits_{\mathbb{R}^d} (\mathbb{1}_K(x + \tilde{y}) - \mathbb{1}_K(x))^2 \, \mathrm{d}x \\ &= \int\limits_{\mathbb{R}^d} \mathbb{1}_K(x + \tilde{y})^2 \, \mathrm{d}x + \int\limits_{\mathbb{R}^d} \mathbb{1}_K(x)^2 \, \mathrm{d}x - 2 \int\limits_{\mathbb{R}^d} \mathbb{1}_K(x + \tilde{y}) \mathbb{1}_K(x) \, \mathrm{d}x \\ &= 2 \, \mathrm{Vol}_d(K) - 2 \, \mathrm{Vol}_d(K \cap (\tilde{y} + K)) = 2 \left(g_K(\boldsymbol{o}) - g_K(y) \right), \end{aligned}$$

where the second to last equality follows from integrating $\mathbb{1}_K(x)$ over $\mathbb{R}^M + \tilde{y}$ instead of $\mathbb{1}_K(x + \tilde{y})$ over $\mathbb{R}^d$ in the first integral. □

The next two propositions are taken from [44], where one may also find their proof.

Proposition 2.6.5 *Let $U \subseteq \mathbb{R}^d$ be open and let $f \in L^1(U)$. Then, the following statements are equivalent:*

(i) $V(f, U) < \infty$,

(ii) $V_u(f, U) < \infty$ for all $u \in \mathbb{S}^{d-1}$,

(iii) $V_{e_i}(f, U) < \infty$ for all vectors e_i of the canonical basis of $\mathbb{R}^d$.

Additionally,

$$\frac{1}{d}V(f,U) \le \frac{1}{d}\sum_{i=1}^{d} V_{e_i}(f,U) \le \sup_{u\in\mathbb{S}^{d-1}} V_u(f,U) \le V(f,U)$$

and

$$V(f,U) = \frac{1}{2\kappa_{d-1}} \int\limits_{\mathbb{S}^{d-1}} V_u(f,U)\lambda_{\mathbb{S}^{d-1}}(\mathrm{d}u)$$

hold.

The second proposition elaborates a method to calculate directional variations of a function f in U by integrals of difference quotients.

Proposition 2.6.6 *Let $u \in \mathbb{S}^{d-1}$ and $f \in L^1(\mathbb{R}^M)$. Then,*

$$\int\limits_{\mathbb{R}^M} \frac{|f(x+ru)-f(x)|}{|r|}\,\mathrm{d}x \le V_u(f),$$

for all $r \neq 0$, and

$$\lim_{r\to 0} \int\limits_{\mathbb{R}^M} \frac{|f(x+ru)-f(x)|}{|r|}\,\mathrm{d}x = V_u(f).$$

We use the two previous propositions to show the following two.

Proposition 2.6.7 *Let $K \subset \mathbb{R}^d$ be Lebesgue measurable, let g_K be its generalized covariogram and let $u \in \mathbb{S}^{d-1}$. Define $y := (u, \boldsymbol{o}, \dots, \boldsymbol{o})$ and let $r \in \mathbb{R}$ with $r \neq 0$. The following statements are equivalent:*

(i) K has finite directional variation $V_u(K)$,

(ii) the derivative $\lim_{r\to 0} \frac{g_K(\boldsymbol{o})-g_K(ry)}{|r|}$ exists and is finite,

(iii) the function $g_K^u : r \mapsto g_K(ry)$ is Lipschitz.

Additionally, the Lipschitz constant of g_K^u is

$$\mathrm{Lip}(g_K^u) = \lim_{r\to 0} \frac{g_K(\boldsymbol{o}) - g_K(ry)}{|r|} = \frac{1}{2}V_u(K).$$

Proof. Lemma 2.6.4 implies

$$\frac{g_K(\boldsymbol{o}) - g_K(ry)}{|r|} = \frac{1}{2}\int_{\mathbb{R}^d} \frac{|\mathbb{1}_K(x+ru) - \mathbb{1}_K(x)|}{|r|}\,\mathrm{d}x.$$

Applying Proposition 2.6.6 with $f = \mathbb{1}_K$, we obtain the equivalence of (i) and (ii) as well as the formula

$$\lim_{r\to 0} \frac{g_K(\boldsymbol{o}) - g_K(ry)}{|r|} = \frac{1}{2}V_u(K).$$

We show now that (i) implies (iii). By Lemma 2.6.3 we get for $r, s \in \mathbb{R} \setminus \{\boldsymbol{o}\}$, that

$$\begin{aligned}
|g_K(ry) - g_K(sy)| &\le g_K(\boldsymbol{o}) - g_K((r-s)y) \\
&= \frac{1}{2}\int_{\mathbb{R}^d} |\mathbb{1}_K(x+(r-s)u) - \mathbb{1}_K(x)|\,\mathrm{d}x \\
&= \frac{1}{2}|r-s|\int_{\mathbb{R}^d} \frac{|\mathbb{1}_K(x+(r-s)u) - \mathbb{1}_K(x)|}{|r-s|}\,\mathrm{d}x \\
&\le \frac{1}{2}V_u(K)|r-s|,
\end{aligned}$$

where the last inequality stems again from applying Proposition 2.6.6 with $f = \mathbb{1}_K$. Hence, $\mathrm{Lip}(g_K^u) \le \frac{1}{2}V_u(K)$.

It remains to show that (iii) implies (i). For all $r \neq 0$ we have

$$\mathrm{Lip}(g_K^u) \ge \frac{g_K(\boldsymbol{o}) - g_K(ry)}{|r|} = \frac{1}{2}\int_{\mathbb{R}^d} \frac{|\mathbb{1}_K(x+ru) - \mathbb{1}_K(x)|}{|r|}\,\mathrm{d}x.$$

By Proposition 2.6.6 the right-hand side converges to $\frac{1}{2}V_u(K)$, as r goes to 0. Hence, K has finite directional variation in the direction of u and $\mathrm{Lip}(g_K^u) \ge \frac{1}{2}V_u(K)$.

Subsequently, (i) and (iii) are equivalent and

$$\mathrm{Lip}(g_K^u) = \frac{1}{2}V_u(K)$$

holds. □

Note that for g_K^u, i.e., the restriction of the generalized covariogram to the first argument along the direction $u \in \mathbb{S}^{d-1}$, the right derivative in 0 can be expressed as:

$$(g_K^u)'(0^+) = \lim_{r \to 0^+} \frac{g_K(ry) - g_K(\boldsymbol{o})}{r} = -\lim_{r \to 0} \frac{g_K(\boldsymbol{o}) - g_K(ry)}{|r|}. \tag{2.44}$$

Proposition 2.6.8 *Let $K \subset \mathbb{R}^d$ be Lebesgue measurable and g_K be its generalized covariogram. Let $u \in \mathbb{S}^{d-1}$, $r \in \mathbb{R}$ with $r \neq 0$, and define $y := (u, \boldsymbol{o}, \dots, \boldsymbol{o})$. The following two statements are equivalent:*

(i) K has finite perimeter $\operatorname{Per}(K)$*,*

(ii) for all $u \in \mathbb{S}^{d-1}$ the derivative $(g_K^u)'(0^+) = \lim_{r \to 0} \frac{g_K(ry) - g_K(\boldsymbol{o})}{r}$ exists and is finite.

Additionally,

$$\operatorname{Per}(K) = -\frac{1}{\kappa_{d-1}} \int_{\mathbb{S}^{d-1}} (g_K^u)'(0^+)\, \lambda_{\mathbb{S}^{d-1}}(\mathrm{d}u). \tag{2.45}$$

Proof. Proposition 2.6.7 and (2.44) yield the identity

$$(g_K^u)'(0^+) = \lim_{r \to 0^+} \frac{g_K(ry) - g_K(\boldsymbol{o})}{r} = -\frac{1}{2} V_u(K).$$

The equivalence of (i) and (ii), as well as (2.45), derive from applying Proposition 2.6.5 with $f = \mathbb{1}_K$ to this identity.

□

Remark 2.6.9 It is known, that if $K \subset \mathbb{R}^M$ is a convex body, then $V_u(K) = 2\operatorname{Vol}_{d-1}(P_{u^\perp} K)$ holds for its directional variation. This result can be found in [98, Eq. (10.1)] and is restated in [44].

Chapter 3

Beta- and beta′-type polytopes

This chapter revolves around beta- and beta′-type polytopes and Poisson-beta- and Poisson-beta′-type polytopes, respectively. In particular this means that they are generated either by a fixed number or a Poisson distributed number of independently and identically distributed random points chosen according to either a beta- or a beta′-type distribution. More precisely, three types of differently generated random polytopes are investigated. Namely, random polytopes that are the convex hull of such random points, random polytopes which are the symmetric convex hull of such random points and random polytopes which are the convex hull of such random points and the origin.

Our attention is focused towards expected values of a number of functionals: The so-called *T-functional*, the *facet number*, the *Lebesgue volume* and the *intrinsic volumes*, encompassing also the *surface area* and the *mean width*. In the special case of the uniform distribution in the ball, which is just a particular beta-type distribution, we are also able to handle the *vertex number*. We give explicit formulae for the expectation of these quantities depending on the dimension, the parameters of the distribution and the number of random points or the intensity of the Poisson-process, respectively.

Furthermore, we show monotonicity of expected facet numbers for these types of polytopes in the number of random points or intensity of the Poisson random variable, respectively.

3.1 Preliminaries

We are considering the following two families of rotationally symmetric probability distributions. The *beta-type distribution* $\mu_{d,\beta,\sigma}$, having density

$$f_{d,\beta,\sigma}(x) := c_{d,\beta,\sigma}\left(1-\frac{\|x\|^2}{\sigma^2}\right)^{\beta}, \quad \|x\|\le\sigma, \quad \beta>-1, \quad \sigma>0, \tag{3.1}$$

and the *beta′-type distribution* $\tilde{\mu}_{d,\beta,\sigma}$, with density

$$\tilde{f}_{d,\beta,\sigma}(x) := \tilde{c}_{d,\beta,\sigma}\left(1+\frac{\|x\|^2}{\sigma^2}\right)^{-\beta}, \quad x\in\mathbb{R}^d, \quad \beta>\frac{d}{2}, \quad \sigma>0, \tag{3.2}$$

with normalizing constants

$$c_{d,\beta,\sigma} = \sigma^d\,\frac{\Gamma\left(\frac{d}{2}+\beta+1\right)}{\pi^{\frac{d}{2}}\Gamma(\beta+1)} \qquad \text{and} \qquad \tilde{c}_{d,\beta,\sigma} = \sigma^d\,\frac{\Gamma(\beta)}{\pi^{\frac{d}{2}}\Gamma\left(\beta-\frac{d}{2}\right)}. \tag{3.3}$$

and distribution functions $F_{d,\beta,\sigma}$ and $\tilde{F}_{d,\beta,\sigma}$, respectively. Note that most of the time we assume that $\sigma=1$, in which case we drop the σ from the notation, i.e., $\mu_{d,\beta}:=\mu_{d,\beta,1}$, $f_{d,\beta}:=f_{d,\beta,1}$, $c_{d,\beta}:=c_{d,\beta,1}$, $F_{d,\beta}:=F_{d,\beta,1}$ and similarly for the beta′-case. So let from now on $\sigma=1$, except if it is stated otherwise. We also consider the *uniform distribution on the unit sphere* $\mathbb{S}^{d-1}$, defined via

$$\sigma_{d-1}(A) := \kappa_d^{-1}\operatorname{Vol}_d(\{tx : x\in A, t\in[0,1]\}), \quad A\subset\mathbb{S}^{d-1} \text{ measurable}, \tag{3.4}$$

which presents itself as a limiting case of the beta-type distribution, see Proposition 3.1.2. Together with the *standard Gaussian distribution*, given by the density

$$\phi(x) := (2\pi)^{-\frac{d}{2}}\exp\left(-\frac{\|x\|^2}{2}\right), \quad x\in\mathbb{R}^d, \tag{3.5}$$

these are the distributions characterized by Ruben and Miles [95], i.e., the Gaussian, the beta-type and the beta′-type distribution all obey that their marginal distributions are again Gaussian, beta-type and beta′-type, respectively (with varying parameters). The uniform distribution on the unit sphere poses an exceptional case, since its marginal distributions are of beta-type. We refrain from considering the Gaussian case, due to the fact that analogous results to ours have already been presented in [62, 60].

We will investigate three differently generated types of random polytopes based on beta- and beta′-type distributions. For $X_1, \dots, X_n \in \mathbb{R}^d$ being independent random points sampled from the beta-type distribution with parameter β, we set

$$P_{n,d}^{\beta} := [X_1, \dots, X_n], \qquad S_{n,d}^{\beta} := [\pm X_1, \dots, \pm X_n], \qquad Q_{n,d}^{\beta} := [\boldsymbol{o}, X_1, \dots, X_n].$$

Likewise, $\tilde{P}_{n,d}^{\beta}$, $\tilde{S}_{n,d}^{\beta}$ and $\tilde{Q}_{n,d}^{\beta}$ denote the similarly generated random polytopes from n independently and identically beta′-type distributed points in $\mathbb{R}^d$ with parameter β. For $X_1, \dots, X_N \in \mathbb{R}^d$ being independent random points sampled from the beta-type distribution with parameter β and N being a Poisson random variable with intensity t, we define the processes

$$\Pi_{t,d}^{\beta} = \sum_{i=1}^{N} \delta_{X_i}, \qquad \Sigma_{t,d}^{\beta} = \sum_{i=1}^{N} \sum_{j \in \{0,1\}} \delta_{(-1)^j X_i}, \qquad \Theta_{t,d}^{\beta} = \Pi_{t,d}^{\beta} + \delta_{\boldsymbol{o}}.$$

By $\mathcal{P}_{t,d}^{\beta} = \operatorname{conv}\left(\Pi_{t,d}^{\beta}\right)$, $\mathcal{S}_{t,d}^{\beta} = \operatorname{conv}\left(\Sigma_{t,d}^{\beta}\right)$ and $\mathcal{Q}_{t,d}^{\beta} = \operatorname{conv}\left(\Theta_{t,d}^{\beta}\right)$ we mean the Poisson-beta-type polytopes generated by these processes. If the points are independently sampled from a beta′-type distribution on $\mathbb{R}^d$, we analogously define the processes $\tilde{\Pi}_{t,d}^{\beta}$, $\tilde{\Sigma}_{t,d}^{\beta}$ and $\tilde{\Theta}_{t,d}^{\beta}$, as well as the Poisson-beta′-type polytopes $\tilde{\mathcal{P}}_{t,d}^{\beta}$, $\tilde{\mathcal{S}}_{t,d}^{\beta}$ and $\tilde{\mathcal{Q}}_{t,d}^{\beta}$ generated by them.

We start our investigation in this section by observing two interesting limiting behaviors for the beta- and beta′ distributions. This shows that of the four classes of distributions identified by Ruben and Miles [95], actually the beta- and beta′ distributions are at the core of them, while the Gaussian distribution and the uniform distribution on the sphere appear in a sense as "extremal" cases of these.

Proposition 3.1.1 *Let $X \in \mathbb{R}^d$ be either beta- or beta′-type distributed with parameter β. Then $\sqrt{2\beta}X$ converges weakly to the standard normal distribution on $\mathbb{R}^d$ as $\beta \to \infty$.*

Proof. From the pointwise convergence of the densities, i.e.,

$$(2\beta)^{-\frac{d}{2}} f_{d,\beta}\left(\frac{x}{\sqrt{2\beta}}\right) = f_{d,\beta,\sqrt{2\beta}}(x) \longrightarrow \phi(x), \qquad \text{as } \beta \to \infty,$$

the assertion follows by Scheffé's Lemma. A modern version of Scheffé's Lemma with streamlined proof can be found in [82]. The result for the beta′-type distribution follows analogously. □

Proposition 3.1.2 *Let $X \in \mathbb{R}^d$ be beta-type distributed with parameter β. Then X converges weakly to the uniform distribution on $\mathbb{S}^{d-1}$ as $\beta \to -1$.*

Proof. Let $(\beta_k)_{k\in\mathbb{N}}$ be a sequence converging to -1 from above. Since the set of probability measures on the unit ball $\mathbb{B}^d$ is compact in the weak topology, there is a weak accumulation point ν of the sequence $(\mu_{d,\beta_k})_{k\in\mathbb{N}}$. Note that, since $\lim_{\beta\downarrow -1}\Gamma(\beta+1) = +\infty$, it is clear that the density $f_{d,\beta}(x)$ converges to 0 uniformly in $x \in K$, for every compact subset K of the open unit ball. It follows that $\mu_{d,\beta_k}(K)$ converges to 0 as $k \to \infty$. Hence, the probability measure ν is concentrated on the sphere $\mathbb{S}^{d-1}$. Since the measures μ_{d,β_k} are invariant under arbitrary orthogonal transformations on $\mathbb{R}^d$, the same is true for the weak limit ν. It follows that ν is a rotationally invariant probability measure on the sphere $\mathbb{S}^{d-1}$, and, hence, $\nu = \sigma_{d-1}$. □

Let us now state some general lemmas about properties of these distributions. The first one is the already mentioned property that the marginal distributions of beta- and beta′-type distributions remain in their respective classes (with different parameters). While the second lemma establishes formulas for the probability content of half-spaces and slabs, respectively, with respect to these distributions.

Lemma 3.1.3 *Let $L \in G(d,k)$ be a k-dimensional linear subspace of $\mathbb{R}^d$.*

(a) If the random variable X has density $f_{d,\beta}$, then $P_L(X)$ has density $f_{k,\beta+\frac{d-k}{2}}$.

(b) If the random variable X has density $\tilde{f}_{d,\beta}$, then $P_L(X)$ has density $\tilde{f}_{k,\beta-\frac{d-k}{2}}$.

Proof. Both, for (a) and (b) it suffices to consider the case $k = d-1$ because then we can argue by induction. Due to the rotational symmetry of the beta- and beta′-type distribution it suffices to consider $L = \{x \in \mathbb{R}^d : x_d = 0\}$, which we identified with $\mathbb{R}^{d-1}$.

Let us prove (a). Fix some $x^* = (x_1^*, \ldots, x_{d-1}^*) \in \mathbb{B}^{d-1}$ with Euclidean norm $r := \|x^*\| \in [0,1)$. The pre-images of x^* under the projection map P_L have the form $x = (x_1^*, \ldots, x_{d-1}^*, x_d)$ with $x_d \in \mathbb{R}$, but since we are interested only in $x \in \mathbb{B}^d$, we obtain the restriction $|x_d| \le \sqrt{1-r^2}$. It holds that $\|x\|^2 = r^2 + x_d^2$. Thus, the density of $P_L(X)$ at x^* is given by

$$c_{d,\beta} \int_{-\sqrt{1-r^2}}^{+\sqrt{1-r^2}} \left(1 - \|x\|^2\right)^\beta \mathrm{d}x_d = c_{d,\beta} \int_{-\sqrt{1-r^2}}^{+\sqrt{1-r^2}} \left(1 - r^2 - x_d^2\right)^\beta \mathrm{d}x_d$$

$$= c_{d,\beta}\left(1-r^2\right)^{\beta}\int\limits_{-\sqrt{1-r^2}}^{+\sqrt{1-r^2}}\left(1-\frac{x_d^2}{1-r^2}\right)^{\beta}\mathrm{d}x_d$$

$$= c_{d,\beta}\left(1-r^2\right)^{\beta+\frac{1}{2}}\int\limits_{-1}^{1}\left(1-y^2\right)^{\beta}\mathrm{d}y,$$

where we used the transformation $y = x_d/\sqrt{1-r^2}$. Hence, $P_L(X)$ has density $f_{d-1,\beta+\frac{1}{2}}$ and we do not even need to check that $c_{d,\beta}\int_{-1}^{1}\left(1-y^2\right)^{\beta}\mathrm{d}y = c_{d-1,\beta+\frac{1}{2}}$, because the outcome must be a probability density. Inductive application of this result yields the desired statement for arbitrary dimensions k.

In the case of the beta′-type distribution we can apply almost the same argument. Fix some $x^* = (x_1^*,\ldots,x_{d-1}^*)\in\mathbb{R}^{d-1}$ with Euclidean norm $r := \|x^*\|\geq 0$. The pre-images of x^* under the projection P_L have the form $x = (x_1^*,\ldots,x_{d-1}^*,x_d)$ with $x_d\in\mathbb{R}$. Then, the density of $P_L(X)$ at x^* is given by

$$\tilde{c}_{d,\beta}\int\limits_{-\infty}^{+\infty}\left(1+\|x\|^2\right)^{-\beta}\mathrm{d}x_d = \tilde{c}_{d,\beta}\int\limits_{-\infty}^{\infty}\left(1+r^2+x_d^2\right)^{-\beta}\mathrm{d}x_d$$

$$= \tilde{c}_{d,\beta}\left(1+r^2\right)^{-\beta}\int\limits_{-\infty}^{\infty}\left(1+\frac{x_d^2}{1+r^2}\right)^{-\beta}\mathrm{d}x_d$$

$$= \tilde{c}_{d,\beta}\left(1+r^2\right)^{-(\beta-\frac{1}{2})}\int\limits_{-\infty}^{\infty}\left(1+y^2\right)^{-\beta}\mathrm{d}y,$$

where we used the transformation $y = x_d/\sqrt{1+r^2}$. It follows that $P_L(X)$ has density $\tilde{f}_{d-1,\beta-\frac{1}{2}}$. The statement in the case of general dimension k again follows by induction.

□

Lemma 3.1.4 *Consider the affine hyperplane $H(u,h)\in A(d,d-1)$ with $h\in\mathbb{R}$ and $u\in\mathbb{S}^{d-1}$. Let X be beta-type distributed on $\mathbb{B}^d$ with parameter $\beta>-1$. Then,*

$$\mathbb{P}(X\in H(u,h)^+) = 1-F_{1,\beta+\frac{d-1}{2}}(h),\quad \mathbb{P}(X\in H(u,h)^-) = F_{1,\beta+\frac{d-1}{2}}(h),\quad h\in[-1,1],$$

and

$$\mathbb{P}(X\in(H(u,h)^-\cap H(u,-h)^+)) = F_{1,\beta+\frac{d-1}{2}}(h)-F_{1,\beta+\frac{d-1}{2}}(-h),\qquad h\in[0,1].$$

Similarly, if X is beta′-type distributed on $\mathbb{R}^d$ with parameter $\beta > \frac{d}{2}$, then

$$\mathbb{P}(X \in H(u,h)^+) = 1 - \tilde{F}_{1,\beta-\frac{d-1}{2}}(h), \qquad \mathbb{P}(X \in H(u,h)^-) = \tilde{F}_{1,\beta-\frac{d-1}{2}}(h), \qquad h \in \mathbb{R},$$

and

$$\mathbb{P}(X \in (H(u,h)^- \cap H(u,-h)^+)) = \tilde{F}_{1,\beta-\frac{d-1}{2}}(h) - \tilde{F}_{1,\beta-\frac{d-1}{2}}(-h), \qquad h \geq 0.$$

Proof. Let X be a random point with density $f_{d,\beta}$. In order to calculate the probability contents of $H(u,h)^+$ and $H(u,h)^-$ we project X onto the line

$$L := H(u,0)^\perp.$$

Clearly, $X \in H(u,h)^+$ is equivalent to $P_L(X) \geq h$, whereas $X \in H(u,h)^-$ is equivalent to $P_L(X) \leq h$. From Lemma 3.1.3 we know that $P_L(X)$ has the one-dimensional density $f_{1,\beta+\frac{d-1}{2}}$. Hence,

$$\mathbb{P}(X \in H(u,h)^-) = \int_{-\infty}^{h} f_{1,\beta+\frac{d-1}{2}}(x)\,\mathrm{d}x = c_{1,\beta+\frac{d-1}{2}} \int_{-1}^{h} \left(1-x^2\right)^{\beta+\frac{d-1}{2}} \mathrm{d}x = F_{1,\beta+\frac{d-1}{2}}(h).$$

The observation that $\mathbb{P}(X \in H(u,h)^+) = 1 - \mathbb{P}(X \in H(u,h)^-)$, and in the case of a slab, that $\mathbb{P}(X \in (H(u,h)^- \cap H(u,-h)^+)) = \mathbb{P}(X \in H(u,h)^-) - \mathbb{P}(X \in H(u,-h)^-)$ finishes the proof in the beta-type case.

Similarly, if X is a random variable with density $\tilde{f}_{d,\beta}$, then, by Lemma 3.1.3, $P_L(X)$ has density $\tilde{f}_{1,\beta-\frac{d-1}{2}}$ and we get the corresponding results for the beta′-type distribution. □

Throughout our investigation, explicit formulas for moments of the volumes of (lower dimensional) simplices and parallelepipeds generated by random points sampled independently and identically from beta- and beta′-type distributions play a crucial role. As mentioned earlier, integer moments of the volumes of (lower dimensional) simplices where provided by Miles [79, Equations (72) and (74)]. Mathai [76, 77] later gave explicit formulas for all non-negative real moments of the volume of (lower dimensional) parallelepipeds. We will use Mathai's result and (2.29) to extend also Miles' formulas to all non-negative real moments. Recall that we denote by $\Delta_k = \Delta_k(x_0, \ldots, x_k)$ and $\nabla_k = \nabla_k(x_1, \ldots, x_k)$ the k-dimensional Lebesgue volumes of the k-simplex and

k-parallelepiped, respectively, generated by the points $x_0, \ldots, x_k \in \mathbb{R}^d$, where $k = 1, \ldots, d$. Moreover, we write the expectation with respect to the beta-type distribution as $\mathbb{E}_\beta$ and the one with respect to the beta′-type distribution as $\widetilde{\mathbb{E}}_\beta$.

Proposition 3.1.5 *(Mathai)*
Let $X_1, \ldots, X_d$ be i.i.d. random points in $\mathbb{R}^d$.

(a) If $X_1, \ldots, X_d$ are distributed according to a beta-type distribution on $\mathbb{B}^d$ with parameter $\beta > -1$, then

$$\mathbb{E}_\beta\left(\nabla_d^\kappa\right) = \left(\frac{\Gamma\left(\beta + \frac{d}{2} + 1\right)}{\Gamma\left(\beta + \frac{d+\kappa}{2} + 1\right)}\right)^d \prod_{i=1}^{d} \frac{\Gamma\left(\frac{d+\kappa-i+1}{2}\right)}{\Gamma\left(\frac{d-i+1}{2}\right)},$$

for all $\kappa \in [0, \infty)$.

(b) If $X_1, \ldots, X_d$ are distributed according to a beta′-type distribution on $\mathbb{R}^d$ with parameter $\beta > \frac{d}{2}$, then

$$\widetilde{\mathbb{E}}_\beta\left(\nabla_d^\kappa\right) = \left(\frac{\Gamma\left(\beta - \frac{d+\kappa}{2}\right)}{\Gamma\left(\beta - \frac{d}{2}\right)}\right)^d \prod_{i=1}^{d} \frac{\Gamma\left(\frac{d+\kappa-i+1}{2}\right)}{\Gamma\left(\frac{d-i+1}{2}\right)},$$

for all $\kappa \in [0, 2\beta - d)$.

Proposition 3.1.6 *(Miles)*
Let $X_0, \ldots, X_d$ be i.i.d. random points in $\mathbb{R}^d$.

(a) If $X_0, \ldots, X_d$ are distributed according to a beta-type distribution on $\mathbb{B}^d$ with parameter $\beta > -1$, then

$$\mathbb{E}_\beta\left(\Delta_d^\kappa\right) = (d!)^{-\kappa} \frac{\Gamma\left(\frac{d+1}{2}(2\beta + d + \kappa) + 1\right)}{\Gamma\left(\frac{d+1}{2}(2\beta + d) + \frac{d\kappa}{2} + 1\right)} \left(\frac{\Gamma\left(\frac{d}{2} + \beta + 1\right)}{\Gamma\left(\frac{d+\kappa}{2} + \beta + 1\right)}\right)^{d+1} \prod_{i=1}^{d} \frac{\Gamma\left(\frac{i+\kappa}{2}\right)}{\Gamma\left(\frac{i}{2}\right)},$$

for all $\kappa \in [0, \infty)$.

(b) If $X_0, \ldots, X_d$ are distributed according to a beta′-type distribution on $\mathbb{R}^d$ with parameter $\beta > \frac{d}{2}$, then

$$\widetilde{\mathbb{E}}_\beta\left(\Delta_d^\kappa\right) = (d!)^{-\kappa} \frac{\Gamma\left(\left(\beta - \frac{d}{2}\right)(d+1) - \frac{d}{2}\kappa\right)}{\Gamma\left(\left(\beta - \frac{d+\kappa}{2}\right)(d+1)\right)} \left(\frac{\Gamma\left(\beta - \frac{d+\kappa}{2}\right)}{\Gamma\left(\beta - \frac{d}{2}\right)}\right)^{d+1} \prod_{i=1}^{d} \frac{\Gamma\left(\frac{i+\kappa}{2}\right)}{\Gamma\left(\frac{i}{2}\right)},$$

for all $\kappa \in [0, 2\beta - d)$.

Remark 3.1.7 Note that in (b) we have indeed $-\frac{d}{2}\kappa$ instead of $+\frac{d}{2}\kappa$, as stated in [79]. This typo has already been observed and corrected by Chu [35], for example.

Before we are able to prove Miles' formula for non-negative real moments, we need to know how the moments of the volume of random simplices chosen according to beta- and beta′-type densities restricted to affine subspaces behave. This will come also in handy in later considerations.

Lemma 3.1.8 *Let $H \in A(d, d-1)$ be an affine hyperplane at distance h from the origin. In the case of the beta-type distribution with parameter $\beta > -1$, for all $h \in [0,1]$ and $\kappa \in [0, \infty)$ we have*

$$\int\limits_{H^d} \Delta_{d-1}^{\kappa}(x_1, \ldots, x_d) \left(\prod_{i=1}^{d} f_{d,\beta}(x_i) \right) \lambda_H^d(\mathrm{d}(x_1, \ldots, x_d)) \\ = \frac{c_{d,\beta}^d}{c_{d-1,\beta}^d} \left(1 - h^2\right)^{d\beta + \frac{d-1}{2}(d+\kappa)} \mathbb{E}_\beta\left(\Delta_{d-1}^{\kappa}\right).$$

Similarly, in the case of the beta′-type distribution with parameter $\beta > \frac{d}{2}$, for all $h \geq 0$ and $\kappa \in [0, 2\beta - d)$ we have

$$\int\limits_{H^d} \Delta_{d-1}^{\kappa}(x_1, \ldots, x_d) \left(\prod_{i=1}^{d} \tilde{f}_{d,\beta}(x_i) \right) \lambda_H^d(\mathrm{d}(x_1, \ldots, x_d)) \\ = \frac{\tilde{c}_{d,\beta}^d}{\tilde{c}_{d-1,\beta}^d} \left(1 + h^2\right)^{-d\beta + \frac{d-1}{2}(d+\kappa)} \widetilde{\mathbb{E}}_\beta\left(\Delta_{d-1}^{\kappa}\right).$$

Proof. Without loss of generality we take $H = H(e_d, h) \in A(d, d-1)$. Consider also the linear hyperplane $L = H(e_d, 0) \in G(d, d-1)$ which is parallel to H. With $z^* = P_L(z)$, for $z \in H$, we have that $\|z\|^2 = \|z^*\|^2 + h^2$, where the euclidean norms are taken in the respective Euclidean spaces. Hence, for all $h \in [0,1]$,

$$\int\limits_{H^d} \Delta_{d-1}^{\kappa}(x_1, \ldots, x_d) \left(\prod_{i=1}^{d} f_{d,\beta}(x_i) \right) \lambda_H^d(\mathrm{d}(x_1, \ldots, x_d)) \\ = c_{d,\beta}^d \int\limits_{\left(H \cap \mathbb{B}^d\right)^d} \Delta_{d-1}^{\kappa}(x_1, \ldots, x_d) \prod_{i=1}^{d} \left(1 - \|x_i\|^2\right)^{\beta} \lambda_H^d(\mathrm{d}(x_1, \ldots, x_d))$$

$$= c_{d,\beta}^d \int\limits_{\left(L\cap\sqrt{1-h^2}\mathbb{B}^d\right)^d} \Delta_{d-1}^\kappa\left(x_1^*,\dots,x_d^*\right)\prod_{i=1}^d \left(1-\|x_i^*\|^2-h^2\right)^\beta \lambda_L^d\left(\mathrm{d}\left(x_1^*,\dots,x_d^*\right)\right)$$

$$= c_{d,\beta}^d \left(1-h^2\right)^{d\beta} \int\limits_{\left(L\cap\sqrt{1-h^2}\mathbb{B}^d\right)^d} \Delta_{d-1}^\kappa\left(x_1^*,\dots,x_d^*\right)\prod_{i=1}^d \left(1-\frac{\|x_i^*\|^2}{1-h^2}\right)^\beta \lambda_L^d\left(\mathrm{d}\left(x_1^*,\dots,x_d^*\right)\right).$$

Introducing the new variable $y_i = \left(1-h^2\right)^{-\frac{1}{2}} x_i^*$ for every $i = 1,\dots,d$ and identifying L with $\mathbb{R}^{d-1}$ leads to

$$\int\limits_{H^d} \Delta_{d-1}^\kappa\left(x_1,\dots,x_d\right)\prod_{i=1}^d f_{d,\beta}(x_i)\lambda_H^d\left(\mathrm{d}\left(x_1,\dots,x_d\right)\right)$$

$$= c_{d,\beta}^d \left(1-h^2\right)^{d\beta+\frac{d-1}{2}(d+\kappa)} \int\limits_{\left(\mathbb{B}^{d-1}\right)^d} \Delta_{d-1}^\kappa\left(y_1,\dots,y_d\right)\prod_{i=1}^d \left(1-\|y_i\|^2\right)^\beta \lambda_L^d\left(\mathrm{d}\left(y_1,\dots,y_d\right)\right)$$

$$= \frac{c_{d,\beta}^d}{c_{d-1,\beta}^d} \left(1-h^2\right)^{d\beta+\frac{d-1}{2}(d+\kappa)} \int\limits_{\left(\mathbb{B}^{d-1}\right)^d} \Delta_{d-1}^\kappa\left(y_1,\dots,y_d\right)\prod_{i=1}^d f_{d-1,\beta}(y_i)\,\lambda_L^d\left(\mathrm{d}\left(y_1,\dots,y_d\right)\right)$$

$$= \frac{c_{d,\beta}^d}{c_{d-1,\beta}^d} \left(1-h^2\right)^{d\beta+\frac{d-1}{2}(d+\kappa)} \mathbb{E}_\beta\left(\Delta_{d-1}^\kappa\right).$$

The result for the beta′-type distribution is derived analogously by using the transformation $y_i = \left(1+h^2\right)^{-\frac{1}{2}} x_i^*$, for $i = 1,\dots,d$, and by suitably adapting the range of integration. □

Now we can go over to proving Miles' formulas for all non-negative real moments:

Proof of Proposition 3.1.6. Let $X_1,\dots,X_d$ be i.i.d. random points distributed according to a beta-type distribution on $\mathbb{B}^d$ with parameter $\beta > -1$. Recall that we denote by $\eta(\{X_1,\dots,X_d\})$ the Euclidean distance of the affine hull $\mathrm{aff}(\{X_1,\dots,X_d\})$ to the origin $\boldsymbol{o}$. From (2.29) and the well-known base-times-height-formula for the volume of simplices we get

$$\begin{aligned}\mathbb{E}_\beta\left(\nabla_d^\kappa(X_1,\dots,X_d)\right) &= (d!)^\kappa \mathbb{E}_\beta\left(\Delta_d^\kappa(\boldsymbol{o},X_1,\dots,X_d)\right)\\ &= (d!)^\kappa \mathbb{E}_\beta\left(d^{-\kappa}\eta^\kappa([X_1,\dots,X_d])\Delta_{d-1}^\kappa(X_1,\dots,X_d)\right)\\ &= ((d-1)!)^\kappa \mathbb{E}_\beta\left(\eta^\kappa([X_1,\dots,X_d])\Delta_{d-1}^\kappa(X_1,\dots,X_d)\right).\end{aligned}$$

Abbreviating $C = ((d-1)!)^{\kappa+1}$, rewriting this as an integral over $(\mathbb{R}^d)^d$ and applying the affine Blaschke-Petkantschin formula yields

$$\begin{aligned}
&\mathbb{E}_\beta\left(\nabla_d^\kappa(X_1,\dots,X_d)\right)\\
&= C\int\limits_{(\mathbb{R}^d)^d} \eta^\kappa([X_1,\dots,X_d])\Delta_{d-1}^\kappa(x_1,\dots,x_d)\left(\prod_{i=1}^d f_{d,\beta}(x_i)\right)\lambda_d^d(\mathrm{d}(x_1,\dots,x_d))\\
&= Cb_{d,d-1}\int\limits_{\mathbb{S}^{d-1}}\int\limits_0^1\int\limits_{E^d} h^\kappa\Delta_{d-1}^{\kappa+1}(x_1,\dots,x_d)\left(\prod_{i=1}^d f_{d,\beta}(x_i)\right)\lambda_E^d(\mathrm{d}(x_1,\dots,x_d))\,\mathrm{d}h\,\sigma_{d-1}(\mathrm{d}u)\\
&= 2Cb_{d,d-1}\left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^d\mathbb{E}_\beta\left(\Delta_{d-1}^{\kappa+1}(X_1,\dots,X_d)\right)\int\limits_0^1 h^\kappa(1-h^2)^{d\beta+\frac{d-1}{2}(d+\kappa+1)}\,\mathrm{d}h\\
&= Cb_{d,d-1}\left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^d\mathbb{E}_\beta\left(\Delta_{d-1}^{\kappa+1}(X_1,\dots,X_d)\right)B\left(\frac{\kappa+1}{2},d\beta+\frac{d-1}{2}(d+\kappa+1)+1\right),
\end{aligned}$$

where the second to last equality follows from Lemma 3.1.8. Recall that $B(\cdot,\cdot)$ stands for the beta function. Thus, rearranging and adjusting for the correct indices gives

$$\mathbb{E}_\beta\left(\Delta_d^\kappa(X_0,\dots,X_d)\right) = \frac{c_{d,\beta}^{d+1}}{c_{d+1,\beta}^{d+1}}\,\frac{\mathbb{E}_\beta\left(\nabla_{d+1}^{\kappa-1}(X_0,\dots,X_d)\right)}{b_{d+1,d}(d!)^\kappa B\left(\frac{\kappa}{2},(d+1)\beta+\frac{d}{2}(d+\kappa+1)+1\right)},\tag{3.6}$$

from which the claim follows by working out the constants; see Proposition 3.1.5. One sees from the last equality that what we have shown is only true for $\kappa \geq 1$. We want to conclude that it in fact holds for all $\kappa \geq 0$ by analytic continuation. To this end, we notice first that the map $\kappa \mapsto \mathbb{E}_\beta\left(\Delta_d^\kappa(X_0,\dots,X_d)\right)$ has the integral representation

$$\mathbb{E}_\beta\left(\Delta_d^\kappa(X_0,\dots,X_d)\right) = \int\limits_{(\mathbb{B}^d)^{d+1}} \Delta_d^\kappa(x_0,\dots,x_d)\left(\prod_{i=0}^d f_{d,\beta}(x_i)\right)\lambda_d^{d+1}(\mathrm{d}(x_0,\dots,x_d)),$$

which is real analytic for all $\kappa \geq 0$. Moreover, the right-hand side of (3.6) is real analytic for all $\kappa > 0$. Hence, by analytic continuation these two expressions must coincide for $\kappa > 0$, since they coincide for all $\kappa > 1$. In addition, for $\kappa = 0$ both sides are equal to 1.

The corresponding result for the beta′-type distribution can be shown in the same way. However, the necessity for an analytic continuation does not arise there. □

3.2 Functionals of beta- and beta′-polytopes

Expected volumes and intrinsic volumes

Theorem 3.2.1 *Let $X_1, \dots, X_n$ be independent beta-type distributed random points in $\mathbb{B}^d$ with parameter $\beta > -1$. Then,*

$$\mathbb{E}\,\mathrm{Vol}_d(P_{n,d}^\beta) = A_{n,d}^\beta \int_{-1}^{1} \left(1-h^2\right)^q F_{1,\beta+\frac{d-1}{2}}(h)^{n-d-1}\,\mathrm{d}h,$$

$$\mathbb{E}\,\mathrm{Vol}_d(S_{n,d}^\beta) = 2^{d+1} A_{n,d}^\beta \int_{0}^{1} \left(1-h^2\right)^q \left(F_{1,\beta+\frac{d-1}{2}}(h) - F_{1,\beta+\frac{d-1}{2}}(-h)\right)^{n-d-1}\,\mathrm{d}h, \tag{3.7}$$

$$\mathbb{E}\,\mathrm{Vol}_d(Q_{n,d}^\beta) = D_{n,d}^\beta + A_{n,d}^\beta \int_{0}^{1} \left(1-h^2\right)^q F_{1,\beta+\frac{d-1}{2}}(h)^{n-d-1}\,\mathrm{d}h,$$

where $q = (d+1)\left(\beta - \frac{1}{2}\right) + \frac{d}{2}(d+3)$ and

$$\begin{aligned} A_{n,d}^\beta &= \frac{(d+1)\kappa_d}{2^d \pi^{\frac{d+1}{2}}} \binom{n}{d+1} \left(\beta + \frac{d+1}{2}\right) \left(\frac{\Gamma\left(\frac{d+2}{2}+\beta\right)}{\Gamma\left(\frac{d+3}{2}+\beta\right)}\right)^{d+1}, \\ D_{n,d}^\beta &= \frac{\kappa_d}{2^n \pi^{\frac{d}{2}}} \binom{n}{d} \left(\frac{\Gamma\left(\frac{d+2}{2}+\beta\right)}{\Gamma\left(\frac{d+3}{2}+\beta\right)}\right)^{d}. \end{aligned} \tag{3.8}$$

Theorem 3.2.2 *Let $X_1, \dots, X_n$ be independent beta′-type distributed random points in $\mathbb{R}^d$ with parameter $\beta > \frac{d+1}{2}$. Then,*

$$\mathbb{E}\,\mathrm{Vol}_d(\tilde{P}_{n,d}^\beta) = \tilde{A}_{n,d}^\beta \int_{-\infty}^{\infty} (1+h^2)^{-\tilde{q}} \tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-d-1}\,\mathrm{d}h,$$

$$\mathbb{E}\,\mathrm{Vol}_d(\tilde{S}_{n,d}^\beta) = 2^{d+1} \tilde{A}_{n,d}^\beta \int_{0}^{\infty} (1+h^2)^{-\tilde{q}} \left(\tilde{F}_{1,\beta-\frac{d-1}{2}}(h) - \tilde{F}_{1,\beta-\frac{d-1}{2}}(-h)\right)^{n-d-1}\,\mathrm{d}h,$$

$$\mathbb{E}\,\mathrm{Vol}_d(\tilde{Q}_{n,d}^\beta) = \tilde{D}_{n,d}^\beta + \tilde{A}_{n,d}^\beta \int_{0}^{\infty} (1+h^2)^{-\tilde{q}} \tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-d-1}\,\mathrm{d}h,$$

where $\tilde{q} = (d+1)(\beta + \frac{1}{2}) - \frac{d}{2}(d+3)$ *and*

$$\tilde{A}^{\beta}_{n,d} = \frac{(d+1)\kappa_d}{2^d \pi^{\frac{d+1}{2}}} \binom{n}{d+1} \left(\beta - \frac{d+1}{2}\right) \left(\frac{\Gamma\left(\beta - \frac{d+1}{2}\right)}{\Gamma\left(\beta - \frac{d}{2}\right)} \right)^{d+1},$$

$$\tilde{D}^{\beta}_{n,d} = \frac{\kappa_d}{2^n \pi^{\frac{d}{2}}} \binom{n}{d} \left(\frac{\Gamma\left(\beta - \frac{d+1}{2}\right)}{\Gamma\left(\beta - \frac{d}{2}\right)} \right)^{d+1}.$$

The formulae for the expected intrinsic volumes are obtained using the next proposition.

Proposition 3.2.3 *The expected intrinsic volumes* $\mathbb{E}V_k(P^{\beta}_{n,d})$ *and* $\mathbb{E}V_k(\tilde{P}^{\beta}_{n,d})$ *for* $k = 1, \ldots, d$ *are given by the formulae*

$$\mathbb{E}V_k(P^{\beta}_{n,d}) = \binom{d}{k} \frac{\kappa_d}{\kappa_k \kappa_{d-k}} \mathbb{E}\,\mathrm{Vol}_k\left(P^{\beta + \frac{d-k}{2}}_{n,k}\right),$$

$$\mathbb{E}V_k(\tilde{P}^{\beta}_{n,d}) = \binom{d}{k} \frac{\kappa_d}{\kappa_k \kappa_{d-k}} \mathbb{E}\,\mathrm{Vol}_k\left(\tilde{P}^{\beta - \frac{d-k}{2}}_{n,k}\right).$$

These formulae hold if $P^{\beta}_{n,d}$ *is replaced by* $S^{\beta}_{n,d}$ *or* $Q^{\beta}_{n,d}$*, respectively* $\tilde{P}^{\beta}_{n,d}$ *by* $\tilde{S}^{\beta}_{n,d}$ *or* $\tilde{Q}^{\beta}_{n,d}$*.*

Expected surface area and expected mean width

Proposition 3.2.3 implies formulae for the expected surface area of $P^{\beta}_{n,d}$, $S^{\beta}_{n,d}$, $Q^{\beta}_{n,d}$, $\tilde{P}^{\beta}_{n,d}$, $\tilde{S}^{\beta}_{n,d}$ and $\tilde{Q}^{\beta}_{n,d}$. Recall, that for a convex set $K \subset \mathbb{R}^d$ we have $S_{d-1}(K) := 2V_{d-1}(K)$.

Corollary 3.2.4 *Let* $X_1, \ldots, X_n$ *be independent beta-type distributed random points in* $\mathbb{B}^d$ *with parameter* $\beta > -1$*. Then,*

$$\mathbb{E}S_{d-1}(P^{\beta}_{n,d}) = \gamma_d A^{\beta+\frac{1}{2}}_{n,d-1} \int_{-1}^{1} \left(1 - h^2\right)^q F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\,\mathrm{d}h,$$

$$\mathbb{E}S_{d-1}(S^{\beta}_{n,d}) = 2^d \gamma_d A^{\beta+\frac{1}{2}}_{n,d-1} \int_{0}^{1} \left(1 - h^2\right)^q \left(F_{1,\beta+\frac{d-1}{2}}(h) - F_{1,\beta+\frac{d-1}{2}}(-h)\right)^{n-d} \mathrm{d}h,$$

$$\mathbb{E}S_{d-1}(Q^{\beta}_{n,d}) = \gamma_d \left(D^{\beta+\frac{1}{2}}_{n,d-1} + A^{\beta+\frac{1}{2}}_{n,d-1} \int_{0}^{1} \left(1 - h^2\right)^q F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\,\mathrm{d}h \right),$$

where $q = d\beta + \frac{d-1}{2}(d+2)$ *and* $\gamma_d = \frac{d\kappa_d}{\kappa_{d-1}}$ *. The constants* $A^{\beta}_{n,d}$ *and* $D^{\beta}_{n,d}$ *are the same as in Theorem 3.2.1.*

We remark that a different representation for $\mathbb{E}S_{d-1}(P_{n,d}^{\beta})$ was previously given by Buchta, Müller and Tichy [32].

Corollary 3.2.5 *Let $X_1, \ldots, X_n$ be independent beta′-type distributed random points in $\mathbb{R}^d$ with parameter $\beta > \frac{d+1}{2}$. Then,*

$$\mathbb{E}S_{d-1}(\tilde{P}_{n,d}^{\beta}) = \gamma_d \tilde{A}_{n,d-1}^{\beta-\frac{1}{2}} \int_{-\infty}^{\infty} (1+h^2)^{-\tilde{q}} \tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-d} \, \mathrm{d}h,$$

$$\mathbb{E}S_{d-1}(\tilde{S}_{n,d}^{\beta}) = 2^d \gamma_d \tilde{A}_{n,d-1}^{\beta-\frac{1}{2}} \int_{0}^{\infty} (1+h^2)^{-\tilde{q}} \left(\tilde{F}_{1,\beta-\frac{d-1}{2}}(h) - \tilde{F}_{1,\beta-\frac{d-1}{2}}(-h)\right)^{n-d} \mathrm{d}h,$$

$$\mathbb{E}S_{d-1}(\tilde{Q}_{n,d}^{\beta}) = \gamma_d \left(\tilde{D}_{n,d-1}^{\beta-\frac{1}{2}} + \tilde{A}_{n,d-1}^{\beta-\frac{1}{2}} \int_{0}^{\infty} (1+h^2)^{-\tilde{q}} \tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-d} \, \mathrm{d}h \right),$$

where $\tilde{q} = d\beta - \frac{d-1}{2}(d+2)$ and $\gamma_d = \frac{d\kappa_d}{\kappa_{d-1}}$. The constants $\tilde{A}_{n,d}^{\beta}$ and $\tilde{D}_{n,d}^{\beta}$ are the same as in Theorem 3.2.2.

Similarly, we can find explicit formulae for the mean width of these random polytopes. We recall that the mean width $\mathcal{W}_d(K)$ of a convex set $K \subset \mathbb{R}^d$ is defined as the expected length of the projection of K onto a uniformly chosen random line. The mean width is related to the first intrinsic volume by the formula $\mathcal{W}_d(K) = \frac{2\kappa_{d-1}}{d\kappa_d} V_1(K)$; see [99, p.223].

Corollary 3.2.6 *Let $X_1, \ldots, X_n$ be independent beta-type distributed random points in $\mathbb{B}^d$ with parameter $\beta > -1$. Then,*

$$\mathbb{E}\mathcal{W}_d(P_{n,d}^{\beta}) = A_{n,1}^{\beta+\frac{d-1}{2}} \int_{-1}^{1} \left(1-h^2\right)^q F_{1,\beta+\frac{d-1}{2}}(h)^{n-2} \, \mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_d(S_{n,d}^{\beta}) = 4A_{n,1}^{\beta+\frac{d-1}{2}} \int_{0}^{1} \left(1-h^2\right)^q \left(F_{1,\beta+\frac{d-1}{2}}(h) - F_{1,\beta+\frac{d-1}{2}}(-h)\right)^{n-2} \mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_d(Q_{n,d}^{\beta}) = D_{n,1}^{\beta+\frac{d-1}{2}} + A_{n,1}^{\beta+\frac{d-1}{2}} \int_{0}^{1} \left(1-h^2\right)^q F_{1,\beta+\frac{d-1}{2}}(h)^{n-2} \, \mathrm{d}h,$$

where $q = 2\beta + d$. The constants $A_{n,d}^{\beta}$ and $D_{n,d}^{\beta}$ are the same as in Theorem 3.2.1.

As in the case of the surface area, a different representation for $\mathbb{E}\mathcal{W}_d(P^\beta_{n,d})$ was previously given by Buchta, Müller and Tichy [32].

Corollary 3.2.7 *Let $X_1, \ldots, X_n$ be independent beta′-type distributed random points in $\mathbb{R}^d$ with parameter $\beta > \frac{d+1}{2}$. Then,*

$$\mathbb{E}\mathcal{W}_d(\tilde{P}^\beta_{n,d}) = \tilde{A}^{\beta-\frac{d-1}{2}}_{n,1} \int_{-\infty}^{\infty} (1+h^2)^{-\tilde{q}} \tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-2}\, \mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_d(\tilde{S}^\beta_{n,d}) = 4\tilde{A}^{\beta-\frac{d-1}{2}}_{n,1} \int_0^{\infty} (1+h^2)^{-\tilde{q}} \left(\tilde{F}_{1,\beta-\frac{d-1}{2}}(h) - \tilde{F}_{1,\beta-\frac{d-1}{2}}(-h)\right)^{n-2} \mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_d(\tilde{Q}^\beta_{n,d}) = \tilde{D}^{\beta-\frac{d-1}{2}}_{n,1} + \tilde{A}^{\beta-\frac{d-1}{2}}_{n,1} \int_0^{\infty} (1+h^2)^{-\tilde{q}} \tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-2}\, \mathrm{d}h,$$

where $\tilde{q} = 2\beta - d$. The constants $\tilde{A}^\beta_{n,d}$ and $\tilde{D}^\beta_{n,d}$ are the same as in Theorem 3.2.2.

Expectation of the T-functional

Fix $a, b \geq 0$. Recall that for a convex polytope $P \subset \mathbb{R}^d$, we are interested in the functional

$$T^{d,k}_{a,b}(P) = \sum_{F \in \mathcal{F}_k(P)} \eta^a(F)\, \mathrm{Vol}^b_k(F),$$

where $\mathcal{F}_k(P)$ is the set of all k-dimensional faces of P, and $\eta(F)$ is the Euclidean distance from the affine hull $\mathrm{aff}(F)$ of the k-dimensional face F to the origin.

Theorem 3.2.8 *Fix $a, b \geq 0$. Let $X_1, \ldots, X_n$ be independent beta-type distributed random points in $\mathbb{B}^d$ with parameter $\beta > -1$. Then,*

$$\mathbb{E}T^{d,d-1}_{a,b}\left(P^\beta_{n,d}\right) = C^{\beta,b}_{n,d} \int_{-1}^{1} |h|^a \left(1-h^2\right)^q F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\, \mathrm{d}h,$$

$$\mathbb{E}T^{d,d-1}_{a,b}\left(S^\beta_{n,d}\right) = 2^d C^{\beta,b}_{n,d} \int_0^1 h^a \left(1-h^2\right)^q \left(F_{1,\beta+\frac{d-1}{2}}(h) - F_{1,\beta+\frac{d-1}{2}}(-h)\right)^{n-d} \mathrm{d}h,$$

$$\mathbb{E}T^{d,d-1}_{a,b}\left(Q^\beta_{n,d}\right) = D^{\beta,a,b}_{n,d} + C^{\beta,b}_{n,d} \int_0^1 h^a \left(1-h^2\right)^q F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\, \mathrm{d}h,$$

where $q = d\beta + \frac{d-1}{2}(d+b+1)$ *and*

$$C_{n,d}^{\beta,b} = \binom{n}{d} d!\kappa_d \mathbb{E}_\beta\left(\Delta_{d-1}^{b+1}\right)\left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^d,$$
$$D_{n,d}^{\beta,a,b} = \mathbb{1}\{a=0\}\binom{n}{d-1}\frac{d\kappa_d \mathbb{E}_\beta\left(\nabla_{d-1}^{b+1}\right)}{2^{n-d+1}((d-1)!)^b}\left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^{d-1}.$$

Remark 3.2.9 Since the polytopes $S_{n,d}^\beta$ and $Q_{n,d}^\beta$ always contain the origin, we can decompose them into simplices of the form $[\boldsymbol{o}, x_1, \dots, x_d]$, where $[x_1, \dots, x_d]$ runs through all facets of the corresponding polytope. It follows that the expected volume of these polytopes is given by

$$\mathbb{E}\,\mathrm{Vol}_d(S_{n,d}^\beta) = \frac{1}{d}\mathbb{E}T_{1,1}^{d,d-1}(S_{n,d}^\beta), \quad \mathbb{E}\,\mathrm{Vol}_d(Q_{n,d}^\beta) = \frac{1}{d}\mathbb{E}T_{1,1}^{d,d-1}(Q_{n,d}^\beta).$$

Then, Theorem 3.2.8 yields the formulae

$$\mathbb{E}\,\mathrm{Vol}_d(S_{n,d}^\beta) = \frac{2^d}{d}C_{n,d}^{\beta,1}\int_0^1 h\left(1-h^2\right)^{d\beta+\frac{d-1}{2}(d+2)}\left(F_{1,\beta+\frac{d-1}{2}}(h) - F_{1,\beta+\frac{d-1}{2}}(-h)\right)^{n-d}\,\mathrm{d}h,$$
$$\mathbb{E}\,\mathrm{Vol}_d(Q_{n,d}^\beta) = \frac{1}{d}D_{n,d}^{\beta,1,1} + \frac{1}{d}C_{n,d}^{\beta,1}\int_0^1 h\left(1-h^2\right)^{d\beta+\frac{d-1}{2}(d+2)} F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\,\mathrm{d}h.$$

This method does not work for $P_{n,d}^\beta$ because this polytope need not contain $\boldsymbol{o}$. However, one can consider the following analogue of the T-functional defined for d-dimensional polytopes $P \subset \mathbb{R}^d$:

$$T_{1,1,\pm}^{d,d-1}(P) = \sum_{F \in \mathcal{F}_{d-1}(P)} \eta_\pm(F)\,\mathrm{Vol}_{d-1}(F),$$

where $\eta_\pm(F)$ is the distance from the affine hull of the face F to the origin $\boldsymbol{o}$ taken positive, if the origin $\boldsymbol{o}$ and the polytope P are on the same side of the face F, and negative otherwise. Then, it is easy to see that

$$\mathbb{E}\,\mathrm{Vol}_d(P_{n,d}^\beta) = \frac{1}{d}\mathbb{E}T_{1,1,\pm}^{d,d-1}(P_{n,d}^\beta).$$

The expected value of $T_{1,1,\pm}^{d,d-1}(P_{n,d}^\beta)$ can be computed in the same way as in Theorem

3.2.8 yielding the following formula:

$$\mathbb{E}\,\mathrm{Vol}_d(P_{n,d}^{\beta}) = \frac{1}{d}C_{n,d}^{\beta,1}\int_{-1}^{1} h\left(1-h^2\right)^{d\beta+\frac{d-1}{2}(d+2)} F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\,\mathrm{d}h.$$

The equivalence of these formulae to those given in Theorem 3.2.1 can be shown using partial integration. Anyway, we shall give an alternative proof of Theorem 3.2.1 later.

The analogue of Theorem 3.2.8 in the case of beta′-type densities can be stated as follows.

Theorem 3.2.10 *Fix $a, b \geq 0$. Let $X_1, \ldots, X_n$ be independent beta′-type distributed random points in $\mathbb{R}^d$ with parameter β that satisfies $2d\beta > (d-1)(d+b+1)+a+1$. Then*

$$\mathbb{E}T_{a,b}^{d,d-1}\left(\tilde{P}_{n,d}^{\beta}\right) = \tilde{C}_{n,d}^{\beta,b}\int_{-\infty}^{\infty} |h|^a\left(1+h^2\right)^{-\tilde{q}}\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-d}\,\mathrm{d}h,$$

$$\mathbb{E}T_{a,b}^{d,d-1}\left(\tilde{S}_{n,d}^{\beta}\right) = 2^d\tilde{C}_{n,d}^{\beta,b}\int_{0}^{\infty} h^a\left(1+h^2\right)^{-\tilde{q}}\left(\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)-\tilde{F}_{1,\beta-\frac{d-1}{2}}(-h)\right)^{n-d}\mathrm{d}h,$$

$$\mathbb{E}T_{a,b}^{d,d-1}\left(\tilde{Q}_{n,d}^{\beta}\right) = \tilde{D}_{n,d}^{\beta,a,b} + \tilde{C}_{n,d}^{\beta,b}\int_{0}^{\infty} h^a\left(1+h^2\right)^{-\tilde{q}}\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-d}\,\mathrm{d}h,$$

where $\tilde{q} = d\beta - \frac{d-1}{2}(d+b+1)$ and

$$\tilde{C}_{n,d}^{\beta,b} = \binom{n}{d}d!\kappa_d\widetilde{\mathbb{E}}_{\beta}\left(\Delta_{d-1}^{b+1}\right)\left(\frac{\tilde{c}_{d,\beta}}{\tilde{c}_{d-1,\beta}}\right)^{d},$$

$$\tilde{D}_{n,d}^{\beta,a,b} = \mathbb{1}\{a=0\}\binom{n}{d-1}\frac{d\kappa_d\widetilde{\mathbb{E}}_{\beta}\left(\nabla_{d-1}^{b+1}\right)}{2^{n-d+1}((d-1)!)^b}\left(\frac{\tilde{c}_{d,\beta}}{\tilde{c}_{d-1,\beta}}\right)^{d-1}.$$

Remark 3.2.11 Theorem 3.2.8 immediately implies exact formulae for the expected facet numbers of random beta-type polytopes. Namely, setting $a = b = 0$, it follows from the definition of $T_{a,b}^{d,d-1}$ that we have

$$\mathbb{E}\mathbf{f}_{d-1}(P_{n,d}^{\beta}) = \mathbb{E}T_{0,0}^{d,d-1}(P_{n,d}^{\beta}). \tag{3.9}$$

Recall that, $\mathbf{f}_k(P) = |\mathcal{F}_k(P)|$ denotes the number of k-dimensional faces of the polytope P. The expected facet numbers of $S_{n,d}^{\beta}$, $Q_{n,d}^{\beta}$, $\tilde{P}_{n,d}^{\beta}$, $\tilde{S}_{n,d}^{\beta}$ and $\tilde{Q}_{n,d}^{\beta}$ follow analogously.

Remark 3.2.12 Since random beta-type polytopes are almost surely simplicial it follows from Theorem 2.2.3 that $\mathbf{f}_{d-2}(P^{\beta}_{n,d}) = \frac{d}{2}\mathbf{f}_{d-1}(P^{\beta}_{n,d})$ almost surely. Hence, Theorem 3.2.8 also implies the expected number of $(d-2)$-dimensional faces of $P^{\beta}_{n,d}$. The expected number of $(d-2)$-dimensional faces of $S^{\beta}_{n,d}$, $Q^{\beta}_{n,d}$, $\tilde{P}^{\beta}_{n,d}$, $\tilde{S}^{\beta}_{n,d}$ and $\tilde{Q}^{\beta}_{n,d}$ follow analogously.

Special case: Uniform distribution in the ball

The beta distribution with $\beta = 0$ is just the uniform distribution in the ball. As a special case of Theorem 3.2.1 we obtain the following

Corollary 3.2.13 *For all $d \in \mathbb{N}$ we have*

$$\mathbb{E}\,\mathrm{Vol}_d(P^0_{n,d}) = \frac{(d+1)^2\kappa_d}{2^{d+1}\pi^{\frac{d+1}{2}}}\binom{n}{d+1}\left(\frac{\Gamma(\frac{d+2}{2})}{\Gamma(\frac{d+3}{2})}\right)^{d+1}\int_{-1}^{1}(1-h^2)^{\frac{d^2+2d-1}{2}}F_{1,\frac{d-1}{2}}(h)^{n-d-1}\,\mathrm{d}h,$$

$$\mathbb{E}\,\mathrm{Vol}_d(S^0_{n,d}) = \frac{(d+1)^2\kappa_d}{\pi^{\frac{d+1}{2}}}\binom{n}{d+1}\left(\frac{\Gamma(\frac{d+2}{2})}{\Gamma(\frac{d+3}{2})}\right)^{d+1}$$
$$\times\int_{0}^{1}(1-h^2)^{\frac{d^2+2d-1}{2}}\left(F_{1,\frac{d-1}{2}}(h)-F_{1,\frac{d-1}{2}}(-h)\right)^{n-d-1}\,\mathrm{d}h.$$

Now, we specialize these formulae to dimensions $d = 2$ and $d = 3$. We start with the 2-dimensional case and remark that the formula for $\mathbb{E}\,\mathrm{Vol}_2(P^0_{n,2})$ has previously been obtained by Efron, see Equation (7.13) in [42].

Corollary 3.2.14 *In dimension $d = 2$, we have*

$$\mathbb{E}\,\mathrm{Vol}_2(P^0_{n,2}) = \frac{16}{3(2\pi)^{n-1}}\binom{n}{3}\int_0^{2\pi}(h-\sin h)^{n-3}\sin^8\left(\frac{h}{2}\right)\,\mathrm{d}h,$$

$$\mathbb{E}\,\mathrm{Vol}_2(S^0_{n,2}) = \frac{32}{3\cdot\pi^{n-1}}\binom{n}{3}\int_0^{\pi}(h+\sin h)^{n-3}\cos^8\left(\frac{h}{2}\right)\,\mathrm{d}h.$$

Proof. Since $F_{1,\frac{1}{2}}(h) = \frac{1}{2}+\frac{1}{\pi}(h\sqrt{1-h^2}+\arcsin h)$, $|h|\le 1$, the first result follows from the substitution $h = -\cos\frac{y}{2}$ and by relabeling of y by h. The second result follows from the observation that $F_{1,\frac{1}{2}}(h) - F_{1,\frac{1}{2}}(-h) = \frac{2}{\pi}(h\sqrt{1-h^2}+\arcsin h)$, $|h|\le 1$, by the substitution $h = \sin\frac{y}{2}$ and by relabeling of y by h. □

As anticipated above, the computation of $\mathbb{E}\,\mathrm{Vol}_2(P^0_{n,2})$ is due to Efron [42]. Buchta [29] obtained the following more elegant representation:

$$\mathbb{E}\,\mathrm{Vol}_2(P^0_{n,2}) = \pi + \frac{1}{3(2\pi)^{n-1}}\int_0^{2\pi}(h-\sin h)^n \sin h\,\mathrm{d}h.$$

We continue with the 3-dimensional case and again remark that the formula for $\mathbb{E}\,\mathrm{Vol}_3(P^0_{n,3})$ corresponds to Equation (7.14) in [42]. We also refer to Buchta [29], who derived another representation for the integral.

Corollary 3.2.15 *In dimension $d = 3$, we have*

$$\mathbb{E}\,\mathrm{Vol}_3(P^0_{n,3}) = \frac{27\pi}{4^{n+1}}\binom{n}{4}\int_{-1}^{1}(1-h)^7(1+h)^{2n-1}(2-h)^{n-4}\,\mathrm{d}h,$$
$$\mathbb{E}\,\mathrm{Vol}_3(S^0_{n,3}) = \frac{27\pi}{2^{n+2}}\binom{n}{4}\int_{0}^{1}(1-h^2)^7 h^{n-4}(3-h^2)^{n-4}\,\mathrm{d}h.$$

Proof. To obtain $\mathbb{E}\,\mathrm{Vol}_3(P^0_{n,3})$ we choose $d = 3$ in Corollary 3.2.13 to obtain $F_{1,1}(h) = \frac{3}{4}(\frac{2}{3} + h - \frac{h^3}{3}) = \frac{1}{4}(h-2)(1+h)^2$, $|h| \le 1$ and hence, by factorization of the function under the integral,

$$\begin{aligned}\mathbb{E}\,\mathrm{Vol}_3(P^0_{n,3}) &= \frac{27\pi}{1024}\frac{1}{4^{n-4}}\binom{n}{4}\int_{-1}^{1}(1-h^2)^7(1+h)^{2(n-4)}(h-2)^{n-4}\,\mathrm{d}h\\ &= \frac{27\pi}{4^{n+1}}\binom{n}{4}\int_{-1}^{1}(1-h)^7(1+h)^{2n-1}(2-h)^{n-4}\,\mathrm{d}h.\end{aligned}$$

Similarly, $F_{1,1}(h) - F_{1,1}(-h) = \frac{1}{2}h(3-h^2)$, $|h| \le 1$, and the result for $\mathbb{E}\,\mathrm{Vol}_3(S^0_{n,3})$ follows. □

Using the classical Efron identity, we can obtain the following formulae for the expected number of vertices.

Proposition 3.2.16 *The expected number of vertices of $P^0_{n,d}$ and $S^0_{n,d}$ is given by*

$$\mathbb{E}\mathbf{f}_0(P^0_{n,d}) = n\left(1 - \frac{\mathbb{E}\,\mathrm{Vol}_d(P^0_{n-1,d})}{\kappa_d}\right), \quad \mathbb{E}\mathbf{f}_0(S^0_{n,d}) = 2n\left(1 - \frac{\mathbb{E}\,\mathrm{Vol}_d(S^0_{n-1,d})}{\kappa_d}\right),$$

where the expected volumes on the right-hand side were given in Corollary 3.2.13.

Proof. The first formula is just the classical Efron identity, see [42]. By

$$\begin{aligned}\mathbb{E}\mathbf{f}_0(S^0_{n,d}) &= 2n\mathbb{P}\left(X_1 \in \mathcal{F}_0\left(S^0_{n,d}\right)\right)\\ &= 2n\mathbb{P}(X_1 \notin [\pm X_2, \dots, \pm X_n])\\ &= 2n\left(1 - \frac{\mathbb{E}\,\mathrm{Vol}_d(S^0_{n-1,d})}{\kappa_d}\right).\end{aligned}$$

the second formula follows. □

Moreover, we state results for the expected surface area and the expected mean width of $P^0_{n,2}$, $S^0_{n,2}$, $P^0_{n,3}$ and $S^0_{n,3}$ that follow from Theorem 3.2.4 and Corollary 3.2.6.

Corollary 3.2.17 *In dimension $d = 2$, we have*

$$\mathbb{E}S_1(P^0_{n,2}) = \frac{1}{3\pi 2^{n-4}}\binom{n}{2}\int_{-\pi/2}^{\pi/2} \cos^5 h\left(1 + \frac{\sin(2h) + 2h}{\pi}\right)^{n-2} \mathrm{d}h,$$

$$\mathbb{E}S_1(S^0_{n,2}) = \frac{64}{3\pi}\binom{n}{2}\int_{0}^{\pi/2} \cos^5 h\left(\frac{\sin(2h) + 2h}{\pi}\right)^{n-2} \mathrm{d}h.$$

Corollary 3.2.18 *In dimension $d = 3$, we have*

$$\mathbb{E}S_2(P^0_{n,3}) = \frac{81\pi}{4^{n+1}}\binom{n}{3}\int_{-1}^{1}(1-h)^5(1+h)^{2n-1}(2-h)^{n-3}\,\mathrm{d}h,$$

$$\mathbb{E}S_2(S^0_{n,3}) = \frac{81\pi}{2^{n+2}}\binom{n}{3}\int_{0}^{1}(1-h^2)^5 h^{n-3}(3-h^2)^{n-3}\,\mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_3(P^0_{n,3}) = \frac{9}{4^n}\binom{n}{2}\int_{-1}^{1}(1-h)^3(1+h)^{2n-1}(2-h)^{n-2}\,\mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_3(S^0_{n,3}) = \frac{9}{2^n}\binom{n}{2}\int_{0}^{1}(1-h^2)^3 h^{n-2}(3-h^2)^{n-2}\,\mathrm{d}h.$$

Proof. Observing that $\gamma_3 A^{\frac{1}{2}}_{n,2} = \frac{81\pi}{256}\binom{n}{3}$ and that $F_{1,1}(h) = \frac{1}{4}(2+3h-h^3) = \frac{1}{4}(1+h)^2(2-h)$, $|h| \le 1$, the formulae for $\mathbb{E}S_2(P^0_{n,3})$ and $\mathbb{E}S_2(S^0_{n,3})$ follow from Theorem 3.2.4. The values for $\mathbb{E}\mathcal{W}_3(P^0_{n,3})$ and $\mathbb{E}\mathcal{W}_3(S^0_{n,3})$ can be deduced from Corollary 3.2.6 and the fact that $A^1_{n,1} = \frac{9}{16}\binom{n}{2}$. □

Finally, let us discuss the explicit formulae for the expected number of facets of $P^0_{n,2}, P^0_{n,3}, S^0_{n,2}$ and $S^0_{n,3}$. They follow by specializing Theorem 3.2.8 to the case implied by (3.9).

Corollary 3.2.19 *In dimension $d = 2$, we have*

$$\mathbb{E}\mathbf{f}_1(P^0_{n,2}) = \frac{1}{3\pi 2^{n-4}}\binom{n}{2}\int\limits_{-\pi/2}^{\pi/2} \cos^4 h \left(1 + \frac{\sin(2h)+2h}{\pi}\right)^{n-2} \mathrm{d}h,$$

$$\mathbb{E}\mathbf{f}_1(S^0_{n,2}) = \frac{64}{3\pi}\binom{n}{2}\int\limits_{0}^{\pi/2} \cos^4 h \left(\frac{\sin(2h)+2h}{\pi}\right)^{n-2} \mathrm{d}h$$

Corollary 3.2.20 *In dimension $d = 3$ it holds that*

$$\mathbb{E}\mathbf{f}_2(P^0_{n,3}) = \frac{630}{4^{n+1}}\binom{n}{3}\int\limits_{-1}^{1} (1-h)^4(1+h)^{2n-2}(2-h)^{n-3}\,\mathrm{d}h,$$

$$\mathbb{E}\mathbf{f}_2(S^0_{n,3}) = \frac{315}{2^{n+1}}\binom{n}{3}\int\limits_{0}^{1} (1-h^2)^4 h^{n-3}(3-h^2)^{n-3}\,\mathrm{d}h.$$

Special case: Uniform distribution on the sphere

Recall that the uniform distribution on the sphere $\mathbb{S}^{d-1}$ is the weak limit of the beta distribution, as $\beta \downarrow -1$. Let $X_1, \ldots, X_n$ be independent random points chosen uniformly on the sphere $\mathbb{S}^{d-1}$. We write $P_{n,d} = [X_1, \ldots, X_n]$ for the polytope generated by $X_1, \ldots, X_n$ and $S_{n,d} = [\pm X_1, \ldots, \pm X_n]$ for its symmetrized analogue. Due to the aforementioned weak limit behavior, the expected volumes of the polytopes $P_{n,d}$ and $S_{n,d}$ can be obtained by formally taking $\beta = -1$ in Theorem 3.2.1.

Corollary 3.2.21 *For all $d \geq 2$ we have*

$$\mathbb{E}\,\mathrm{Vol}_d(P_{n,d}) = \frac{(d^2-1)\kappa_d}{2^{d+1}\pi^{\frac{d+1}{2}}}\binom{n}{d+1}\left(\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\right)^{d+1}\int_{-1}^{1}(1-h^2)^{\frac{d^2-3}{2}}F_{1,\frac{d-3}{2}}(h)^{n-d-1}\,\mathrm{d}h,$$

$$\mathbb{E}\,\mathrm{Vol}_d(S_{n,d}) = \frac{(d^2-1)\kappa_d}{\pi^{\frac{d+1}{2}}}\binom{n}{d+1}\left(\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\right)^{d+1}$$
$$\times\int_{0}^{1}(1-h^2)^{\frac{d^2-3}{2}}\left(F_{1,\frac{d-3}{2}}(h)-F_{1,\frac{d-3}{2}}(-h)\right)^{n-d-1}\,\mathrm{d}h.$$

An equivalent formula for $\mathbb{E}\,\mathrm{Vol}_d(P_{n,d})$ was obtained in [32, p. 228] by a different method; see also [81] for an asymptotic result. A different representation for $\mathbb{E}\,\mathrm{Vol}_d(P_{n,d})$ was obtained by Affentranger [2]. Again, let us specialize the result to the 2- and 3-dimensional case.

Corollary 3.2.22 *In dimension $d = 2$, we have*

$$\mathbb{E}\,\mathrm{Vol}_2(P_{n,2}) = \frac{3}{\pi^2}\binom{n}{3}\int_{0}^{\pi}\sin^2 h\left(\frac{h}{\pi}\right)^{n-3}\mathrm{d}h,$$

$$\mathbb{E}\,\mathrm{Vol}_2(S_{n,2}) = \frac{24}{\pi^2}\binom{n}{3}\int_{0}^{\pi/2}\sin^2 h\left(1-\frac{2h}{\pi}\right)^{n-3}\mathrm{d}h.$$

Proof. We have that $F_{1,-1/2}(h) = \frac{2}{\pi}\arcsin\sqrt{\frac{h+1}{2}}$ for $|h| \leq 1$. The claim follows by the change of variables $h = -\cos y$ (for the first integral) or $h = \cos y$ (for the second integral), by elementary transformations and by renaming y by h. □

Corollary 3.2.23 *In dimension $d = 3$, we have*

$$\mathbb{E}\,\mathrm{Vol}_3(P_{n,3}) = \frac{4\pi}{3}\frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}, \quad \mathbb{E}\,\mathrm{Vol}_3(S_{n,3}) = \frac{4\pi}{3}\frac{n(n-2)}{(n+1)(n+3)}.$$

The first of these formulae is due to Affentranger [2].

Proof. We used that $F_{1,0}(h) = \frac{1}{2}(1+h)$ for $|h| \le 1$ together with the formulae

$$\int_{-1}^{1} (1-h^2)^3 2^{-A}(1+h)^A \mathrm{d}h = \frac{3 \cdot 2^8}{(4+A)(5+A)(6+A)(7+A)},$$
$$\int_{0}^{1} (1-h^2)^3 h^A \mathrm{d}h = \frac{48}{(1+A)(3+A)(5+A)(7+A)},$$

where $A > 0$, and straightforward transformations. □

We are also able to give explicit formulae for the expected surface areas and the mean widths in dimensions $d = 2$ and $d = 3$.

Corollary 3.2.24 *In dimension $d = 2$, we have*

$$\mathbb{E}S_1(P_{n,2}) = \frac{4}{\pi}\binom{n}{2} \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} + \frac{h}{\pi}\right)^{n-2} \cos h \, \mathrm{d}h,$$
$$\mathbb{E}S_1(S_{n,2}) = \frac{2^{n+2}}{\pi^{n-1}}\binom{n}{2} \int_{0}^{\pi/2} h^{n-2} \cos h \, \mathrm{d}h.$$

Proof. In view of Theorem 3.2.4, the formula for $\mathbb{E}S_1(P_{n,2})$ follows from the fact that $q = 0$, $\gamma_2 A_{n,1}^{-\frac{1}{2}} = \frac{2}{\pi} n(n-1)$, $F_{1,-1/2}(h) = \frac{1}{2} + \frac{1}{\pi}\arcsin h$, $|h| \le 1$, by applying the substitution $h = \sin y$ and by renaming y by h. The case of $\mathbb{E}S_1(S_{n,2})$ is similar. □

Corollary 3.2.25 *In dimension $d = 3$, we have*

$$\mathbb{E}S_2(P_{n,3}) = 4\pi \frac{(n-1)(n-2)}{(n+1)(n+2)}, \qquad \mathbb{E}S_2(S_{n,3}) = 4\pi \frac{n-1}{n+2},$$
$$\mathbb{E}\mathcal{W}_3(P_{n,3}) = 2\,\frac{n-1}{n+1}, \qquad \mathbb{E}\mathcal{W}_3(S_{n,3}) = 2\,\frac{n}{n+1}.$$

Proof. To obtain $\mathbb{E}S_2(P_{n,3})$ and $\mathbb{E}S_2(S_{n,3})$ we note that $\gamma_3 A_{n,2}^{-\frac{1}{2}} = \frac{\pi}{16} n(n-1)(n-2)$ and $F_{1,0}(h) = \frac{h+1}{2}$, $|h| \le 1$. Since $q = 2$ in this case and since

$$\int_{-1}^{1} (1-h^2)^2 \left(\frac{h+1}{2}\right)^{n-3} \mathrm{d}h = \frac{64}{n(n+1)(n+2)},$$

$$\int_0^1 (1-h^2)^2\, h^{n-3}\,\mathrm{d}h = \frac{8}{n(n-2)(n+2)},$$

the formulae follow again from Theorem 3.2.4. The computations for $\mathbb{E}\mathcal{W}_3(P_{n,3})$ and $\mathbb{E}\mathcal{W}_3(S_{n,3})$ are similar. □

Note that particular values of $\mathbb{E}S_1(P_{n,2})$, $\mathbb{E}S_2(P_{n,3})$ and $\mathbb{E}\mathcal{W}_3(P_{n,3})$, for $n = 2,3,4$ and $n = 3,4,5$, respectively, were calculated by Buchta, Müller and Tichy [32]. We have recovered these special values and found general simple closed expressions for the 3-dimensional case that are valid for all $n \geq 4$ and that were not available before.

As above, we finally present formulae for the expected number of facets of $P_{n,2}, P_{n,3}, S_{n,2}$ and $S_{n,3}$. Before, we notice that, with probability 1,

$$\mathbf{f}_0(P_{n,d}) = n \qquad \text{and} \qquad \mathbf{f}_0(S_{n,d}) = 2n,$$

since each of the n points $X_1, \ldots, X_n$ is almost surely a vertex of $P_{n,d}$ and each of the $2n$ points $\pm X_1, \ldots, \pm X_n$ is almost surely a vertex of the symmetric random polytope $S_{n,d}$.

Corollary 3.2.26 *In dimension $d = 2$, we have*

$$\begin{aligned} \mathbf{f}_1(P_{n,2}) &= n \qquad \textit{almost surely}, \\ \mathbf{f}_1(S_{n,2}) &= 2n \qquad \textit{almost surely}. \end{aligned}$$

Proof. Indeed, $\mathbf{f}_1(P_{n,2}) = \mathbf{f}_0(P_{n,2}) = n$ and $\mathbf{f}_1(S_{n,2}) = \mathbf{f}_0(S_{n,2}) = 2n$, as argued above. □

Corollary 3.2.27 *In dimension $d = 3$ it holds that*

$$\begin{aligned} \mathbf{f}_2(P_{n,3}) &= 2(n-2) \qquad \textit{almost surely}, \\ \mathbf{f}_2(S_{n,3}) &= 4(n-1) \qquad \textit{almost surely}. \end{aligned}$$

Proof. We observe that with probability one the random beta-polytope $P_{n,3}$ is a simplicial polytope, that is, all faces of $P_{n,3}$ are almost surely triangles. This implies that $2\mathbf{f}_1(P_{n,3}) = 3\mathbf{f}_2(P_{n,3})$, which together with Euler's polyhedron formula $\mathbf{f}_0(P_{n,3}) -$

$\mathbf{f}_1(P_{n,3}) + \mathbf{f}_2(P_{n,3}) = 2$ leads to

$$\mathbf{f}_2(P_{n,3}) = 2(\mathbf{f}_0(P_{n,3}) - 2) = 2(n-2).$$

Similarly, we have that

$$\mathbf{f}_2(S_{n,3}) = 2(\mathbf{f}_0(S_{n,3}) - 2) = 2(2n-2) = 4(n-1).$$

with probability one. □

Proof of Theorem 3.2.8 and 3.2.10

Let $X_1, \dots, X_n$ be independent and beta-type distributed random points in $\mathbb{B}^d$ with parameter β. We start with the polytope $P_{n,d}^{\beta} = [X_1, \dots, X_n]$. We have

$$\begin{aligned}
&\mathbb{E}T_{a,b}^{d,d-1}\left(P_{n,d}^{\beta}\right) \\
&= \mathbb{E} \sum_{1 \le i_1 < \dots < i_d \le n} \mathbb{1}\left\{[X_{i_1}, \dots, X_{i_d}] \in \mathcal{F}_{d-1}\left(P_{n,d}^{\beta}\right)\right\} \eta^a\left([X_{i_1}, \dots, X_{i_d}]\right) \Delta_{d-1}^b\left(X_{i_1}, \dots, X_{i_d}\right) \\
&= \binom{n}{d} \mathbb{E}\left(\mathbb{1}\left\{[X_1, \dots, X_d] \in \mathcal{F}_{d-1}\left(P_{n,d}^{\beta}\right)\right\} \eta^a\left([X_1, \dots, X_d]\right) \Delta_{d-1}^b\left(X_1, \dots, X_d\right)\right) \\
&= \binom{n}{d} \int_{\left(\mathbb{R}^d\right)^d} \mathbb{P}\left([x_1, \dots, x_d] \in \mathcal{F}_{d-1}\left(P_{n,d}^{\beta}\right) \middle| X_1 = x_1, \dots, X_d = x_d\right) \eta^a\left([x_1, \dots, x_d]\right) \\
&\qquad \times \Delta_{d-1}^b\left(x_1, \dots, x_d\right) \left(\prod_{i=1}^d f_{d,\beta}(x_i)\right) \lambda_d^d\left(\mathrm{d}\left(x_1, \dots, x_d\right)\right),
\end{aligned}$$

where in the last step we conditioned on the event $\{X_1 = x_1, \dots, X_d = x_d\}$ and used the formula for the total probability. Applying the affine Blaschke-Petkantschin formula stated in Lemma 2.3.2 with $q = d-1$, we obtain

$$\begin{aligned}
\mathbb{E}T_{a,b}^{d,d-1}\left(P_{n,d}^{\beta}\right) &= \binom{n}{d}(d-1)!\frac{d\kappa_d}{2} \\
&\times \int_{A(d,d-1)} \int_{H^d} \mathbb{P}\left([x_1, \dots, x_d] \in \mathcal{F}_{d-1}\left(P_{n,d}^{\beta}\right) \middle| X_1 = x_1, \dots, X_d = x_d\right) \\
&\times \eta^a\left([x_1, \dots, x_d]\right) \Delta_{d-1}^{b+1}\left(x_1, \dots, x_d\right) \left(\prod_{i=1}^d f_{d,\beta}(x_i)\right) \lambda_H^d\left(\mathrm{d}\left(x_1, \dots, x_d\right)\right) \mu_{d-1}(\mathrm{d}H).
\end{aligned}$$

We denote by h the distance from $H = \operatorname{aff}(x_1, \dots, x_d)$ to the origin $\boldsymbol{o}$. Note that the conditional probability on the right-hand side is the probability that all $X_{d+1}, \dots, X_n$ lie in either the half-space H^+ or the half-space H^-. By using rotational invariance of the density $f_{d,\beta}$ we may assume that H has the form H_h and then apply Lemma 3.1.4 to get

$$\mathbb{P}\left([x_1, \dots, x_d] \in \mathcal{F}_{d-1}\left(P^{\beta}_{n,d}\right) \middle| X_1 = x_1, \dots, X_d = x_d\right) = \left(1 - F_{1,\beta+\frac{d-1}{2}}(h)\right)^{n-d} + F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}.$$

Since the integrand is rotationally invariant we can use formula (2.25) to rewrite the integration over $A(d, d-1)$ as

$$\begin{aligned}\mathbb{E}T^{d,d-1}_{a,b}\left(P^{\beta}_{n,d}\right) &= \binom{n}{d} d!\kappa_d \int_0^1 \left(\left(1 - F_{1,\beta+\frac{d-1}{2}}(h)\right)^{n-d} + F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\right) h^a \\ &\quad\times \int_{H^d} \Delta^{b+1}_{d-1}(x_1, \dots, x_d) \left(\prod_{i=1}^d f_{d,\beta}(x_i)\right) \lambda^d_H \left(\mathrm{d}(x_1, \dots, x_d)\right) \mathrm{d}h \\ &= \binom{n}{d} d!\kappa_d \left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^d \mathbb{E}_\beta\left(\Delta^{b+1}_{d-1}\right) \int_0^1 h^a \left(1 - h^2\right)^{d\beta + \frac{d-1}{2}(d+b+1)} \\ &\quad\times \left(\left(1 - F_{1,\beta+\frac{d-1}{2}}(h)\right)^{n-d} + F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\right) \mathrm{d}h,\end{aligned}$$

where the second equality follows from Lemma 3.1.8. In the next step we exploit the identities $f_{1,\beta+\frac{d-1}{2}}(h) = f_{1,\beta+\frac{d-1}{2}}(-h)$ and $1 - F_{1,\beta+\frac{d-1}{2}}(h) = F_{1,\beta+\frac{d-1}{2}}(-h)$ to rewrite the integral as

$$\mathbb{E}T^{d,d-1}_{a,b}\left(P^{\beta}_{n,d}\right) = C^{\beta,b}_{n,d} \int_{-1}^1 |h|^a \left(1 - h^2\right)^{d\beta + \frac{d-1}{2}(d+b+1)} F_{1,\beta+\frac{d-1}{2}}(h)^{n-d} \mathrm{d}h,$$

where

$$C^{\beta,b}_{n,d} = \binom{n}{d} d!\kappa_d \mathbb{E}_\beta\left(\Delta^{b+1}_{d-1}\right) \left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^d. \tag{3.10}$$

Slight adaptation of this proof yields the result for the symmetric polytope $S^{\beta}_{n,d} =$

$[\pm X_1, \ldots, \pm X_d]$ as follows:

$$\begin{aligned}
&\mathbb{E}T_{a,b}^{d,d-1}\left(S_{n,d}^{\beta}\right)\\
&=\mathbb{E}\Bigg(\sum_{1\le i_1<\cdots<i_d\le n}\ \sum_{j_1,\ldots,j_d\in\{0,1\}} \mathbb{1}\left\{\left[(-1)^{j_1}X_{i_1},\ldots,(-1)^{j_d}X_{i_d}\right]\in\mathcal{F}_{d-1}\left(S_{n,d}^{\beta}\right)\right\}\\
&\qquad\times\eta^a\left(\left[(-1)^{j_1}X_{i_1},\ldots,(-1)^{j_d}X_{i_d}\right]\right)\Delta_{d-1}^b\left((-1)^{j_1}X_{i_1},\ldots,(-1)^{j_d}X_{i_d}\right)\Bigg)\\
&=2^d\mathbb{E}\sum_{1\le i_1<\cdots<i_d\le n}\mathbb{1}\left\{[X_{i_1},\ldots,X_{i_d}]\in\mathcal{F}_{d-1}(S_{n,d}^{\beta})\right\}\eta^a\left([X_{i_1},\ldots,X_{i_d}]\right)\Delta_{d-1}^b\left(X_{i_1},\ldots,X_{i_d}\right)\\
&=2^d\binom{n}{d}\mathbb{E}\left(\mathbb{1}\left\{[X_1,\ldots,X_d]\in\mathcal{F}_{d-1}\left(S_{n,d}^{\beta}\right)\right\}\eta^a\left([X_1,\ldots,X_d]\right)\Delta_{d-1}^b\left(X_1,\ldots,X_d\right)\right)\\
&=2^d\binom{n}{d}\int\limits_{\left(\mathbb{R}^d\right)^d}\mathbb{P}\left([x_1,\ldots,x_d]\in\mathcal{F}_{d-1}\left(S_{n,d}^{\beta}\right)\Big|X_1=x_1,\ldots,X_d=x_d\right)\eta^a\left([x_1,\ldots,x_d]\right)\\
&\qquad\times\Delta_{d-1}^b\left(x_1,\ldots,x_d\right)\left(\prod_{i=1}^d f_{d,\beta}(x_i)\right)\lambda_d^d\left(\mathrm{d}\left(x_1,\ldots,x_d\right)\right).
\end{aligned}$$

Applying the affine Blaschke-Petkantschin formula, see Theorem 2.3.2, we get

$$\begin{aligned}
\mathbb{E}T_{a,b}^{d,d-1}\left(S_{n,d}^{\beta}\right)&=2^d\binom{n}{d}(d-1)!\frac{d\kappa_d}{2}\\
&\times\int\limits_{A(d,d-1)}\int\limits_{H^d}\mathbb{P}\left([x_1,\ldots,x_d]\in\mathcal{F}_{d-1}\left(S_{n,d}^{\beta}\right)\Big|X_1=x_1,\ldots,X_d=x_d\right)\\
&\times\eta^a\left([x_1,\ldots,x_d]\right)\Delta_{d-1}^{b+1}\left(x_1,\ldots,x_d\right)\left(\prod_{i=1}^d f_{d,\beta}(x_i)\right)\lambda_H^d\left(\mathrm{d}\left(x_1,\ldots,x_d\right)\right)\mu_{d-1}(\mathrm{d}H).
\end{aligned}$$

Let $h\in[0,1]$ be the distance from $H=\operatorname{aff}(x_1,\ldots,x_d)$ to the origin. The conditional probability on the right-hand side is the probability that the points $X_{d+1},\ldots,X_n$ lie between the hyperplanes H and $-H$. Therefore, by Lemma 3.1.4,

$$\begin{aligned}
&\mathbb{P}\left([X_1,\ldots,X_d]\in\mathcal{F}_{d-1}\left(S_{n,d}^{\beta}\right)\Big|X_1=x_1,\ldots,X_d=x_d\right)\\
&\qquad\qquad=\left(F_{1,\beta+\frac{d-1}{2}}(h)-F_{1,\beta+\frac{d-1}{2}}(-h)\right)^{n-d}.
\end{aligned}$$

All the remaining steps are exactly the same as before (except for the last step, where we cannot exploit symmetry this time). Hence,

$$\mathbb{E}T_{a,b}^{d,d-1}\left(S_{n,d}^{\beta}\right) = 2^d C_{n,d}^{\beta,b} \int_0^1 h^a \left(1-h^2\right)^{d\beta+\frac{d-1}{2}(d+b+1)} \times \left(F_{1,\beta+\frac{d-1}{2}}(h) - F_{1,\beta+\frac{d-1}{2}}(-h)\right)^{n-d} \mathrm{d}h,$$

where $C_{n,d}^{\beta,b}$ is the same as in (3.10).

The derivation of the result for the polytope $Q_{n,d}^{\beta} = [\boldsymbol{o}, X_1, \ldots, X_n]$ needs a case distinction. Namely, we need to distinguish between facets that contain $\boldsymbol{o}$ as a vertex and facets which do not. Furthermore, facets containing $\boldsymbol{o}$ only contribute to $\mathbb{E}T_{a,b}^{d,d-1}\left(Q_{n,d}^{\beta}\right)$ in the case that the parameter a equals zero. Hence,

$$\begin{aligned}
\mathbb{E}T_{a,b}^{d,d-1}\left(Q_{n,d}^{\beta}\right) &= \mathbb{E}\Bigg(\mathbb{1}\{a=0\} \\
&\times \sum_{1\le i_1<\cdots<i_{d-1}\le n} \mathbb{1}\left\{[\boldsymbol{o}, X_{i_1}, \ldots, X_{i_{d-1}}] \in \mathcal{F}_{d-1}(Q_{n,d}^{\beta})\right\} \Delta_{d-1}^b\left(\boldsymbol{o}, X_{i_1}, \ldots, X_{i_{d-1}}\right) \\
&+ \sum_{1\le i_1<\cdots<i_d\le n} \mathbb{1}\left\{[X_{i_1}, \ldots, X_{i_d}] \in \mathcal{F}_{d-1}(Q_{n,d}^{\beta})\right\} \eta^a\left([X_{i_1}, \ldots, X_{i_d}]\right) \Delta_{d-1}^b\left(X_{i_1}, \ldots, X_{i_d}\right)\Bigg) \\
&= \mathbb{1}\{a=0\}\binom{n}{d-1}\mathbb{E}\left(\mathbb{1}\left\{[\boldsymbol{o}, X_1, \ldots, X_{d-1}] \in \mathcal{F}_{d-1}(Q_{n,d}^{\beta})\right\} \Delta_{d-1}^b\left(\boldsymbol{o}, X_1, \ldots, X_{d-1}\right)\right) \\
&+ \binom{n}{d}\mathbb{E}\left(\mathbb{1}\left\{[X_1, \ldots, X_d] \in \mathcal{F}_{d-1}(Q_{n,d}^{\beta})\right\} \eta^a\left([X_1, \ldots, X_d]\right) \Delta_{d-1}^b\left(X_1, \ldots, X_d\right)\right) \\
&= \mathbb{1}\{a=0\}\binom{n}{d-1} \int_{\left(\mathbb{R}^d\right)^{d-1}} \mathbb{P}\left([\boldsymbol{o}, x_1, \ldots, x_{d-1}] \in \mathcal{F}_{d-1}(Q_{n,d}^{\beta}) \middle| X_i = x_i, i \in [d-1]\right) \\
&\qquad \times \Delta_{d-1}^b(\boldsymbol{o}, x_1, \ldots, x_{d-1}) \left(\prod_{i=1}^{d-1} f_{d,\beta}(x_i)\right) \lambda_d^{d-1}\left(\mathrm{d}(x_1, \ldots, x_{d-1})\right) \\
&+ \binom{n}{d} \int_{\left(\mathbb{R}^d\right)^{d}} \mathbb{P}\left([x_1, \ldots, x_d] \in \mathcal{F}_{d-1}(Q_{n,d}^{\beta}) \middle| X_i = x_i, i \in [d-1]\right) \eta^a\left([x_1, \ldots, x_d]\right) \\
&\qquad \times \Delta_{d-1}^b(x_1, \ldots, x_d) \left(\prod_{i=1}^{d} f_{d,\beta}(x_i)\right) \lambda_d^{d}\left(\mathrm{d}(x_1, \ldots, x_d)\right).
\end{aligned}$$

Now observe that if $X_1 = x_1, \ldots, X_{d-1} = x_{d-1}$, then $[\boldsymbol{o}, x_1, \ldots, x_{d-1}]$ is a face of $Q_{n,d}^{\beta}$ if and only if the points $X_d, \ldots, X_n$ are on the same side of the hyperplane passing through $\boldsymbol{o}, x_1, \dots, x_{d-1}$. It immediately follows that

$$\mathbb{P}\left([0, x_1, \dots, x_{d-1}] \in \mathcal{F}_{d-1}\left(Q_{n,d}^{\beta}\right) \Big| X_1 = x_1, \dots, X_{d-1} = x_{d-1}\right) = 2 \cdot 2^{-(n-d+1)} = 2^{-(n-d)}.$$

Furthermore, by introducing the hyperplane $H = \operatorname{aff}(x_1, \dots, x_d) \in A(d, d-1)$, we immediately see that the probability $\mathbb{P}\left([x_1, \dots, x_d] \in \mathcal{F}_{d-1}\left(Q_{n,d}^{\beta}\right) \Big| X_1 = x_1, \dots, X_d = x_d\right)$ is the same as the probability that all points $\pi_{H^\perp}(X_{d+1}), \dots, \pi_{H^\perp}(X_n)$ lie on the same side of $\pi_{H^\perp}(H)$ on which $\boldsymbol{o}$ lies. By Lemma 3.1.3, we therefore have

$$\mathbb{P}\left([x_1, \dots, x_d] \in \mathcal{F}_{d-1}\left(Q_{n,d}^{\beta}\right) \Big| X_1 = x_1, \dots, X_d = x_d\right) = F_{1,\beta+\frac{d-1}{2}}(h)^{n-d},$$

where $h \in [0,1]$ is the distance from H to the origin $\boldsymbol{o}$. Using these observations, the relation between the volumes $\Delta_{d-1}(\boldsymbol{o}, x_1, \dots, x_{d-1})$ and $\nabla_{d-1}(x_1, \dots, x_{d-1})$ in Equation (2.29), the linear Blaschke-Petkantschin formula (see Lemma 2.3.1) for the first summand, the affine Blaschke-Petkantschin formula for the second summand (see Lemma 2.3.2), and exploiting the rotational symmetry of the density, we get

$$\begin{aligned}
\mathbb{E}T_{a,b}^{d,d-1}\left(Q_{n,d}^{\beta}\right) &= \mathbb{1}\{a=0\}\binom{n}{d-1}\frac{d\kappa_d}{2^{n-d+1}((d-1)!)^b}\left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^{d-1} \\
&\times \int\limits_{G(d,d-1)}\int\limits_{L^{d-1}} \nabla_{d-1}^{b+1}(x_1, \dots, x_{d-1})\left(\prod_{i=1}^{d-1} f_{d-1,\beta}(x_i)\right)\lambda_L^{d-1}\left(\mathrm{d}(x_1, \dots, x_{d-1})\right)\nu_{d-1}(\mathrm{d}L) \\
&+ \binom{n}{d}\frac{d!\kappa_d}{2}\int\limits_{A(d,d-1)}\int\limits_{H^d} \mathbb{P}\left([x_1, \dots, x_d] \in \mathcal{F}_{d-1}\left(Q_{n,d}^{\beta}\right) \Big| X_1 = x_1, \dots, X_d = x_d\right) \\
&\times \eta^a\left([x_1, \dots, x_d]\right)\Delta_{d-1}^{b+1}(x_1, \dots, x_d)\left(\prod_{i=1}^{d} f_{d,\beta}(x_i)\right)\lambda_H^d\left(\mathrm{d}(x_1, \dots, x_d)\right)\mu_{d-1}(\mathrm{d}H) \\
&= \mathbb{1}\{a=0\}\binom{n}{d-1}\frac{d\kappa_d}{2^{n-d+1}((d-1)!)^b}\left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^{d-1}\mathbb{E}_\beta\left(\nabla_{d-1}^{b+1}\right) \\
&+ \binom{n}{d} d!\kappa_d \int\limits_0^1 h^a F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\int\limits_{H^d} \Delta_{d-1}^{b+1}(x_1, \dots, x_d)\prod_{i=1}^{d} f_{d,\beta}(x_i)\,\lambda_H^d\left(\mathrm{d}(x_1, \dots, x_d)\right)\mathrm{d}h \\
&= D_{n,d}^{\beta,a,b} + C_{n,d}^{\beta,b}\int\limits_0^1 h^a(1-h^2)^{d\beta+\frac{d-1}{2}(d+b+1)}F_{1,\beta+\frac{d-1}{2}}(h)^{n-d}\,\mathrm{d}h,
\end{aligned}$$

for all $a, b \geq 0$. For the last equation we followed again along the lines of the proof for the polytope $P_{n,d}^{\beta}$.

One can do the analogous computations for the beta′-type distribution. In this case one has to pay attention to the different range of integration, probability contents provided in Lemma 3.1.4 and transformation provided in Lemma 3.1.8. □

Expected Lebesgue volume and intrinsic volumes: Proof of Theorems 3.2.1 and 3.2.2

Proof of Theorems 3.2.1 and 3.2.2. We start by investigating the case of a beta-type distribution with parameter $\beta > -1$. Let first $\beta > -\frac{1}{2}$. For an arbitrary linear hyperplane $L \in G(d+1, d)$, Lemma 3.1.3 implies

$$P_L\left(P_{n,d+1}^{\beta-\frac{1}{2}}\right) \overset{d}{=} P_{n,d}^{\beta}, \tag{3.11}$$

where $\overset{d}{=}$ indicated equality in distribution. By Kubota's formula stated in (2.26), we have

$$V_d\left(P_{n,d+1}^{\beta-\frac{1}{2}}\right) = (d+1)\frac{\kappa_{d+1}}{2\kappa_d}\int\limits_{G(d+1,d)} \mathrm{Vol}_d\left(P_L\left(P_{n,d+1}^{\beta-\frac{1}{2}}\right)\right)\,\nu_d(\mathrm{d}L).$$

Taking the expectation, using Fubini's theorem to change the order of integration, and applying (3.11), we obtain

$$\begin{aligned}\mathbb{E}V_d\left(P_{n,d+1}^{\beta-\frac{1}{2}}\right) &= (d+1)\frac{\kappa_{d+1}}{2\kappa_d}\int\limits_{G(d+1,d)} \mathbb{E}\,\mathrm{Vol}_d\left(P_L\left(P_{n,d+1}^{\beta-\frac{1}{2}}\right)\right)\,\nu_d(\mathrm{d}L)\\ &= (d+1)\frac{\kappa_{d+1}}{2\kappa_d}\int\limits_{G(d+1,d)} \mathbb{E}\,\mathrm{Vol}_d\left(P_{n,d}^{\beta}\right)\,\nu_d(\mathrm{d}L)\\ &= (d+1)\frac{\kappa_{d+1}}{2\kappa_d}\mathbb{E}\,\mathrm{Vol}_d\left(P_{n,d}^{\beta}\right).\end{aligned}$$

Since the dth intrinsic volume V_d of a $(d+1)$-dimensional polytope is half its surface area, we can write the above in terms of the T-functional with $a = 0$ and $b = 1$ as follows:

$$\mathbb{E}\,\mathrm{Vol}_d\left(P_{n,d}^{\beta}\right) = \frac{2\kappa_d}{(d+1)\kappa_{d+1}}\mathbb{E}V_d\left(P_{n,d+1}^{\beta-\frac{1}{2}}\right) = \frac{\kappa_d}{(d+1)\kappa_{d+1}}\mathbb{E}T_{0,1}^{d+1,d}\left(P_{n,d+1}^{\beta-\frac{1}{2}}\right). \tag{3.12}$$

Using Theorem 3.2.8, we obtain

$$\mathbb{E}\,\mathrm{Vol}_d\left(P_{n,d}^{\beta}\right) = A_{n,d}^{\beta}\int\limits_{-1}^{1}\left(1-h^2\right)^{(d+1)\left(\beta-\frac{1}{2}\right)+\frac{d}{2}(d+3)} F_{1,\beta+\frac{d-1}{2}}(h)^{n-d-1}\,\mathrm{d}h, \tag{3.13}$$

where

$$A_{n,d}^{\beta} = \frac{\kappa_d}{(d+1)\kappa_{d+1}} C_{n,d+1}^{\beta-\frac{1}{2},1} = \frac{(d+1)\kappa_d}{2^d\pi^{\frac{d+1}{2}}}\binom{n}{d+1}\left(\beta+\frac{d+1}{2}\right)\left(\frac{\Gamma\left(\frac{d+2}{2}+\beta\right)}{\Gamma\left(\frac{d+3}{2}+\beta\right)}\right)^{d+1}.$$

In order to derive the last formula we used elementary transformations involving Lemma 3.1.6 and the Legendre duplication formula for the gamma function. So far, we established formula (3.13) for $\beta > -\frac{1}{2}$ only because the proof was based on representation (3.11). In order to prove that (3.13) holds in the full range $\beta > -1$, we argue again by analytic continuation. First of all, the function $\beta \mapsto \mathbb{E}\,\mathrm{Vol}_d(P_{n,d}^{\beta})$ is real analytic in $\beta > -1$ as one can see from the integral representation

$$\mathbb{E}\,\mathrm{Vol}_d\left(P_{n,d}^{\beta}\right) = \int\limits_{(\mathbb{B}^d)^n} \mathrm{Vol}_d([x_1,\dots,x_n])\left(\prod_{i=1}^{n} f_{d,\beta}(x_i)\right)\lambda_d^n\left(\mathrm{d}\left(x_1,\dots,x_n\right)\right).$$

Secondly, the function on the right-hand side of (3.13) is also real analytic in $\beta > -1$. Since these functions coincide for $\beta > -\frac{1}{2}$, they must coincide in the full range $\beta > -1$.

Similarly, we obtain in the case of a beta′-type distribution with parameter β the intrinsic volume V_d of $\tilde{P}_{n,d}^{\beta}$ as

$$\mathbb{E}V_d\left(\tilde{P}_{n,d}^{\beta}\right) = \frac{\kappa_d}{(d+1)\kappa_{d+1}}\mathbb{E}T_{0,1}^{d+1,d}\left(\tilde{P}_{n,d+1}^{\beta+\frac{1}{2}}\right), \tag{3.14}$$

from which we get by means of Theorem 3.2.10

$$\mathbb{E}V_d\left(\tilde{P}_{n,d}^{\beta}\right) = \tilde{A}_{n,d}^{\beta}\int\limits_{-\infty}^{\infty}\left(1+h^2\right)^{-(d+1)\left(\beta+\frac{1}{2}\right)+\frac{d}{2}(d+3)} \tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-d-1}\,\mathrm{d}h \tag{3.15}$$

with

$$\tilde{A}_{n,d}^{\beta} = \frac{\kappa_d}{(d+1)\kappa_{d+1}} \tilde{C}_{n,d+1}^{\beta+\frac{1}{2},1} = \frac{(d+1)\kappa_d}{2^d\pi^{\frac{d+1}{2}}}\binom{n}{d+1}\left(\beta-\frac{d+1}{2}\right)\left(\frac{\Gamma\left(\beta-\frac{d+1}{2}\right)}{\Gamma\left(\beta-\frac{d}{2}\right)}\right)^{d+1}.$$

To obtain the results for $S^{\beta}_{n,d}$, $Q^{\beta}_{n,d}$, $\tilde{S}^{\beta}_{n,d}$ and $\tilde{Q}^{\beta}_{n,d}$ we only need to replace the corresponding constants and indices from Theorems 3.2.8, 3.2.10 with the ones obtained here. □

Proof of Proposition 3.2.3. The idea is to represent the expected intrinsic volume as the expected volume of the random projection by means of Kubota's formula. For every linear subspace $L \in G(d,k)$, Lemma 3.1.3 yields the representation

$$P_L\left(P^{\beta}_{n,d}\right) \stackrel{d}{=} P^{\beta+\frac{d-k}{2}}_{n,k}.$$

Using Kubota's formula, see (2.26), in conjunction with Fubini's theorem and the above representation, we get

$$\begin{aligned}\mathbb{E}V_k\left(P^{\beta}_{n,d}\right) &= \binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}\int\limits_{G(d,k)} \mathbb{E}\,\mathrm{Vol}_k\left(P_L\left(P^{\beta}_{n,d}\right)\right)\nu_k(\mathrm{d}L)\\ &= \binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}\int\limits_{G(d,k)} \mathbb{E}\,\mathrm{Vol}_k\left(P^{\beta+\frac{d-k}{2}}_{n,k}\right)\nu_k(\mathrm{d}L)\\ &= \binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}\mathbb{E}\,\mathrm{Vol}_k\left(P^{\beta+\frac{d-k}{2}}_{n,k}\right).\end{aligned}$$

In the beta' case, by Lemma 3.1.3, i.e., the representation

$$P_L\left(\tilde{P}^{\beta}_{n,d}\right) \stackrel{d}{=} \tilde{P}^{\beta-\frac{d-k}{2}}_{n,k}$$

for every linear subspace $L \in G(d,k)$, and the same arguments as before, we have

$$\mathbb{E}V_k\left(\tilde{P}^{\beta}_{n,d}\right) = \binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}\mathbb{E}\,\mathrm{Vol}_k\left(\tilde{P}^{\beta-\frac{d-k}{2}}_{n,k}\right).$$

The corresponding results for $S^{\beta}_{n,d}$, $Q^{\beta}_{n,d}$, $\tilde{S}^{\beta}_{n,d}$ and $\tilde{Q}^{\beta}_{n,d}$ hold with the same argumentation. □

3.3 Functionals of Poisson-beta- and Poisson-beta′-polytopes

In this section we consider the Poissonized versions of beta- and beta′-type polytopes. We will only prove Theorem 3.3.8 and 3.3.9, since all the other proofs follow just analogously as in the non-Poisson case. Furthermore, we refrain from giving formulas for the special cases of the uniform distribution in the ball or on the sphere.

Expected volumes and intrinsic volumes

Theorem 3.3.1 *Let $X_1, \dots, X_N$ be independent beta-type distributed random points in $\mathbb{B}^d$ with parameter $\beta > -1$ and N be a Poisson distributed random variable with intensity t. Then,*

$$\begin{aligned}
\mathbb{E}\,\mathrm{Vol}_d(\mathcal{P}_{t,d}^{\beta}) &= \mathcal{A}_{t,d}^{\beta}\int_{-1}^{1}\left(1-h^2\right)^q \exp\left(-tF_{1,\beta+\frac{d-1}{2}}(h)\right)\,\mathrm{d}h,\\
\mathbb{E}\,\mathrm{Vol}_d(\mathcal{S}_{t,d}^{\beta}) &= 2^{d+1}\mathcal{A}_{t,d}^{\beta}\int_{0}^{1}\left(1-h^2\right)^q\\
&\qquad\qquad\times\exp\left(-t\left(F_{1,\beta+\frac{d-1}{2}}(h)-F_{1,\beta+\frac{d-1}{2}}(-h)\right)\right)\,\mathrm{d}h,\\
\mathbb{E}\,\mathrm{Vol}_d(\mathcal{Q}_{t,d}^{\beta}) &= \mathcal{D}_{t,d}^{\beta}+\mathcal{A}_{t,d}^{\beta}\int_{0}^{1}\left(1-h^2\right)^q \exp\left(-tF_{1,\beta+\frac{d-1}{2}}(h)\right)\,\mathrm{d}h,
\end{aligned}\tag{3.16}$$

where $q = (d+1)\left(\beta-\frac{1}{2}\right)+\frac{d}{2}(d+3)$ and

$$\begin{aligned}
\mathcal{A}_{t,d}^{\beta} &= \frac{\kappa_d\, t^d}{2^d\pi^{\frac{d+1}{2}}d!}\left(\beta+\frac{d+1}{2}\right)\left(\frac{\Gamma\left(\frac{d+2}{2}+\beta\right)}{\Gamma\left(\frac{d+3}{2}+\beta\right)}\right)^{d+1},\\
\mathcal{D}_{t,d}^{\beta} &= \frac{\kappa_d\, t^d}{\pi^{\frac{d}{2}}\mathrm{e}^{\frac{t}{2}}}\left(\frac{\Gamma\left(\frac{d+2}{2}+\beta\right)}{\Gamma\left(\frac{d+3}{2}+\beta\right)}\right)^{d}.
\end{aligned}\tag{3.17}$$

Theorem 3.3.2 *Let $X_1, \dots, X_N$ be independent beta′-type distributed random points in $\mathbb{R}^d$ with parameter $\beta > \frac{d+1}{2}$ and N be a Poisson distributed random variable with intensity t. Then,*

$$\mathbb{E}\,\mathrm{Vol}_d(\tilde{\mathcal{P}}^{\beta}_{t,d}) = \tilde{\mathcal{A}}^{\beta}_{t,d} \int\limits_{-\infty}^{\infty} (1+h^2)^{-\tilde{q}} \exp\left(-t\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

$$\mathbb{E}\,\mathrm{Vol}_d(\tilde{\mathcal{S}}^{\beta}_{t,d}) = 2^{d+1}\tilde{\mathcal{A}}^{\beta}_{t,d} \int\limits_{0}^{\infty} (1+h^2)^{-\tilde{q}} \exp\left(-t\left(\tilde{F}_{1,\beta-\frac{d-1}{2}}(h) - \tilde{F}_{1,\beta-\frac{d-1}{2}}(-h)\right)\right) \mathrm{d}h,$$

$$\mathbb{E}\,\mathrm{Vol}_d(\tilde{\mathcal{Q}}^{\beta}_{t,d}) = \tilde{\mathcal{D}}^{\beta}_{t,d} + \tilde{\mathcal{A}}^{\beta}_{t,d} \int\limits_{0}^{\infty} (1+h^2)^{-\tilde{q}} \exp\left(-t\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

where $\tilde{q} = (d+1)(\beta+\frac{1}{2}) - \frac{d}{2}(d+3)$ and

$$\tilde{\mathcal{A}}^{\beta}_{t,d} = \frac{\kappa_d\, t^d}{2^d \pi^{\frac{d+1}{2}} d!}\left(\beta - \frac{d+1}{2}\right)\left(\frac{\Gamma\left(\beta-\frac{d+1}{2}\right)}{\Gamma\left(\beta-\frac{d}{2}\right)}\right)^{d+1},$$

$$\tilde{\mathcal{D}}^{\beta}_{t,d} = \frac{\kappa_d\, t^d}{\pi^{\frac{d}{2}} e^{\frac{t}{2}}}\binom{n}{d}\left(\frac{\Gamma\left(\beta-\frac{d+1}{2}\right)}{\Gamma\left(\beta-\frac{d}{2}\right)}\right)^{d+1}.$$

The formulae for the expected intrinsic volumes can be obtained using the following proposition.

Proposition 3.3.3 *The expected intrinsic volumes $\mathbb{E}V_k(\mathcal{P}^{\beta}_{t,d})$ and $\mathbb{E}V_k(\tilde{\mathcal{P}}^{\beta}_{t,d})$ for $k = 1, \dots, d$ are given by the formulae*

$$\mathbb{E}V_k(\mathcal{P}^{\beta}_{t,d}) = \binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}\mathbb{E}\,\mathrm{Vol}_k\left(\mathcal{P}^{\beta+\frac{d-k}{2}}_{t,k}\right),$$

$$\mathbb{E}V_k(\tilde{\mathcal{P}}^{\beta}_{t,d}) = \binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}\mathbb{E}\,\mathrm{Vol}_k\left(\tilde{\mathcal{P}}^{\beta-\frac{d-k}{2}}_{t,k}\right).$$

These formulae hold if $\mathcal{P}^{\beta}_{t,d}$, respectively $\tilde{\mathcal{P}}^{\beta}_{t,d}$, is replaced by $\mathcal{S}^{\beta}_{t,d}$ or $\mathcal{Q}^{\beta}_{t,d}$, respectively $\tilde{\mathcal{S}}^{\beta}_{t,d}$ or $\tilde{\mathcal{Q}}^{\beta}_{t,d}$.

Expected surface area and expected mean width

In particular, Proposition 3.3.3 implies formulae for the expected surface area of the polytopes $\mathcal{P}_{t,d}^{\beta}$, $\mathcal{S}_{t,d}^{\beta}$, $\mathcal{Q}_{t,d}^{\beta}$, $\tilde{\mathcal{P}}_{t,d}^{\beta}$, $\tilde{\mathcal{S}}_{t,d}^{\beta}$ and $\tilde{\mathcal{Q}}_{t,d}^{\beta}$.

Corollary 3.3.4 *Let $X_1, \dots, X_N$ be independent beta-type distributed random points in $\mathbb{B}^d$ with parameter $\beta > -1$ and N be a Poisson distributed random variable with intensity t. Then,*

$$\mathbb{E}S_{d-1}(\mathcal{P}_{t,d}^{\beta}) = \gamma_d \mathcal{A}_{t,d-1}^{\beta+\frac{1}{2}} \int_{-1}^{1} \left(1-h^2\right)^q \exp\left(-tF_{1,\beta+\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

$$\mathbb{E}S_{d-1}(\mathcal{S}_{t,d}^{\beta}) = 2^d \gamma_d \mathcal{A}_{t,d-1}^{\beta+\frac{1}{2}} \int_{0}^{1} \left(1-h^2\right)^q \exp\left(-t\left(F_{1,\beta+\frac{d-1}{2}}(h) - F_{1,\beta+\frac{d-1}{2}}(-h)\right)\right) \mathrm{d}h,$$

$$\mathbb{E}S_{d-1}(\mathcal{Q}_{t,d}^{\beta}) = \gamma_d \left(\mathcal{D}_{t,d-1}^{\beta+\frac{1}{2}} + \mathcal{A}_{t,d-1}^{\beta+\frac{1}{2}} \int_{0}^{1} \left(1-h^2\right)^q \exp\left(-tF_{1,\beta+\frac{d-1}{2}}(h)\right) \mathrm{d}h\right),$$

where $q = d\beta + \frac{d-1}{2}(d+2)$ and $\gamma_d = \frac{d\kappa_d}{\kappa_{d-1}}$. The constants $\mathcal{A}_{t,d}^{\beta}$ and $\mathcal{D}_{t,d}^{\beta}$ are the same as in Theorem 3.3.1.

Corollary 3.3.5 *Let $X_1, \dots, X_N$ be independent beta′-type distributed random points in $\mathbb{R}^d$ with parameter $\beta > \frac{d+1}{2}$ and N be a Poisson distributed random variable with intensity t. Then,*

$$\mathbb{E}S_{d-1}(\tilde{\mathcal{P}}_{t,d}^{\beta}) = \gamma_d \tilde{\mathcal{A}}_{t,d-1}^{\beta-\frac{1}{2}} \int_{-\infty}^{\infty} (1+h^2)^{-\tilde{q}} \exp\left(-t\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

$$\mathbb{E}S_{d-1}(\tilde{\mathcal{S}}_{t,d}^{\beta}) = 2^d \gamma_d \tilde{\mathcal{A}}_{t,d-1}^{\beta-\frac{1}{2}} \int_{0}^{\infty} (1+h^2)^{-\tilde{q}} \exp\left(-t\left(\tilde{F}_{1,\beta-\frac{d-1}{2}}(h) - \tilde{F}_{1,\beta-\frac{d-1}{2}}(-h)\right)\right) \mathrm{d}h,$$

$$\mathbb{E}S_{d-1}(\tilde{\mathcal{Q}}_{t,d}^{\beta}) = \gamma_d \left(\tilde{\mathcal{D}}_{t,d-1}^{\beta-\frac{1}{2}} + \tilde{\mathcal{A}}_{t,d-1}^{\beta-\frac{1}{2}} \int_{0}^{\infty} (1+h^2)^{-\tilde{q}} \exp\left(-t\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)\right) \mathrm{d}h\right),$$

where $\tilde{q} = d\beta - \frac{d-1}{2}(d+2)$ and $\gamma_d = \frac{d\kappa_d}{\kappa_{d-1}}$. The constants $\tilde{\mathcal{A}}_{t,d}^{\beta}$ and $\tilde{\mathcal{D}}_{t,d}^{\beta}$ are the same as in Theorem 3.3.2.

Similarly, we can find explicit formulae for the expectation of the width of these random polytopes.

Corollary 3.3.6 *Let $X_1, \ldots, X_N$ be independent beta-type distributed random points in $\mathbb{B}^d$ with parameter $\beta > -1$ and N be a Poisson distributed random variable with intensity t. Then,*

$$\mathbb{E}\mathcal{W}_d(\mathcal{P}_{t,d}^{\beta}) = \mathcal{A}_{t,1}^{\beta+\frac{d-1}{2}} \int_{-1}^{1} \left(1-h^2\right)^q \exp\left(-tF_{1,\beta+\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_d(\mathcal{S}_{t,d}^{\beta}) = 4\mathcal{A}_{t,1}^{\beta+\frac{d-1}{2}} \int_{0}^{1} \left(1-h^2\right)^q \exp\left(-t\left(F_{1,\beta+\frac{d-1}{2}}(h) - F_{1,\beta+\frac{d-1}{2}}(-h)\right)\right) \mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_d(\mathcal{Q}_{t,d}^{\beta}) = \mathcal{D}_{t,1}^{\beta+\frac{d-1}{2}} + \mathcal{A}_{n,1}^{\beta+\frac{d-1}{2}} \int_{0}^{1} \left(1-h^2\right)^q \exp\left(-tF_{1,\beta+\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

where $q = 2\beta + d$. The two constants $\mathcal{A}_{t,d}^{\beta}$ and $\mathcal{D}_{t,d}^{\beta}$ are the same as the ones appearing in Theorem 3.3.1.

Corollary 3.3.7 *Let $X_1, \ldots, X_N$ be independent beta′-type distributed random points in $\mathbb{R}^d$ with parameter $\beta > \frac{d+1}{2}$ and N be a Poisson distributed random variable with intensity t. Then,*

$$\mathbb{E}\mathcal{W}_d(\tilde{\mathcal{P}}_{t,d}^{\beta}) = \tilde{\mathcal{A}}_{t,1}^{\beta-\frac{d-1}{2}} \int_{-\infty}^{\infty} (1+h^2)^{-\tilde{q}} \exp\left(-t\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_d(\tilde{\mathcal{S}}_{t,d}^{\beta}) = 4\tilde{\mathcal{A}}_{t,1}^{\beta-\frac{d-1}{2}} \int_{0}^{\infty} (1+h^2)^{-\tilde{q}} \exp\left(-t\left(\tilde{F}_{1,\beta-\frac{d-1}{2}}(h) - \tilde{F}_{1,\beta-\frac{d-1}{2}}(-h)\right)\right) \mathrm{d}h,$$

$$\mathbb{E}\mathcal{W}_d(\tilde{\mathcal{Q}}_{t,d}^{\beta}) = \tilde{\mathcal{D}}_{t,1}^{\beta-\frac{d-1}{2}} + \tilde{\mathcal{A}}_{t,1}^{\beta-\frac{d-1}{2}} \int_{0}^{\infty} (1+h^2)^{-\tilde{q}} \exp\left(-t\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

where $\tilde{q} = 2\beta - d$. The two constants $\tilde{\mathcal{A}}_{t,d}^{\beta}$ and $\tilde{\mathcal{D}}_{t,d}^{\beta}$ are the same as the ones appearing in Theorem 3.3.2.

Expectation of the T-functional

Theorem 3.3.8 *Fix $a, b \geq 0$. Let $X_1, \dots, X_N$ be independent beta-type distributed random points in $\mathbb{B}^d$ with parameter $\beta > -1$ and N be a Poisson distributed random variable with intensity t. Then,*

$$\mathbb{E}T_{a,b}^{d,d-1}\left(\mathcal{P}_{t,d}^{\beta}\right) = \mathcal{C}_{t,d}^{\beta,b}\int_{-1}^{1} |h|^a \left(1-h^2\right)^q \exp\left(-tF_{1,\beta+\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

$$\mathbb{E}T_{a,b}^{d,d-1}\left(\mathcal{S}_{t,d}^{\beta}\right) = 2^d\mathcal{C}_{t,d}^{\beta,b}\int_{0}^{1} h^a \left(1-h^2\right)^q \exp\left(-t\left(F_{1,\beta+\frac{d-1}{2}}(h) - F_{1,\beta+\frac{d-1}{2}}(-h)\right)\right) \mathrm{d}h,$$

$$\mathbb{E}T_{a,b}^{d,d-1}\left(\mathcal{Q}_{t,d}^{\beta}\right) = \mathcal{D}_{t,d}^{\beta,a,b} + \mathcal{C}_{t,d}^{\beta,b}\int_{0}^{1} h^a \left(1-h^2\right)^q \exp\left(-tF_{1,\beta+\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

where $q = d\beta + \frac{d-1}{2}(d+b+1)$ and

$$\mathcal{C}_{t,d}^{\beta,b} = t^d\kappa_d\mathbb{E}_\beta\left(\Delta_{d-1}^{b+1}\right)\left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^d,$$

$$\mathcal{D}_{t,d}^{\beta,a,b} = \mathbb{1}\{a=0\}\frac{d\kappa_d\, t^{d-1}}{((d-1)!)^{b+1}\mathrm{e}^{\frac{t}{2}}}\mathbb{E}_\beta\left(\nabla_{d-1}^{b+1}\right)\left(\frac{c_{d,\beta}}{c_{d-1,\beta}}\right)^{d-1}.$$

Theorem 3.3.9 *Fix $a, b \geq 0$. Let $X_1, \dots, X_N$ be independent beta′-type distributed random points in $\mathbb{R}^d$ with parameter β, that satisfies $2d\beta > (d-1)(d+b+1)+a+1$, and N be a Poisson distributed random variable with intensity t. Then,*

$$\mathbb{E}T_{a,b}^{d,d-1}\left(\tilde{\mathcal{P}}_{t,d}^{\beta}\right) = \tilde{\mathcal{C}}_{t,d}^{\beta,b}\int_{-\infty}^{\infty} |h|^a \left(1+h^2\right)^{-\tilde{q}} \exp\left(-t\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

$$\mathbb{E}T_{a,b}^{d,d-1}\left(\tilde{\mathcal{S}}_{t,d}^{\beta}\right) = 2^d\tilde{\mathcal{C}}_{t,d}^{\beta,b}\int_{0}^{\infty} h^a \left(1+h^2\right)^{-\tilde{q}} \exp\left(-t\left(\tilde{F}_{1,\beta-\frac{d-1}{2}}(h) - \tilde{F}_{1,\beta-\frac{d-1}{2}}(-h)\right)\right) \mathrm{d}h,$$

$$\mathbb{E}T_{a,b}^{d,d-1}\left(\tilde{\mathcal{Q}}_{t,d}^{\beta}\right) = \tilde{\mathcal{D}}_{t,d}^{\beta,a,b} + \tilde{\mathcal{C}}_{n,d}^{\beta,b}\int_{0}^{\infty} h^a \left(1+h^2\right)^{-\tilde{q}} \exp\left(-t\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)\right) \mathrm{d}h,$$

where $\tilde{q} = d\beta - \frac{d-1}{2}(d+b+1)$ and

$$\tilde{\mathcal{C}}_{t,d}^{\beta,b} = t^d\kappa_d\widetilde{\mathbb{E}}_\beta\left(\Delta_{d-1}^{b+1}\right)\left(\frac{\tilde{c}_{d,\beta}}{\tilde{c}_{d-1,\beta}}\right)^d,$$

$$\tilde{\mathcal{D}}_{t,d}^{\beta,a,b} = \mathbb{1}\{a=0\}\frac{d\kappa_d\, t^{d-1}}{((d-1)!)^{b+1}\mathrm{e}^{\frac{t}{2}}}\widetilde{\mathbb{E}}_\beta\left(\nabla_{d-1}^{b+1}\right)\left(\frac{\tilde{c}_{d,\beta}}{\tilde{c}_{d-1,\beta}}\right)^{d-1}.$$

Remark 3.3.10 As in the non-Poissonized case, Theorem 3.3.8 immediately implies exact formulae for the expected facet numbers and, together with Theorem 2.2.3, expected number of $(d-2)$-dimensional faces of random Poisson-beta-type polytopes. Namely, setting $a = b = 0$, it follows from the definition of $T_{a,b}^{d,d-1}$ that we have

$$\mathbb{E} f_{d-1}(\mathcal{P}_{t,d}^{\beta}) = \mathbb{E} T_{0,0}^{d,d-1}(\mathcal{P}_{t,d}^{\beta}) \tag{3.18}$$

and

$$\mathbb{E} f_{d-2}(\mathcal{P}_{t,d}^{\beta}) = \frac{d}{2}\, \mathbb{E} T_{0,0}^{d,d-1}(\mathcal{P}_{t,d}^{\beta}) \tag{3.19}$$

The expected facet numbers and expected numbers of $(d-2)$-dimensional faces of $\mathcal{S}_{t,d}^{\beta}$, $\mathcal{Q}_{t,d}^{\beta}$, $\tilde{\mathcal{P}}_{t,d}^{\beta}$, $\tilde{\mathcal{S}}_{t,d}^{\beta}$ and $\tilde{\mathcal{Q}}_{t,d}^{\beta}$ follow analogously.

Proof of Theorem 3.3.8 and 3.3.9

From Mecke's formula (2.38) we have

$$\begin{aligned}
&\mathbb{E} T_{a,b}^{d,d-1}\left(\mathcal{P}_{t,d}^{\beta}\right) \\
&= \mathbb{E} \sum_{1 \le i_1 < \ldots < i_d \le N} \mathbb{1}\left\{[X_{i_1}, \ldots, X_{i_d}] \in \mathcal{F}_{d-1}(\mathcal{P}_{t,d}^{\beta})\right\} \eta^a\left([X_{i_1}, \ldots, X_{i_d}]\right) \Delta_{d-1}^b\left(X_{i_1}, \ldots, X_{i_d}\right) \\
&= \frac{t^d}{d!} \int_{(\mathbb{R}^d)^d} \mathbb{P}\left([x_1, \ldots, x_d] \in \mathcal{F}_{d-1}\left(\operatorname{conv}\left(\Pi_{t,d}^{\beta} + \sum_{i=1}^{d} \delta_{x_i}\right)\right)\right) \eta^a\left([x_1, \ldots, x_d]\right) \\
&\qquad \times \Delta_{d-1}^b\left(x_1, \ldots, x_d\right) \left(\prod_{i=1}^{d} f_{d,\beta}(x_i)\right) \lambda_d^d(\mathrm{d}(x_1, \ldots, x_d)).
\end{aligned}$$

Denote $E = \operatorname{aff}([x_1, \ldots, x_d])$ and $h = P_{E^{\perp}}(E)$. Then, we have

$$\begin{aligned}
&\mathbb{P}\left([x_1, \ldots, x_d] \in \mathcal{F}_{d-1}\left(\operatorname{conv}\left(\Pi_{t,d}^{\beta} + \sum_{i=1}^{d} \delta_{x_i}\right)\right)\right) \\
&\quad = \mathbb{P}\left(\Pi_{t,d}^{\beta} \cap E^+ = \varnothing\right) + \mathbb{P}\left(\Pi_{t,d}^{\beta} \cap E^- = \varnothing\right) \\
&\quad = \mathbb{P}\left(\Pi_{t,1}^{\beta + \frac{d-1}{2}} \cap [h, \infty) = \varnothing\right) + \mathbb{P}\left(\Pi_{t,1}^{\beta + \frac{d-1}{2}} \cap (-\infty, h] = \varnothing\right) \\
&\quad = \exp\left(-t F_{1,\beta+\frac{d-1}{2}}(h)\right) + \exp\left(-t\left(1 - F_{1,\beta+\frac{d-1}{2}}(h)\right)\right).
\end{aligned}$$

The rest of the proof follows analogously as in the proof of Theorem 3.2.8. Similarly, we can derive the formulas for the polytopes $\mathcal{S}_{t,d}^{\beta}$, $\mathcal{Q}_{t,d}^{\beta}$, $\tilde{\mathcal{P}}_{t,d}^{\beta}$, $\tilde{\mathcal{S}}_{t,d}^{\beta}$ and $\tilde{\mathcal{Q}}_{t,d}^{\beta}$. □

3.4 Monotonicity of expected facet numbers

Denote by P_n and S_n the random convex hulls, respectively symmetric convex hull, generated by random vectors $X_1, \ldots, X_n \in \mathbb{R}^d$ sampled identically and independently from the standard Gaussian distribution, a beta-type distribution, a beta$'$-type distribution or the uniform distribution on the Euclidean unit sphere. $\mathcal{P}_t$ and $\mathcal{S}_t$ denote their Poissonized versions. We are interested in the following monotonicity questions:

Is it true that $\mathbb{E}f_{d-1}(P_n)$ and $\mathbb{E}f_{d-1}(S_n)$ are monotonously growing in n and $\mathbb{E}f_{d-1}(\mathcal{P}_t)$ and $\mathbb{E}f_{d-1}(\mathcal{S}_t)$ are monotonously growing in t?

For the historic development and results achieved so far regarding this question we refer the reader to Section 1.2. Recall that it was answered for Gaussian polytopes and polytopes generated from the uniform distribution in the unit ball by Beermann [21]. We extend this result to beta-type, beta$'$-type polytopes and polytopes generated by the uniform distribution on the unit sphere, as well as for their symmetrized and Poissonized versions. In fact, we even show that the stronger strict monotonicity holds.

Theorem 3.4.1 *Let $X_1, \ldots, X_n \in \mathbb{R}^d$, $n \geq d+1$, be independent and identically distributed according to either the Gaussian distribution, a beta-type distribution, a beta$'$-type distribution or the uniform distribution on the sphere. Then,*

$$\mathbb{E}f_{d-1}(P_n) > \mathbb{E}f_{d-1}(P_{n-1}) \qquad \textit{and} \qquad \mathbb{E}f_{d-1}(S_n) > \mathbb{E}f_{d-1}(S_{n-1}).$$

We emphasize that strict monotonicity of $n \mapsto f_{d-1}(P_n)$ cannot hold pathwise (except for the trivial case $n = d+1$), since the addition of a further random point can reduce the facet number arbitrarily as the additional point might "see" much more than d vertices of the already constructed random convex hull. For this reason, the expectation in Theorem 3.4.1 is essential. Furthermore, also the expected facet number of the Poissonized versions of these polytopes are strictly monotonous.

Theorem 3.4.2 *Let $X_1, \ldots, X_N \in \mathbb{R}^d$ be independent and identically distributed according to either the Gaussian distribution, a beta-type distribution, a beta$'$-type distribution or the uniform distribution on the sphere and let N be Poisson distributed with intensity t. Then,*

$$\mathbb{E}f_{d-1}(\mathcal{P}_t) \qquad \textit{and} \qquad \mathbb{E}f_{d-1}(\mathcal{S}_t)$$

are strictly monotonously growing in t.

Remark 3.4.3 As already argued in Remark 3.2.12 and Remark 3.3.10, it follows from Theorem 2.2.3 that these monotonicity results also hold for the expected number of $(d-2)$-dimensional faces $\mathbb{E}f_{d-2}(P_n)$, $\mathbb{E}f_{d-2}(S_n)$, $\mathbb{E}f_{d-2}(\mathcal{P}_t)$ and $\mathbb{E}f_{d-2}(\mathcal{S}_t)$ due to these polytopes being almost surely simplicial.

Remark 3.4.4 We want to mention once more that the strict monotonicity of the expectation of the whole **f**-vector of Gaussian polytopes was previous already shown by Kabluchko and Thäle [60]. Furthermore, postdating this work Kabluchko, Thäle and Zaporozhets [61] also showed the strict monotonicity of the whole **f**-vector for beta- and beta′-type polytopes, as well as for polytopes coming from the uniform distribution on the sphere. However, these result do not apply to the symmetric versions of the polytopes.

Proof of Theorem 3.4.1. We start by considering the beta′-type distribution on $\mathbb{R}^d$, where $\beta > d/2$. From Theorem 3.2.10 we know

$$\begin{aligned}\mathbb{E}f_{d-1}\left(\tilde{P}_{n,d}^{\beta}\right) &= \mathbb{E}T_{0,0}^{d,d-1}\left(\tilde{P}_{n,d}^{\beta}\right)\\ &= c\binom{n}{d}\int_{-\infty}^{\infty}(1+h^2)^{-d\beta+\frac{d-1}{2}(d+1)}\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)^{n-d}\,\mathrm{d}h\\ &= c\binom{n}{d}\int_{-\infty}^{\infty}(1+h^2)^{\frac{d-1}{2}}\tilde{f}_{1,\beta-\frac{d-1}{2}}(h)^{d}\left(1-\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)\right)^{n-d}\,\mathrm{d}h,\end{aligned}$$

where, for the sake of brevity, we collected all constants independent of n into the new constant c. Furthermore, we define $f(h) := \tilde{f}_{1,\beta-\frac{d-1}{2}}(h)$ and $F(h) := \tilde{F}_{1,\beta-\frac{d-1}{2}}(h)$.

Write now $s = F(h)$ and $L(s) = f\left(F^{-1}(s)\right)\sqrt{1+\left(F^{-1}(s)\right)^2}$ to obtain

$$\mathbb{E}f_{d-1}\left(\tilde{P}_{n,d}^{\beta}\right) = c\binom{n}{d}\int_{0}^{1}(1-s)^{n-d}L(s)^{d-1}\,\mathrm{d}s.$$

Thus, this yields the representation

$$\begin{aligned}&\mathbb{E}f_{d-1}\left(\tilde{P}_{n,d}^{\beta}\right)-\mathbb{E}f_{d-1}\left(\tilde{P}_{n-1,d}^{\beta}\right)\\ &\qquad = c\int_{0}^{1}\left[\binom{n}{d}(1-s)-\binom{n-1}{d}\right](1-s)^{n-d-1}L(s)^{d-1}\,\mathrm{d}s. \qquad (3.20)\end{aligned}$$

In order to apply Lemma 2.6.1, we have to verify that $L(s)$ is strictly concave on $(0,1)$. We prove this by showing that the first derivative of $L(s)$ is strictly decreasing. From the definition of F it follows that

$$\left(F^{-1}(s)\right)' = \frac{1}{f\left(F^{-1}(s)\right)} = \frac{1}{\tilde{c}_{1,\beta-\frac{d-1}{2}}\left(1+(F^{-1}(s))^2\right)^{-\beta+\frac{d-1}{2}}} \tag{3.21}$$

We recall that

$$L(s) = f\left(F^{-1}(s)\right)\sqrt{1+(F^{-1}(s))^2} = \tilde{c}_{1,\beta-\frac{d-1}{2}}\left(1+\left(F^{-1}(s)\right)^2\right)^{-\beta+\frac{d}{2}}.$$

Hence, using (3.21), the first derivative of $L(s)$ is

$$\begin{aligned} L'(s) &= \tilde{c}_{1,\beta-\frac{d-1}{2}}\left(\frac{d}{2}-\beta\right)\left(1+\left(F^{-1}(s)\right)^2\right)^{-\beta+\frac{d-2}{2}} 2F^{-1}(s)\left(F^{-1}(s)\right)' \\ &= 2\tilde{c}_{1,\beta-\frac{d-1}{2}}\left(\frac{d}{2}-\beta\right)\left(1+\left(F^{-1}(s)\right)^2\right)^{-\frac{1}{2}} F^{-1}(s) \end{aligned}$$

Clearly, $\left(1+(F^{-1}(s))^2\right)^{-\frac{1}{2}}$ is strictly decreasing, as well as, $\left(\frac{d}{2}-\beta\right)F^{-1}(s)$, and, hence, $L(s)$ is a strictly concave function.

As a consequence, we can apply Lemma 2.6.1 to deduce that

$$\begin{aligned} &\mathbb{E}f_{d-1}\left(\tilde{P}^{\beta}_{n,d}\right) - \mathbb{E}f_{d-1}\left(\tilde{P}^{\beta}_{n-1,d}\right) \\ &= c\int_0^1 \left[\binom{n}{d}(1-s) - \binom{n-1}{d}\right](1-s)^{n-d-1}L(s)^{d-1}\,\mathrm{d}s \\ &> c\left(\frac{L(d/n)}{d/n}\right)^{d-1}\binom{n}{d}\int_0^1 (1-s)^{n-d-1}s^{d-1}\left((1-s)-\frac{n-d}{n}\right)\mathrm{d}s \\ &= c\left(\frac{L(d/n)}{d/n}\right)^{d-1}\binom{n}{d}\left(B(d,n-d+1)-\frac{n-d}{n}B(d,n-d)\right) \\ &= 0, \end{aligned}$$

where we used the well-known relation $B(d,n-d+1) = \frac{n-d}{n}B(d,n-d)$ for the beta function.

Let us turn now to the symmetric polytope $\tilde{S}^{\beta}_{n,d}$. By Theorem 3.2.10 we have

$$\mathbb{E}f_{d-1}\left(\tilde{S}^{\beta}_{n,d}\right) = c\binom{n}{d}\int_0^\infty (1+h^2)^{\frac{d-1}{2}}\tilde{f}_{1,\beta-\frac{d-1}{2}}(h)^d\left(\tilde{F}_{1,\beta-\frac{d-1}{2}}(h)-\tilde{F}_{1,\beta-\frac{d-1}{2}}(-h)\right)^{n-d}\,\mathrm{d}h.$$

We define the function $G(h) = F(h) - F(-h)$. Set $s = 1 - G(h)$ and $L(s) = f\left(G^{-1}(1-s)\right)\sqrt{1+\left(G^{-1}(1-s)\right)^2}$. Hence, we have as before.

$$\begin{aligned}\mathbb{E}f_{d-1}&\left(\tilde{S}^{\beta}_{n,d}\right) - \mathbb{E}f_{d-1}\left(\tilde{S}^{\beta}_{n-1,d}\right)\\ &= c\int_0^1\left[\binom{n}{d}(1-s)-\binom{n-1}{d}\right](1-s)^{n-d-1}L(s)^{d-1}\,\mathrm{d}s.\end{aligned}\tag{3.22}$$

One easily computes

$$L'(s) = -\tilde{c}_{1,\beta-\frac{d-1}{2}}\left(\frac{d}{2}-\beta\right)\left(1+\left(G^{-1}(1-s)\right)^2\right)^{-\frac{1}{2}}G^{-1}(1-s).$$

Again, since $G^{-1}(1-s)$ and $\left(1+\left(G^{-1}(1-s)\right)^2\right)^{-\frac{1}{2}}$ are both strictly decreasing, $L(s)$ is strictly concave and the assertions follows with Lemma 2.6.1.

As the next case we consider the class of beta-type distributions on the unit ball $\mathbb{B}^d$ with density $f_{d,\beta}$ for some $\beta > -1$. In this case the proof follows almost line by line the proof of the beta′-type case, up to some minor modifications. Let now $f(h) := f_{1,\beta+\frac{d-1}{2}}(h)$ and $F(h) := F_{1,\beta+\frac{d-1}{2}}(h)$. For the polytope $P^{\beta}_{n,d}$ (3.20) stays the same except that now $L(s) = f\left(F^{-1}(s)\right)\sqrt{1-\left(F^{-1}(s)\right)^2}$, while for $S^{\beta}_{n,d}$ we have that (3.22) with $L(s) = f\left(G^{-1}(1-s)\right)\sqrt{1-\left(G^{-1}(1-s)\right)^2}$, where $G(h) = F(h) - F(-h)$.

Finally, we consider the case of the uniform distribution on $\mathbb{S}^{d-1}$. Recall that we can take $b = -1$ in Theorem 3.2.8 to obtain the values of the expected T-functional for this case. Hence,

$$\mathbb{E}f_{d-1}\left(P_{n,d}\right) = c\binom{n}{d}\int_{-1}^1 (1-h^2)^{\frac{d-1}{2}}f_{1,\frac{d-3}{2}}(h)^d\left(1-F_{1,\frac{d-3}{2}}(h)\right)^{n-d}\,\mathrm{d}h$$

and

$$\mathbb{E}f_{d-1}\left(S_{n,d}\right) = c\binom{n}{d}\int_{0}^{1}(1-h^2)^{\frac{d-1}{2}} f_{1,\frac{d-3}{2}}(h)^d \left(F_{1,\frac{d-3}{2}}(h) - F_{1,\frac{d-3}{2}}(-h)\right)^{n-d} \,\mathrm{d}h.$$

From here on out the result follows from the beta-type distributed case.

The Gaussian case was successfully solved by Beermann [21, Theorem 5.3.1] as already mentioned before. However, these results only prove monotonicity. For the strict monotonicity one has to apply Lemma 2.6.1 as was done above. To obtain the result for symmetric Gaussian polytopes one has to follow along the lines of Beermann's prove and adapt it to the symmetric case. Since a blueprint for such an adaptation has been given in the proof of Theorem 3.2.8, we refrain from working out the details. □

Proof of Theorem 3.4.2. We have

$$\mathbb{E}f_{d-1}(\mathcal{P}_t) = \sum_{n=0}^{\infty}\mathbb{P}(N=n)\mathbb{E}f_{d-1}(P_n) = \sum_{n=0}^{\infty}\frac{\mathrm{e}^{-t}t^n}{n!}\mathbb{E}f_{d-1}(P_n)$$

and, hence,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}f_{d-1}(\mathcal{P}_t) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(n\mathrm{e}^{-t}t^{n-1} - \mathrm{e}^{-t}t^n\right)\mathbb{E}f_{d-1}(P_n)\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{e}^{-t}t^n}{n!}\mathbb{E}f_{d-1}(P_{n+1}) - \sum_{n=0}^{\infty}\frac{\mathrm{e}^{-t}t^n}{n!}\mathbb{E}f_{d-1}(P_n)\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{e}^{-t}t^n}{n!}\left(\mathbb{E}f_{d-1}(P_n) - \mathbb{E}f_{d-1}(P_{n+1})\right)\\
&> 0,
\end{aligned}$$

where the last inequality follows from $(\mathbb{E}f_{d-1}(P_n) - \mathbb{E}f_{d-1}(P_{n+1})) > 0$, for all $n \geq d+1$, as implied by Theorem 3.4.1, and $(\mathbb{E}f_{d-1}(P_n) - \mathbb{E}f_{d-1}(P_{n+1})) = 0$, for all $n < d+1$. The case of the polytope $\mathcal{S}_t$ follows analogously. □

3.5 Particular values for special cases in low dimension

We close this chapter by collecting some particular mean values for the random polytopes $P_{n,d}^{\beta}$ and $S_{n,d}^{\beta}$ for $d = 2$, $d = 3$ and with $\beta = 0$ (uniform distribution in the unit ball) and $\beta = -1$ (uniform distribution on the sphere).

	$\mathbb{E}\,\mathrm{Vol}_2(P_{n,2}^0)$	$\mathbb{E}\,\mathrm{Vol}_2(S_{n,2}^0)$	$\mathbb{E}S_1(P_{n,2}^0)$	$\mathbb{E}S_1(S_{n,2}^0)$
$n=3$	$\frac{35}{48\pi}$	$\frac{35}{12\pi}$	$\frac{128}{15\pi}$	$\frac{512}{15\pi} - \frac{104704}{1575\pi^2}$
$n=4$	$\frac{35}{24\pi}$	$\frac{35}{6\pi} - \frac{2816}{135\pi^3}$	$\frac{256}{15\pi} - \frac{11075584}{165375\pi^3}$	$\frac{1024}{15\pi} - \frac{88604672}{165375\pi^3}$
$n=5$	$\frac{175}{72\pi} - \frac{23023}{6912\pi^3}$	$\frac{175}{18\pi} - \frac{23023}{432\pi^3}$	$\frac{256}{9\pi} - \frac{5537792}{33075\pi^3}$	$\frac{1024}{9\pi} - \frac{88604672}{33075\pi^3} + \frac{204130238464}{38201625\pi^4}$

TABLE 3.1: Mean area and perimeter length of a random polygon and a symmetric random polygon generated by n points uniformly distributed in the unit disc.

	$\mathbb{E}\,\mathrm{Vol}_2(P_{n,2})$	$\mathbb{E}\,\mathrm{Vol}_2(S_{n,2})$	$\mathbb{E}S_1(P_{n,2})$	$\mathbb{E}S_1(S_{n,2})$
$n=3$	$\frac{3}{2\pi}$	$\frac{6}{\pi}$	$\frac{12}{\pi}$	$\frac{48}{\pi} - \frac{96}{\pi^2}$
$n=4$	$\frac{3}{\pi}$	$\frac{12}{\pi} - \frac{48}{\pi^3}$	$\frac{24}{\pi} - \frac{96}{\pi^3}$	$\frac{96}{\pi} - \frac{768}{\pi^3}$
$n=5$	$\frac{5}{\pi} - \frac{15}{2\pi^3}$	$\frac{20}{\pi} - \frac{120}{\pi^3}$	$\frac{40}{\pi} - \frac{240}{\pi^3}$	$\frac{160}{\pi} - \frac{3840}{\pi^3} + \frac{7680}{\pi^4}$

TABLE 3.2: Mean area and perimeter length of a random polygon and a symmetric random polygon generated by n points uniformly distributed on the unit circle.

	$\mathbb{E}V_3(P_{n,3}^0)$	$\mathbb{E}V_3(S_{n,3}^0)$	$\mathbb{E}S_2(P_{n,3}^0)$	$\mathbb{E}S_2(S_{n,3}^0)$	$\mathbb{E}\mathcal{W}_3(P_{n,3}^0)$	$\mathbb{E}\mathcal{W}_3(S_{n,3}^0)$
$n=4$	$\frac{12\pi}{715}$	$\frac{96\pi}{715}$	$\frac{36\pi}{77}$	$\frac{135\pi}{112}$	$\frac{666}{715}$	$\frac{6408}{5005}$
$n=5$	$\frac{6\pi}{143}$	$\frac{195\pi}{1024}$	$\frac{11448\pi}{17017}$	$\frac{24048\pi}{17017}$	$\frac{1044}{1001}$	$\frac{2421}{1792}$
$n=6$	$\frac{2070\pi}{29393}$	$\frac{77472\pi}{323323}$	$\frac{1314\pi}{1547}$	$\frac{5661\pi}{3584}$	$\frac{33102}{29393}$	$\frac{454140}{323323}$

TABLE 3.3: Mean volume, surface area and mean width of a random polytope and a symmetric random polytope generated by n points uniformly distributed in the 3-dimensional unit ball.

	$\mathbb{E}V_3(P_{n,3})$	$\mathbb{E}V_3(S_{n,3})$	$\mathbb{E}S_2(P_{n,3})$	$\mathbb{E}S_2(S_{n,3})$	$\mathbb{E}\mathcal{W}_3(P_{n,3})$	$\mathbb{E}\mathcal{W}_3(S_{n,3})$
$n=4$	$\frac{4\pi}{105}$	$\frac{32\pi}{105}$	$\frac{4\pi}{5}$	2π	$\frac{6}{5}$	$\frac{8}{5}$
$n=5$	$\frac{2\pi}{21}$	$\frac{5\pi}{12}$	$\frac{8\pi}{7}$	$\frac{16\pi}{7}$	$\frac{4}{3}$	$\frac{5}{3}$
$n=6$	$\frac{10\pi}{63}$	$\frac{32\pi}{63}$	$\frac{10\pi}{7}$	$\frac{5\pi}{2}$	$\frac{10}{7}$	$\frac{12}{7}$

TABLE 3.4: Mean volume, surface area and mean width of a random polytope and a symmetric random polytope generated by n points uniformly distributed on the 3-dimensional unit sphere.

	$\mathbb{E}\mathbf{f}_1(P^0_{n,2})$	$\mathbb{E}\mathbf{f}_1(S^0_{n,2})$	$\mathbb{E}\mathbf{f}_2(P^0_{n,3})$	$\mathbb{E}\mathbf{f}_2(S^0_{n,3})$	$\mathbf{f}_2(P_{n,3})$	$\mathbf{f}_2(S_{n,3})$
$n=3$	3	$6-\frac{32}{3\pi^2}$	–	8	–	8
$n=4$	$4-\frac{35}{12\pi^2}$	$8-\frac{70}{3\pi^2}$	4	$\frac{357}{32}$	4	12
$n=5$	$5-\frac{175}{24\pi^2}$	$10-\frac{175}{3\pi^2}+\frac{5632}{27\pi^4}$	$\frac{840}{143}$	$\frac{2000}{143}$	6	16
$n=6$	$6-\frac{175}{12\pi^2}+\frac{23023}{1152\pi^4}$	$12-\frac{350}{3\pi^2}+\frac{23023}{36\pi^4}$	$\frac{1090}{143}$	$\frac{8485}{512}$	8	20

TABLE 3.5: Mean number of edges and facets of a random polytope and a symmetric random polytope generated by n points uniformly distributed in the unit ball. The last two columns collect the a.s. number of facets of a random polytope and a symmetric random polytope generated by n random points uniformly distributed on the 3-dimensional unit sphere.

	$\mathbb{E}\mathbf{f}_1(\hat{P}^0_{n,2})$	$\mathbb{E}\mathbf{f}_2(\hat{P}^0_{n,3})$	$\mathbb{E}\mathbf{f}_3(\hat{P}^0_{n,4})$
$n=3$	3	–	–
$n=4$	$6-\frac{24}{\pi^2}$	4	–
$n=5$	$10-\frac{60}{\pi^2}$	$\frac{20}{3}-\frac{10}{\pi^2}$	5
$n=6$	$15-\frac{180}{\pi^2}+\frac{720}{\pi^4}$	$10-\frac{30}{\pi^2}$	$15-\frac{200}{3\pi^2}$
$n=7$	$21-\frac{420}{\pi^2}+\frac{2520}{\pi^4}$	$14-\frac{70}{\pi^2}+\frac{105}{\pi^4}$	$35-\frac{700}{3\pi^2}$
$n=8$	$28-\frac{840}{\pi^2}+\frac{10080}{\pi^4}-\frac{40320}{\pi^6}$	$\frac{56}{3}-\frac{140}{\pi^2}+\frac{420}{\pi^4}$	$70-\frac{2800}{3\pi^2}+\frac{101920}{27\pi^4}$

TABLE 3.6: The mean number of (spherical) facets of a random polytope generated by n points uniformly distributed on the 2-/3-/4-dimensional upper half-sphere.

Chapter 4

Random polytopes on half spheres

In this chapter, we turn our attention to a certain class of random polytopes on half-sphere which are tightly connected to beta′-type polytopes. This class of random polytopes are generated as spherical convex hulls of n independently and identically distributed random points on a d-dimensional half-sphere. Their distribution follows a power-law density with respect to the uniform distribution on that half-sphere.

We make use of the correspondence between spherical polytopes on half-spheres and polyhedral cones. Namely, we intersect these cones with the tangent hyperplane at the north pole of the half-sphere and, after appropriate rescaling, show weak limit theorems of these intersections, in the space of compact convex subsets, as the number of points generating the spherical polytopes tends to infinity. As it turns out, these intersections are beta′-type polytopes and the limiting polytope is generated by a Poisson point process with power law intensity. Similarly, a weak limit theorem for the $\mathbf{f}$-vector of the spherical polytopes is uncovered.

This allows us to carry over monotonicity results form beta′-type polytopes, find limits for the moments of the number of faces of these spherical polytopes as the number of points goes to infinity, give an Efron-type identity for all expected Grassmann angles and expected number of faces, and give the asymptotics for expected conic intrinsic volumes, Grassmann angles and conic mean projection volumes of these spherical poyltopes as the number of points tends to infinity.

Furthermore, we investigate the expected T-functional, volume, intrinsic volumes and facet numbers of the limiting polytope coming from a Poisson point process with power law intensity as well as its symmetrized analogue.

4.1 Preliminaries

We fix a space dimension $d \geq 2$ and let $\bar{\mu}_{d,\gamma}$, $\gamma > -1$, be the probability measure with *power-law density* $\bar{f}_{d,\gamma}(x) = \bar{c}_{d,\gamma} x_{d+1}^{\gamma}$, $x \in \mathbb{S}_+^d$, with respect to the uniform distribution $\bar{\sigma}_{d-1}$ on the d-dimensional *upper half-sphere* $\mathbb{S}_+^d$. Let $U_1, U_2, \ldots$ be independent random points distributed according to $\bar{\mu}_{d,\gamma}$. The constant $\bar{c}_{d,\gamma}$ is easily calculated and satisfies $\bar{c}_{d,\gamma} = \tilde{c}_{d,\frac{d+\gamma+1}{2}}$, with $\tilde{c}_{d,\frac{d+\gamma+1}{2}}$ as introduced in (3.3). We are interested in the random convex cone in $\mathbb{R}^{d+1}$, defined as the *positive hull* of $U_1, \ldots, U_n$, $n \geq d+1$, that is,

$$C_{n,\gamma} = \mathrm{pos}(\{U_1, \ldots, U_n\}).$$

As already discussed in the introduction, the random cone, or, more precisely, the random spherical polytope $C_{n,\gamma} \cap \mathbb{S}_+^d$, has been studied in [16] for the special case $\gamma = 0$. Some of their results concern the expected $\mathbf{f}$-vector of $C_{n,0}$, that is, the expected number $\mathbb{E} f_k(C_{n,0})$ of k-dimensional faces of $C_{n,0}$, $k \in \{1, \ldots, d\}$. Recall that the $\mathbf{f}$-vector of the cone $C_{n,\gamma}$ is related to the $\mathbf{f}$-vector of the spherical polytope $C_{n,\gamma} \cap \mathbb{S}_+^d$ by $\mathbf{f}_k(C_{n,\gamma}) = \mathbf{f}_{k-1}(C_{n,\gamma} \cap \mathbb{S}_+^d)$. For our purposes, it is more convenient to work with cones rather than with spherical polytopes. Let us briefly state a few known results of relevance. By [16, Theorem 3.1], the expected number of facets $\mathbb{E}\mathbf{f}_d(C_{n,0})$ of $C_{n,0}$ is explicitly given by

$$\mathbb{E}\mathbf{f}_d(C_{n,0}) = \frac{2\omega_d}{\omega_{d+1}} \binom{n}{d} \int_0^{\pi} \left(1 - \frac{\alpha}{\pi}\right)^{n-d} \sin(\alpha)^{d-1} \, \mathrm{d}\alpha. \tag{4.1}$$

Moreover, it has been shown in [16, Theorem 3.1] that

$$\lim_{n \to \infty} \mathbb{E}\mathbf{f}_d(C_{n,0}) = 2^{-d} d! \kappa_d^2. \tag{4.2}$$

Regarding the expected number of one-dimensional faces, i.e., edges, of $C_{n,0}$ (or, equivalently, vertices of $C_{n,0} \cap \mathbb{S}_+^d$), [16, Theorem 7.1] says that

$$\lim_{n \to \infty} \mathbb{E}\mathbf{f}_1(C_{n,0}) = C(d) \pi^{d+1} \left(\frac{2}{\omega_{d+1}}\right)^{d+1} \omega_d \tag{4.3}$$

for a certain positive constant $C(d)$ given in form of a multiple integral, see [16, Equation (22)]. Let us also mention that cones generated by random points with uniform distribution on the whole sphere $\mathbb{S}^d$ were studied in [36] and [56].

In our investigation, we heavily employ the so-called *gnomonic projection*, that is, the map $\mathcal{P} : \mathbb{S}^d_+ \cap \{x_0 > 0\} \to \mathbb{R}^d$, defined by

$$\mathcal{P}(x_0, x_1, \dots, x_d) := \left(\frac{x_1}{x_0}, \dots, \frac{x_d}{x_0}\right). \tag{4.4}$$

We need to know the image measure of the measure $\bar{\mu}_{d,\gamma}$ under $\mathcal{P}$, which is a consequence of [23, Proposition 4.2] and, in a more general set-up, has been proved in the argument of [26, Theorem 7].

Proposition 4.1.1 *Let $(\xi_0, \dots, \xi_d)$ be a random vector distributed according to $\bar{\mu}_{d,\gamma}$. Then, the vector $\mathcal{P}(\xi_0, \xi_1, \dots, \xi_d) := (\xi_1/\xi_0, \dots, \xi_d/\xi_0)$ has the beta′-type density*

$$x \mapsto \frac{\Gamma\left(\frac{\gamma+d+1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)} \frac{1}{(1+\|x\|^2)^{\frac{\gamma+d+1}{2}}}, \qquad x \in \mathbb{R}^d.$$

Proof. Note that the inverse of the map $\mathcal{P}$ is given by

$$\mathcal{P}^{-1}(y) = \left(\frac{x_1}{\sqrt{1+\|x\|^2}}, \dots, \frac{x_d}{\sqrt{1+\|x\|^2}}, \frac{1}{\sqrt{1+\|x\|^2}}\right).$$

Let $\mathrm{D}\mathcal{P}^{-1}$ be the Jacobian matrix of $\mathcal{P}^{-1}$ and put $J_{\mathcal{P}^{-1}}(x) := \sqrt{\det \mathrm{D}\mathcal{P}^{-1}(x)^{\intercal}\mathrm{D}\mathcal{P}^{-1}(x)}$. Then, it holds that

$$J_{\mathcal{P}^{-1}}(x) = (1+\|x\|^2)^{-\frac{d+1}{2}},$$

see [23, Proposition 4.2]. Moreover, for a measurable subset $A \subset \mathbb{R}^d$ and a measurable function $f : A \to \mathbb{R}$, the area formula [43, Theorem 3.2.3] says that

$$\int_A f(x)\,\mathrm{d}x = \int_{g(A)} f \circ \mathcal{P}(y)(J_g \circ \mathcal{P}(y))^{-1}\,\bar{\sigma}_d(\mathrm{d}y).$$

Next, we notice that $1+\|\mathcal{P}(y)\|^2 = y_{d+1}^{-2}$ and apply the formula with $f(x) = \tilde{f}_{d,\frac{\gamma+d+1}{2}}(x)$:

$$\int_A \tilde{c}_{d,\frac{\gamma+d+1}{2}}\,(1+\|x\|^2)^{-\frac{\gamma+d+1}{2}}\,\mathrm{d}x = \int_{g(A)} \tilde{c}_{d,\frac{\gamma+d+1}{2}}\,y_{d+1}^{\gamma}\,\bar{\sigma}_d(\mathrm{d}y).$$

As a result, we see that the density $\bar{f}_{d,\gamma}$ of $\bar{\mu}_{d,\gamma}$ with respect to the uniform measure $\bar{\sigma}_{d-1}$ on $\mathbb{S}^d_+$ is the push-forward of the density $\tilde{f}_{d,\frac{\gamma+d+1}{2}}$ of $\bar{\mu}_{d,\frac{\gamma+d+1}{2}}$ with respect to the Lebesgue measure on $\mathbb{R}^d$ under $\mathcal{P}^{-1}$. □

This proposition immediately implies a result on the *monotonicity in expectation* for the **f**-vector of the spherically convex polytope $C_{n,\gamma} \cap \mathbb{S}^d_+$. By Theorem 3.4.1, the result follows for the expected number of facets and $(d-2)$-dimensional faces of $C_{n,\gamma} \cap \mathbb{S}^d_+$. However, since the monotonicity in expectation of all the components of the **f**-vector for beta′-type polytopes was shown in [61], we even have the following theorem.

Theorem 4.1.2 *Let $X_1, \ldots, X_n$, $n \geq d+1$, be independent and identically distributed random points distributed according to the distribution $\bar{\mu}_{d,\gamma}$. Then,*

$$\mathbb{E}\mathbf{f}_k \left(C_{n,\gamma} \cap \mathbb{S}^d_+\right) > \mathbb{E}\mathbf{f}_k \left(C_{n-1,\gamma} \cap \mathbb{S}^d_+\right).$$

for all $k = 0, 1, \ldots, d-1$.

Proof. Let $\beta = \frac{\gamma+d+1}{2}$ and let $P^\beta_{n,d}$ be the random convex hull in $\mathbb{R}^d$ generated by n independent points with density $\tilde{f}_{d,\beta}$. Then, the push-forward of $P^\beta_{n,d}$ has the same distribution as the spherical random polytope $C_{n,\gamma} \cap \mathbb{S}^d_+$ and their facets are in one-to-one correspondence. As a consequence, the mean number of k-dimensional faces of the spherical random polytope $C_{n,\gamma} \cap \mathbb{S}^d_+$ is the same as the mean number of k-dimensional faces of the random convex hull $P^\beta_{n,d}$, i.e.,

$$\mathbb{E}\mathbf{f}_k \left(C_{n,\gamma} \cap \mathbb{S}^d_+\right) = \mathbb{E}\mathbf{f}_k \big(P^\beta_{n,d}\big).$$

Thus, the monotonicity follows from Theorem 3.4.1 for the expected number of facets and $(d-2)$-dimensional faces, and from [61, Theorem 1.14] for general k. $\square$

Secondly, Proposition 4.1.1 also allows us to extend equation (4.1) to arbitrary $\gamma > -1$.

Theorem 4.1.3 *Let $\gamma > -1$ and put $\beta = \frac{1}{2}(\gamma + d + 1)$. Then,*

$$\mathbb{E}\mathbf{f}_{d-1} \left(C_{n,\gamma} \cap \mathbb{S}^d_+\right) = \mathbb{E}\mathbf{f}_{d-1}\big(\tilde{P}^\beta_{n,d}\big) = \tilde{C}^{\frac{\gamma+d+1}{2},0}_{n,d} \int_0^\pi \sin(h)^{\gamma d+d-1} \tilde{F}_{1,\frac{\gamma}{2}+1}(\cot h)^{n-d} \,\mathrm{d}h.$$

In particular, if $\gamma = 0$, then,

$$\mathbb{E}\mathbf{f}_{d-1} \left(C_{n,0} \cap \mathbb{S}^d_+\right) = \mathbb{E}\mathbf{f}_{d-1} \left(\tilde{P}^{\frac{d+1}{2}}_{n,d}\right) = \binom{n}{d} \frac{2\omega_d}{\omega_{d+1}} \int_0^\pi \left(1 - \frac{h}{\pi}\right)^{n-d} \sin(h)^{d-1} \,\mathrm{d}h.$$

For $\gamma = 0$ this formula coincides with (4.1).

Proof. The first formula follows from Theorem 3.2.10 with $b = 0$ there together with the substitution $h = \cot y$ and, then, by renaming y by h. The second formula follows from the first one by observing that $\tilde{F}_{1,1}(\cot h) = 1 - \frac{h}{\pi}$, $h \in [0, \pi]$, and that $\tilde{C}_{n,d}^{\frac{d+1}{2},0} = \binom{n}{d} \frac{2\omega_d}{\omega_{d+1}}$. □

The remainder of this section will be devoted to stating a number of auxiliary lemmas regarding properties of the uniform distribution $\bar{\sigma}_{d-1}$ on $\mathbb{S}_+^d$ and the beta′-type distribution $\bar{\mu}_{d,\frac{\gamma+d+1}{2}}$ from the previous proposition.

Lemma 4.1.4 *Let $U := (\xi_0, \xi_1, \ldots, \xi_d)$ be a random vector with distribution $\bar{\mu}_{d,\gamma}$. Then, ξ_0 has probability density function*

$$t \mapsto \frac{2\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}(1-t^2)^{\frac{d}{2}-1}t^{\gamma}, \qquad t \in [0,1], \tag{4.5}$$

with respect to the Lebesgue measure on $\mathbb{R}$.

Proof. The result follows similarly as Lemma 2.3.5 by using $\bar{\mu}_{d,\gamma}(\mathrm{d}u) = \bar{c}_{d,\gamma} u_{d+1}^{\gamma} \bar{\sigma}_d(\mathrm{d}u)$ and the fact that the uniform measure σ_d on $\mathbb{S}^d$ and $\bar{\sigma}_d$ on $\mathbb{S}_+^d$ are related via $\bar{\sigma}_d(A) = 2\sigma_d(A)$ for any measurable $A \subset \mathbb{S}_+^d$. □

Lemma 4.1.5 *Let $U := (\xi_0, \xi_1, \ldots, \xi_d)$ be a random vector with distribution $\bar{\mu}_{d,\gamma}$ on the d-dimensional half-sphere $\mathbb{S}_+^d$. Then, the distribution of the vector $\mathcal{P}(U) = (\xi_1/\xi_0, \ldots, \xi_d/\xi_0)$ is regularly varying in $\mathbb{R}^d$ and we have the vague convergence*

$$n\mathbb{P}\big(n^{-\frac{1}{\gamma+1}}\mathcal{P}(U) \in \cdot\big) \xrightarrow{\text{v}} \nu(\cdot) \tag{4.6}$$

on $\mathcal{M}_{\mathbb{R}^d \setminus \{\boldsymbol{o}\}}$, as $n \to \infty$, where ν is a measure on $\mathbb{R}^d \setminus \{\boldsymbol{o}\}$ with density $\frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}}\|x\|^{-(d+\gamma+1)}$, $x \in \mathbb{R}^d \setminus \{\boldsymbol{o}\}$, with respect to the Lebesgue measure on $\mathbb{R}^d$.

Proof. From Proposition 4.1.1 we know that the distribution of $\mathcal{P}(U)$ is spherically symmetric in $\mathbb{R}^d$. Whence, (4.6) is equivalent to

$$\lim_{n\to\infty} n\mathbb{P}\left(n^{-\frac{1}{\gamma+1}}\|\mathcal{P}(U)\| > r\right) = \nu(\{x \in \mathbb{R}^d : \|x\| > r\}) = \frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}} \int\limits_{\{\|x\|>r\}} \frac{\mathrm{d}x}{\|x\|^{d+\gamma+1}},$$

for every $r > 0$. We have

$$n\mathbb{P}\left(n^{-\frac{1}{\gamma+1}}\|\mathcal{P}(U)\| > r\right) = n\mathbb{P}\left(\xi_1^2 + \cdots + \xi_d^2 > n^{\frac{2}{\gamma+1}} r^2 \xi_0^2\right) = n\mathbb{P}\left(1 - \xi_0^2 > n^{\frac{2}{\gamma+1}} r^2 \xi_0^2\right)$$

$$= n\mathbb{P}\left(\xi_0 < (n^{\frac{2}{\gamma+1}} r^2 + 1)^{-1/2}\right) \longrightarrow \frac{\Gamma\left(\frac{d+\gamma+1}{2}\right)}{(\gamma+1)\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)} \frac{1}{r^{\gamma+1}},$$

as $n \to \infty$, having utilized formula (4.5) in the last passage.

On the other hand we obtain by transformation into spherical coordinates:

$$\begin{aligned}
\frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}} \int\limits_{\{\|x\|>r\}} \frac{\mathrm{d}x}{\|x\|^{d+\gamma+1}} &= \frac{\omega_{\gamma+1}\omega_d}{\omega_{d+\gamma+1}} \int\limits_r^\infty \frac{\mathrm{d}s}{s^{\gamma+2}} \\
&= \frac{\omega_{\gamma+1}\omega_d}{(\gamma+1)\omega_{\gamma+1}} \frac{1}{r^{\gamma+1}} \\
&= \frac{\Gamma\left(\frac{d+\gamma+1}{2}\right)}{(\gamma+1)\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)} \frac{1}{r^{\gamma+1}},
\end{aligned}$$

where we used the definition of ω_d. □

Lemma 4.1.6 *Fix $\varepsilon_2, \dots, \varepsilon_d \in \{-1, +1\}$ and let $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ be a random vector with beta′-type distribution as in Proposition 4.1.1. Then, for all $r > 0$ and $n \in \mathbb{N}$ with $rn^{\frac{1}{\gamma+1}} > 1$ we have*

$$\mathbb{P}\left(\frac{\xi}{n^{\frac{1}{\gamma+1}}} \in A_{\varepsilon_2,\dots,\varepsilon_d}(r)\right) \geq \left(\frac{1}{2}\right)^d \frac{\left(1 + r^2 n^{\frac{2}{\gamma+1}}\right)^{-\frac{\gamma+1}{2}}}{\sqrt{\pi\left(\frac{\gamma}{2}+1\right)}}.$$

Proof. Every coordinate of ξ is beta′-type distributed with density $\tilde{c}_{1,\frac{\gamma}{2}+1}(1+t^2)^{-\frac{\gamma}{2}+1}$, $t \in \mathbb{R}$, see Lemma 3.1.3. Hence,

$$\begin{aligned}
\mathbb{P}\left(\frac{\xi}{n^{\frac{1}{\gamma+1}}} \in A_{\varepsilon_2,\dots,\varepsilon_d}(r)\right) &= \left(\frac{1}{2}\right)^{d-1} \mathbb{P}\left(\xi_1 > rn^{\frac{1}{\gamma+1}}\right) \\
&= \left(\frac{1}{2}\right)^{d-1} \tilde{c}_{1,\frac{\gamma}{2}+1} \int\limits_{rn^{\frac{1}{\gamma+1}}}^\infty (1+t^2)^{-\frac{\gamma}{2}+1}\,\mathrm{d}t \\
&\geq \left(\frac{1}{2}\right)^d \frac{\left(1 + r^2 n^{\frac{2}{\gamma+1}}\right)^{-\frac{\gamma+1}{2}}}{\sqrt{\pi\left(\frac{\gamma}{2}+1\right)}},
\end{aligned}$$

where the lower bound is provided in [25, Remark 2.3]. □

Lemma 4.1.7 *Let $\xi^{(1)}, \dots, \xi^{(n)} \in \mathbb{R}^d$ be independent random vectors with a beta'-type distribution as in Proposition 4.1.1. Then, there exist constants $c_1, c_2 > 0$ only depending on d such that, for all $r > 0$ and $n \in \mathbb{N}$ with $rn^{\frac{1}{\gamma+1}} > 1$,*

$$\mathbb{P}\bigg(re_1 \notin \bigg[\frac{\xi^{(1)}}{n^{\frac{1}{\gamma+1}}}, \dots, \frac{\xi^{(n)}}{n^{\frac{1}{\gamma+1}}}\bigg]\bigg) \leq c_1 \exp\left(-c_2\left(r^2 + n^{-\frac{2}{\gamma+1}}\right)^{-\frac{\gamma+1}{2}}\right).$$

Proof. We let $c_2 = \left(2^d\sqrt{\pi\left(\frac{\gamma}{2}+1\right)}\right)^{-1}$ be the constant from Lemma 4.1.6. Combining Lemma 2.2.6 with Lemma 4.1.6 yields

$$\begin{aligned}
&\mathbb{P}\bigg(re_1 \notin \bigg[\frac{\xi^{(1)}}{n^{\frac{1}{\gamma+1}}}, \dots, \frac{\xi^{(n)}}{n^{\frac{1}{\gamma+1}}}\bigg]\bigg) \leq 2^d\,\mathbb{P}\bigg(\bigg\{\frac{\xi^{(1)}}{n^{\frac{1}{\gamma+1}}}, \dots, \frac{\xi^{(n)}}{n^{\frac{1}{\gamma+1}}}\bigg\} \cap A_{+1,\dots,+1}(r) = \varnothing\bigg) \\
&= 2^d\left(1 - \mathbb{P}\left(\frac{\xi^{(1)}}{n^{\frac{1}{\gamma+1}}} \in A_{+1,\dots,+1}(r)\right)\right)^n \leq 2^d\left(1 - c_2\left(r^2 n^{\frac{2}{\gamma+1}} + 1\right)^{-\frac{\gamma+1}{2}}\right)^n \\
&\leq 2^d \exp\left(-c_2 n\left(r^2 n^{\frac{2}{\gamma+1}} + 1\right)^{-\frac{\gamma+1}{2}}\right),
\end{aligned}$$

where the last inequality follows since $\log(1-x) \leq -x$, for $x < 1$. Putting $c_1 := 2^d$ completes the proof. □

Lemma 4.1.8 *Let $\xi^{(1)}, \dots, \xi^{(n)} \in \mathbb{R}^d$ be as in Lemma 4.1.7. Then, there exist constants $c_1, c_2, c_3 > 0$ only depending on d such that, for all $r > 0$ and $n \in \mathbb{N}$ with $rn^{\frac{1}{\gamma+1}} > d^{-\frac{1}{2}}$,*

$$\mathbb{P}\bigg(\bigg[\frac{\xi^{(1)}}{n^{\frac{1}{\gamma+1}}}, \dots, \frac{\xi^{(n)}}{n^{\frac{1}{\gamma+1}}}\bigg] \not\supset \mathbb{B}^d(\boldsymbol{o}, r)\bigg) \leq c_1 \exp\left(-\left(c_2 r^2 + c_3 n^{-\frac{2}{\gamma+1}}\right)^{-\frac{\gamma+1}{2}}\right).$$

Proof. Recall that $e_1, \dots, e_d$ is the standard basis of $\mathbb{R}^d$. For all $r > 0$ the cross-polytope $\operatorname{conv}\{\pm d^{\frac{1}{2}} re_j, j = 1, \dots, d\}$ contains $\mathbb{B}^d(\boldsymbol{o}, r)$. Then,

$$\begin{aligned}
&\mathbb{P}\bigg(\bigg[\frac{\xi^{(1)}}{n^{\frac{1}{\gamma+1}}}, \dots, \frac{\xi^{(n)}}{n^{\frac{1}{\gamma+1}}}\bigg] \not\supset \mathbb{B}^d(\boldsymbol{o}, r)\bigg) \\
&\leq \mathbb{P}\bigg(\varepsilon d^{\frac{1}{2}} re_j \notin \bigg[\frac{\xi^{(1)}}{n^{\frac{1}{\gamma+1}}}, \dots, \frac{\xi^{(n)}}{n^{\frac{1}{\gamma+1}}}\bigg] \text{ for some } j = 1, \dots, d \text{ and } \varepsilon \in \{+1, -1\}\bigg) \\
&\leq 2d\,\mathbb{P}\bigg(d^{\frac{1}{2}} re_1 \notin \bigg[\frac{\xi^{(1)}}{n^{\frac{1}{\gamma+1}}}, \dots, \frac{\xi^{(n)}}{n^{\frac{1}{\gamma+1}}}\bigg]\bigg).
\end{aligned}$$

The claim now follows from Lemma 4.1.7 with $d^{\frac{1}{2}} r$ in place of r. □

4.2 Weak convergence of the random cone

The weak convergence theorem

In what follows, we shall present a weak limit theorem for the random cone $C_{n,\gamma}$. It is clear that, for large n, the cone $C_{n,\gamma}$ is close to the half-space $\{x_0 > 0\}$, so that in order to obtain a non-trivial limit for $C_{n,\gamma}$ we need an appropriate rescaling. This is achieved by the linear operator $T_{n,\gamma} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ defined by

$$T_{n,\gamma}(x_0, x_1, \ldots, x_d) := \left(n^{\frac{1}{\gamma+1}} x_0, x_1, \ldots, x_d\right).$$

Let H_1 be the hyperplane $\{x_0 = 1\}$ in $\mathbb{R}^{d+1}$. Note that H_1 is tangent to the half-sphere $\mathbb{S}^d_+$ at its north pole. We shall prove that the random polytope $(T_{n,\gamma}C_{n,\gamma} \cap H_1) - e_0$, which can be viewed as the "horizontal" section of the cone $T_{n,\gamma}C_{n,\gamma}$, converges in distribution on the space of compact convex subsets of $H_1 - e_0$ that we identify with $\mathbb{R}^d$.

To describe the limit, take some $\gamma, c > 0$, and let $\Pi_{d,\gamma}(c)$ be a Poisson point process on $\mathbb{R}^d \setminus \{\boldsymbol{o}\}$ whose intensity measure is absolutely continuous with respect to the Lebesgue measure and whose density function with respect to the Lebesgue measure is given by

$$x \mapsto \frac{c}{\omega_{d+\gamma}} \frac{1}{\|x\|^{d+\gamma}}, \qquad x \in \mathbb{R}^d \setminus \{\boldsymbol{o}\}. \tag{4.7}$$

Note that the number of points of $\Pi_{d,\gamma}(c)$ outside any ball centered at the origin having strictly positive radius is almost surely finite (because the intensity is integrable near ∞), while the number of points inside any such ball is infinite with probability one (because the integral of the intensity over such balls diverges). Even though $\Pi_{d,\gamma}(c)$ almost surely consists of infinitely many points, the random convex set $\operatorname{conv}(\Pi_{d,\gamma}(c))$ turns out to be almost surely a polytope, see Corollary 4.2.7 below. The next theorem identifies the weak limit of the rescaled random polytopes $(T_{n,\gamma}C_{n,\gamma} \cap H_1) - e_0$ in terms of a Poisson point process of the type just discussed.

Theorem 4.2.1 *As $n \to \infty$, the random polytopes $(T_{n,\gamma}C_{n,\gamma} \cap H_1) - e_0$ converge in distribution to $\operatorname{conv}(\Pi_{d,\gamma+1}(\omega_{\gamma+1}))$ on the space $\mathcal{K}^d$ of compact convex subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric.*

Proof. Recall the gnomonic projection $\mathcal{P} : \mathbb{S}^d_+ \cap \{x_0 > 0\} \to \mathbb{R}^d$ defined by equality

(4.4). For each $i \in \{1, \ldots, n\}$, let ℓ_i be the line in $\mathbb{R}^{d+1}$ passing through the origin and the point U_i. This line intersects the hyperplane $H_1 := \{x_0 = 1\}$ at the point $(1, \mathcal{P}(U_i)) \in H_1$. This observation implies that

$$C_{n,\gamma} \cap H_1 = \operatorname{conv}(\{(1, \mathcal{P}(U_i))\colon i = 1, \ldots, n\})$$

and, therefore,

$$(T_{n,\gamma} C_{n,\gamma} \cap H_1) - e_0 = \operatorname{conv}\left(\left\{n^{-\frac{1}{\gamma+1}} \mathcal{P}(U_i) : i = 1, \ldots, n\right\}\right). \tag{4.8}$$

Hence, it is enough to show that

$$\operatorname{conv}\left(\left\{n^{-\frac{1}{\gamma+1}} \mathcal{P}(U_i)\colon i = 1, \ldots, n\right\}\right) \xrightarrow{\text{w}} \operatorname{conv}\left(\Pi_{d,\gamma+1}(\omega_{\gamma+1})\right) \tag{4.9}$$

on the space $\mathcal{K}^d$. To prove this, we first note that as a consequence of Lemma 4.1.5 and [93, Proposition 3.21] we have

$$\sum_{i=1}^{n} \delta_{n^{-\frac{1}{\gamma+1}} \mathcal{P}(U_i)} \xrightarrow{\text{w}} \Pi_{d,\gamma+1}(\omega_{\gamma+1}), \qquad \text{as } n \to \infty, \tag{4.10}$$

weakly on the space $\mathcal{N}_{\mathbb{R}^d \setminus \{\boldsymbol{o}\}}$. Now, we can use the Skorokhod representation theorem, i.e., Lemma 2.4.4, to pass to the a.s. convergence on a new probability space, and, then, apply Lemma 2.4.5 pointwise. Going back to the original probability space, we get the required convergence (4.9). The proof of Theorem 4.2.1 is thus complete. □

Remark 4.2.2 For $d = 2$ the convergence (4.9) also follows from [38, Theorem 3.1].

Convergence of the f-vector

From Theorem 4.2.1 we derive the following result on the distributional convergence of the $\mathbf{f}$-vector of the random spherical polytope $C_{n,\gamma} \cap \mathbb{S}^d_+$. We remind the reader that $\mathbf{f}_k(C_{n,\gamma} \cap \mathbb{S}^d_+) = \mathbf{f}_{k+1}(C_{n,\gamma})$.

Theorem 4.2.3 *As $n \to \infty$, we have that*

$$\mathbf{f}\left(C_{n,\gamma} \cap \mathbb{S}^d_+\right) \xrightarrow{\text{d}} \mathbf{f}\left(\operatorname{conv}\left(\Pi_{d,\gamma+1}\left(\omega_{\gamma+1}\right)\right)\right),$$

where $\xrightarrow{\text{d}}$ denotes convergence in distribution.

Proof of Theorem 4.2.3. Let $k \in \{1, \dots, d\}$. From (4.8) we obtain the almost sure equality

$$\begin{aligned}\mathbf{f}_{k-1}\left(C_{n,\gamma} \cap \mathbb{S}_+^d\right) = \mathbf{f}_k(C_{n,\gamma}) &= \mathbf{f}_{k-1}\left((T_{n,\gamma}C_{n,\gamma} \cap H_1) - e_0\right) \\ &= \mathbf{f}_{k-1}\left(\mathrm{conv}\left(\left\{n^{-\frac{1}{\gamma+1}}\mathcal{P}(U_i) \colon i = 1, \dots, n\right\}\right)\right).\end{aligned}$$

Passing in (4.10) to the a.s. convergence by the Skorokhod representation theorem, i.e., Lemma 2.4.4, using Lemma 2.4.5 pointwise, and returning back to the original probability space, yields

$$\mathbf{f}\left(\mathrm{conv}\left(\left\{n^{-\frac{1}{\gamma+1}}\mathcal{P}(U_i) \colon i = 1, \dots, n\right\}\right)\right) \xrightarrow{\mathrm{d}} \mathbf{f}(\mathrm{conv}(\Pi_{d,\gamma+1}(\omega_{\gamma+1}))),$$

which proves the desired statement. □

The following Theorem generalizes a result from [16] discussed above and answers - in an extended form - a question raised in [16, Section 9].

Theorem 4.2.4 *For every $k \in \{1, \dots, d\}$ and every $m \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \mathbb{E}\mathbf{f}_k^m(C_{n,\gamma}) = \lim_{n\to\infty} \mathbb{E}\mathbf{f}_{k-1}^m\left(C_{n,\gamma} \cap \mathbb{S}_+^d\right) = \mathbb{E}\mathbf{f}_{k-1}^m(\mathrm{conv}(\Pi_{d,\gamma+1}(\omega_{\gamma+1}))).$$

For $m = 1$ the limits of the expectations are

$$\lim_{n\to\infty} \mathbb{E}\mathbf{f}_k(C_{n,\gamma}) = \lim_{n\to\infty} \mathbb{E}\mathbf{f}_{k-1}\left(C_{n,\gamma} \cap \mathbb{S}_+^d\right) = \mathbb{E}\mathbf{f}_{k-1}(\mathrm{conv}(\Pi_{d,\gamma+1}(\omega_{\gamma+1}))) = \frac{2}{k!}B_{\gamma,k,d},$$

where $B_{\gamma,1,d}, \dots, B_{\gamma,d,d}$ are constants given by

$$\begin{aligned}B_{\gamma,k,d} &= \frac{1}{2}\Big(\frac{2\omega_{\gamma+1}}{\omega_{d+\gamma+1}}\Big)^k \\ &\times \int\limits_{(\mathbb{R}^d)^k} \mathbb{P}(\mathrm{conv}(\Pi_{d,\gamma+1}(\omega_{\gamma+1})) \cap \mathrm{aff}(\{x_1, \dots, x_k\}) = \varnothing) \prod_{i=1}^k \frac{\mathrm{d}x_i}{\|x_i\|^{d+\gamma+1}} < \infty. \quad (4.11)\end{aligned}$$

Remark 4.2.5 We shall prove at the very end of this chapter that

$$B_{\gamma,d,d} = \frac{(d-1)!(\gamma+1)^{d-1}}{\sqrt{\pi}} \left(\frac{\omega_{\gamma+2}}{\omega_{\gamma+1}}\right)^d \frac{\Gamma\left(\frac{\gamma+1}{2}d + \frac{1}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}d\right)}, \quad (4.12)$$

which in the special case of $\gamma = 0$ simplifies to

$$B_{0,d,d} = (2\pi)^{d-1}\Gamma\left(\frac{d+1}{2}\right)^2. \tag{4.13}$$

In the case of $\gamma = 0$ this recovers, together with Theorem 4.2.4 and Legendre's duplication formula, equation (4) of [16], where it was proved that $\lim_{n\to\infty}\mathbb{E}\mathbf{f}_d(C_n) = 2^{-d}d!\kappa_d^2$. In Proposition 4.2.11, we shall compute the value of $B_{0,2,d}$, yielding the formula

$$\lim_{n\to\infty}\mathbb{E}\mathbf{f}_2(C_{n,0}) = B_{0,2,d} = \frac{1}{2}\binom{d+1}{3}\pi^2.$$

Proof of Theorem 4.2.4. In view of Theorem 4.2.3, we need to show that the sequence $(\mathbf{f}_k^m(C_n))_{n\in\mathbb{N}}$ is uniformly integrable for every $k = 1, \ldots, d$ and $m \in \mathbb{N}$. This is equivalent to

$$\sup_{n\in\mathbb{N}}\mathbb{E}\mathbf{f}_k^m(C_{n,\gamma}) < \infty \tag{4.14}$$

for every $k = 1, \ldots, d$ and $m \in \mathbb{N}$, since, for a fixed m, (4.14) implies uniform integrability of $(\mathbf{f}_k^\ell(C_{n,\gamma}))_{n\in\mathbb{N}}$ for $0 \le \ell < m$.

To prove (4.14) we note that for an arbitrary (spherical) polytope P_n the number $\mathbf{f}_k(P_n)$ of its k-dimensional faces satisfies

$$\mathbf{f}_k(P_n) \le \binom{\mathbf{f}_0(P_n)}{k+1} \le \mathbf{f}_0^{k+1}(P_n), \qquad k = 0, \ldots, d-1.$$

From this observation it follows that (4.14) is equivalent to

$$\sup_{n\in\mathbb{N}}\mathbb{E}\mathbf{f}_0^m\left(C_{n,\gamma}\cap\mathbb{S}_+^d\right) < \infty, \tag{4.15}$$

for every $m \in \mathbb{N}$. Recall that $\mathcal{P} : \mathbb{S}_+^d \cap \{x_0 > 0\} \to \mathbb{R}^d$ is the map defined by (4.4). Clearly, $\mathbf{f}_0(C_{n,\gamma}\cap\mathbb{S}_+^d)$ coincides with the number of vertices of the convex hull of $\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)$ in $\mathbb{R}^d$. Write

$$\begin{aligned}
&\mathbb{E}\mathbf{f}_0^m\left(C_{n,\gamma}\cap\mathbb{S}_+^d\right)\\
&\quad = \mathbb{E}\left(\sum_{i=1}^n \mathbb{1}\{\mathcal{P}(U_i) \notin (\mathrm{conv}\{\mathcal{P}(U_j), j \ne i, j = 1, \ldots, n\})\}\right)^m\\
&\quad = \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \mathbb{P}\left(\mathcal{P}(U_{i_k}) \notin \mathrm{conv}(\{\mathcal{P}(U_j), j \ne i_k, j = 1, \ldots, n\}), k = 1, \ldots, m\right)
\end{aligned}$$

$$\leq \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \mathbb{P}\left(\mathcal{P}(U_{i_1}), \mathcal{P}(U_{i_2}), \ldots, \mathcal{P}(U_{i_m}) \notin \operatorname{conv}(\{\mathcal{P}(U_j), j \notin \{i_1, i_2, \ldots, i_m\}\})\right).$$

In view of this representation, the inequality (4.15) follows once we can show that

$$\mathbb{P}\left(\mathcal{P}(U_1), \mathcal{P}(U_2), \ldots, \mathcal{P}(U_k) \notin [\mathcal{P}(U_{k+1}), \ldots, \mathcal{P}(U_n)]\right) = \mathcal{O}(n^{-k}),$$

as $n \to \infty$, for every fixed $k \in \mathbb{N}$, where the constant in the Landau term $\mathcal{O}(\,\cdot\,)$ might depend on k. Denote by $K_n \subset \mathbb{R}^d$ the convex hull of the random points $\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)$. Fix $k \in \mathbb{N}$ and let $Y_1, Y_2, \ldots, Y_k$ be independently and identically distributed according to the beta$'$-type distribution described in Proposition 4.1.1. Assume also that $Y_1, \ldots, Y_k$ are independent of K_n. We are going to show that, as $n \to \infty$,

$$n^k \mathbb{P}\left(Y_1, \ldots, Y_k \notin K_n\right) = \mathcal{O}(1).$$

Note that the left-hand side can be written as

$$\begin{aligned} n^k \mathbb{P}\left(Y_1, \ldots, Y_k \notin K_n\right) &= n^k \mathbb{E}\left(\mathbb{P}^k\left(Y_1 \notin K_n | K_n\right)\right) \\ &= \mathbb{E}\left(\frac{n\Gamma\left(\frac{d+\gamma+1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)} \int\limits_{\mathbb{R}^d \setminus K_n} \frac{\mathrm{d}x}{(1+\|x\|^2)^{\frac{d+\gamma+1}{2}}}\right)^k. \end{aligned}$$

Thus, it suffices to show that

$$\mathbb{E}\left[\left(\frac{n\Gamma\left(\frac{d+\gamma+1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)} \int\limits_{\mathbb{R}^d \setminus K_n} \frac{\mathrm{d}x}{(1+\|x\|^2)^{\frac{d+\gamma+1}{2}}}\right)^k\right] = \mathcal{O}(1),$$

as $n \to \infty$, because $\mathbb{P}(\boldsymbol{o} \notin K_n) = \mathcal{O}(\mathrm{e}^{-cn})$ by Lemma 4.1.8, with $r \downarrow 0$. To bound the latter integral, introduce the random variable

$$\theta_n := \min_{x \in \partial K_n} \|x\|$$

and note that

$$\mathbb{E}\left[\left(\frac{n\Gamma\left(\frac{d+\gamma+1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)} \int\limits_{\mathbb{R}^d \setminus K_n} \frac{\mathrm{d}x}{(1+\|x\|^2)^{\frac{d+\gamma+1}{2}}}\right)^k \mathbb{1}\{\boldsymbol{o} \in K_n\}\right]$$

$$\leq \mathbb{E}\left(\frac{n\Gamma\left(\frac{d+\gamma+1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}\int\limits_{\mathbb{R}^d\setminus\mathbb{B}^d(\boldsymbol{o},\theta_n)}\frac{\mathrm{d}x}{(1+\|x\|^2)^{\frac{d+\gamma+1}{2}}}\right)^k,$$

where $\mathbb{B}^d(\boldsymbol{o},\theta_n)$ is the ball of radius θ_n centered at the origin $\boldsymbol{o}$. From now on, for the sake of brevity, any constants only depending on d and k will be denoted by c_1, c_2 etc.

Passing to polar coordinates in the expression for the above expectation, we obtain

$$I(n) := \mathbb{E}\left(\frac{n\Gamma\left(\frac{d+\gamma+1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}\int\limits_{\mathbb{R}^d\setminus\mathbb{B}^d(\boldsymbol{o},\theta_n)}\frac{\mathrm{d}x}{(1+\|x\|^2)^{\frac{d+\gamma+1}{2}}}\right)^k = \mathbb{E}\left(c_1 n\int\limits_{\theta_n}^{\infty}\frac{r^{d-1}\mathrm{d}r}{(1+r^2)^{\frac{d+\gamma+1}{2}}}\right)^k.$$

Note that

$$\frac{r^{d-1}}{(1+r^2)^{\frac{d+\gamma+1}{2}}} \leq \frac{1}{\max\{r^{\gamma+2},1\}}, \qquad r>0,$$

and, therefore,

$$\int\limits_{\theta_n}^{\infty}\frac{r^{d-1}\mathrm{d}r}{(1+r^2)^{\frac{d+\gamma+1}{2}}} \leq \int\limits_{\theta_n}^{\infty}\frac{\mathrm{d}r}{\max\{r^{\gamma+2},1\}} = \begin{cases}\frac{\gamma+2}{\gamma+1}-\theta_n, & \theta_n\leq 1,\\ \frac{1}{(\gamma+1)\theta_n^{\gamma+1}}, & \theta_n>1.\end{cases}$$

Hence,

$$\begin{aligned}
I(n) &\leq \left(\frac{\gamma+2}{\gamma+1}\right)^k c_1^k n^k \mathbb{P}(\theta_n<1) + c_1^k\mathbb{E}\left[\left(\frac{n}{(\gamma+1)\theta_n^{\gamma+1}}\right)^k \mathbb{1}\{\theta_n\geq 1\}\right]\\
&\leq \left(\frac{\gamma+2}{\gamma+1}\right)^k c_1^k n^k \mathbb{P}(K_n\not\supset\mathbb{B}^d) + c_1^k\int\limits_0^{\infty}\mathbb{P}\left(\left(\frac{n}{(\gamma+1)\theta_n^{\gamma+1}}\right)^k\mathbb{1}\{\theta_n\geq 1\}>x\right)\mathrm{d}x\\
&= \left(\frac{\gamma+2}{\gamma+1}\right)^k c_1^k n^k \mathbb{P}(K_n\not\supset\mathbb{B}^d) + c_1^k\int\limits_0^{\left(\frac{n}{\gamma+1}\right)^k}\mathbb{P}\left(1\leq\theta_n<\left(\frac{nx^{-\frac{1}{k}}}{\gamma+1}\right)^{\frac{1}{\gamma+1}}\right)\mathrm{d}x\\
&\leq \left(\frac{\gamma+2}{\gamma+1}\right)^k c_1^k n^k \mathbb{P}(K_n\not\supset\mathbb{B}^d) + c_1^k\int\limits_0^{\left(\frac{n}{\gamma+1}\right)^k}\mathbb{P}\left(K_n\not\supset\mathbb{B}^d\left(\boldsymbol{o},\left(\frac{nx^{-1/k}}{\gamma+1}\right)^{\frac{1}{\gamma+1}}\right)\right)\mathrm{d}x\\
&= \left(\frac{\gamma+2}{\gamma+1}\right)^k c_1^k n^k \mathbb{P}\left(\frac{K_n}{n^{\frac{1}{\gamma+1}}}\not\supset\mathbb{B}^d\left(\boldsymbol{o},n^{-\frac{1}{\gamma+1}}\right)\right)
\end{aligned}$$

$$+ c_1^k \int_0^{\left(\frac{n}{\gamma+1}\right)^k} \mathbb{P}\left(\frac{K_n}{n^{\frac{1}{\gamma+1}}} \not\supset \mathbb{B}^d\left(\boldsymbol{o}, \left(\frac{x^{-1/k}}{\gamma+1}\right)^{\frac{1}{\gamma+1}}\right)\right) \mathrm{d}x.$$

We want to apply Lemma 4.1.8 to both summands. For the first summand we have $r = n^{-\frac{1}{\gamma+1}}$ and, therefore, $rn^{\frac{1}{\gamma+1}} = 1$. In case of the second summand $r = \left(\frac{x^{-1/k}}{\gamma+1}\right)^{\frac{1}{\gamma+1}}$ and, since $x \leq \left(\frac{n}{\gamma+1}\right)^k$, we have that $rn^{\frac{1}{\gamma+1}} \geq 1$. Now we can apply Lemma 4.1.8 to bound both summands to conclude that

$$I(n) \leq c_2 n^k \exp(-c_3 n) + c_4 \int_0^{\left(\frac{n}{\gamma+1}\right)^k} \exp\left(-\left(c_5 x^{-\frac{2}{k(\gamma+1)}} + c_6 n^{-\frac{2}{\gamma+1}}\right)^{-\frac{\gamma+1}{2}}\right) \mathrm{d}x.$$

The first summand, clearly, converges to zero and it remains to show that the integral on the right-hand side is bounded by a constant not depending on n. If $x \leq \left(c_5^{\frac{\gamma+1}{2}} c_6^{-\frac{\gamma+1}{2}} n\right)^k$, then, $\left(c_5 x^{-\frac{2}{k(\gamma+1)}} + c_6 n^{-\frac{2}{\gamma+1}}\right)^{\frac{\gamma+1}{2}} \leq (2c_5)^{\frac{\gamma+1}{2}} x^{-1/k}$ and we have

$$\begin{aligned}
&\int_0^{\left(c_5^{\frac{\gamma+1}{2}} c_6^{-\frac{\gamma+1}{2}} n\right)^k} \exp\left(-\left(c_5 x^{-\frac{2}{k(\gamma+1)}} + c_6 n^{-\frac{2}{\gamma+1}}\right)^{-\frac{\gamma+1}{2}}\right) \mathrm{d}x \\
&\leq \int_0^{\left(c_5^{\frac{\gamma+1}{2}} c_6^{-\frac{\gamma+1}{2}} n\right)^k} \exp\left(-\frac{1}{(2c_5)^{\frac{\gamma+1}{2}} x^{-1/k}}\right) \mathrm{d}x \\
&\leq \int_0^{\infty} \exp\left(-\frac{1}{c_7 x^{-1/k}}\right) \mathrm{d}x < \infty.
\end{aligned}$$

On the other hand, if $x \in \left(\left(c_5^{\frac{\gamma+1}{2}} c_6^{-\frac{\gamma+1}{2}} n\right)^k, \left(\frac{n}{\gamma+1}\right)^k\right]$ (provided this interval is not empty), we have

$$\int_{\left(c_5^{\frac{\gamma+1}{2}} c_6^{-\frac{\gamma+1}{2}} n\right)^k}^{\left(\frac{n}{\gamma+1}\right)^k} \exp\left(-\left(c_5 x^{-\frac{2}{k(\gamma+1)}} + c_6 n^{-\frac{2}{\gamma+1}}\right)^{-\frac{\gamma+1}{2}}\right) \mathrm{d}x$$

$$\leq \int_{\left(c_5^{\frac{\gamma+1}{2}} c_6^{-\frac{\gamma+1}{2}} n\right)^k}^{\left(\frac{n}{\gamma+1}\right)^k} \exp\left(-\frac{1}{c_8 n^{-1} + c_5 n^{-1}}\right) \mathrm{d}x$$
$$= \mathcal{O}(n^k e^{-n/(c_8+c_5)}),$$

as $n \to \infty$. This completes the proof of the moment convergence.

The formula for $\mathbb{E}\mathbf{f}_{k-1}(\mathrm{conv}(\Pi_{d,\gamma+1}(\omega_{\gamma+1})))$ in Theorem 4.2.4 follows from the Mecke equation (2.38) applied with $f(x_1, \dots, x_k; \Pi) = \mathbb{1}\{(x_1, \dots, x_k) \in \mathcal{F}_{k-1}(\mathrm{conv}(\Pi))\}$. The proof of Theorem 4.2.4 is complete. □

Conic intrinsic volumes

We start off with a theorem about the distributional convergence of the solid angle $\alpha(C_{n,\gamma})$. Clearly, we have that $\alpha(C_{n,\gamma})$ almost surely converges to $1/2$, as $n \to \infty$. Theorem 7.1 in [16] provides for the case $\gamma = 0$ a more delicate asymptotic result, namely,

$$\mathbb{E}\left(\frac{1}{2} - \alpha(C_{n,0})\right) = C(d)\pi^{d+1}\left(\frac{2}{\omega_{d+1}}\right)^{d+1} \frac{\omega_d}{\omega_{d+1}} \frac{1}{n} + \mathcal{O}(n^{-2}), \tag{4.16}$$

as $n \to \infty$, where $C(d)$ is the same constant as in (4.3). We show the distributional counterpart to this formula for general $\gamma > -1$.

Theorem 4.2.6 *As $n \to \infty$, we have that*

$$n\left(\frac{1}{2} - \alpha(C_{n,\gamma})\right) \xrightarrow{\mathrm{d}} \frac{\omega_{\gamma+1}}{2\omega_{d+\gamma+1}} \int_{\mathbb{R}^d \setminus \mathrm{conv}(\Pi_{d,\gamma+1}(\omega_{\gamma+1}))} \frac{\mathrm{d}v}{\|v\|^{d+\gamma+1}}.$$

Before we prove this theorem let us briefly observe that for each $\gamma > 0$ and $c > 0$, $\mathrm{conv}(\Pi_{d,\gamma}(c))$ is an element of the space $\mathcal{N}$, and, therefore, Lemma 2.4.5 immediately yields the following result.

Corollary 4.2.7 *For each $\gamma > 0$ and $c > 0$,* $\mathrm{conv}(\Pi_{d,\gamma}(c))$ *is almost surely a convex polytope containing the origin $\boldsymbol{o}$ in its interior.*

Proof of Theorem 4.2.6. We shall use the following alternative definition of the solid angle. For a convex cone $C \subset \{x_0 \geq 0\} \subset \mathbb{R}^{d+1}$, the solid angle equals

$$\alpha(C) = \frac{1}{2}\,\mathbb{P}\left(U \in C \cap \mathbb{S}^d_+\right),$$

where U is a random vector with the uniform distribution on the half-sphere $\mathbb{S}^d_+$. We have

$$2n\left(\frac{1}{2} - \alpha(C_{n,\gamma})\right) = n\left(1 - \mathbb{P}\left(U \in C_{n,\gamma} \cap \mathbb{S}^d_+ \middle| C_{n,\gamma}\right)\right) = n\mathbb{P}\left(U \notin C_{n,\gamma} \cap \mathbb{S}^d_+ \middle| C_{n,\gamma}\right),$$

where U is independent of $C_{n,\gamma}$ and $\mathbb{P}(\,\cdot\,|\,\cdot\,)$ denotes conditional probability. Further,

$$\begin{aligned} n\mathbb{P}\left(U \notin C_{n,\gamma} \cap \mathbb{S}^d_+ \middle| C_{n,\gamma}\right) &= n\mathbb{P}\big((1, \mathcal{P}(U)) \notin C_{n,\gamma} \cap H_1 | C_{n,\gamma}\big) \\ &= n\mathbb{P}\big(\mathcal{P}(U) \notin \operatorname{conv}(\{\mathcal{P}(U_i) : i = 1, \dots, n\}) | U_1, \dots, U_n\big) \\ &= \hat{\mu}_{n,\gamma}\left(\mathbb{R}^d \setminus \operatorname{conv}\left(\left\{n^{-\frac{1}{\gamma+1}}\mathcal{P}(U_i) : i = 1, \dots, n\right\}\right)\right), \end{aligned}$$

where the measure $\hat{\mu}_{n,\gamma}$ is given by $\hat{\mu}_{n,\gamma}(\cdot) := n\mathbb{P}\left(n^{-\frac{1}{\gamma+1}}\mathcal{P}(U) \in \cdot\,\right)$. As a consequence of Proposition 4.1.1, the Lebesgue density of $\hat{\mu}_{n,\gamma}$ is given by

$$\hat{f}_{n,\gamma}(x) = \frac{\Gamma\left(\frac{d+\gamma+1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)} \frac{n^{\frac{d}{\gamma+1}+1}}{(1 + n^{\frac{2}{\gamma+1}}\|x\|^2)^{\frac{d+\gamma+1}{2}}}.$$

Denoting the random polytope $\operatorname{conv}\left(\left\{n^{-\frac{1}{\gamma+1}}\mathcal{P}(U_i) : i = 1, \dots, n\right\}\right)$ by $L_{n,\gamma}$, we can write

$$2n\left(\frac{1}{2} - \alpha(C_{n,\gamma})\right) = \int_{\mathbb{R}^d} (1 - \mathbb{1}_{L_{n,\gamma}}(x))\hat{f}_{n,\gamma}(x)\mathrm{d}x.$$

Let also $L_{0,\gamma} := \operatorname{conv}(\Pi_{d,\gamma+1}(2\omega_{\gamma+1}))$. From (4.9), we know that $L_{n,\gamma}$ converges to $L_{0,\gamma}$ weakly on the space $\mathcal{K}^d$. By the Skorokhod representation theorem, i.e., Lemma 2.4.4, on a new probability space we can define random convex sets $(L'_{n,\gamma})_{n\in\mathbb{N}_0}$ such that $L'_{n,\gamma}$ has the same distribution as $L_{n,\gamma}$, for all $n \in \mathbb{N}_0$, and with probability one $L'_{n,\gamma} \to L'_{0,\gamma}$ holds in the Hausdorff metric. Let us fix some outcome ω in the new probability space outside the event where the convergence fails to hold or where $L'_{0,\gamma}$ is not a polytope containing the origin $\boldsymbol{o}$ in its interior. The probability of this exceptional event is zero; see Corollary 4.2.7. With this convention, the deterministic polytopes $L'_{n,\gamma}(\omega)$ converge to $L'_{0,\gamma}(\omega)$ in the Hausdorff metric. From Lemma 2.4.3 it follows that

$$\lim_{n\to\infty} \mathbb{1}_{L'_{n,\gamma}(\omega)}(x) = \mathbb{1}_{L'_{0,\gamma}(\omega)}(x), \text{ for all } x \in \mathbb{R}^d \backslash \partial L'_{0,\gamma}(\omega).$$

Note, that the Lebesgue measure of $\partial L'_{0,\gamma}(\omega)$ is zero because $L'_{0,\gamma}(\omega)$ is a polytope. The density $\hat{f}_{n,\gamma}(x)$ satisfies

$$\lim_{n\to\infty} \hat{f}_{n,\gamma}(x) = \frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}} \frac{1}{\|x\|^{d+\gamma+1}} \qquad \text{and} \qquad \hat{f}_{n,\gamma}(x) \leq \frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}} \frac{1}{\|x\|^{d+\gamma+1}},$$

for all $x \in \mathbb{R}^d \backslash \{\boldsymbol{o}\}$. Combining everything, we obtain that for Lebesgue-a.e. $x \in \mathbb{R}^d$,

$$\lim_{n\to\infty} (1 - \mathbb{1}_{L'_{n,\gamma}(\omega)}(x))\hat{f}_{n,\gamma}(x) = (1 - \mathbb{1}_{L'_{0,\gamma}(\omega)}(x)) \frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}} \frac{1}{\|x\|^{d+\gamma+1}}.$$

Also, we have the integrable bound

$$(1 - \mathbb{1}_{L'_{n,\gamma}(\omega)}(x))\hat{f}_{n,\gamma}(x) \leq \mathbb{1}\{\|x\| \geq r(\omega)\} \frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}} \frac{1}{\|x\|^{d+\gamma+1}},$$

where $r(\omega) > 0$ is the distance from the origin $\boldsymbol{o}$ to the boundary of $L'_{0,\gamma}(\omega)$. The dominated convergence theorem yields

$$\begin{aligned} \int_{\mathbb{R}^d} (1 - \mathbb{1}_{L'_{n,\gamma}(\omega)}(x))\hat{f}_{n,\gamma}(x) \mathrm{d}x \to & \int_{\mathbb{R}^d} (1 - \mathbb{1}_{L'_{0,\gamma}(\omega)}(x)) \frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}} \frac{\mathrm{d}x}{\|x\|^{d+\gamma+1}} \\ &= \frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}} \int_{\mathbb{R}^d \backslash L'_{0,\gamma}(\omega)} \frac{\mathrm{d}x}{\|x\|^{d+\gamma+1}}, \end{aligned}$$

as $n \to \infty$. We recall that this convergence holds for every outcome ω outside some event with probability zero. In particular, it implies the distributional convergence of the corresponding random variables. Returning back to the original probability space, we can replace $L'_{n,\gamma}$ by $L_{n,\gamma}$ for all $n \in \mathbb{N}_0$, thus obtaining

$$2n\left(\frac{1}{2} - \alpha(C_{n,\gamma})\right) = \int_{\mathbb{R}^d} (1 - \mathbb{1}_{L_{n,\gamma}}(x))\hat{f}_{n,\gamma}(x) \mathrm{d}x \xrightarrow{\mathrm{d}} \frac{\omega_{\gamma+1}}{\omega_{d+\gamma+1}} \int_{\mathbb{R}^d \backslash L_{0,\gamma}} \frac{\mathrm{d}x}{\|x\|^{d+\gamma+1}},$$

as $n \to \infty$. Recall finally that $L_{0,\gamma} = \mathrm{conv}(\Pi_{d,\gamma+1}(\omega_{\gamma+1}))$, which finishes the proof. $\square$

The next result relates the expected Grassmann angles h_k of the random cone $C_{n,0}$ to its expected **f**-vector.

Theorem 4.2.8 *For all $k \in \{1, \ldots, d\}$, we have*

$$2\binom{n+d+1-k}{d+1-k}\Big(\frac{1}{2} - \mathbb{E}h_{k+1}(C_{n,0})\Big) = \mathbb{E}\mathbf{f}_{d-k+1}(C_{n+d+1-k,0}).$$

The above formula should be compared to the well-known Efron identity [42] that states that for random points $Q_1, Q_2, \ldots$ sampled uniformly and independently from a convex body $K \subset \mathbb{R}^d$ and all $n \geq d+1$ we have

$$\frac{\mathbb{E}\,\mathrm{Vol}_d([Q_1, \ldots, Q_n])}{\mathrm{Vol}_d(K)} = 1 - \frac{\mathbb{E}\mathbf{f}_0([Q_1, \ldots, Q_{n+1}])}{n+1}.$$

Buchta [30] obtained an analogue of this identity for higher moments of the volume, but no identity relating the expected **f**-vector of random polytopes to their intrinsic volumes is known in the Euclidean case, to the best of our knowledge (however, we refer to [53, 97] for results in this direction for the zero cells of Poisson hyperplane tessellations).

Proof of Theorem 4.2.8. We shall derive formulae for the expectations of Grassmann angles and the **f**-vectors of $C_{n,0}$ and, then, obtain Theorem 4.2.8 by comparing these formulae.

Step 1. We are interested in the expected Grassmann angle

$$\mathbb{E}h_{k+1}(C_{n,0}) = \frac{1}{2}\mathbb{P}(C_{n,0} \cap L \neq \{\boldsymbol{o}\}),$$

where $L \in G(d+1, d+1-k)$ is a random subspace with distribution ν_{d+1-k}, and $k \in \{1, \ldots, d\}$. Recall that $C_{n,0} = \mathrm{pos}(\{U_1, \ldots, U_n\})$, where $U_1, \ldots, U_n$ are i.i.d. random points distributed uniformly on $\mathbb{S}^d_+$. Observe that L can be generated as a linear hull of $d+1-k$ i.i.d. random points $V_1, \ldots, V_{d+1-k}$, are distributed uniformly on $\mathbb{S}^d_+$ and independent of the U_i's.

Applying the mapping $\mathcal{P}$ defined by (4.4), together with Proposition 4.1.1, we see that

$$\mathbb{E}h_{k+1}(C_{n,0}) = \frac{1}{2}\mathbb{P}\big([\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)] \cap \mathrm{aff}(\{Z_1, \ldots, Z_{d+1-k}\}) \neq \varnothing\big).$$

Here $Z_1 := \mathcal{P}(V_1), \ldots, Z_{d+1-k} := \mathcal{P}(V_{d+1-k})$ are independent random points in $\mathbb{R}^d$ distributed according to the beta′-type distribution described in Proposition 4.1.1 with

$\gamma = 0$. Thus,

$$\frac{1}{2} - \mathbb{E}h_{k+1}(C_{n,0}) = \frac{1}{2} \int\limits_{(\mathbb{R}^d)^{d+1-k}} \mathbb{P}(\operatorname{aff}(\{x_1, \ldots, x_{d+1-k}\}) \cap [\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)] = \varnothing) \times \prod_{i=1}^{d+1-k} \frac{(2/\omega_{d+1})\,\mathrm{d}x_i}{(1+\|x_i\|^2)^{\frac{d+1}{2}}}.$$

Step 2. We now derive a formula for $\mathbb{E}\mathbf{f}_k(C_{n,0})$ or, equivalently, the expected number of $(k-1)$-dimensional faces of the random polytope $K_{n,0} := [\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)]$. We have

$$\mathbb{E}\mathbf{f}_k(C_{n,0}) = \mathbb{E}\mathbf{f}_{k-1}(K_{n,0}) = \mathbb{E} \sum_{1 \le i_1 < \ldots < i_k \le n} \mathbb{1}\{[\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)] \in \mathcal{F}_{k-1}(K_{n,0})\}.$$

Since $\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)$ are independent and identically distributed according to the beta′-type distribution described in Proposition 4.1.1 with $\gamma = 0$, we have that

$$\mathbb{E}\mathbf{f}_k(C_{n,0}) = \binom{n}{k} \int\limits_{(\mathbb{R}^d)^k} \mathbb{P}([x_1, \ldots, x_k]) \in \mathcal{F}_{k-1}(K_n) \,|\, \mathcal{P}(U_1) = x_1, \ldots, \mathcal{P}(U_k) = x_k) \times \prod_{i=1}^{k} \frac{(2/\omega_{d+1})\,\mathrm{d}x_i}{(1+\|x_i\|^2)^{\frac{d+1}{2}}}.$$

Observe that conditionally on $\mathcal{P}(U_1) = x_1, \ldots, \mathcal{P}(U_k) = x_k$, we got $[x_1, \ldots, x_k] \in \mathcal{F}_{k-1}(K_{n,0})$ if and only if $\operatorname{aff}(\{x_1, \ldots, x_k\}) \cap [\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)] = \varnothing$. Therefore,

$$\mathbb{E}\mathbf{f}_k(C_{n,0}) = \binom{n}{k} \int\limits_{(\mathbb{R}^d)^k} \mathbb{P}(\operatorname{aff}(\{x_1, \ldots, x_k\}) \cap [\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)] = \varnothing) \times \prod_{i=1}^{k} \frac{(2/\omega_{d+1})\,\mathrm{d}x_i}{(1+\|x_i\|^2)^{\frac{d+1}{2}}}. \tag{4.17}$$

Step 3. Comparing the formulae obtained in step 1 and step 2, we arrive at

$$2\binom{n+d+1-k}{d+1-k}\Big(\frac{1}{2} - \mathbb{E}h_{k+1}(C_{n,0})\Big) = \mathbb{E}\mathbf{f}_{d-k+1}(C_{n+d+1-k,0}),$$

which completes the proof. □

The next result identifies asymptotically the expected conic intrinsic volumes, the Grassmann angles and the conic mean projection volumes of the random cones $C_{n,0}$. As already mentioned, this completely settles in an extended form the conjecture of Bárány et al. stated in [16, Section 9].

Theorem 4.2.9 *For every $k \in \{0, 1, \dots, d\}$, we have*

$$\lim_{n\to\infty} n^{d+1-k}\Big(\frac{1}{2} - \mathbb{E}h_{k+1}(C_{n,0})\Big) = B_{0,d+1-k,d}, \tag{4.18}$$

where $B_{0,1,d}, \dots, B_{0,d,d}$ are given by (4.11), and $B_{0,d+1,d} = 0$. Moreover, for all $\ell, r \in \{0, 1, \dots, d-1\}$, we have

$$\lim_{n\to\infty} n^{d-1-\ell}\,\mathbb{E}v_\ell(C_{n,0}) = B_{0,d-1-\ell,d}, \tag{4.19}$$

and

$$\lim_{n\to\infty} n^{d+1-r}\big(1 - \mathbb{E}w_{r+1}(C_{n,0})\big) = B_{0,d+1-r,d}. \tag{4.20}$$

Remark 4.2.10 Note that $v_d(C_{n,0}) = h_d(C_{n,0}) \to 1/2$ and $v_{d+1}(C_{n,0}) = h_{d+1}(C_{n,0}) \to 1/2$, as $n \to \infty$. Hence, we have restricted ourselves to the conic intrinsic volumes $v_l(C_{n,0})$ of orders $\ell \in \{0, \dots, d-1\}$ in (4.19). Similarly, $w_{d+1}(C_{n,0}) = h_{d+1}(C_{n,0})$, which is why we omitted the case $r = d$ in (4.20).

Proof of Theorem 4.2.9. We first prove the asymptotic formula for h_{k+1}. For $k = 0$, the result is trivial since $h_1(C_{n,0}) = 1/2$, so let $k \in \{1, \dots, d\}$. We use Theorem 4.2.8 together with Theorem 4.2.4 to obtain

$$\begin{aligned} n^{d+1-k}\Big(\frac{1}{2} - \mathbb{E}h_{k+1}(C_{n,0})\Big) &= \frac{1}{2}\frac{n^{d+1-k}}{\binom{n+d+1-k}{d+1-k}}\mathbb{E}\mathbf{f}_{d-k+1}(C_{n+d+1-k,0}) \\ &\to B_{0,d-k+1,d}, \end{aligned} \tag{4.21}$$

as $n \to \infty$. To deduce the result for the conic intrinsic volumes, recall the conic Crofton formula (2.22) and note that it implies, for $\ell \in \{0, 1, \dots, d-1\}$,

$$\mathbb{E}v_\ell(C_{n,0}) = \mathbb{E}h_\ell(C_{n,0}) - \mathbb{E}h_{\ell+2}(C_{n,0}) = \Big(\frac{1}{2} - \mathbb{E}h_{\ell+2}(C_{n,0})\Big) - \Big(\frac{1}{2} - \mathbb{E}h_\ell(C_{n,0})\Big).$$

So, we get that

$$\lim_{n\to\infty} n^{d-1-\ell}\,\mathbb{E}v_\ell(C_{n,0}) = \lim_{n\to\infty} n^{d-1-\ell}\Big(\frac{1}{2} - \mathbb{E}h_{\ell+2}(C_{n,0})\Big) - \lim_{n\to\infty} n^{d-1-\ell}\Big(\frac{1}{2} - \mathbb{E}h_\ell(C_{n,0})\Big).$$

According to (4.21), the first limit equals $B_{0,d-1-\ell,d}$, while the second one is zero (indeed, the sequence goes to zero like a constant multiple of n^{-1}, as $n \to \infty$).

Finally, the asymptotic formulae for the mean projection volumes can be deduced in a similar way from the conic Kubota formula (2.24). Namely, for all $r \in \{0, 1, \dots, d-1\}$, we have $w_{r+1}(C_{n,0}) = h_{r+1}(C_{n,0}) + h_{r+2}(C_{n,0})$, hence

$$\begin{aligned}\lim_{n\to\infty} & n^{d+1-r}\big(1 - \mathbb{E}w_{r+1}(C_{n,0})\big) \\ &= \lim_{n\to\infty} n^{d+1-r}\Big(\frac{1}{2} - \mathbb{E}v_{r+1}(C_{n,0})\Big) + \lim_{n\to\infty} n^{d+1-r}\Big(\frac{1}{2} - \mathbb{E}v_{r+2}(C_{n,0})\Big).\end{aligned}$$

By (4.21), the first limit equals $B_{0,d+1-r,d}$, whereas the second one is zero. □

Proposition 4.2.11 *For all $d \geq 2$, we have*

$$B_{0,2,d} = \frac{1}{2}\binom{d+1}{3}\pi^2.$$

Proof. For the expected surface area, i.e. $(d-1)$-dimensional Hausdorff measure, of the spherical polytope $C_{n,0} \cap \mathbb{S}^d$, it was shown in [16, Theorem 5.1] that

$$\mathbb{E}S_{d-1}(C_{n,0} \cap \mathbb{S}^d) = \omega_d\left(1 - \binom{d+1}{3}\pi^2 n^{-2} + \mathcal{O}(n^{-3})\right),$$

where $S_{d-1}(K)$ denotes the surface area of the spherical polytope K. On the other hand, the relation $2\omega_d h_d(C_{n,0}) = 2\omega_d v_d(C_{n,0}) = S\left(C_{n,0} \cap \mathbb{S}^d\right)$ and Theorem 4.2.9 with $k = d-1$ yield

$$\mathbb{E}S_{d-1}\left(C_{n,0} \cap \mathbb{S}^d\right) = \omega_d\left(1 - 2B_{0,2,d}n^{-2} + o(n^{-2})\right).$$

Comparing both asymptotic relations, we obtain the required formula for $B_{0,2,d}$. □

Let us consider the special case $d = 2$, where $B_{0,2,2} = \frac{1}{2}\pi^2$ and, hence,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\mathbf{f}_0\left(C_{n,0} \cap \mathbb{S}^2_+\right) &= \lim_{n\to\infty} \mathbb{E}\mathbf{f}_1\left(C_{n,0} \cap \mathbb{S}^2_+\right) = \mathbb{E}\mathbf{f}_0(\mathrm{conv}(\Pi_{2,1}(c))) \\ &= \mathbb{E}\mathbf{f}_1(\mathrm{conv}(\Pi_{2,1}(c))) = \frac{1}{2}\pi^2,\end{aligned}$$

with $c > 0$ being arbitrary. For $d = 3$, the identities $B_{0,3,3} = 4\pi^2$ and $B_{0,2,3} = 2\pi^2$ (following from (4.12) and Proposition 4.2.11) combined with the Euler relation $\mathbf{f}_0 - \mathbf{f}_1 + \mathbf{f}_2 = 2$ yield

$$\begin{aligned}&\lim_{n\to\infty} \left(\mathbb{E}\mathbf{f}_0\left(C_{n,0}\cap\mathbb{S}^3_+\right), \mathbb{E}\mathbf{f}_1\left(C_{n,0}\cap\mathbb{S}^3_+\right), \mathbb{E}\mathbf{f}_2\left(C_{n,0}\cap\mathbb{S}^3_+\right)\right)\\ &\qquad = (\mathbb{E}\mathbf{f}_0(\mathrm{conv}(\Pi_{3,1}(c))), \mathbb{E}\mathbf{f}_1(\mathrm{conv}(\Pi_{3,1}(c))), \mathbb{E}\mathbf{f}_2(\mathrm{conv}(\Pi_{3,1}(c))))\\ &\qquad = \left(2 + \frac{2}{3}\pi^2, 2\pi^2, \frac{4}{3}\pi^2\right).\end{aligned}$$

Remark 4.2.12 Using the same methods as in the proof of Theorems 4.2.9 and 4.2.6, it is possible to prove the following distributional convergence for all $k \in \{0, 1, \dots, d\}$:

$$\begin{aligned}n^{d+1-k}\Big(\frac{1}{2} - h_{k+1}(C_{n,0})\Big) &\to \frac{1}{2}\left(\frac{2}{\omega_{d+1}}\right)^{d+1-k}\\ &\times \int\limits_{(\mathbb{R}^d)^{d+1-k}} \mathbb{1}\{\mathrm{conv}(\Pi_{d,1}(2)) \cap \mathrm{aff}(\{x_1, \dots, x_{d+1-k}\}) = \varnothing\} \prod_{i=1}^{d+1-k} \frac{\mathrm{d}x_i}{\|x_i\|^{d+1}},\end{aligned}$$

as $n \to \infty$. Observe that since $h_{d+1}(C_{n,0})$ coincides with the solid angle $\alpha(C_{n,0})$, we recover the case $\gamma = 0$ of Theorem 4.2.6 as a special case of this relation with $k = d$.

4.3 Convex hulls of Poisson point processes

We start by stating the following projection stability. It says that the projection of a Poisson point process with a power-law intensity measure as in (4.7) onto a linear subspace is again a Poisson point process of the same type within this subspace. We will make heavy use of this fact.

Lemma 4.3.1 *Let $\gamma, c > 0$ and $k \in \{1, \dots, d-1\}$. The orthogonal projection of $\Pi_{d,\gamma}(c)$ onto any k-dimensional linear subspace L of $\mathbb{R}^d$ has the same law as $\Pi_{k,\gamma}(c)$, where we identify L with $\mathbb{R}^k$.*

Proof. First suppose that $k = d-1$. By rotational symmetry, we may assume that we project onto the hyperplane $\{x_1 = 0\}$. The intensity of the projected Poisson point process at $(0, x_2, \dots, x_d)$ with $x_2^2 + \ldots + x_d^2 = a^2$ equals

$$\frac{c}{\omega_{d+\gamma}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}x_1}{(a^2 + x_1^2)^{\frac{d+\gamma}{2}}} = \frac{c}{\omega_{d+\gamma}} \int_{-\infty}^{+\infty} \frac{a\mathrm{d}y}{a^{d+\gamma}(1+y^2)^{\frac{d+\gamma}{2}}} = \frac{c\, a^{1-d-\gamma}}{\omega_{d+\gamma}} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{(1+y^2)^{\frac{d+\gamma}{2}}},$$

where we used the change of variables $y = x_1/a$. Applying the substitution $y^2 = t$, the last integral equals

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}y}{(1+y^2)^{\frac{d+\gamma}{2}}} = \int_{0}^{\infty} \frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{d+\gamma}{2}}}\,\mathrm{d}t = \sqrt{\pi}\,\frac{\Gamma(\frac{d+\gamma-1}{2})}{\Gamma(\frac{d+\gamma}{2})}$$

by definition of the beta function and its relationship to the gamma function. Hence, the intensity of the projected Poisson point process is

$$\frac{c\, a^{1-d-\gamma}}{\omega_{d+\gamma}} \sqrt{\pi}\,\frac{\Gamma(\frac{d+\gamma-1}{2})}{\Gamma(\frac{d+\gamma}{2})} = \frac{c}{\omega_{d+\gamma-1}} \frac{1}{a^{d+\gamma-1}}$$

by definition of $\omega_{d+\gamma}$ and $\omega_{d+\gamma-1}$. Arguing now inductively, we arrive at the desired claim. □

Expectation of the T-functional

We are now going to state explicit formulae for the expected values of some functionals of the random polytopes $\mathrm{conv}(\Pi_{d,\gamma}(c))$. The results are most conveniently expressed via the *T-functional* which we already used in Chapter 3. The next theorem provides an explicit formula for the expected T-functional with $k = d-1$ of the random polytopes $\mathrm{conv}(\Pi_{d,\gamma}(c))$.

Theorem 4.3.2 *For every $\gamma, c > 0$ and all $a, b \geq 0$ such that $(\gamma - b)d + b - a > 0$ and $\gamma - b > 0$, we have that*

$$\mathbb{E}T_{a,b}^{d,d-1}(\mathrm{conv}(\Pi_{d,\gamma}(c))) = \frac{c^d \omega_d}{\gamma\, d!\, \omega_{\gamma+1}^d} \left(\frac{c}{\gamma \omega_{\gamma+1}}\right)^{\frac{a-b+(b-\gamma)d}{\gamma}} \Gamma\left(\frac{(\gamma-b)d+b-a}{\gamma}\right)$$
$$\times \frac{1}{((d-1)!)^b} \frac{\Gamma\left(\frac{\gamma-b}{2}d + \frac{b+1}{2}\right)}{\Gamma\left(\frac{\gamma-b}{2}d\right)} \left(\frac{\Gamma\left(\frac{\gamma-b}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^d \prod_{i=1}^{d-1} \frac{\Gamma\left(\frac{i+b+1}{2}\right)}{\Gamma\left(\frac{i}{2}\right)}.$$

If $(\gamma - b)d + b - a \leq 0$ or $\gamma - b \leq 0$, then, the expectation equals $+\infty$.

Before we prove this, let us take a closer look at special values for the parameters a and b which lead to some interesting consequences. We relegate all proofs to the end of this section.

Expected number of facets

Recall that for $a = b = 0$, it almost surely holds that

$$T_{0,0}^{d,d-1}(\mathrm{conv}(\Pi_{d,\gamma}(c))) = \mathbf{f}_{d-1}(\mathrm{conv}(\Pi_{d,\gamma}(c))).$$

After simplification, this yields the following result for the mean number of facets of $\mathrm{conv}(\Pi_{d,\gamma}(c))$.

Corollary 4.3.3 *For every $\gamma > 0, c > 0$, we have that*

$$\mathbb{E}\mathbf{f}_{d-1}(\mathrm{conv}(\Pi_{d,\gamma}(c))) = \frac{2}{d}\gamma^{d-1}\pi^{\frac{d-1}{2}} \frac{\Gamma(\frac{\gamma d+1}{2})}{\Gamma(\frac{\gamma d}{2})} \left(\frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\gamma+1}{2})}\right)^d,$$

independently of the parameter $c > 0$.

Remark 4.3.4 Again, all faces of the polytope $\mathrm{conv}(\Pi_{d,\gamma}(c))$ are simplices with probability 1. Hence, the Dehn–Sommerville relation, i.e., Theorem 2.2.3, implies

$$d\mathbf{f}_{d-1}(\mathrm{conv}(\Pi_{d,\gamma}(c))) = 2\mathbf{f}_{d-2}(\mathrm{conv}(\Pi_{d,\gamma}(c)))$$

for the expected number of $(d-2)$-faces of $\operatorname{conv}(\Pi_{d,\gamma}(c))$. However, computing the expected number of k-faces for general k remains an open problem.

In particular, for $\gamma = 1$ we obtain

$$\mathbb{E}\mathbf{f}_{d-1}(\operatorname{conv}(\Pi_{d,1}(c))) = \frac{2\pi^{d-\frac{1}{2}}}{d}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} = \pi^{d-\frac{1}{2}}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})}$$

for all $c > 0$. Using Legendre's duplication formula for the gamma function, this can be rewritten as follows:

$$\begin{aligned}\pi^{d-\frac{1}{2}}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})} &= \pi^{d-\frac{1}{2}}\frac{\Gamma(\frac{d+1}{2})}{\Gamma(1+\frac{d}{2})}\frac{\Gamma(1+\frac{d}{2})}{\Gamma(1+\frac{d}{2})} = \frac{d\pi^{d-\frac{1}{2}}}{2}\frac{\Gamma(\frac{d+1}{2})\Gamma(\frac{d}{2})}{\Gamma(1+\frac{d}{2})^2}\\ &= \frac{d\pi^{d-\frac{1}{2}}}{2}\frac{\Gamma(d)\sqrt{2\pi}\,2^{-d+\frac{1}{2}}}{\Gamma(1+\frac{d}{2})^2} = \frac{2^{-d}\pi^d\,d!}{\Gamma(1+\frac{d}{2})^2} = 2^{-d}d!\kappa_d^2.\end{aligned}$$

This coincides with the limit in (4.2) and is consistent with Theorem 4.2.3. More generally, for any $a \in [0, d)$, we have the explicit formula

$$\mathbb{E}T_{a,0}^{d,d-1}(\operatorname{conv}(\Pi_{d,1}(c))) = 2^{1-2a}c^a\left(\frac{\pi}{2}\right)^{d-a}\frac{\Gamma(d-a)}{\Gamma(1+\frac{d}{2})\Gamma(\frac{d}{2})}.$$

Another special case in which the formula from Corollary 4.3.3 simplifies is $\gamma = 2$. After similar transformations as before, we obtain

$$\mathbb{E}\mathbf{f}_{d-1}(\operatorname{conv}(\Pi_{d,2}(c))) = \binom{2d}{d}.$$

In dimension $d = 2$ this means that the expected number of edges (or vertices) of the convex hull of the Poisson point process with intensity $\|x\|^{-4}$ in $\mathbb{R}^2$ is 6, a fact due to Rogers [94]. For $d = 3$, we obtain that the expected number of faces of the convex hull of the Poisson point process with intensity $\|x\|^{-5}$ is 20. Since the faces are simplices a.s., the relation $3\mathbf{f}_2 = 2\mathbf{f}_1$ holds, which together with the Euler relation $\mathbf{f}_0 - \mathbf{f}_1 + \mathbf{f}_2 = 2$ yields that the expected number of edges (respectively, vertices) is 30 (respectively, 12). To summarize, the expected $\mathbf{f}$-vector of $\operatorname{conv}(\Pi_{3,2})$ is the same as the $\mathbf{f}$-vector of the regular icosahedron. Finally, observe that in the case $d = 2$ and for arbitrary $\gamma > 0$, Corollary 4.3.3 can be written as

$$\mathbb{E}\mathbf{f}_1(\operatorname{conv}(\Pi_{2,\gamma}(c))) = \mathbb{E}\mathbf{f}_0(\operatorname{conv}(\Pi_{2,\gamma}(c))) = 4\pi\frac{B\left(\frac{1}{2},\gamma+\frac{1}{2}\right)}{B^2\left(\frac{1}{2},\frac{\gamma+1}{2}\right)},$$

where B denotes the beta function. This formula is due to [38, Theorem 4.4], see also [34], where a similar formula is derived for convex hulls of i.i.d. samples with spherically symmetric regularly varying distributions.

Expected volume, intrinsic volumes and the symmetric convex hull

Let us compute the expected volume of $\mathrm{conv}(\Pi_{d,\gamma}(c))$. Since the origin is a.s. in the interior of $\mathrm{conv}(\Pi_{d,\gamma}(c))$, we have that

$$\mathrm{Vol}_d(\mathrm{conv}(\Pi_{d,\gamma}(c))) = \frac{1}{d} T_{1,1}^{d,d-1}(\mathrm{conv}(\Pi_{d,\gamma}(c))),$$

which together with Theorem 4.3.2 leads to the following result for the mean volume of the convex hull of $\Pi_{d,\gamma}(c)$.

Corollary 4.3.5 *For every $\gamma > 1$ and $c > 0$, we have that*

$$\mathbb{E}\,\mathrm{Vol}_d(\mathrm{conv}(\Pi_{d,\gamma}(c))) = \frac{c^{\frac{d}{\gamma}}}{d!2^{d(1+\frac{1}{\gamma})}\pi^{\frac{d}{2\gamma}}} \left(\frac{\gamma}{\Gamma(\frac{\gamma+1}{2})}\right)^{\frac{d(\gamma-1)}{\gamma}} \frac{\Gamma(1+d-\frac{d}{\gamma})\Gamma(\frac{\gamma-1}{2})^d}{\Gamma(1+\frac{d}{2})}.$$

For $0 < \gamma \leq 1$, we have $\mathbb{E}\,\mathrm{Vol}_d(\mathrm{conv}(\Pi_{d,\gamma}(c))) = +\infty$.

In the special case $\gamma = 2$, the formula becomes especially simple, namely:

$$\mathbb{E}\,\mathrm{Vol}_d(\mathrm{conv}(\Pi_{d,2}(c))) = \frac{1}{d!}\left(\frac{c}{2}\right)^{d/2}.$$

Next, we compute the expected values of the intrinsic volumes $V_k(\mathrm{conv}(\Pi_{d,\gamma}(c)))$, $k \in \{0, \dots, d\}$, of the random polytopes $\mathrm{conv}(\Pi_{d,\gamma}(c))$.

Corollary 4.3.6 *For every $\gamma > 1$, $c > 0$ and $k \in \{1, \dots, d\}$, we have that*

$$\mathbb{E}V_k(\mathrm{conv}(\Pi_{d,\gamma}(c))) = \binom{d}{k} \frac{\kappa_d}{\kappa_k\kappa_{d-k}} \frac{c^{\frac{k}{\gamma}}}{2^{k(1+\frac{1}{\gamma})}\pi^{\frac{k}{2\gamma}}} \left(\frac{\gamma}{\Gamma(\frac{\gamma+1}{2})}\right)^{\frac{k(\gamma-1)}{\gamma}} \frac{\Gamma(1+k-\frac{k}{\gamma})\Gamma(\frac{\gamma-1}{2})^k}{\Gamma(1+\frac{k}{2})}.$$

For $0 < \gamma \leq 1$ and all $k \in \{1, \dots, d\}$, we have $\mathbb{E}V_k(\mathrm{conv}(\Pi_{d,1}(c))) = +\infty$.

Finally, we give a theorem that evaluates the expected T-functional of $\mathrm{sconv}(\Pi_{d,\gamma}(c))$.

Theorem 4.3.7 *For every $\gamma, c > 0$ and all $a, b \geq 0$ such that $(\gamma - b)d + b - a > 0$ and $\gamma - b > 0$, we have that*

$$\mathbb{E}T_{a,b}^{d,d-1}(\mathrm{sconv}(\Pi_{d,\gamma}(c))) = \mathbb{E}T_{a,b}^{d,d-1}(\mathrm{conv}(\Pi_{d,\gamma}(2c))).$$

It is now straightforward to state the formulae for the expected facet number, volume, and intrinsic volumes of the symmetric convex hull of $\Pi_{d,\gamma}(c)$.

Proofs of Theorems 4.3.2, 4.3.7 and Corollary 4.3.6

Proof of Theorem 4.3.2. To simplify the notation, we shall write $\Pi_{d,\gamma}$ for $\Pi_{d,\gamma}(c)$ in this proof and keep $c > 0$ fixed. Recall that $\mathrm{conv}(\Pi_{d,\gamma})$ denotes the convex hull of all points of the Poisson process $\Pi_{d,\gamma}$. By Corollary 4.2.7, $\mathrm{conv}(\Pi_{d,\gamma})$ is almost surely a convex polytope. We denote by $H = H(x_1, \ldots, x_k) \in A(d, k-1)$ the $(k-1)$-dimensional affine subspace spanned by the points $x_1, \ldots, x_k$. Let also $\eta(H)$ be the distance from H to the origin. By the multivariate Mecke formula for Poisson point processes (2.38), we have

$$\mathbb{E}T_{a,b}^{d,k-1}(\mathrm{conv}(\Pi_{d,\gamma})) = \frac{1}{k!} \int\limits_{(\mathbb{R}^d)^k} \Delta_{k-1}^b(x_1, \ldots, x_k)\, \eta^a(E) \times \mathbb{P}\Big([x_1, \ldots, x_k] \in \mathcal{F}_{k-1}(\mathrm{conv}(\hat{\Pi}_{d,\gamma}))\Big) \prod_{i=1}^{k} \frac{c\,\mathrm{d}x_i}{\omega_{d+\gamma}\|x_i\|^{d+\gamma}},$$

where $\hat{\Pi}_{d,\gamma} := \Pi_{d,\gamma} + \sum_{i=1}^{k} \delta_{x_i}$. Recall that $H^\perp$ is the orthogonal complement of H and $P_{H^\perp}$ the orthogonal projection onto $H^\perp$. Note that $P_{H^\perp}x_1 = \ldots = P_{H^\perp}x_k$. Clearly, the simplex $[x_1, \ldots, x_k]$ is a $(k-1)$-dimensional face of $\mathrm{conv}(\hat{\Pi}_{d,\gamma})$ if and only if $P_{H^\perp}x_1$ is not contained in $P_{H^\perp}\,\mathrm{conv}(\Pi_{d,\gamma})$. Define the non-absorption probability

$$p_{d,\gamma}(R) := \mathbb{P}(Re_1 \notin \mathrm{conv}(\Pi_{d,\gamma})), \quad R > 0, \tag{4.22}$$

where e_1 is any vector of unit length in $\mathbb{R}^d$. By Lemma 4.3.1, $P_{H^\perp}\Pi_{d,\gamma}$ has the same distribution as $\Pi_{d-k+1,\gamma}$, where we identify $H^\perp$ with $\mathbb{R}^{d-k+1}$. Hence,

$$\mathbb{E}T^{d,k-1}_{a,b}(\operatorname{conv}(\Pi_{d,\gamma})) = \frac{1}{k!}\int\limits_{(\mathbb{R}^d)^k} \Delta^b_{k-1}(x_1,\dots,x_k)\,\eta^a(H) \times p_{d-k+1,\gamma}(\eta(H))\prod_{i=1}^k \frac{c\,\mathrm{d}x_i}{\omega_{d+\gamma}\|x_i\|^{d+\gamma}}. \tag{4.23}$$

Next, we use the affine Blaschke–Petkantschin formula (2.31):

$$\mathbb{E}T^{d,k-1}_{a,b}(\operatorname{conv}(\Pi_{d,\gamma})) = \frac{c^k((k-1)!)^{d-k+1}b_{d,k-1}}{k!\,\omega^k_{d+\gamma}}\int\limits_{A(d,k-1)}\int\limits_{H^k}\Delta^{b+d-k+1}_{k-1}(x_1,\dots,x_k) \times \eta^a(H)\,p_{d-k+1,\gamma}(\eta(H))\left(\prod_{i=1}^k\frac{1}{\|x_i\|^{d+\gamma}}\right)\mathrm{d}\lambda^k_H(x_1,\dots,x_k)\mu_{k-1}(\mathrm{d}H).$$

Writing

$$h(\eta(H)) := \int\limits_{H^k}\Delta^{b+d-k+1}_{k-1}(x_1,\dots,x_k)\left(\prod_{i=1}^k\frac{1}{\|x_i\|^{d+\gamma}}\right)\mathrm{d}\lambda^k_H(x_1,\dots,x_k), \tag{4.24}$$

we arrive at

$$\mathbb{E}T^{d,k-1}_{a,b}(\operatorname{conv}(\Pi_{d,\gamma})) = \frac{c^k(k-1)!^{d-k+1}b_{d,k-1}}{k!\,\omega^k_{d+\gamma}} \times \int\limits_{A(d,k-1)}\eta^a(H)\,p_{d-k+1,\gamma}(\eta(H))h(\eta(H))\,\mu_{k-1}(\mathrm{d}H).$$

Let $\beta := b+d-k+1$. We compute

$$\begin{aligned}h(r) &= \int\limits_{(\mathbb{R}^{k-1})^k}\Delta^\beta_{k-1}(y_1,\dots,y_k)\prod_{i=1}^k\frac{\mathrm{d}y_i}{(r^2+\|y_i\|^2)^{\frac{d+\gamma}{2}}}\\ &= \int\limits_{(\mathbb{R}^{k-1})^k} r^{(k-1)\beta}\Delta^\beta_{k-1}(z_1,\dots,z_k)\prod_{i=1}^k\frac{r^{k-1}\mathrm{d}z_i}{r^{d+\gamma}(1+\|z_i\|^2)^{\frac{d+\gamma}{2}}}\\ &= r^{(k-1)k-(d+\gamma)k+\beta(k-1)}\int\limits_{(\mathbb{R}^{k-1})^k}\Delta^\beta_{k-1}(z_1,\dots,z_k)\prod_{i=1}^k\frac{\mathrm{d}z_i}{(1+\|z_i\|^2)^{\frac{d+\gamma}{2}}},\end{aligned}$$

where we have used the change of variables $y_i = rz_i$. Thus, the function h satisfies the scaling property

$$h(r) = r^{(k-1)k-(d+\gamma)k+\beta(k-1)}h(1).$$

To compute the value of $h(1)$, let $Z_1, \dots, Z_k$ be independent random variables on $\mathbb{R}^{k-1}$ with the beta′-type density $\tilde{f}_{d,\frac{d+\gamma}{2}}(x)$, that is,

$$\tilde{f}_{d,\frac{d+\gamma}{2}}(x) = \frac{\omega_{d-k+1+\gamma}}{\omega_{d+\gamma}}(1+\|x\|^2)^{-\frac{d+\gamma}{2}}, \qquad x \in \mathbb{R}^{k-1}.$$

Recall that $\Delta_{k-1}(Z_1, \dots, Z_k)$ is the volume of the simplex with vertices $Z_1, \dots, Z_k$. Then, we can interpret $h(1)$ as follows:

$$h(1) = \frac{\omega_{d+\gamma}^k}{\omega_{d-k+1+\gamma}^k}\mathbb{E}\Delta_{k-1}^{\beta}(Z_1, \dots, Z_k).$$

The moments of $\Delta_{k-1}(Z_1, \dots, Z_k)$ are just the ones in Miles' formula, i.e., Lemma 3.1.6.

Let us consider the case $k = d$. Then, $\beta = b + 1$ and the above formulae simplify to

$$h(r) = r^{(b-\gamma)d-b-1}h(1) \tag{4.25}$$

and

$$h(1) = \left(\frac{\omega_{d+\gamma}}{\omega_{1+\gamma}}\right)^d \frac{1}{((d-1)!)^{b+1}} \frac{\Gamma\left(\frac{\gamma-b}{2}d + \frac{b+1}{2}\right)}{\Gamma\left(\frac{\gamma-b}{2}d\right)} \left(\frac{\Gamma\left(\frac{\gamma-b}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^d \prod_{i=1}^{d-1} \frac{\Gamma\left(\frac{i+b+1}{2}\right)}{\Gamma\left(\frac{i}{2}\right)}, \tag{4.26}$$

provided that $\gamma - b > 0$. We also have $h(r) = +\infty$, $r > 0$, if $\gamma \le b$. Since

$$b_{d,d-1} = \frac{\omega_d}{2} = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2})},$$

the above formulae yield

$$\mathbb{E}T_{a,b}^{d,d-1}(\mathrm{conv}(\Pi_{d,\gamma})) = \frac{c^d(d-1)!\omega_d}{2d!\,\omega_{d+\gamma}^d} \int\limits_{A(d,d-1)} \eta^a(H)\, p_{1,\gamma}(\eta(H))h(\eta(H))\, \mu_{d-1}(\mathrm{d}H).$$

Now, recalling the definition of $p_{1,\gamma}(R)$ from (4.22), we obtain

$$\begin{aligned} p_{1,\gamma}(R) &= \mathbb{P}(R \notin \operatorname{conv}(\Pi_{1,\gamma})) = \mathbb{P}(\Pi_{1,\gamma}[R,\infty) = 0) \\ &= \mathrm{e}^{-\frac{c}{\omega_{\gamma+1}}\int_R^\infty \frac{\mathrm{d}x}{x^{\gamma+1}}} = \mathrm{e}^{-\frac{c}{\gamma\omega_{\gamma+1}}R^{-\gamma}}. \end{aligned} \tag{4.27}$$

Hence,

$$\begin{aligned} \mathbb{E}T_{a,b}^{d,d-1}(\operatorname{conv}(\Pi_{d,\gamma})) &= \frac{c^d(d-1)!\omega_d}{2d!\,\omega_{d+\gamma}^d}h(1) \\ &\times \int\limits_{A(d,d-1)} \eta^{a-b-1+(b-\gamma)d}(H)\,\mathrm{e}^{-\frac{c}{\gamma\omega_{\gamma+1}}\eta^{-\gamma}(H)}\,\mu_{d-1}(\mathrm{d}H). \end{aligned}$$

The properties of the measure μ_{d-1} imply

$$\mathbb{E}T_{a,b}^{d,d-1}(\operatorname{conv}(\Pi_{d,\gamma})) = \frac{c^d(d-1)!\omega_d}{d!\,\omega_{d+\gamma}^d}h(1)\int\limits_0^\infty x^{a-b-1+(b-\gamma)d}\mathrm{e}^{-\frac{c}{\gamma\omega_{\gamma+1}}x^{-\gamma}}\,\mathrm{d}x. \tag{4.28}$$

Evaluating the integral, we get

$$\mathbb{E}T_{a,b}^{d,d-1}(\operatorname{conv}(\Pi_{d,\gamma})) = \frac{c^d(d-1)!\omega_d}{d!\,\omega_{d+\gamma}^d}h(1)\gamma^{-1}\left(\frac{c}{\gamma\omega_{\gamma+1}}\right)^{\frac{a-b+(b-\gamma)d}{\gamma}}\Gamma\left(\frac{(\gamma-b)d+b-a}{\gamma}\right)$$

under the condition $(\gamma-b)d+b-a>0$. Otherwise, the integral equals $+\infty$. Applying formula (4.26) completes the proof. □

Proof of Corollary 4.3.6. Lemma 4.3.1 implies that for any $L \in G(d,k)$, the projected random polytope $P_L \operatorname{conv}(\Pi_{d,\gamma})$ has the same distribution as $\operatorname{conv}(\Pi_{k,\gamma})$, if we identify L with $\mathbb{R}^k$. Using this together with the definition of intrinsic volumes and Fubini's theorem, we get

$$\begin{aligned} \mathbb{E}V_k(\operatorname{conv}(\Pi_{d,\gamma})) &= \binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}\mathbb{E}\int\limits_{G(d,k)} \operatorname{Vol}_k(P_L(\operatorname{conv}(\Pi_{d,\gamma})))\,\nu_k(\mathrm{d}L) \\ &= \binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}\int\limits_{G(d,k)} \mathbb{E}\operatorname{Vol}_k(P_L(\operatorname{conv}(\Pi_{d,\gamma})))\,\nu_k(\mathrm{d}L) \\ &= \binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}\mathbb{E}\operatorname{Vol}_k(\operatorname{conv}(\Pi_{k,\gamma})), \end{aligned}$$

since ν_k is a probability measure. Now, Corollary 4.3.5 can be used to complete the proof. □

Proof of Theorem 4.3.7. We keep the notation $\Pi_{d,\gamma}$ for $\Pi_{d,\gamma}(c)$. By Corollary 4.2.7, $\mathrm{sconv}(\Pi_{d,\gamma})$ is almost surely a convex polytope. Its $(k-1)$-dimensional faces have the form $\mathrm{conv}\{\varepsilon_1 x_1, \dots, \varepsilon_k x_k\}$, where $x_1, \dots, x_k$ are distinct points from $\Pi_{d,\gamma}$ and $\varepsilon_1, \dots, \varepsilon_k \in \{+1, -1\}$. We can write

$$\mathbb{E}T_{a,b}^{d,k-1}(\mathrm{sconv}(\Pi_{d,\gamma})) = \frac{1}{k!}\,\mathbb{E} \sum_{(\varepsilon_1,\dots,\varepsilon_k)\in\{+1,-1\}^k} \sum_{(x_1,\dots,x_k)\in\Pi^k_{d,\gamma,\neq}} \eta^a(\mathrm{aff}(\{\varepsilon_1 x_1, \dots, \varepsilon_k x_k\})) \\ \times \Delta^b_{k-1}(\varepsilon_1 x_1, \dots, \varepsilon_k x_k)\mathbb{1}\{[\varepsilon_1 x_1, \dots, \varepsilon_k x_k] \in \mathcal{F}_{k-1}(\mathrm{sconv}(\Pi_{d,\gamma}))\}.$$

Interchanging the expectation and the sum over $(\varepsilon_1, \dots, \varepsilon_k)$ and using the Mecke formula (2.38), we obtain

$$\mathbb{E}T_{a,b}^{d,k-1}(\mathrm{sconv}(\Pi_{d,\gamma})) = \frac{1}{k!} \sum_{(\varepsilon_1,\dots,\varepsilon_k)\in\{+1,-1\}^k} \mathbb{E} \int\limits_{(\mathbb{R}^d)^k} \eta^a(\mathrm{aff}(\{\varepsilon_1 x_1, \dots, \varepsilon_k x_k\})) \\ \times \Delta^b_{k-1}(\varepsilon_1 x_1, \dots, \varepsilon_k x_k)\mathbb{1}\{[\varepsilon_1 x_1, \dots, \varepsilon_k x_k] \in \mathcal{F}_{k-1}(\mathrm{sconv}(\hat{\Pi}_{d,\gamma}))\} \prod_{i=1}^k \frac{c\,\mathrm{d}x_i}{\omega_{d+\gamma}\|x_i\|^{d+\gamma}},$$

where $\hat{\Pi}_{d,\gamma} = \Pi_{d,\gamma} + \delta_{x_1} + \ldots + \delta_{x_k}$. Interchanging the integral and the expectation and noting that the expectation of an indicator function is the probability of the corresponding event, we get

$$\mathbb{E}T_{a,b}^{d,k-1}(\mathrm{sconv}(\Pi_{d,\gamma})) = \frac{1}{k!} \sum_{(\varepsilon_1,\dots,\varepsilon_k)\in\{+1,-1\}^k} \int\limits_{(\mathbb{R}^d)^k} \eta^a(\mathrm{aff}(\{\varepsilon_1 x_1, \dots, \varepsilon_k x_k\})) \\ \times \Delta^b_{k-1}(\varepsilon_1 x_1, \dots, \varepsilon_k x_k)\, \mathbb{P}\left([\varepsilon_1 x_1, \dots, \varepsilon_k x_k] \in \mathcal{F}_{k-1}(\mathrm{sconv}(\hat{\Pi}_{d,\gamma}))\right) \prod_{i=1}^k \frac{c\,\mathrm{d}x_i}{\omega_{d+\gamma}\|x_i\|^{d+\gamma}}.$$

Now, observe that

$$\mathrm{sconv}(\hat{\Pi}_{d,\gamma}) = \mathrm{sconv}(\Pi_{d,\gamma} + \delta_{x_1} + \ldots + \delta_{x_k}) = \mathrm{sconv}(\Pi_{d,\gamma} + \delta_{\varepsilon_1 x_1} + \ldots + \delta_{\varepsilon_k x_k}).$$

Noting that the integral remains invariant under the change of variables $\varepsilon_1 x_1 \mapsto x_1, \dots, \varepsilon_k x_k \mapsto x_k$, we arrive at

$$\mathbb{E}T_{a,b}^{d,k-1}(\mathrm{sconv}(\Pi_{d,\gamma})) = \frac{2^k}{k!} \int\limits_{(\mathbb{R}^d)^k} \eta^a(\mathrm{aff}(\{x_1,\dots,x_k\}))$$
$$\times \Delta_{k-1}^b(x_1,\dots,x_k)\, \mathbb{P}\Big([x_1,\dots,x_k] \in \mathcal{F}_{k-1}(\mathrm{sconv}(\hat{\Pi}_{d,\gamma}))\Big) \prod_{i=1}^k \frac{c\,\mathrm{d}x_i}{\omega_{d+\gamma}\|y_i\|^{d+\gamma}}.$$

From now on we can argue exactly as in the proof of Theorem 4.3.2, but an additional factor of 2^k appears throughout and the non-absorption probability $p_{d,\gamma}(R)$ has to be replaced by its symmetrized version

$$q_{d,\gamma}(R) := \mathbb{P}(Re_1 \notin \mathrm{sconv}(\Pi_{d,\gamma})), \quad R > 0.$$

In particular, in the special case $k = d$, we arrive at

$$\mathbb{E}T_{a,b}^{d,d-1}(\mathrm{sconv}(\Pi_{d,\gamma})) = \frac{(2c)^d(d-1)!\omega_d}{2d!\,\omega_{d+\gamma}^d} \int\limits_{A(d,d-1)} \eta^a(H)\, q_{1,\gamma}(\eta(H))h(\eta(H))\,\mu_{d-1}(\mathrm{d}H).$$

The non-absorption probability can easily be calculated as follows:

$$\begin{aligned} q_{1,\gamma}(R) &= \mathbb{P}(R \notin \mathrm{sconv}(\Pi_{1,\gamma})) = \mathbb{P}(\Pi_{1,\gamma}[R,\infty) = \Pi_{1,\gamma}(-\infty,-R] = 0) \\ &= \mathbb{P}(\Pi_{1,\gamma}[R,\infty) = 0)^2 = \mathrm{e}^{-\frac{2c}{\omega_{\gamma+1}}\int_R^\infty \frac{\mathrm{d}x}{x^{\gamma+1}}} = \mathrm{e}^{-\frac{2c}{\gamma\omega_{\gamma+1}}R^{-\gamma}}. \end{aligned}$$

By the definition of the measure μ_{d-1}, we obtain

$$\mathbb{E}T_{a,b}^{d,d-1}(\mathrm{sconv}(\Pi_{d,\gamma})) = \frac{(2c)^d(d-1)!\omega_d}{d!\,\omega_{d+\gamma}^d} h(1) \int\limits_0^\infty x^{a-b-1+(b-\gamma)d} \mathrm{e}^{-\frac{2c}{\gamma\omega_{\gamma+1}}x^{-\gamma}}\mathrm{d}x, \qquad (4.29)$$

where $h(1)$ is given by (4.26). Now, a comparison of (4.29) with (4.28) completes the proof. □

It remains to calculate $B_{\gamma,d,d}$ as provided in (4.12).

Proof of (4.12). We compute the constant $B_{\alpha,d,d}$ (where we use α instead of γ to avoid a clash of notation). Using the Blaschke–Petkantschin formula (2.31) with $k = d-1$, we see that

$$B_{\alpha,d,d} = \frac{1}{2}\Big(\frac{\omega_{\alpha+1}}{\omega_{d+\alpha+1}}\Big)^d \frac{\omega_d}{2}(d-1)! \int\limits_{A(d,d-1)} \int\limits_{H^d} \mathbb{P}(\mathrm{conv}(\Pi_{d,\alpha+1}(\omega_{\alpha+1})) \cap H = \varnothing)$$

$$\times \Delta_{d-1}(x_1, \dots, x_d) \prod_{i=1}^{d} \frac{\mathrm{d}x_i}{\|x_i\|^{d+\alpha+1}} \, \mu_{d-1}(\mathrm{d}H).$$

The probability has already been computed in (4.27):

$$\mathbb{P}(\mathrm{conv}(\Pi_{d,\alpha+1}(\omega_{\alpha+1})) \cap H = \varnothing) = \exp\left(-\frac{\omega_{\alpha+1}}{(\alpha+1)\omega_{\alpha+2}} r^{-(\alpha+1)}\right),$$

if $r > 0$ denotes the distance of H to the origin. Thus, using the definition (4.26) of $h(1)$ and the scaling relation (4.25) (with $b = 0$ and $\gamma = \alpha + 1$), we conclude that

$$\begin{aligned}
B_{\alpha,d,d} &= \left(\frac{\omega_{\alpha+1}}{\omega_{d+\alpha+1}}\right)^d \frac{\omega_d}{2} (d-1)! \, h(1) \int_0^\infty \exp\left(-\frac{\omega_{\alpha+1}}{(\alpha+1)\omega_{\alpha+2}} r^{-(\alpha+1)}\right) r^{-((\alpha+1)d+1)} \, \mathrm{d}r \\
&= \left(\frac{\omega_{\alpha+1}}{\omega_{d+\alpha+1}}\right)^d \frac{\omega_d}{2} (d-1)! \, h(1) \left(\frac{(\alpha+1)\omega_{\alpha+2}}{\omega_{\alpha+1}}\right)^d \frac{1}{\alpha+1} \int_0^\infty \exp(-s) s^{d-1} \, \mathrm{d}s \\
&= \left(\frac{\omega_{\alpha+1}}{\omega_{d+\alpha+1}}\right)^d \frac{\omega_d}{2} (d-1)! \, h(1) \left(\frac{(\alpha+1)\omega_{\alpha+2}}{\omega_{\alpha+1}}\right)^d \frac{1}{\alpha+1} (d-1)! \\
&= \frac{(d-1)!(\alpha+1)^{d-1}}{\sqrt{\pi}} \left(\frac{\omega_{\alpha+2}}{\omega_{\alpha+1}}\right)^d \frac{\Gamma\left(\frac{\alpha+1}{2} d + \frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2} d\right)},
\end{aligned}$$

where in the second equality we have used the transformation $s = \frac{\omega_{\alpha+1}}{(\alpha+1)\omega_{\alpha+2}} r^{-(\alpha+1)}$ and in the last equality simplified the expression. In the case of $\alpha = 0$, we obtain

$$B_{0,d,d} = (d-1)! \frac{\pi^{d-\frac{1}{2}} \Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} = (2\pi)^{d-1} \Gamma\Big(\frac{d+1}{2}\Big)^2,$$

where we have used Legendre's duplication formula. This completes the proof. □

Chapter 5

Empty simplices

5.1 Preliminaries

Let $d \geq 2$ and $n \in \mathbb{N}$ with $n \geq d+1$. Let $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$ be a finite point set in general position. Denote by $\left[\begin{smallmatrix} X \\ d \end{smallmatrix}\right]$ the set of all d-element subsets of X. For $\{x_{i_1}, \ldots, x_{i_d}\} \in \left[\begin{smallmatrix} X \\ d \end{smallmatrix}\right]$ we define the degree of that subset, denoted by $\deg(x_{i_1}, \ldots, x_{i_d}; X)$, as the number of $(d+1)$-element subsets $\{x_{i_1}, \ldots, x_{i_{d+1}}\} \subset X \setminus \{x_{i_1}, \ldots, x_{i_d}\}$ for which the property $\operatorname{int}([x_{i_1}, \ldots, x_{i_{d+1}}]) \cap X = \varnothing$ holds. This definition can be written as

$$\deg(x_{i_1}, \ldots, x_{i_d}; X) = \sum_{x_{d+1} \in X \setminus \{x_{i_1}, \ldots, x_{i_d}\}} \mathbb{1}\{\operatorname{int}([x_{i_1}, \ldots, x_{i_{d+1}}]) \cap X = \varnothing\}. \tag{5.1}$$

The degree of X, denoted as $\deg(X)$, is defined as the maximum of the degrees of the d-element subsets of X, i.e.,

$$\deg(X) = \max_{\{x_{i_1}, \ldots, x_{i_d}\} \in \left[\begin{smallmatrix} X \\ d \end{smallmatrix}\right]} \deg(x_{i_1}, \ldots, x_{i_d}; X). \tag{5.2}$$

An illustration of these definitions is given in Figure 1.7 and 1.8 in the introduction. Up to this point all the definitions are of deterministic nature. As already mentioned in the introduction Erdős originally asked the question whether the degree of a deterministic point set $X \subset \mathbb{R}^2$ in general position goes to infinity as the number of points go to infinity. Although Bárány and Valtr in [19] gave a prescription how to construct a set $X \subset \mathbb{R}^2$ in general position for which $\deg(X) = 4\sqrt{n}(1 + o(1))$, the general case remains open. However, if we let X be a random set of independently and uniformly distributed random points from a compact set K we can confirm this conjecture.

Let in the following $c(\ldots)$ denote a generic constants, only depending on what appears in the bracket and which may vary from line to line.

Theorem 5.1.1 *Let $\xi_n = \{X_1, \ldots, X_n\}$ be a set of n independently and uniformly chosen random points from a compact set $K \subset \mathbb{R}^d$ with $\mathrm{Vol}_d(K) > 0$. Then,*

(a) $c(d, K)n \leq \mathbb{E}\deg(\xi_n) \leq n$, for some positive constant $c(d, K)$.

(b) $\deg(\xi_n) \to \infty$ in probability, as $n \to \infty$.

Remark 5.1.2 Numerical computations by M. Meckes and by the author suggest that even the constant may be surprisingly large. For $K = B^2$ or the planar square the results suggest that

$$\frac{\mathbb{E}\deg_d(\xi_n)}{n} \in [0.70, 0.95].$$

Remark 5.1.3 By Jensen's inequality, Theorem 5.1.1 implies the asymptotic behavior of all the moments of $\deg \xi_n$. Namely, as $n \to \infty$, we have

$$\mathbb{E}\deg(\xi_n)^k = \Theta(n^k).$$

The upper bound for $\mathbb{E}\deg(\xi_n)$ in Theorem 5.1.1 (a) follows trivially from the fact that one can form at most $d - k$ simplices with nonempty interior from a d-element subset of an n-element point set. Furthermore, Theorem 5.1.1 (b) can be shown just as it's 2-dimensional analogue was shown in [18]. All one has to do is to replace the mesh in the plane with mesh width $n^{-1/2}$ by a mesh in $\mathbb{R}^d$ with mesh width $n^{-1/d}$ and every instance where $n - 2$ appears by $n - d$.

For the lower bound we present two proof approaches. The first approach was given by the author in [105] and generalizes the two-dimensional approach from Bárány, Marckert and Reitzner [18]. In this proof we will show a lower bound for all the moments of $\deg(\xi_n)$ and not make use of Jensen's inequality for the higher moments as suggested in Remark 5.1.3. However, this proof only provides the lower bounds $\mathbb{E}\deg(\xi_n)^k \geq c(d, k, K)n^k(\log n)^{-1}$ and only works for convex bodies. An alteration of the proof idea and certain functionals that are involved, presented by Reitzner and the author in [90], will be the content of the second approach. As it turns out, these modifications allow to uncover the correct asymptotic behavior of $\mathbb{E}\deg(\xi_n)$, significantly simplify the proof and extend the result to arbitrary, not necessarily convex, compact sets. It will also allow us to invoke Remark 5.1.3 to get the correct asymptotic behavior of all integer moments of $\deg(\xi_n)$.

5.2 Prove approach 1

We start by defining two functionals. So let $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ be a deterministic point set in general position. Let $t > 0$ and define

$$N_t(X) := \sum_{\{x_{i_1},\dots,x_{i_d}\} \in \binom{X}{d}} \mathbb{1}\left\{\exists j \in [d] : \{x_{i_1}, \dots, x_{i_d}\} \subset \mathbb{B}^d\left(x_{i_j}, t\right)\right\}$$

and

$$F_t^{(k)}(X) := \sum_{\{x_{i_1},\dots,x_{i_d}\} \in \binom{X}{d}} \mathbb{1}\left\{\exists j \in [M] : \{x_{i_1}, \dots, x_{i_d}\} \subset \mathbb{B}^d\left(x_{i_j}, t\right)\right\} \deg(x_{i_1}, \dots, x_{i_d}; X)^k$$

for all $k \in \mathbb{N}$.

The core idea of the proof will be to use the inequality

$$F_t^{(k)}(X) \le N_t(X) \deg(X)^k, \tag{5.3}$$

which holds for all $k \in \mathbb{N}$ and $t > 0$.

From now on let $X = \xi_n = \{X_1, \dots, X_n\}$ be a random point set of n independent and uniformly chosen point from a convex body $K \subset \mathbb{R}^d$. Hence, if we take the expectation on both sides of (5.3) and rearrange, we get

$$\mathbb{E} \deg(\xi_n)^k \ge \mathbb{E}\left(\frac{F_t^{(k)}(\xi_n)}{N_t(\xi_n)} \mathbb{1}\{N_t(\xi_n) \ge 1\}\right).$$

This in turn implies for any $T > 0$

$$\mathbb{E} \deg(\xi_n)^k \ge \frac{1}{T} \mathbb{E}\left(F_t^{(k)}(\xi_n) \mathbb{1}\{0 < N_t(\xi_n) \le T\}\right). \tag{5.4}$$

Before we go on, note that the degree deg is invariant under non-degenerate affine transformations. By John's Theorem, see [98, Theorem 10.12.2], there exists an ellipsoid E such that $E \subset K \subset dE$. First we apply an affine transformation so that the area of K becomes equal to one, making the Lebesgue measure thereon coincide with the uniform measure, and second, we apply a volume preserving affine transformation that carries E to $r\mathbb{B}^d$ and dE to $dr\mathbb{B}^d$. Assume from now on that K is in this position, i.e., $\mathrm{Vol}_d(K) = 1$ and $r\mathbb{B}^d \subset K \subset dr\mathbb{B}^d$, where $r\mathbb{B}^d$ is the maximal volume ellipsoid contained in K. We refer to this property by saying: *K is in appropriate position.*

Reconsider now (5.4). As we can see, the idea is to average only over the degrees of a small subset of $\left[\begin{smallmatrix}\xi_n\\ d\end{smallmatrix}\right]$. Clearly we want to choose the parameter t in $N_t(\xi_n)$ in such a way that $\mathbb{E}N_t(\xi_n)$ is as small as possible, but at the same time does not converge to zero as n goes to infinity. Let us determine this particular choice of t:

Proposition 5.2.1 *Let $d \geq 2$ and $K \subset \mathbb{R}^d$ be a convex body in appropriate position. For all $t > 0$ we have*

$$c\, t^{d(d-1)}(1+\mathcal{O}(t)) \leq \mathbb{E}N_t(\xi_n) \leq c\, d\, t^{d(d-1)}(1+\mathcal{O}(t)), \tag{5.5}$$

as $t \to 0$, where $c = \kappa_d^{d-1}\binom{n}{d}$

Proof. We have

$$\begin{aligned}
\mathbb{E}N_t(\xi_n) &= \mathbb{E} \sum_{\{X_{i_1},\dots,X_{i_d}\}\in\left[\begin{smallmatrix}\xi_n\\ d\end{smallmatrix}\right]} \mathbb{1}\left\{\exists j \in [d] : \{X_{i_1},\dots,X_{i_d}\} \subset \mathbb{B}^d\left(X_{i_j}, t\right)\right\} \\
&= \binom{n}{d} \int\limits_{K^d} \mathbb{1}\left\{\exists i \in [d] : \{x_1,\dots,x_d\} \subset \mathbb{B}^d\left(x_i, t\right)\right\} \mathrm{d}x_1 \dots \mathrm{d}x_d \\
&\leq \sum_{i=1}^{d} \binom{n}{d} \int\limits_{K^d} \mathbb{1}\left\{\{x_1,\dots,x_d\} \subset \mathbb{B}^d\left(x_i, t\right)\right\} \mathrm{d}x_1 \dots \mathrm{d}x_d \\
&= d\binom{n}{d} \int\limits_{\left(\mathbb{R}^d\right)^{d-1}} \left(\prod_{i=1}^{d-1} \mathbb{1}\{\|x_i\| \leq t\}\right) \\
&\quad\times \int\limits_{\mathbb{R}^d} \mathbb{1}\{x_d \in K, x_d - x_1 \in K, \dots, x_d - x_{d-1} \in K\} \,\mathrm{d}x_d \,\mathrm{d}x_1 \dots \mathrm{d}x_{d-1} \\
&= d\binom{n}{d} \int\limits_{\left(\mathbb{B}^d(\boldsymbol{o},t)\right)^{d-1}} g_K(x_1,\dots,x_{d-1}) \,\mathrm{d}x_1 \dots \mathrm{d}x_{d-1} \\
&= d\binom{n}{d} \int\limits_{[0,t]^{d-1}} \left(\prod_{i=1}^{d-1} r_i^{d-1}\right) \\
&\quad\times \int\limits_{\left(\mathbb{S}^{d-1}\right)^{d-1}} g_K(r_1u_1,\dots,r_{d-1}u_{d-1}) \,\sigma_{d-1}(\mathrm{d}u_1)\dots\sigma_{d-1}(\mathrm{d}u_{d-1}) \,\mathrm{d}r_1 \dots \mathrm{d}r_{d-1},
\end{aligned}$$

where we introduced polar coordinates in each of the $d-1$ integrals. For a fixed vector

$u = (u_1, \ldots, u_{d-1}) \in \left(\mathbb{S}^{d-1}\right)^{d-1}$ the Taylor expansion of the generalized covariogram $g_K(r_1u_1, \ldots, r_{d-1}u_{d-1})$ in $r = (r_1, \ldots, r_{d-1}) = (0, \ldots, 0)$ gives

$$\begin{aligned} &g_K(r_1u_1, \ldots, r_{d-1}u_{d-1}) \\ &\qquad = g_K(\boldsymbol{o}) + \sum_{i=1}^{d-1} r_i \frac{\partial}{\partial r_i} g_K(r_1u_1, \ldots, r_{d-1}u_{d-1})\Big|_{r=(0,\ldots,0)} + o(r_1, \ldots, r_{d-1}). \end{aligned}$$

Recall (2.43), i.e., the invariance of the covariogram under permutation of its arguments. We can conclude that the integration of the Taylor expansion is possible since due to (2.43) and Proposition 2.6.8 (ii) the partial derivative of $g_K(r_1u_1, \ldots, r_{d-1}u_{d-1})$ with respect to r_i at $r = (r_1, \ldots, r_{d-1}) = 0$ exists and is finite for each $i = 1, \ldots, d-1$. We integrate term by term. First, due to $g_K(0, \ldots, 0) = 1$,

$$\int\limits_{\left(\mathbb{S}^{d-1}\right)^{d-1}} g_K(\boldsymbol{o})\, \sigma_{d-1}(\mathrm{d}u_1) \ldots \sigma_{d-1}(\mathrm{d}u_{d-1}) = (d\kappa_d)^{d-1}.$$

Second, again by (2.43) and Proposition 2.6.8, we have

$$\begin{aligned} &\int\limits_{\left(\mathbb{S}^{d-1}\right)^{d-1}} \frac{\partial}{\partial r_i} g_K(r_1u_1, \ldots, r_{d-1}u_{d-1})\Big|_{r=(0,\ldots,0)} \sigma_{d-1}(\mathrm{d}u_1) \ldots \sigma_{d-1}(\mathrm{d}u_{d-1}) \\ &\qquad = -\kappa_{d-1}(d\kappa_d)^{d-2}\, \mathrm{Per}(W) \end{aligned}$$

for each $i = 1, \ldots, d-1$. From (2.43) we know that $g_K(\boldsymbol{o}, \ldots, \boldsymbol{o}, r_iu_i, \boldsymbol{o}, \ldots, \boldsymbol{o}) = g_K(r_iu_i, \boldsymbol{o}, \ldots, \boldsymbol{o}) = g_K^{u_i}(r_i)$ is a Lipschitz function in r_i. By Proposition 2.6.7 its Lipschitz constant is half of the bounded directional variation $V_{u_i}(K)$ of K in direction u_i, which, by Remark 2.6.9, is, due to the convexity of K, the $(d-1)$-dimensional Lebesgue volume $\mathrm{Vol}_{d-1}(P_{u_i^\perp}K)$. Hence,

$$\begin{aligned} |g_K(r_1u_1, \ldots, r_{d-1}u_{d-1}) - g_K(\boldsymbol{o})| &= g_K(\boldsymbol{o}) - g_K(r_1u_1, \ldots, r_{d-1}u_{d-1}) \\ &= \mathrm{Vol}_d\left(\bigcup_{i=1}^{d-1} K \setminus (r_iu_i + K)\right) \le \sum_{i=1}^{d-1} \mathrm{Vol}_d(K \setminus (r_iu_i + K)) \\ &= \sum_{i=1}^{d-1}(g_K(\boldsymbol{o}) - g_K(\boldsymbol{o}, \ldots, \boldsymbol{o}, r_iu_i, \boldsymbol{o}, \ldots, \boldsymbol{o}) \le \sum_{i=1}^{d-1} \mathrm{Lip}(g_K^{u_i})\, r_i \\ &= \sum_{i=1}^{d-1} \mathrm{Vol}_{d-1}(P_{u_i^\perp}K). \end{aligned}$$

Therefore, $1 \geq g_K(r_1u_1, \ldots, r_{d-1}u_{d-1}) \geq 1 - \sum_{i=1}^{d-1} \mathrm{Vol}_{d-1}(P_{u_i^\perp}K)r_i$ holds. Furthermore, by Cauchy's surface area formula, see (2.27), we get

$$\begin{aligned}(d\kappa_d)^{d-1}\,\mathrm{Vol}_d(K) &\geq \int\limits_{(\mathbb{S}^{d-1})^{d-1}} g_K(r_1u_1, \ldots, r_{d-1}u_{d-1})\,\sigma_{d-1}(\mathrm{d}u_1)\ldots\sigma_{d-1}(\mathrm{d}u_{d-1}) \\ &\geq (d\kappa_d)^{d-1} - (d\kappa_d)^{d-2}\kappa_{d-1}\,\mathrm{Per}(K)\sum_{i=1}^{d-1} r_i.\end{aligned}$$

Thus, the $o(r_1, \ldots, r_{d-1})$-term is positive and bounded by $(d\kappa_d)^{d-2}\kappa_{d-1}\,\mathrm{Per}(K)\sum_{i=1}^{d-1} r_i$, allowing us to infer the relation

$$\begin{aligned}&\int\limits_{(\mathbb{S}^{d-1})^{d-1}} g_K(r_1u_1, \ldots, r_{d-1}u_{d-1})\,\sigma_{d-1}(\mathrm{d}u_1)\ldots\sigma_{d-1}(\mathrm{d}u_{d-1}) \\ &\qquad = (d\kappa_d)^{d-2}\left(d\kappa_d - \kappa_{d-1}\,\mathrm{Per}(K)\sum_{i=1}^{d-1} r_i\right) + o(r_1, \ldots, r_{d-1})\end{aligned}$$

for all Lebesgue measurable sets K with finite perimeter. Thus, integration gives

$$\begin{aligned}&d\binom{n}{d}\int\limits_{[0,t]^{d-1}}\left(\prod_{i=1}^{d-1} r_i^{d-1}\right) \\ &\quad\times\int\limits_{(\mathbb{S}^{d-1})^{d-1}} g_K(r_1u_1, \ldots, r_{d-1}u_{d-1})\,\sigma_{d-1}(\mathrm{d}u_1)\ldots\sigma_{d-1}(\mathrm{d}u_{d-1})\,\mathrm{d}r_1\ldots\mathrm{d}r_{d-1} \\ &= d\binom{n}{d}\int\limits_{[0,t]^{d-1}}(d\kappa_d)^{d-1}\prod_{i=1}^{d-1} r_i^{d-1} \\ &\quad - (d\kappa_d)^{d-2}\kappa_{d-1}\,\mathrm{Per}(K)\sum_{j=1}^{d-1} r_j^d\prod_{\substack{i=1\\ i\neq j}}^{d-1} r_i^{d-1} + o(r_1^d, \ldots, r_{d-1}^d)\,\mathrm{d}r_1\ldots\mathrm{d}r_{d-1} \\ &= d\binom{n}{d}\left((d\kappa_d)^{d-1}\frac{t^{d(d-1)}}{d^{d-1}} - (d\kappa_d)^{d-2}\kappa_{d-1}\,\mathrm{Per}(K)\frac{(d-1)t^{d(d-1)+1}}{(d+1)d^{d-2}} + o\left(t^{d^2-1}\right)\right) \\ &= d\binom{n}{d}\left(\kappa_d^{d-1}t^{d(d-1)} - \frac{d-1}{d+1}\kappa_{d-1}\kappa_d^{d-2}\,\mathrm{Per}(K)t^{d(d-1)+1} + o\left(t^{d^2-1}\right)\right) \\ &= d\kappa_d^{d-1}\binom{n}{d}t^{d(d-1)}(1 + \mathcal{O}(t)).\end{aligned}$$

Hence, as $t \to 0$,

$$\mathbb{E}N_t(\xi_n) \leq d\kappa_d^{d-1}\binom{n}{d}t^{d(d-1)}(1+\mathcal{O}(t)).$$

Since $\mathbb{1}\{\{x_1,\dots,x_d\} \subset \mathbb{B}^d(x_1,t)\} = 1$ implies $\mathbb{1}\{\exists i \in [d] : \{x_1,\dots,x_d\} \subset \mathbb{B}^d(x_i,t)\} = 1$ one can show with the same proof, that

$$\mathbb{E}N_t(\xi_n) \geq \kappa_d^{d-1}\binom{n}{d}t^{d(d-1)}(1+\mathcal{O}(t)),$$

as $t \to 0$. □

Remark 5.2.2 We immediately see that for the choice of $t = n^{-1/(d-1)}$ we have that

$$\kappa_d^{d-1}\binom{n}{d}n^{-d}(1+\mathcal{O}(t)) \leq \mathbb{E}N_{n^{-1/(d-1)}}(\xi_n) \leq d\kappa_d^{d-1}\binom{n}{d}n^{-d}(1+\mathcal{O}(t)),$$

as $n \to \infty$, and therefore that $\mathbb{E}N_{n^{-1/(d-1)}}(\xi_n)$ behaves asymptotically like a constant.

So let from here on out $t = n^{-1/(d-1)}$ and reconsider (5.4). We have

$$\begin{aligned}
&\frac{1}{T}\mathbb{E}\left(F_t^{(k)}(\xi_n)\mathbb{1}\left\{0 < N_t(\xi_n) \leq T\right\}\right) \\
&= \frac{\binom{n}{d}}{T}\mathbb{E}\Big(\mathbb{1}\left\{\exists i \in [d] : \{X_1,\dots,X_d\} \subset \mathbb{B}^d(X_i, n^{-1/(d-1)})\right\} \\
&\quad \times \deg(X_1,\dots,X_d;\xi_n)^k \mathbb{1}\left\{0 < N_t(\xi_n) \leq T\right\}\Big) \\
&= \frac{\binom{n}{d}}{T}\int\limits_{K^d} \mathbb{1}\left\{\exists i \in [d] : \{x_1,\dots,x_d\} \subset \mathbb{B}^d(x_i, n^{-1/(d-1)})\right\} \\
&\quad \times \mathbb{E}\left(\deg(x_1,\dots,x_d;\xi_n')^k \mathbb{1}\left\{0 < N_t(\xi_n') \leq T\right\}\right) \mathrm{d}x_1 \dots \mathrm{d}x_d,
\end{aligned} \tag{5.6}$$

where $\xi_n' = \xi_{n-d} \cup \{x_1,\dots,x_d\}$. By the elementary equality $\mathbb{E}(X\mathbb{1}\{A\}) = \mathbb{E}(X) - \mathbb{E}(X\mathbb{1}\{A^c\})$, which holds for any random variable X with $\mathbb{E}|X| < \infty$ and any event A, we can further bound this by

$$\begin{aligned}
&\mathbb{E}\left(\deg(x_1,\dots,x_d;\xi_n')^k \mathbb{1}\left\{0 < N_t(\xi_n') \leq T\right\}\right) \\
&= \mathbb{E}\left(\deg(x_1,\dots,x_d;\xi_n')^k\right) - \mathbb{E}\left(\deg(x_1,\dots,x_d;\xi_n')^k \mathbb{1}\left\{N_t(\xi_n') \in \{0\} \cup (T,\infty)\right\}\right) \\
&\geq \mathbb{E}\left(\deg(x_1,\dots,x_d;\xi_n')^k\right) - n^k\mathbb{E}\left(\mathbb{1}\left\{N_t(\xi_n') \in \{0\} \cup (T,\infty)\right\}\right) \\
&= \mathbb{E}\left(\deg(x_1,\dots,x_d;\xi_n')^k\right) - n^k\mathbb{P}\left(N_t(\xi_n') \in \{0\} \cup (T,\infty)\right) \\
&= \mathbb{E}\left(\deg(x_1,\dots,x_d;\xi_n')^k\right) - n^k\mathbb{P}\left(N_t(\xi_n') > T\right) - n^k\mathbb{P}\left(N_t(\xi_n') = 0\right).
\end{aligned} \tag{5.7}$$

Let $\rho = r/2$. Then the ball $\rho\mathbb{B}^d$ is contained in $r\mathbb{B}^d \subset K$ and every point of $\rho\mathbb{B}^d$ is farther than ρ away from the boundary of K. Hence,

$$\begin{aligned} N_t(\xi_n') &\le N_t(\xi_{n-d}) + \sum_{i=1}^{d} |\xi_{n-d} \cap \mathbb{B}^d(x_i, t)| + 1 \\ &\le N_t(\xi_{n-d}) + dN_{2t}(\xi_{n-d}) + 1 \\ &\le (d+1)N_{2t}(\xi_{n-d}) + 1 \\ &\le (d+1)N_{2t}(\xi_n) + 1, \end{aligned}$$

where we used the fact that, if d points lie in $\mathbb{B}^d(x_i, t)$, then their pairwise distance is at most $2t$. Choose now $T = T_n = 2(d+1)\log n$ to get

$$\mathbb{P}(N_t(\xi_n') > T_n) \le \mathbb{P}((d+1)N_{2t}(\xi_n) + 1 > T_n) \le \mathbb{P}(N_{2t}(\xi_n) \ge \log n),$$

from which, with the choice of $t = n^{-1/(d-1)}$, Markov's inequality and Proposition 5.2.1, we have

$$\mathbb{P}(N_t(\xi_n) > K_n) \le \mathbb{P}(N_{2t}(\xi_n) > \log n) \le \frac{\mathbb{E}(N_{2t}(\xi_n))}{\log n} \le c(\log n)^{-1}, \tag{5.8}$$

for some constant $c > 0$. Furthermore, one sees that $\mathbb{1}\{\exists i \in [d] : \{x_1, \dots, x_d\} \subset \mathbb{B}^d(x_i, t)\} = 1$ implies that $N_t(\xi_n') \ge 1$, and subsequently, that $\mathbb{P}(N_t(\xi_n') = 0) = 0$.

Recalling (5.7) it remains to bound $\mathbb{E}\left(\deg(x_1, \dots, x_d; \xi_n')^k\right)$ from below. We have

Proposition 5.2.3 *Let $d \ge 2$, $K \subset \mathbb{R}^d$ be a convex body in appropriate position and $\rho > 0$. If there exist an $i \in \{1, \dots, d\}$ such that $\{x_1, \dots, x_d\} \subset \mathbb{B}^d(x_i, n^{-1/(d-1)})$ and $\{x_1, \dots, x_d\} \subset \rho\mathbb{B}^d$, then*

$$\mathbb{E}\left(\deg(x_1, \dots, x_d; \xi_n')^k\right) \ge n^k \left(\frac{\rho^{d-1} d!}{2^{d-1}} \left(1 - \exp\left(-\frac{2^{d-1}\rho}{d!}\right)\right)\right)^k$$

for all $k \in \mathbb{N}$ and sufficiently large n.

Proof. Let $\rho > 0$ and $x_1, \dots, x_d \in \rho\mathbb{B}^d$ be fixed vectors such that there exists and

$i \in [d]$ with $\{x_1, \dots, x_d\} \subset \mathbb{B}^d(x_i, n^{-1/(d-1)})$. Then,

$$\begin{aligned}\mathbb{E}\left(\deg(x_1, \dots, x_d; \xi_n')\right) &= \mathbb{E}\left(\sum_{z \in \xi_{n-d}} \mathbb{1}\left\{\xi_n' \cap \operatorname{int}([x_1, \dots, x_d, z]) = \varnothing\right\}\right) \\ &= (n-d)\mathbb{P}\left(\xi_n'' \cap \operatorname{int}([x_1, \dots, x_d, Y]) = \varnothing\right),\end{aligned}$$

where $\xi_n'' = \xi_{n-d-1} \cup \{x_1, \dots, x_d, Y\}$ and Y is a uniformly distributed random variable in K independent of ξ_{n-d-1}. This gives

$$\mathbb{E}\left(\deg(x_1, \dots, x_d; \xi_n')\right) = (n-d) \int_K \left(1 - \operatorname{Vol}_d([x_1, \dots, x_d, y])\right)^{n-m-1} \, \mathrm{d}y. \tag{5.9}$$

Let $Q(x_1, \dots, x_d)$ be the d-dimensional cube with side length ρ, centered at $x_i \in \rho\mathbb{B}^d$, with one side parallel to the hyperplane spanned by $x_1, \dots, x_d$. Instead of integrating with respect to y over K, the integration will be restricted to the cube $Q(x_1, \dots, x_d)$. Due to the fact that $x_1, \dots, x_d \in \mathbb{B}^d\left(x_i, n^{-1/(d-1)}\right)$, the pairwise distances between the points $x_1, \dots, x_d$ are less than $2n^{-1/(d-1)}$. Additionally, it holds that $\operatorname{Vol}_{d-1}([x_1, \dots, x_d])$ is smaller than the $(d-1)$-dimensional volume of the regular d-simplex with side length $2n^{-1/(d-1)}$, which can be estimated from above by $\frac{2^{d-1}}{(d-1)!}n^{-1}$. Hence, the estimate $\operatorname{Vol}_{d-1}([x_1, \dots, x_d]) \le \frac{2^{d-1}}{(d-1)!}n^{-1}$ can be used to get the lower bound

$$\begin{aligned}&\int_K \left(1 - \operatorname{Vol}_d([x_1, \dots, x_d, y])\right)^{n-d-1} \, \mathrm{d}y \\ &\quad\ge \int_{Q(x_1,\dots,x_d)} \left(1 - \operatorname{Vol}_d([x_1, \dots, x_d, y])\right)^{n-d-1} \, \mathrm{d}y \\ &\quad\ge \int_{[0,\rho]^d} \left(1 - \frac{\operatorname{Vol}_{d-1}([x_1, \dots, x_d])}{d} y_d\right)^{n-d-1} \mathrm{d}y_d \dots \mathrm{d}y_1 \\ &\quad= \rho^{d-1} \int_0^\rho \left(1 - \frac{\operatorname{Vol}_{d-1}([x_1, \dots, x_d])}{d} y_d\right)^{n-d-1} \mathrm{d}y_d \\ &\quad= \frac{\rho^{d-1} d}{\operatorname{Vol}_{d-1}([x_1, \dots, x_d])} \frac{1}{n-d} \left(1 - \left(1 - \frac{\operatorname{Vol}_{d-1}([x_1, \dots, x_d])\rho}{d}\right)^{n-d}\right) \\ &\quad\ge \frac{\rho^{d-1} n d!}{2^{d-1}(n-d)} \left(1 - \left(1 - \frac{2^{d-1}\rho}{nd!}\right)^{n-d}\right)\end{aligned}$$

$$\geq \frac{\rho^{d-1} nd!}{2^{d-1}(n-d)} \left(1 - \exp\left(-\frac{(n-d)2^{d-1}\rho}{nd!}\right)\right)$$
$$\geq \frac{\rho^{d-1} nd!}{2^{d-1}(n-d)} \left(1 - \exp\left(-\frac{2^{d-1}\rho}{d!}\right)\right)$$

for large enough n. Combining this result with

$$\mathbb{E}\left[\deg(x_1, \ldots, x_d; \xi_n')^k\right] \geq (n-d)^k \left(\int\limits_K (1 - \mathrm{Vol}_d([x_1, \ldots, x_d, y]))^{n-d-1}\, \mathrm{d}y\right)^k,$$

which follows from (5.9) by applying Jensen's inequality, finishes the proof. □

Now we plug the result from Proposition 5.2.3 and (5.8) back into (5.7), which we in turn invoke in (5.6). We have

$$\frac{1}{T_n} \mathbb{E}\left(F_t^{(k)}(\xi_n) \mathbb{1}\left\{0 < N_t(\xi_n) \leq T\right\}\right)$$
$$\geq \frac{\binom{n}{d}}{2(d+1)\log n} \left(n^k \left(\frac{\rho^{d-1} d!}{2^{d-1}} \left(1 - \exp\left(-\frac{2^{d-1}\rho}{d!}\right)\right)\right)^k - c\frac{n^k}{\log n}\right)$$
$$\times \int\limits_{\left(\rho\mathbb{B}^d\right)^d} \int \left\{\exists i \in [d] : \{x_1, \ldots, x_d\} \subset \mathbb{B}^d\left(x_i, n^{-1/(d-1)}\right)\right\} \mathrm{d}x_1 \ldots \mathrm{d}x_d.$$

The value of the integral is c/n^d, for some constant $c > 0$. Therefore, we conclude

$$\mathbb{E}\deg(\xi_n)^k \geq cn^k (\log n)^{-1},$$

for n large enough, with some constant $c > 0$. □

5.3 Prove approach 2

Let again $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$ be a deterministic point set in general position. Let $t > 0$ and redefine now the two previously used functionals by

$$N_t(X) := \sum_{\{x_{i_1},\ldots,x_{i_d}\} \in \binom{X}{d}} \mathbb{1}\left\{\forall j,k \in [d] : \|x_{i_j} - x_{i_k}\| \le t\right\}$$

and

$$F_t(X) := \sum_{\{x_{i_1},\ldots,x_{i_d}\} \in \binom{X}{d}} \mathbb{1}\left\{\forall j,k \in [d] : \|x_{i_j} - x_{i_k}\| \le t\right\} \deg(x_{i_1}, \ldots, x_{i_d}; X).$$

Similarly, the core idea of the proof will be to use the inequality

$$F_t(X) \le N_t(X) \deg(X), \tag{5.10}$$

which holds for all $t > 0$. Again, by considering $X = \xi_n = \{X_1, \ldots, X_n\}$ to be a random point set of n independent and uniformly chosen point from a compact set $K \subset \mathbb{R}^d$ and taking expectations on both sides, we arrive at

$$\mathbb{E} \deg(\xi_n) \ge \mathbb{E}\left(\frac{F_t(\xi_n)}{N_t(\xi_n)} \mathbb{1}\{N_t(\xi_n) \ge 1\}\right) \ge \mathbb{E}\left(F_t(\xi_n) \mathbb{1}\{N_t(\xi_n) = 1\}\right).$$

We write this as an integral and obtain

$$\begin{aligned}\mathbb{E} \deg(\xi_n) \ge \frac{(n)_d}{\mathrm{Vol}_d(K)^d} \int_{K^d} &\mathbb{1}\left\{\forall j,k \in [d] : \|x_{i_j} - x_{i_k}\| \le t\right\} \\ &\times \mathbb{E}\left(\deg(x_1, \ldots, x_d; \xi_n') \mathbb{1}\{N_t(\xi_n') = 1\}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_d,\end{aligned} \tag{5.11}$$

where $\xi_n' = \xi_{n-d} \cup \{x_1, \ldots, x_d\}$. Next, we need to lower bound the expectation in the integral. We have

$$\begin{aligned}&\mathbb{E}\left(\deg(x_1, \ldots, x_d; \xi_n') \mathbb{1}\{N_t(\xi_n') = 1\}\right) \\ &\quad = \mathbb{E} \sum_{X \in \xi_{n-d}} \mathbb{1}\left\{\mathrm{int}([x_1, \ldots, x_d, X]) \cap \xi_n' = \varnothing\right\} \mathbb{1}\{N_t(\xi_n') = 1\} \\ &\quad = \frac{n-d}{\mathrm{Vol}_d(K)} \int_K \mathbb{P}\left(\mathrm{int}([x_1, \ldots, x_d, x_{d+1}]) \cap \xi_n'' = \varnothing,\ N_t(\xi_n'') = 1\right) \mathrm{d}x_{d+1},\end{aligned} \tag{5.12}$$

where $\xi_n''' = \xi_{n-d-1} \cup \{x_1, \dots, x_{d+1}\}$. Now set

$$A(x_1, \dots, x_{d+1}) = \left([x_1, \dots, x_{d+1}] \cup \bigcup_{i=1}^{d+1} \mathbb{B}^d(x_i, t) \right) \cap K.$$

The base $[x_1, \dots, x_d]$ of the simplex $[x_1, \dots, x_{d+1}]$ has edge length bounded by t and thus is contained in $\mathbb{B}^d(x_i, t)$, it's height is bounded by $\operatorname{diam}(K)$, the diameter of K. This implies

$$\operatorname{Vol}_d(A(x_1, \dots, x_{d+1})) \leq \frac{1}{d}\kappa_d t^{d-1} \operatorname{diam}(K) + (d+1)\kappa_d t^d \leq c(d, K) t^{d-1},$$

for sufficiently small t. Hence, for sufficiently small t, the probability that at least one point of ξ_{n-d-1} is contained in $A(x_1, \dots, x_{d+1})$ can be estimated via

$$\begin{aligned}
&\mathbb{P}(\xi_{n-d-1} \cap \operatorname{int}(A(x_1, \dots, x_{d+1})) \neq \varnothing) \\
&\qquad = 1 - \left(1 - \frac{\operatorname{Vol}_d(A(x_1, \dots, x_{d+1}))}{\operatorname{Vol}_d(K)}\right)^{n-d-1} \\
&\qquad \leq 1 - \left(1 - \frac{c(d, K)t^{d-1}}{\operatorname{Vol}_d(K)}\right)^{n-d-1} \\
&\qquad \leq c(d, K) t^{d-1}(n - d - 1).
\end{aligned} \tag{5.13}$$

In the next step we will need to know what $\mathbb{P}(N_t(\xi_{n-d-1}) = 0)$ is. We will do this by first calculating $\mathbb{E}N_t(\xi_n)$ and determining a t such that this expectation converges to a constant for $t \to 0$. Let us introduce two lemmas for this.

Lemma 5.3.1 *Let ξ_n be a set of n independent and uniformly chosen random points from a compact set $K \subset \mathbb{R}^d$. Then,*

$$\mathbb{E}N_t(\xi_n) = \frac{(n)_d}{\operatorname{Vol}_d(K)^d} t^d \int_K \int_{(K')^{d-1}} \prod_{1 \leq i < j < d} \mathbb{1}\{\|y_i - y_j\| \leq 1\} \prod_{1 \leq i < d} \mathbb{1}\{\|y_i\| \leq 1\} \prod_{i=1}^{d-1} \mathrm{d}y_i \, \mathrm{d}x_d,$$

where $K' = t(K - x_d)$.

Proof. We have

$$\mathbb{E}N_t(\xi_n) = \mathbb{E} \sum_{\{X_{i_1}, \dots, X_{i_d}\} \in \binom{\xi_n}{d}} \int \{\forall j, k \in [d] : \|X_{i_j} - X_{i_k}\| \leq t\}$$

$$= \frac{(n)_d}{\mathrm{Vol}_d(K)^d} \int\limits_{K^d} \mathbb{1}\left\{\forall j,k \in [d] : \|x_i - x_j\| \le t\right\} \mathrm{d}x_1 \ldots \mathrm{d}x_d. \tag{5.14}$$

For $i \in [d-1]$ we transform $x_i = ty_i + x_d$ and use the notation $K' = t(K - x_d)$, to obtain

$$\mathbb{E}N_t(\xi_n)$$
$$= \frac{(n)_d\, t^d}{\mathrm{Vol}_d(K)^d} \int\limits_K \int\limits_{(K')^{d-1}} \prod_{1\le i<j<d} \mathbb{1}\{\|y_i - y_j\| \le 1\} \prod_{1\le i<d} \mathbb{1}\{\|y_i\| \le 1\} \prod_{i=1}^{d-1} \mathrm{d}y_i\, \mathrm{d}x_d. \tag{5.15}$$

finishing the proof. □

Remark 5.3.2 Let $t = (\gamma n)^{-1/(d-1)}$ with some $\gamma > 0$. Then, $K' \to \mathbb{R}^d$, as $n \to \infty$, if x_d is in the interior of K, this shows that

$$\lim_{n\to\infty} \mathbb{E}N_{(\gamma n)^{-1/(d-1)}}(\xi_n) = \gamma^{-d}\,\mathrm{Vol}_d(K)^{-(d-1)} \int\limits_{\left(\mathbb{B}^d\right)^{d-1}} \prod_{1\le i<j<d} \mathbb{1}\{\|y_i - y_j\| \le 1\} \prod_{i=1}^{d-1} \mathrm{d}y_i,$$

which is also clearly an upper bound for $\mathbb{E}N_{n^{-1/(d-1)}}(\xi_n)$.

So let from now on $t = (\gamma n)^{-1/(d-1)}$. We turn now to bounding $\mathbb{P}(N_t(\xi_{n-d-1}) = 0)$.

Lemma 5.3.3 *Let ξ_n be a set of n independent and uniformly chosen random points from a compact set $K \subset \mathbb{R}^d$. Then,*

$$\mathbb{P}\left(N_{(\gamma n)^{-1/(d-1)}}(\xi_n) = 0\right) \ge 1 - c(d)\gamma^{-d}\,\mathrm{Vol}_d(K)^{-(d-1)}.$$

Proof. Since $N_{n^{-1/(d-1)}}(\xi_n) \ge 0$, we have

$$\begin{aligned}
\mathbb{P}\left(N_{(\gamma n)^{-1/(d-1)}}(\xi_n) = 0\right) &= 1 - \sum_{k\ge 1} \mathbb{P}\left(N_{(\gamma n)^{-1/(d-1)}}(\xi_n) = k\right) \\
&\ge 1 - \sum_{k\ge 1} k\mathbb{P}\left(N_{(\gamma n)^{-1/(d-1)}}(\xi_n) = k\right) \\
&= 1 - \mathbb{E}N_{(\gamma n)^{-1/(d-1)}}(\xi_n).
\end{aligned}$$

combining this with Lemma 5.3.1, respectively Remark 5.3.2, concludes the proof. □

Now we return to the probability in (5.12). Using (5.13) and Lemma 5.3.3 we obtain

$$\begin{aligned}
&\mathbb{P}\left(\operatorname{int}([x_1,\dots,x_d,x_{d+1}]) \cap \xi_n''' = \varnothing,\ N_{(\gamma n)^{-1/(d-1)}}(\xi_n''') = 1\right)\\
&\quad\geq \mathbb{P}\left(\xi_{n-d-1} \cap \operatorname{int}(A(x_1,\dots,x_{d+1})) \neq \varnothing,\ N_{(\gamma n)^{-1/(d-1)}}(\xi_n) = 0\right)\\
&\quad\geq \mathbb{P}\left(N_{(\gamma n)^{-1/(d-1)}}(\xi_n) = 0\right) - \mathbb{P}\left(\xi_{n-d-1} \cap \operatorname{int}(A(x_1,\dots,x_{d+1})) = \varnothing\right)\\
&\quad\geq c(d)\gamma^{-d}\operatorname{Vol}_d(K)^{-(d-1)} + c(d,K)(\gamma n)^{-1}(n-d-1).
\end{aligned}$$

Because the term in brackets decreases to zero for increasing γ, there exists a $\gamma' \in \mathbb{R}$ such that

$$\mathbb{P}\left(N_{(\gamma' n)^{-1/(d-1)}}(\xi_n) = 0\right) \geq \frac{1}{2}.$$

Plugging this into (5.12), we obtain for $\gamma = \gamma'$

$$\mathbb{E}\left(\deg(x_1,\dots,x_d;\xi_n')\mathbb{1}\{N_t(\xi_n') = 1\}\right) \geq \frac{n-d}{\operatorname{Vol}_d(K)}\int_K \frac{1}{2}\,\mathrm{d}x_{d+1} = \frac{n-d}{2}.$$

Thus, we can conclude by using (5.14) with $t = (\gamma n)^{-1/(d-1)}$ into (5.11)

$$\begin{aligned}
\mathbb{E}\deg(\xi_n) &\geq (n)_d \int_{K^d} \mathbb{1}\left\{\forall i,j \in [d] : \|x_i - x_j\| \leq \gamma^{-\frac{1}{d-1}}\right\}\frac{n-d}{2}\,\mathrm{d}x_1\dots\mathrm{d}x_d\\
&= \frac{n-d}{2}(n)_d \int_{K^d} \mathbb{1}\left\{\forall i,j \in [d] : \|x_i - x_j\| \leq \gamma^{-\frac{1}{d-1}}\right\}\mathrm{d}x_1\dots\mathrm{d}x_d\\
&= \frac{n-d}{2}\operatorname{Vol}_d(K)^d\mathbb{E}N_{(\gamma n)^{-1/(d-1)}}(\xi_n)\\
&\geq c(d,K)n,
\end{aligned}$$

to finish the proof. □

Symbol index

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