1 Introduction

This thesis deals with Gamma approximation for elements of a Wiener chaos (chapter 2), fine asymptotics for certain models that have moments of Gamma type (chapter 3 and 4), as well as moment estimates of Rosenthal type with applications to a variety of models (chapter 4).

Chapter 2 of this thesis combines the techniques of Stein’s method and Malliavin calculus to obtain quantitative non-central limit theorems for certain functionals of Gaussian fields.

Stein’s Method was first introduced by Charles Stein in his paper [103], and then further developed in [104]. His crucial observation was that a real-valued random variable $N$ follows a standard normal distribution $\mathcal{N}(0,1)$ if and only if

$$\mathbb{E}[Nf(N)] = \mathbb{E}[f'(N)]$$

for a certain class of test functions $f$. This is also known as Stein’s lemma. He then heuristically concluded, that if the quantity $\mathbb{E}[Xf(X) - f'(X)]$ is close to zero for some class of test functions $f$, then the random variable $X$ must be close to a standard normal random variable. The proximity of two random variables $X$ and $Y$ is usually described in terms of the quantity $\sup_{h \in \mathcal{F}}|\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|$, where $\mathcal{F}$ is a class of test functions. Stein’s method for normal approximation now considers the following ordinary differential equation, the so-called Stein equation

$$f'(x) - xf(x) = h(x) - \mathbb{E}[h(N)]. \quad (1.1)$$

It is clear that when plugging in a random variable $X$ for $x$ and taking expectations on both sides, the right hand side is exactly the term we are interested in. If for any given $h$ from a class of test functions, we are able to find bounds on the solution $f_h$ to (1.1), then we are able to bound the distance between our random variable $X$ and the standard normal target $N$. Note that the left hand side does not involve the target random variable $N$ anymore. Interestingly enough, it is typically easier to bound than the right hand side.

Stein’s method has since become a popular tool for showing quantitative limit theorems and has been applied in many fields of probability theory. For a comprehensive overview on Stein’s method for normal approximation, the reader is referred to the textbook [18].

Stein’s method for gamma approximation has first been considered by Luk [70] and
then been further refined by Pickett [95] in their Ph.D. theses. The Stein equation we will be using has been introduced by Döbler and Pecatti in [24]. We use this equation, because it avoids the positive part that is present in the Stein equation that was used e.g. in [82], and because the solution has better smoothness properties.

Malliavin calculus of variations, on the other hand, is an infinite dimensional differential calculus that was first introduced by Malliavin in [72]. Its operators, some of which we will introduce in the preliminaries of the second chapter, act on functionals of Gaussian Processes (e.g. a Brownian motion). For a detailed overview on Malliavin calculus, the reader is referred to the textbook [87].

In 2005, Nualart and Peccati proved the famous fourth moment theorem using Malliavin calculus (see [88]). It states that for a sequence of certain functionals of a Gaussian field, a central limit theorem is equivalent to convergence of the third moments. More precisely, if \((X_n)_{n \in \mathbb{N}}\) is a sequence of centered random variables with unit variance that lies inside a fixed finite sum of Wiener chaoses, and \(N \sim \mathcal{N}(0,1)\), then

\[ X_n \xrightarrow{D} N \quad \text{if and only if} \quad \mathbb{E}[X_n^4] \xrightarrow{n \to \infty} 3 = \mathbb{E}[N^4]. \]

This can be seen as a substantial simplification of the method of moments/cumulants, where instead of just the fourth moment, one would have to check convergence of all corresponding moments or, equivalently, cumulants.

A combination of Stein’s method and Malliavin calculus was first exploited by Nourdin and Peccati in [82]. Merging these two techniques, the authors were able to provide explicit quantitative bounds in the fourth moment theorem in terms of the fourth moment / fourth cumulant of the sequence. In particular, they showed that if \(F\) is Malliavin derivable, then

\[ d_W(F, N) \leq \sqrt{\mathbb{E}[1 - \Gamma_1(F)]}, \]

where \(d_W\) denotes the 1-Wasserstein distance (see section 2.1.2) and \(\Gamma_1\) is the iterated Gamma operator introduced in section 2.2.1. The authors then expressed the right hand side in terms of contraction norms, which can in turn be bounded by the fourth cumulant.

In the recent paper [85], Nourdin and Peccati showed that the exact convergence rate (in total variation distance) in the fourth moment theorem is determined by the fourth and third cumulant. More precisely, if \((F_n)_{n \in \mathbb{N}}\) is an element of a fixed Wiener chaos converging in distribution to \(N \sim \mathcal{N}(0,1)\), then there exist two positive constants \(c < C\), not depending on \(n\), such that

\[ c \times \max\{|\kappa_3(F_N)|, |\kappa_4(F_N)|\} \leq d_{TV}(F_n, N) \leq C \times \max\{|\kappa_3(F_N)|, |\kappa_4(F_N)|\}. \]

Note that whilst the third cumulant comes into play, the square root from the previous result has been removed in the upper bound. The proof is based on iterating both Stein’s method and the Malliavin integration-by-parts formula. To this end, the authors
employed the notion of iterated Gamma operators, albeit different ones than those that were used in previous publications.

Similar results are also available when the target distribution is replaced by a centered Gamma distribution (see section 2.1.1). A *four moments theorem* (not “fourth”) for Gamma approximation has already been considered in [81], [82] and [86]. For a quantitative version, we shall refer to Theorem 1.7 of [24], where the authors proved a bound in 1-Wasserstein distance, namely

\[ d_W(F,G(\nu)) \leq \max \left( 1, \frac{2}{\nu} \right) E \left[ (2F + 2\nu - \Gamma_1(F))^2 \right]^{1/2}. \]

Here \( G(\nu) \) denotes a centered Gamma distribution with parameter \( \nu > 0 \). From [86, Theorem 3.6], for any random variable \( F \) in the \( q \)-th Wiener chaos with \( E[F^2] = 2\nu \), we have the estimate

\[ E \left[ (2F + 2\nu - \Gamma_1(F))^2 \right] \leq \frac{q-1}{3q} |\kappa_4(F) - \kappa_4(G(\nu)) - 12\kappa_3(F) + 12\kappa_3(G(\nu))| \]

\[ \leq \text{const.} \times \max \left\{ |\kappa_3(F) - \kappa_3(G(\nu))|, |\kappa_4(F) - \kappa_4(G(\nu))| \right\}. \]

Combining these two results, we obtain an upper bound similar to the one in the fourth moment theorem, but worse by a whole square root, namely

\[ d_W(F,G(\nu)) \leq \text{const.} \times \max \left\{ |\kappa_3(F) - \kappa_3(G(\nu))|, |\kappa_4(F) - \kappa_4(G(\nu))| \right\}^{1/2}. \ (1.2) \]

A natural question that arises, and that will be the main focus of the second chapter is if, and under which conditions, we can remove the square root employing techniques similar to the ones used in [85].

The second chapter is organized as follows. Section 2.1 introduces our target of interest, the centered Gamma distribution, as well as the various probability metrics we will consider. We then provide a basic account of the two main techniques we are using: Stein’s method for Gamma approximation and Malliavin calculus.

In section 2.2, we first introduce the concept of iterated Gamma operators. These operators, which have been used in the proof of the main theorem of [85], have not been discussed in full detail before, so we take a closer look at them. In particular, we point out the difference between these new \( \Gamma \)-operators and the classical ones. With all these tools at hand, we are then able to proof our main upper bound in terms of Gamma operators, which is Theorem 2.2.2. We then briefly discuss why in general it is almost impossible to translate these bounds into cumulants as it has been done in [83] for a Gaussian target.

Section 2.3 then focuses solely on the case, where the examined sequence lies in the second Wiener chaos. Here, we have additional techniques at our disposal, namely that we can consider the eigenvalues of the so-called Hilbert-Schmidt operator associated to
the random element of the second Wiener chaos. After some toy examples, it becomes clear, that the square root in the upper bound in (1.2) cannot always be removed using our Stein-Malliavin bound from the previous section. In order to do so, we would need two variance estimates, both of which are considered separately. We will see that the first one always holds, but in order to show the second one, we need an additional condition on the underlying sequence of random variables, or rather the eigenvalues of the corresponding Hilbert-Schmidt operators. Under this additional condition, we are then able to state and prove our optimal Theorem 2.3.18. In the next part of this section, we then consider a technical condition on the Hilbert-Schmidt operator itself, under which our optimality theorem continues to hold. With this we are even able to state a non-asymptotic version of it. In the final part of this section, we come back to our toy example, where the removal of the square root could not be achieved. Using explicit computations with the corresponding densities, we are able to show that in total variation, the desired upper bound indeed does hold.

The next section 2.4 goes back to the general setting and removes the restriction of being in the second Wiener chaos. Using techniques that date back to Tikhomirov (105), we apply the Stein equation on the level of characteristic functions to derive an upper bound for convergence rate in Kolmogorov distance. This result is completely independent from the findings of the previous sections.

Finally, section 2.5 presents an outlook on what could be done in the future to remove the technical condition in our optimality theorem. Chapter two is based on the preprint


Chapter 3 presents fine asymptotics for random variables with moments of Gamma type. In the survey [57] (see also [58]), a positive random variable \(X\) is defined to have moments of Gamma type if, for \(s\) in some interval,

\[
\mathbb{E}[X^s] = CD^s \prod_{j=1}^{J} \Gamma(a_j s + b_j) \prod_{k=1}^{K} \Gamma(a'_k s + b'_k)
\]

for some integers \(J, K \geq 0\) and some real constants \(C, D > 0, a_j, b_j, a'_k, b'_k\). In [57] and [58] a rich class of examples was presented. This includes many standard distributions, among them the Gamma distribution, the Beta distribution, stable distributions, the Mittag-Leffler distribution, extreme value distributions, the Fejér distribution, and many more probability laws. For our purposes we will choose \(J = K = p\), which might depend on a parameter, say \(p(n)\). Moreover, we choose \(b_j = b'_j = \alpha(j + l)\) for some \(l\) which might depend on \(n\) and \(p(n)\) and a constant \(\alpha\), and we will choose \(a'_j = 0\). The random variables with moments of Gamma type that we focus on in this chapter are mainly determinants of random matrices, as well as volumes of random parallelotopes.
and simplices.

In order to obtain our results for these models, we use the framework of mod-$\phi$ convergence. Mod-Gaussian convergence (a special case of mod-$\phi$ convergence) was first studied in \[53\], inspired by theorems and conjectures in random matrix theory and number theory concerning moments of values of characteristic polynomials or zeta functions. The basic idea is to look for a natural renormalization of the characteristic functions of a sequence of random variables that does not converge in distribution. This sequence of characteristic functions then converges to some non-trivial limit.

Mod-Gaussian convergence immediately implies a central limit theorem for a properly rescaled version of the sequence under consideration. But in fact, there is much more encoded in mod-$\phi$ convergence. Under some extra conditions, one can show an extended central limit theorem, precise deviations, local limit theorems, large and moderate deviation principles and Berry-Esseen bounds, see Theorems 3.1.4, 3.1.5, 3.1.9 and 3.1.10 in section 3.1.

Recently, in \[16\], second-order refinements of central limit theorems for log-determinants of certain random matrix ensembles were considered. The authors provide an asymptotic expansion of the Laplace transforms of the log-determinants and apply the framework of mod-Gaussian convergence. Their results include mod-Gaussian convergence, extended central limit theorems, precise moderate deviations, Berry-Esseen bounds, as well as local limit theorems. Moreover, they were able to apply the techniques to random characteristic polynomials evaluated at 1 for circular and circular Jacobi beta ensembles.

In this chapter, we study precise asymptotics for log-determinants of $\beta$-Laguerre ensembles for $p(n) \times p(n)$ random matrices $A^\dagger A$, where $A$ is a certain $p(n) \times n$ matrix and $A^\dagger$ denotes the transpose, the Hermitian conjugate or the dual of $A$ when $A$ is real, complex and quaternion respectively. In mathematical statistics, $p(n)$ typically is the number of variables, each of which is observed $n$ times. Therefore, it is natural to consider the case $n \neq p(n)$. The case $n = p(n)$ has already been covered in \[16\]. Depending on the behavior of the sequence $n - p(n)$, we observe mod-Gaussian convergence or mod-stable convergence on $i\mathbb{R}$ (see Theorem 3.5.1).

An important observation of our findings is that the asymptotic behavior of the determinants of $\beta$-Laguerre ensembles for varying dimensions is sufficient to be able to study the asymptotics of determinants of $\beta$-Jacobi ensembles, of Ginibre ensembles, of 7 further matrix ensembles within the tenfold way of mesoscopic physics, and of the determinant of certain Gram matrices with respect to certain distributions in $\mathbb{R}^n$, representing the volume of parallelotopes. Hence we will provide similar fine asymptotics for all of these models.

The outline of chapter 3 is as follows. Section 3.1 establishes the concept of mod-$\phi$ convergence and presents the various results it entails such as extended central limit
theorems, precise deviations, local limit theorems and Berry-Esseen bounds. Section 3.2 then introduces all the different models we are considering. Based on Selberg integration, we provide explicit formulas for the Mellin transform of the corresponding random determinant. Inspired by the $\beta$-Laguerre case, we define our function $L(p, l, \alpha; z)$, and notice that it also appears in all of the other models. This is due to the fact that all of these determinants have moments of Gamma type.

In section 3.3 we prove our main theorem, which is nothing but an expansion of our function $L(p, l, \alpha; z)$. In section 3.4 we use this expansion to analyze the asymptotic behavior of $L(p(n), r(n), \beta/2; z)$ as $n$ tends to infinity. This is enough to cover all the models we consider.

Sections 3.5 and 3.6 then gathers all the mod-$\phi$ results for our models with all their consequences. Note that in some particular cases, no mod-$\phi$ converges is observed, for example in the $\beta$-Jacobi ensemble when $p(n)$ is fixed. In other cases we observe non-Gaussian mod-$\phi$ convergence on $i\mathbb{R}$, for instance in the $\beta$-Laguerre ensemble when the number of variables $p(n)$ is fixed.

Chapter four is based on the preprint


In chapter 4, we deduce inequalities of Rosenthal type under a certain condition on the cumulants of a given sequence of random variables $(Z_n)_{n\in\mathbb{N}}$. This kind of bounds on the cumulants we impose on our random variables are classic and have been studied before, for instance in [102], [37] and [25]. For independent random variables, the Rosenthal inequalities relate moments of order higher than 2 of partial sums of random variables to the variance of partial sums. In [96] it was proved that for $(X_k)_{k\in\mathbb{N}}$ being an independent and centered sequence of real valued random variables with finite moments of order $p$, $p \geq 2$, one obtains for every positive integer $n$

$$E\left[\left(\sum_{j=1}^{n} X_j\right)^p\right] \leq \sum_{j=1}^{n} E[|X_j|^p] + \left(\sum_{j=1}^{n} E[X_j^2]\right)^{\frac{p}{2}}.$$ 

Here $a_n \ll b_n$ means that there exists a numerical constant $C_p$, depending only on $p$ (neither on the underlying random variables nor on $n$), such that $a_n \leq C_p b_n$ for all positive integers $n$. A first Rosenthal-type inequality for weakly dependent random variables was derived in [28]. In [29] cumulant estimates are employed for deriving inequalities of Rosenthal type for weakly dependent random variables. Our abstract result, Theorem 4.2.1 is motivated by this work. We will prove moment estimates for a couple of statistics applying Theorem 4.2.1.

The chapter is structured as follows. Section 4.1 recalls the notion of cumulants and lists
some of their most important properties. We also present some of the classical results that follow from the kind of cumulant bounds we are considering.

Then next section 4.2 then presents our main Theorem 4.2.1 which is an upper bound on the difference between the moments of our considered random variable and standard Gaussian variable. As an initial example, we apply our theorem to a partial sum of independent, non-identically distributed random variables.

Section 4.3 then presents applications to several models: Dependency graphs, weighted dependency graphs, non-degenerate U-statistics, characteristic polynomials in the circular ensembles and determinants of random matrix ensembles and random simplices. Note that the last class of examples has already been studied in the previous chapter. Nonetheless, we would like to point out that these kinds of moment estimates do not follow from the asymptotics derived in chapter 2.

Chapter four is based on the preprint