High-Dimensional Asymptotics for Random Polytopes

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Contents

Chapter 1

Introduction

In this chapter we introduce the topic of random polytopes and some of the problems that this subject handles. From a methodical point of view, the study of random polytopes combines ideas and techniques from several areas of mathematics such as convex and discrete geometry, geometric functional analysis and probability theory. We refer to the surveys [15, 49, 66] for further discussions and results.

In the first part of the current chapter we provide an overview of the main problems that this text approaches, together with a short historical context. This directly serves the purpose of giving an appropriate background for the topic faced, starting from Chapter 3.

In the second part of this chapter we provide a summary of the results presented in the rest of the text, listing the research papers they are based on.

1.1 Overview of the problems and their motivations

The study of random polytopes constitutes an important branch of stochastic geometry, standing at a crossroads between convex geometry and probability theory.

Although for many centuries geometry was only treated within the context of a deterministic environment, modern mathematics has more and more dealt with the concept of randomness throughout the past decades. Actually, this trend started even earlier than 1933, year in which Kolmogorov posed the axioms of probability theory [55]. To confirm this statement, let us check an extract from possibly the first world-known problem concerning random polytopes, the so called *Sylvester's four-point problem* of 1864. To the ears of a modern mathematician, it makes no rigorous sense. Published in the magazine *Educational Times* [81], it reads:

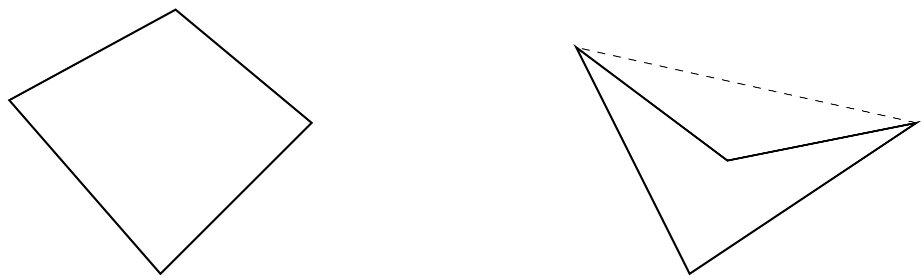

Figure 1.1: A convex quadrilateral (on the left side) and a reentrant quadrilateral (on the right). Note that, in the latter case, one of the four vertices of the quadrilateral falls inside the convex hull of the other three, as highlighted by the dashed segment.

"Show that the chance of four points forming the apices of a reentrant quadrilateral is 1/4 *if they be taken at random in an indefinite plane."*

By *reentrant quadrilateral*, Sylvester meant that one of the four points must fall inside the convex hull of the other three, in this way having the four points for vertices one can draw a non-convex quadrilateral, see Figure 1.1.

One can see how the aforementioned problem is ill-posed, as Sylvester's words do not specify which probability distribution on the plane he intends to use. Also, there is no such thing as a uniform probability distribution on the plane.

Among all the proposed solutions to the problem, we mention the one by Woolhouse [88]. He decided to pick the points uniformly at random in a circle of radius r and then let $r \to \infty$ to have some sort of approximation of a uniform distribution on the plane. Note though that the required probability is invariant under affine transformations, so we know from starters that the result will not depend on r. Woolhouse obtained the value $35/12\pi^2 \approx 0.296$ for this probability.

Naturally, one could use the same argument with any shape instead of a circle. In such a case, the problem boils down to choosing a particular compact subset of the plane from which one draws independent random points, and see what value it gives.

In view of this, we can formalize the problem in modern terms. When $X_1, \dots, X_n$ are random points in $\mathbb{R}^d$, their convex hull is a *random polytope* contained in $\mathbb{R}^d$. Let now K be a convex, compact subset of $\mathbb{R}^d$ with non-empty interior. K is usually referred to as a *convex body* in $\mathbb{R}^d$. Therefore, one possible way to make sense of Sylvester's four-point problem, is the following:

Fix a convex body K in $\mathbb{R}^2$, and draw 4 points $X_1, \dots, X_4$ independently and uniformly at random inside K. Denoting by P_4^K the random convex hull of these points, *compute*

$$\mathbf{P}(P_4^K \text{ is a triangle }).$$

It is easy to compute that

$$\mathbf{P}(P_4^K \text{ is a triangle }) = \frac{4\,\mathbf{E}\,\mathrm{vol}_2(P_3^K)}{\mathrm{vol}_2(K)},$$

where $\mathbf{E}$ denotes the expectation taken with respect to the probability measure $\mathbf{P}$ and $\mathrm{vol}_2(\cdot)$ stands for the Lebesgue measure of $\mathbb{R}^2$.

Therefore, if the volume of K is prescribed, Sylvester's problem is equivalent to computing $\mathbf{E}\,\mathrm{vol}_2(P_3^K)$, which depends only on the shape of K, and, being it an integral over the plane, can be explicitly computed as long as K possesses enough symmetries.

Around fifty years after the problem was first posed, Blaschke proved in [21] that the equilateral triangle S and the circle B represent the extremal convex bodies for this quantity in dimension 2, i.e.

$$\frac{35}{12\pi^2} = \mathbf{E}\,\mathrm{vol}_2(P_3^B) \le \mathbf{E}\,\mathrm{vol}_2(P_3^K) \le \mathbf{E}\,\mathrm{vol}_2(P_3^S) = \frac{1}{3}.$$

Moreover, Blaschke stated that the proof could be carried out to higher dimensions, where now S and B stand for the regular simplex and the Euclidean ball, respectively. Unfortunately, it turned out that the issue is not quite as simple: while, on the one hand, Groemer proved in [43] and [44] not only that for every dimension d,

$$\mathbf{E}\,\mathrm{vol}_d(P_{d+1}^B) \le \mathbf{E}\,\mathrm{vol}_d(P_{d+1}^K),$$

but also that for every n,

$$\mathbf{E}\,\mathrm{vol}_d(P_n^B) \le \mathbf{E}\,\mathrm{vol}_d(P_n^K), \tag{1.1}$$

on the other hand, Blaschke's argument is false when attempting to prove the upper bound. In fact, the upper bound has only been proven so far for $d = 2$ in [35]. We refer to [62] for further details on the history of Sylvester four-point problem.

The problem of showing whether for any K

$$\mathbf{E}\,\mathrm{vol}_d(P_n^K) \le \mathbf{E}\,\mathrm{vol}_d(P_n^S),$$

is true, still seems of difficult solution. So much in fact that it has been shown that a positive answer would imply a solution for the so-called *hyperplane conjecture* which is one of the most important standing conjectures in convex geometry.

The hyperplane conjecture asks whether it is true that every convex body of unitary volume in any dimension admits an hyperplane which cuts a section of the body whose volume is at least an absolute constant. For this reason, this question is also known as *slicing problem*. Since it is hard to compute exactly the required quantities in a deterministic general setting, it makes sense to see if it can be verified on a random model with high probability. For instance, one can look for a counterexample when drawing independent random points inside a sequence of convex bodies in increasing dimension, and estimating the isotropic constants of the resulting random polytopes. It turns out that for many classes of convex bodies, and different kinds of probability distributions on them, the isotropic constant of the resulting random polytopes is bounded with high probability.

Going back to original matter, one can observe that if we denote by $f_0(P)$ the number of vertices of a polytope P, then Efron's identity, see [39],

$$\mathbf{E}\, f_0(P_n^K) = \frac{n(1 - \mathbf{E}\,\mathrm{vol}_d(P_{n-1}^K))}{\mathrm{vol}_d(K)},$$

translates Groemer's inequality (1.1) into

$$\mathbf{E}\, f_0(P_n^K) \leq \mathbf{E}\, f_0(P_n^B),$$

for any dimension $d \geq 2$. Inspired by such an inequality, one can ask whether it is true that, fixing a prescribed convex body $K \subset \mathbb{R}^d$, increasing the number of random points also increases the expected number of i-dimensional faces $f_i(P_n^K)$ of the corresponding random polytope, i.e.

$$\mathbf{E}\, f_i(P_n^K) \leq \mathbf{E}\, f_i(P_{n+1}^K).$$

This inquiry goes back at least to Van Vu [83]. As one may expect, precise computations of geometric functionals, such as the faces number, of random polytopes with a fixed number of vertices are of difficult solution. However, when increasing the amount of random points inside a convex body K, the resulting random polytope P_n will eventually tend to fill the whole space of K. In particular, the volume of P_n will approach the whole volume of K, i.e.

$$\mathrm{vol}_d(P_n) \to \mathrm{vol}_d(K),$$

as $n \to \infty$. Therefore, it would be interesting to know how fast this convergence occurs, i.e. if one can give an asymptotic estimate for the distribution of $\mathrm{vol}_d(K) - \mathrm{vol}_d(P_n)$ as $n \to \infty$, for example in terms of its moments, or see whether they satisfy a central limit theorem. In this regard, we cite the seminal work [71] of Rényi and Sulanke, where the authors studied the expectations of $\mathrm{vol}_2(K) - \mathrm{vol}_2(P_n)$ in the Euclidean plane.

Furthermore, as a generalization of the concept of volume, one can study further geometric functionals of a convex body $K \subseteq \mathbb{R}^d$, the so called *intrinsic volumes* $V_\ell(K)$, $\ell \in \{0, \dots, d\}$ which describe the lower dimensional volumetric features of K. One can think of $V_\ell(K)$ as the average, over all the possible ℓ-dimensional linear subspaces L, of the volume of the orthogonal projection of K on L. In particular, up to multiplicative constants, $V_d(K)$ is the usual volume of K, i.e. its Lebesgue measure, $V_{d-1}(K)$ is its surface area, $V_1(K)$ is the mean-width of K and $V_0(K)$ is its Euler characteristic, which is a constant for every polytope in any fixed dimension.

The importance of the intrinsic volumes arises also from the celebrated Hadwiger's theorem, first proved in [46]. It states that any real valuation ψ taking argument in the set of convex bodies, which is continuous and motion-invariant, is necessarily a linear combination of intrinsic volumes. Namely, there exists $(\lambda_0, \dots, \lambda_d) \subset \mathbb{R}^{d+1}$, such that for every convex body $K \subset \mathbb{R}^d$,

$$\psi(K) = \sum_{\ell=0}^{d} \lambda_\ell V_\ell(K).$$

Hence, one may ask for the asymptotic features of the distribution of $V_\ell(K) - V_\ell(P_n)$, in particular how fast its expectation approaches 0, and its fluctuations.

Leaving the setting of a prescribed convex body in a fixed dimension for a high-dimensional setting represents a different kind of asymptotics that one might want to examine. For example, if we let now the ambient dimension grow to infinity, and we take a sequence of convex bodies $K_n \subset \mathbb{R}^n$, one in each dimension, how many points $N = N(n)$ do we have to pick inside of each of them, in such a way that the volumes of the respective random polytopes $P_N \subset K_n$ approach, in the dimensional limit, the whole volume of K_n, i.e.

$$\frac{\mathrm{vol}_n(P_N)}{\mathrm{vol}_n(K_n)} \to 1,$$

as $n \to \infty$?

In the next section we explain in more detail how we deal with each of the aforementioned subjects, and the answers provided by this text.

1.2 Guideline

Chapter 2 is a collection of basic and general results of probability theory, integral and convex geometry, which will be needed, as a common resource, in several points of the dissertation.

In Chapter 3 we discuss limit theorems for intrinsic volumes of random polytopes in smooth convex bodies. In the earliest periods of studies, intrinsic volumes have been investigated extensively in the setting of random polytopes that arise as convex hulls of points chosen uniformly at random *inside* a prescribed convex body K in $\mathbb{R}^d$. We denote by P_n the convex hull of $X_1, \ldots, X_n$.

Results concerning the expectations of $V_\ell(P_n)$, $\ell \in \{1, \ldots, d\}$, have been studied, for example, by Reitzner [69], variance bounds can be found in Böröczky, Fodor, Reitzner and Vígh [24] and Bárány, Fodor and Vígh [11], and central limit theorems were treated in Reitzner [70], Vu [85] and Lachièze-Rey, Schulte and Yukich [58].

Using estimates for floating bodies, in combination with a general normal approximation bound obtained by Chatterjee [32] and Lachièze-Rey and Peccati [57] originating in Stein's method, our contribution is a quick, transparent and direct proof of the central limit theorems for the intrinsic volumes $V_\ell(P_n)$, $\ell \in \{1, \ldots, d\}$, as $n \to \infty$. More precisely, while the traditional methods (see [61, 70, 85]) first use a conditioning argument to compare P_n with the floating body and to prove the central limit theorem for a Poissonized version of the random polytopes, before pushing this result to the original model by de-Poissonization, we give a direct proof without making the detour just described. In this way we also avoid the more technical theory of stabilizing functionals developed in [58].

Furthermore, the approximation of a convex body K by means of a sequence of random polytopes K_n is improved whenever the vertices of K_n are restricted to lie on the boundary of K, making it a model of particular interest. Indeed, in this framework, the expectations of $V_\ell(K_n)$, $\ell \in \{1, \ldots, d\}$, have been studied, for example, by Buchta, Müller and Tichy [29], Reitzner [67], Schütt and Werner [78], and Böröczky, Fodor and Hug [25].

However, more detailed information about moments of the intrinsic volumes is only known for the usual volume $\mathrm{vol}_d(K_n)$. In particular, an upper variance bound was found by Reitzner [68] and a lower variance bound together with concentration inequalities by Richardson, Vu and Wu [72]. Only recently, Thäle [82] obtained a quantitative central limit theorem for $\mathrm{vol}_d(K_n)$ based on Stein's method.

In view of this, in the second part of Chapter 3 we generalize the results obtained in [68, 72] regarding the central limit theorem to the full regime of intrinsic volumes $V_\ell(K_n)$, $\ell \in \{1, \dots, d\}$. In fact, we prove a lower variance bound following the ideas of [11, 70, 72] and an upper variance bound in the manner of [11], making use of a version of the Efron-Stein jackknife inequality formulated in [68]. In particular, the upper variance bound implies a strong law of large numbers as in [11]. Secondly, we prove a quantitative central limit theorem for $V_\ell(K_n)$, $\ell \in \{1, \dots, d\}$, using a normal approximation bound obtained in [57], extending the result of [82].

Chapter 3 is based on the papers: C. Thäle, N. Turchi and F. Wespi. "Random polytopes: central limit theorems for intrinsic volumes" *Proceedings of the American Mathematical Society* 146, 3063–3071 (2018); and N. Turchi and F. Wespi "Limit theorems for random polytopes with vertices on convex surfaces" *Advances in Applied Probability* 50 (4), 1227-1245 (2018).

In Chapter 4 we study the monotonicity of the facetes number of the convex hull P_n of an increasing number of points, drawn independently at random in the space according to different probability distributions, in particular the so called *beta distribution* and *beta-prime distribution* in $\mathbb{R}^d$.

More specifically, we ask the following monotonicity question:

$$\textit{Is it true that } \mathbf{E}\, f_{d-1}(P_{n-1}) \le \mathbf{E}\, f_{d-1}(P_n)\textit{?}$$

This question has been put forward and answered positively by Devillers, Glisse, Goaoc, Moroz and Reitzner [36] for random points that are uniformly distributed in a convex body $K \subset \mathbb{R}^d$ if $d = 2$ and, if $d \ge 3$, under the additional assumptions that the boundary of K is twice differentiable with strictly positive Gaussian curvature and that n is sufficiently large, that is, $n \ge n(K)$, where $n(K)$ is a constant depending on K. Moreover, an affirmative answer was obtained by Beermann [19] in the case that the random points are chosen with respect to the standard Gaussian distribution on $\mathbb{R}^d$ or according to the uniform distribution in the d-dimensional unit ball for all $d \ge 2$. Beermann's proof relies on the Blaschke-Petkantschin formula, a well known change-of-variables formula in integral geometry. Generalizing her approach, we are able to answer positively to the other original question, in the setting where the underlying probability distributions are those classified by [59] (see p. 376 there) and Ruben and Miles, [73], for which a certain scaling property is satisfied. In particular, we can apply our results for the beta and beta-prime setting, to study similar monotonicity questions for a class of spherical convex hulls generated by random points on a half-sphere, which comprises as, a special case, the model recently studied by Bárány, Hug, Reitzner and Schneider [12].

Chapter 4 is based on the paper: G. Bonnet, J. Grote, D. Temesvari, C. Thäle, N. Turchi and F. Wespi). "Monotonicity of facet numbers of random convex hulls" *Journal of Mathematical Analysis and Applications* 455, 1351-1364 (2017).

In Chapter 5 we switch our interest to the high-dimensional setting, meaning that this time we aim to study objects in Euclidean spaces of increasing dimension. Once we abandon the study of convex hulls of increasing number of points in a fixed convex body, we have to deal with different problems. One possible question is to consider the convex hull $\mathrm{conv}(X_1, \ldots, X_N)$ of a finite number of points chosen randomly from the interior of a convex body K_n in $\mathbb{R}^n$, and investigate how many points $N = N(n)$ are needed in order to catch a certain portion of the whole volume of K_n as n tends to infinity. The first seminal work in this direction was done by Dyer, Füredi and McDiarmid [37], who proved that the expected volume of the convex hull $C_N = \mathrm{conv}(X_1, \ldots, X_N)$ of $N > n$ points chosen uniformly and independently from the vertices of the n-dimensional cube $[-1, 1]^n$, exhibits a phase transition when N is taken to be exponential in the dimension n, namely, that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\mathbf{E}\,\mathrm{vol}_n(C_N)}{\mathrm{vol}_n([-1,1]^n)} = \begin{cases} 0 & \text{if } N \leq (2e^{-1/2} - \varepsilon)^n \\ 1 & \text{if } N \geq (2e^{-1/2} + \varepsilon)^n, \end{cases}$$

where vol_n denotes the n-dimensional volume of a set. The method introduced in [37] influenced a number of later works, like for instance the approach that Bárány and Pór [17] used to prove the existence of ± 1 polytopes with a super-exponential number of facets. Subsequently, new volume threshold results were established by Gatzouras and Giannopoulos [41] for random polytopes generated by a wide class of probability measures μ in $\mathbb{R}^n$, as well as Pivovarov [63], who treated the case of independent points with respect to the Gaussian measure in $\mathbb{R}^n$ and the uniform measure on the Euclidean sphere. We stress that the authors in both [41] and [63] exploit the method of [37], which due to its geometric viewpoint seems to be applicable for a wide variety of probability distributions.

In the fifth chapter, we establish thresholds for the volume of beta and beta-prime random polytopes introduced in the previous chapter. The high-dimensional geometry of sets arising from these models of randomness have been studied extensively; for instance, in terms of properties of their volume [45], facet numbers [23] or intrinsic volumes [51]. Asymptotic estimates on the expected volume of the beta polytope in fixed dimension were derived by Affentranger [1]. Note also, that the gnomonic projection of a uniformly distributed point on the half-sphere is beta-prime distributed, which is exploited in [23] and [50]. One of the main results presented is that the threshold in the beta model consists of a super-exponential number N of random points.

Chapter 5 is based on the paper: G. Bonnet, G. Chasapis, J. Grote, D. Temesvari and N. Turchi "Threshold phenomena for high-dimensional random polytopes" to appear in *Communications in Contemporary Mathematics* (2018+).

In Chapter 6 we maintain the high dimensional setting, turning our interest towards the slicing problem for random polytopes whose vertices lie on the boundary of isotropic convex bodies, i.e. convex bodies $K \subset \mathbb{R}^n$ of unit volume whose barycenter is at the origin and inertia matrices are constant multiples L_K^2 of the identity matrix. The constant L_K is the isotropic constant of the body K and the question is whether or not there exists an absolute constant $C \in (0, \infty)$ such that $L_K \leq C$ for all space dimensions $n \in \mathbb{N}$ and all isotropic convex bodies $K \subset \mathbb{R}^n$. The hyperplane or isotropic constant conjecture is one of the outstanding open problems that first appeared explicitly in a work of Bourgain [26]. The currently best bound $L_K \leq C\sqrt[4]{n}$, which is due to Klartag [52], improves by a logarithmic factor the previous bound of Bourgain [27]. While this problem is still open in its general form, the isotropic constant of several special classes of convex bodies is in fact known to be bounded. Examples include zonoids and duals of zonoids [10], unconditional convex bodies [26, 56] and unit balls of Schatten classes [56]. Against this background, Klartag and Kozma [54] started to investigate the isotropic constant of random convex sets, as it is known since the groundbreaking work of Gluskin on the Banach-Mazur compactum [42] that random constructions often display some kind of extremal behaviour. Their ideas were taken up by Alonso-Gutiérrez [3], Alonso-Gutiérrez, Litvak and Tomczak-Jaegermann [5], Dafnis, Giannopoulos and Guédon [33] and Hörrmann, Prochno and Thäle [48] to prove boundedness of the isotropic constant for several classes of random polytopes with probability tending to 1, as the space dimension tends to infinity. An entirely different approach was used by Hörrmann, Hug, Reitzner and Thäle [47] for zero cells of a class of Poisson hyperplane tessellations.

This chapter acts as a natural continuation of [3] and [48], where random polytopes generated by random points on ℓ_p-spheres have been investigated. Here we take a more general point of view and consider random convex hulls whose points are distributed according to the cone (probability) measure on a convex surface, i.e. on the boundary of an arbitrary (isotropic) convex body $K \subset \mathbb{R}^n$. This chapter can also be regarded as a complement to [5, 33], where the random points were selected uniformly at random from the interior of K. More precisely, we prove

(i) that the isotropic constant L_{K_N} of a random polytope generated by $n < N < e^{\sqrt{n}}$

independent random points on the boundary of an isotropic convex body $K \subset \mathbb{R}^n$ satisfies

$$L_{K_N} \leq C\sqrt{\log \frac{2N}{n}}$$

with probability at least $1 - c_1 e^{-c_2 n} - e^{-c_3\sqrt{N}}$ for absolute constants $C, c_1, c_2, c_3 \in (0, \infty)$;

(ii) that if K is in addition symmetric with respect to all coordinate hyperplanes (i.e. if K is unconditional), we even have that

$$L_{K_N} \leq C$$

with probability bounded below by $1 - c_1 e^{-c_2 n}$ for all $N > n$.

The result (i) for general K resembles the so-far best known upper bound for the isotropic constant of random convex hulls in [5], where the generating points were selected with respect to the uniform distribution on K. Similarly, our result (ii) is the analogue to the main finding in [33], where boundedness of the isotropic constant of random convex hulls was obtained in the unconditional case. However, we emphasize that as in [3, 48] our bounds cannot be concluded from those in the existing literature, since the cone probability measure on the boundary of an isotropic convex body is not log-concave. To study the isotropic constant of random polytopes for which the generating measure is not log-concave was in fact the main source of motivation for this work and its predecessors [3, 48].

Chapter 6 is based on the paper: J. Prochno, C. Thäle and N. Turchi "The isotropic constant of random polytopes with vertices on convex surfaces" to appear in *Journal of Complexity* (2019+).

Chapter 2

Preliminaries

In this chapter we introduce the notation, the mathematical tools from geometry and probability theory, and some preparatory general results that we need in multiple parts of the rest of text.

2.1 General notation

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers. For $d \in \mathbb{N}$, we work in the d-dimensional Euclidean space $\mathbb{R}^d$ with standard inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|_2$. We uIe' sually indicate by $e_1, \ldots, e_d$ the standard orthonormal base of $\mathbb{R}^d$.

More generally, for $p \in [1, \infty]$ we introduce the p-norm of $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ by putting

$$\|x\|_p := \begin{cases} \Big(\sum_{i=1}^{d} |x_i|^p\Big)^{1/p} & \text{if } p < \infty, \\ \max\{|x_1|, \ldots, |x_d|\} & \text{if } p = \infty. \end{cases}$$

By $\mathbb{B}_p^d$ we denote the unit ball in $\mathbb{R}^d$ with respect to the p-norm and we let $\mathbb{S}_p^{d-1}$ denote its boundary. $\mathbb{B}_p^d$ and $\mathbb{S}_p^{d-1}$ are usually referred to as ℓ_p-ball and ℓ_p-sphere, respectively. For the special case $p = 2$, we may write $\|\cdot\|$, $\mathbb{B}^d$ and $\mathbb{S}^{d-1}$ instead of $\|\cdot\|_2$, $\mathbb{B}_2^d$ and $\mathbb{S}_2^{d-1}$, respectively.

For any set A, we indicate by $\mathbf{1}_A$ the indicator function of A, i.e.

$$\mathbf{1}_A(x) = \begin{cases} 0 & \text{if } x \notin A, \\ 1 & \text{if } x \in A. \end{cases}$$

With a slight abuse of notation, given an event E, we may indicate

$$\mathbf{1}\{E\} = \begin{cases} 0 & \text{if } E \text{ does not occur,} \\ 1 & \text{if } E \text{ occurs.} \end{cases}$$

For example, for the event "$x \in A$", the notation $\mathbf{1}\{x \in A\}$ means $\mathbf{1}_A(x)$. The cardinality of A is then defined as

$$\#A = |A| := \sum_{x \in A} \mathbf{1}_A(x),$$

i.e. the number of its elements.

For a set $A \subseteq \mathbb{R}^d$ we indicate by ∂A its boundary and by int$A$ its interior.

If $A \subseteq \mathbb{R}^d$ is also Lebesgue measurable, we denote by $\mathrm{vol}_d(A)$ its Lebesgue measure, i.e.

$$\mathrm{vol}_d(A) = \int_{\mathbb{R}^d} \mathbf{1}_A(x)\, \mathrm{d}x.$$

When more clarity of notation is required, we may refer to the Lebesgue measure as λ instead of $\mathrm{d}x$. We will write $A \in \mathcal{B}(\mathbb{R}^d)$ to indicate that A is a Borel set of $\mathbb{R}^d$.

Given sets $A \subset \mathbb{R}^d$ and $I \subset [0, \infty)$, we define the set $IA \subset \mathbb{R}^d$ as

$$IA := \big\{rx \in \mathbb{R}^n : r \in I, x \in A\big\}.$$

When $I = \{r\}, r \in [0, \infty)$ we also write rA instead of $\{r\}A$. Moreover, conv(A) will denote the convex hull of A.

Given two sequences of numbers positive real numbers $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we will use the notation $a_n \ll b_n$ for $a_n = o(b_n)$, meaning that $a_n/b_n \to 0$, as $n \to \infty$. Analogously, we will use $a_n \gg b_n$ meaning $a_n/b_n \to +\infty$, as $n \to \infty$. Furthermore, we write $a_n \sim b_n$, if $a_n/b_n \to 1$, as $n \to \infty$. We also write $a_n \lesssim b_n$ for $a_n = O(b_n)$, i.e. if there exists a constant $C \in (0, +\infty)$ such that $a_n \leq Cb_n$ for all n large enough. Whenever $b_n \lesssim a_n \lesssim b_n$, we may write $b_n = \Theta(a_n)$.

Finally, we denote the set $\{1, \ldots, n\}$ by $[n]$.

2.2 Basic definitions of probability

Fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Given a random vector on it, $X\colon \Omega \to \mathbb{R}^d$, the *expectation* of X, is defined whenever $X \in \mathrm{L}^1(\mathbf{P})$ as

$$\mathbf{E}\, X \coloneqq \int_\Omega X(\omega)\, \mathrm{d}\mathbf{P}(\omega).$$

The probability measure $\mu \coloneqq \mathbf{P} \circ X^{-1}$ is referred to as the *probability distribution* of X, or law of X, and it holds that

$$\mathbf{E}\, X = \int_{\mathbb{R}^d} x\, \mathrm{d}\mu(x).$$

We say that X is centred whenever $\mathbf{E}\, X = 0$. The *variance* of X is then defined as

$$\mathbf{Var}\, X \coloneqq \mathbf{E}\, \|X - \mathbf{E}\, X\|_2^2.$$

The *comulative distribution function* of a real random variable X is defined as

$$F_X(x) \coloneqq \mathbf{P}(X \leq x), \quad x \in \mathbb{R}.$$

We say that X is a continuous random variable whenever its law μ is absolutely continuous with respect to the Lebesgue measure λ, is which case we call *probability density*, or just density, the Radon-Nikodym derivative $f \geq 0$ of μ with respect to λ, i.e. $\mathrm{d}\mu = f\, \mathrm{d}\lambda$, in which case

$$\mathbf{E}\, g(X) = \int_{\mathbb{R}^d} g(x) f(x)\, \mathrm{d}x.$$

We say that a random vector has a *standard Gaussian distribution*, or standard normal distribution, if it has density

$$f(x) = \frac{1}{(2\pi)^{d/2}} \exp\Big(-\frac{\|x\|_2^2}{2}\Big), \quad x \in \mathbb{R}^d.$$

A sequence of real random variables $(X_n)_{n\in\mathbb{N}}$ is said to *converge in distribution* to the real random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

for any x at which F_X is continuous. Analogously, a sequence of random vectors $(X_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ converges in distribution to the random vector X if

$$\lim_{n\to\infty} \mathbf{P}(X_n \in A) = \mathbf{P}(X \in A), \tag{2.1}$$

for any Borel set A such that $\mathbf{P}(\partial A) = 0$.

2.3 Notions of convex and integral geometry

For every $A \subseteq \mathbb{R}^d$, we indicate with $\text{conv}(A)$ the *convex hull* of A in $\mathbb{R}^d$, meaning the smallest convex set $C \subseteq \mathbb{R}^d$, such that $A \subseteq C$. Here, the term "smallest" refer to the property of being the minimal set C, with respect to partial order given by the inclusion of sets in $\mathbb{R}^d$, satisfying the property that $A \subseteq C$. It can be shown that such minimal set is, in fact, unique.

Equivalently, $\text{conv}(A)$ is the intersection of all the convex sets containing A, namely:

$$\text{conv}(A) = \bigcap \{C \subseteq \mathbb{R}^d : A \subseteq C,\, C \text{ is convex}\}.$$

Whenever $A = \{x_1, \dots, x_d\} \subset \mathbb{R}^d$, we will also write $\text{conv}(x_1, \dots, x_d)$ and $[x_1, \dots, x_d]$, meaning $\text{conv}(\{x_1, \dots, x_d\})$. In such a case, A is called a *convex polytope*, or simply a *polytope*.

We say that $K \subseteq \mathbb{R}^d$ is a *convex body* if it is a compact convex set with non empty interior $\text{int}K$, and we indicate it's boundary by ∂K, i.e. $\partial K = K \setminus \text{int}K$. Note that a convex set of $\mathbb{R}^d$ is measurable, hence $\text{vol}_d(K)$ is always well defined without further assumptions.

The support function of K is defined by

$$h_K(u) = \sup\{\langle x, u\rangle : x \in K\}, \quad u \in \mathbb{S}^{d-1},$$

and it uniquely characterizes K in $\mathbb{R}^d$.

Let $u \in \mathbb{R}^d$ and $h \in \mathbb{R}$. We denote by $H(u, h)$ the hyperplane $\{x \in \mathbb{R}^d : \langle x, u\rangle = h\}$. The corresponding halfspaces $\{x \in \mathbb{R}^d : \langle x, u\rangle \geq h\}$ and $\{x \in \mathbb{R}^d : \langle x, u\rangle \leq h\}$ are denoted by $H^+(u, h)$ and $H^-(u, h)$, respectively.

There exists a norm associated to any symmetric convex body K, called the Minkowski functional of K. It is defined for every $x \in \mathbb{R}^d$ as

$$\|x\|_K \coloneqq \inf\{r > 0 : x \in rK\}.$$

Note, in particular, that $\|x\|_K = 1$ if and only if $x \in \partial K$. In the case where K is an an ℓ_p-ball it is possible to compute that

$$\|x\|_{\mathbb{B}_p^d} = \|x\|_p$$

for every $x \in \mathbb{R}^d$. However, in general such norm does not admit a close expression in the coordinates of x.

Let P be a polytope and H be an hyperplane of $\mathbb{R}^d$. If $F \coloneqq P \cap H \neq \emptyset$, we say that F is a *face* of P if either $P \subseteq H^+$ or $P \subseteq H^-$. If the affine hull of F is i-dimensional, $i \in \{0, \dots, d-1\}$, then we say that F is a i-dimensional face of P. In particular 0-dimensional faces are called *vertices*, 1-dimensional faces are *edges* and $(d-1)$-dimensional faces are called *facets*.

The information about the amount of faces that a polytope $P \subseteq \mathbb{R}^d$ possesses is encoded in a d-dimensional vector called the *f-vector* of P, namely,

$$f(P) = \big(f_0(P), \dots, f_{d-1}(P)\big),$$

where, for any $\in \{0, \dots, d-1\}$,

$$f_i(P) \coloneqq \#\{F \subseteq \mathbb{R}^d : F \text{ is a } i\text{-dimensional face of } P\}.$$

We indicate by $\mathcal{K}_2^+$ the set of convex bodies whose boundary is twice differentiable and has positive Gaussian curvature everywhere.

For a set $K \subset \mathbb{R}^d$, we shall write $\mathcal{H}_K^q$ for the q-dimensional Hausdorff measure on K.

We will use the notation $\Delta_{d-1}(x_1, \dots, x_d)$ to indicate the $(d-1)$-dimensional volume of the convex hull of d points $x_1, \dots, x_d$.

The volume of $\mathbb{B}^d$ is denoted by κ_d and it holds

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma\big(1 + \frac{d}{2}\big)}.$$

Analogously, we indicate by ω_d the Hausdorff measure of $\mathbb{S}^{d-1}$ and it holds $\omega_d = d\kappa_d$. We indicate with $\sphericalangle(u, v)$ the angle between two vectors $u, v \in \mathbb{R}^d$. For a linear subspace V of $\mathbb{R}^d$, we define $\sphericalangle(u, V) \coloneqq \inf\{\sphericalangle(u, v) : v \in V\}$. Given a subset $U \subseteq \mathbb{R}^d$, its projection onto $\mathbb{R}^{d-1}$ is denoted by $\mathrm{proj}_{\mathbb{R}^{d-1}} U = \{x \in \mathbb{R}^{d-1} : (x, y) \in U \text{ for some } y \in \mathbb{R}\}$.

2.3.1 Intrinsic volumes.

Let $\ell \in \{0, \ldots, d\}$, we denote by $G(d, \ell)$ the Grassmannian of all ℓ-dimensional linear subspaces of $\mathbb{R}^d$, which is supplied with the unique Haar probability measure ν_ℓ, see [74].

Let $K \subset \mathbb{R}^d$ be a convex body. For $L \in G(d, \ell)$, we write $K|L$ to indicate the the orthogonal projection of K onto L. Then $K|L$ is an ℓ-dimensional convex set, so it remains well-defined its Lebsegue measure $\mathrm{vol}_\ell(K|L)$. In view of this, we define for any $\ell \in \{0, \ldots, d\}$, the *$\ell$-th intrinsic volume* of K as a suitable normalization of the average ℓ-dimensional volume of its orthogonal projections, more specifically

$$V_\ell(K) := \binom{d}{\ell} \frac{\kappa_d}{\kappa_\ell \kappa_{d-\ell}} \int_{G(d,\ell)} \mathrm{vol}_\ell(K|L)\, \nu_\ell(\mathrm{d}L)\,, \tag{2.2}$$

which is known as Kubota's formula, see [76, Equations (6.11) and (5.5)].

The intrinsic volume can be equivalently defined as the non-negative coefficients of the polynomial in t that arises from Steiner's formula, (see e.g. [74, Equation (4.2.27)]), namely

$$\mathrm{vol}_d(K + t\mathbb{B}^d) = \sum_{\ell=0}^{d} t^{d-\ell} \kappa_{d-\ell} V_\ell(K).$$

In particular, $V_d(K)$ is the ordinary volume $\mathrm{vol}_d(K)$, $V_{d-1}(K)$ is half of the surface area, $V_1(K)$ is a constant multiple of the mean width and $V_0(K)$ is the Euler-characteristic of K.

2.3.2 Isotropic convex bodies

A convex body $K \subset \mathbb{R}^d$ is called *isotropic* whenever $\mathrm{vol}_d(K) = 1$, it's centred, i.e. for any $\theta \in \mathbb{S}^{n-1}$

$$\int_K \langle x, \theta \rangle \,\mathrm{d}x = 0$$

and there exists a constant L_K^2 such that

$$\int_K \langle x, \theta \rangle^2 \,\mathrm{d}x = L_K^2$$

for all $\theta \in \mathbb{S}^{d-1}$.

L_K is then called the *isotropic constant* of K. Note that an isotropic convex body satisfies

$$\int_K \|x\|_2^2 \,\mathrm{d}x = \sum_{i=1}^{d} \int_K \langle x, e_i\rangle^2 \,\mathrm{d}x = dL_K^2.$$

Let GL(d) indicate the for the group of invertible linear transformations on $\mathbb{R}^d$ (see [8, Definition 10.1.6].). It is a well know fact (see [28, pag. 73])that every centred convex body admits an isotropic linear image, i.e. for any K centered, there exists $T \in \mathrm{GL}(d)$ such that TK is an isotropic convex body. Moreover, for any convex body K, the quantity

$$\min\left\{ \frac{1}{d \operatorname{vol}_d TK^{1+2/d}} \int_{z+TK} \|x\|_2^2 \,\mathrm{d}x : z \in \mathbb{R}^d, T \in \mathrm{GL}(d) \right\}, \tag{2.3}$$

only depends on the affine class of K, so we can define the isotropic constant L_K^2 of every convex body as the term in Equation (2.3) (see [28, Definition 2.3.6]).

This means that the study of the isotropic constant of convex bodies can be restricted to the class of isotropic convex bodies. A class of convex bodies of particular interest is constituted by the so called *unconditional* isotropic convex bodies. They are defined as those convex bodies of unitary volume, which are invariant under reflection with respect to every coordinate hyperplane. Geometrically, this also means that if K contains x, then every $y \in K$ as long as $y_i \in [-x_i, x_i]$ for every $i \in \{1, \ldots, d\}$. Normalized ℓ^p-balls are therefore an example of unconditional isotropic convex body.

The hyperplane conjecture states that there exists an absolute constant $c > 0$ such that for every dimension $d \in \mathbb{N}$ and every centered convex body $K \subset \mathbb{R}^d$ with $\mathrm{vol}_d(K) = 1$, there exists $\theta \in \mathbb{S}^{d-1}$ such that

$$\mathrm{vol}_{d-1}(K \cap \theta^\perp) \geq c,$$

where $\theta^\perp$ indicates the hyperplane orthogonal to θ and passing through the origin.

Since there exist absolute constants $c_1, c_2 > 0$ such that for every isotropic convex body $K \subset \mathbb{R}^d$ and for any $\theta \in \mathbb{S}^{d-1}$ (see [**c**]o BGVV),

$$\frac{c_1}{L_K} \leq \mathrm{vol}_{d-1}(K \cap \theta^\perp) \leq \frac{c_2}{L_K},$$

the hyperplane conjecture is equivalent to conjecturing that there exists $C > 0$ such that

$$L_K \leq C,$$

for every convex body K in any dimension.

2.3.3 Isotropic log-concave probability measures

A probability measure μ on $\mathbb{R}^n$ is called *log-concave*, if for all compact subsets A, B of $\mathbb{R}^n$ and all $\lambda \in [0,1]$

$$\mu((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda}\mu(B)^{\lambda}.$$

μ is called *isotropic*, if its barycentre is at the origin, i.e.

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle \, \mathrm{d}\mu(x) = 0$$

holds for every $\theta \in \mathbb{S}^{n-1}$, and satisfies the isotropic condition, that is,

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 \, \mathrm{d}\mu(x) = 1$$

for all $\theta \in \mathbb{S}^{n-1}$.

Note that isotropic log-concave probability measures generalize the concept of uniform probability distribution on an isotropic convex body $K \subset \mathbb{R}^n$, for the probability measure $L_K^n \mathbf{1}_{L_K^{-1}K} \, \mathrm{d}x$ is isotropic log-concave.

An important property of isotropic log-concave probability measures is represented by the so-called thin-shell concentration property of isotropic log-concave probability measures. Answering a central question in asymptotic convex geometry (see [6]), Klartag [53, Theorem 1.4] proved that an isotropic log-concave measure is typically concentrated on a "thin spherical shell" around the Euclidean ball of radius $\sqrt{n}$. The statement reads as follows.

Theorem 2.1. *Let μ be an isotropic log-concave probability measure in $\mathbb{R}^n$. Then, for every $\varepsilon \in (0,1)$,*

$$\mu\big(\big\{x \in \mathbb{R}^n : \big|\|x\|_2 - \sqrt{n}\big| \geq \varepsilon\sqrt{n}\big\}\big) \leq C n^{-c\varepsilon^2}, \tag{2.4}$$

for some absolute constants $c, C > 0$.

Results of this type are closely linked to the long-standing thin shell conjecture. It asks, whether it's true that exists and absolute constants $C > 0$ such that

$$\mathbf{E}\big(\|X\|_2 - \sqrt{n}\big)^2 \leq C$$

for any random vector X distributed according to an isotropic and log-concave probability measure on $\mathbb{R}^n$. We refer to the monograph [28] for further information on the history of this problem, recent improvements of Theorem 2.1, as well as the general theory of isotropic log-concave probability measures.

2.4 Special functions

The Gamma function is defined as

$$\Gamma(x) := \int_0^\infty s^{x-1} e^{-s} \, \mathrm{d}s \qquad x > 0.$$

Its relevance is highlighted by the following characterization: the Gamma function is the unique function on the positive reals such that $\Gamma(1) = 1$, $\Gamma(x+1) = x\Gamma(x)$ and it is logarithmically convex, i.e. its logarithm is a convex function on the positive reals. Hence, the Gamma function coincides with a shift of the factorial function on the natural numbers, i.e. $\Gamma(n+1) = n!$ for any $n \in \mathbb{N}$.

The following inequality on the ratio of Gamma function will be of special interest, and, in the form stated here, is a particular case of Wendel's inequality (see e.g. eq. (7) in [86]), but written in a similar form already earlier in [7].

Lemma 2.2. *For every $x > 1$,*

$$\frac{1}{\sqrt{x}} < \frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})} < \frac{1}{\sqrt{x-1}}.$$

A direct consequence of the previous lemma is that $\Gamma(x+1/2) \sim \sqrt{x}\Gamma(x)$ as x tends to infinity.

Starting with the Gamma function, one can define the Beta function by means of the formula

$$\mathrm{B}(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x, y > 0.$$

Note that by definition, the Beta function is symmetric in its arguments. Moreover, it is well-known that it admits the following integral representation,

$$\mathrm{B}(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} \, \mathrm{d}s, \qquad a, b > 0.$$

Chapter 3

Intrinsic Volumes of Random Polytopes in Convex Bodies

In this chapter we discuss the fluctuations of the intrinsic volumes of random polytopes generated by independent random points, uniformly distributed inside a smooth convex body, or on its surface.

Fix a space dimension $d \geq 2$, let $n \geq d+1$. Let $K \subset \mathbb{R}^d$ be a prescribed convex body , which we assume to have a boundary ∂K which is twice differentiable and has positive Gaussian curvature everywhere. We summarize these conditions by writing $K \in \mathcal{K}^2_+$. We will establish limit theorems for the intrinsic volumes of random polytopes in K, according to two different models:

1. *uniform inside K:* We let $X_1, \dots, X_n$ be independent identically distributed random points, uniformly distributed in K, namely

$$\mathbf{P}(X_1 \in A) = \frac{\mathrm{vol}_d(A \cap K)}{\mathrm{vol}_d(K)}$$

for any $A \in \mathcal{B}(\mathbb{R}^d)$. Notice that from this definition we can then assume that $\mathrm{vol}_d(K) = 1$ without loss of generality, so we will from now on. We denote as P_n the random polytope that is the convex hull of these random points, namely

$$P_n \coloneqq \mathrm{conv}(X_1, \dots, X_n).$$

2. *uniform on the surface of K:* $X_1, \dots, X_n$ are independent identically distributed

random points, uniformly distributed on ∂K, namely

$$\mathbf{P}(X_1 \in B) = \frac{\mathcal{H}^{d-1}_{\partial K}(B)}{\mathcal{H}^{d-1}_{\partial K}(\partial K)}$$

for any $B \in \mathcal{B}(\partial K)$. With a slight abuse of notation we will indicate in this chapter $\mathcal{H}^{d-1}(\cdot)$ the aforementioned probability measure. We denote by K_n the random polytope that arises as the convex hull of the aforementioned random points, namely

$$K_n \coloneqq \operatorname{conv}(X_1, \dots, X_n).$$

Note that in this model, unlike in the previous one, every random point is a vertex of the polytope, due to the curvature of the boundary.

For $\ell \in \{1, \dots, d\}$, we indicate by $V_\ell(P_n)$ and $V_\ell(K_n)$ the ℓ-th intrinsic volume of P_n and K_n, respectively.

The purpose of this chapter is to prove central limit theorems for $V_\ell(P_n)$, as $n \to \infty$ and upper variance bounds, law of large numbers and central limit theorems for $V_\ell(K_n)$, filling this way some of the gaps in the existing literature.

3.1 Background material

3.1.1 Floating bodies

We recall the concept of the floating body, that was introduced independently in [13] and [79].

We define the function $v : K \to \mathbb{R}$ by

$$v(x) \coloneqq \min\{\operatorname{vol}_d(K \cap H) : H \text{ is a half space in } \mathbb{R}^d \text{ containing } x\}.$$

Then, the set

$$K_{(t)} = K(v \geq t) \coloneqq \{x \in K : v(x) \geq t\}$$

is called the *floating body* of K with parameter $t > 0$. The *wet part* of K is defined by

$$K(t) = K(v \leq t) \coloneqq \{x \in K : v(x) \leq t\}.$$

In a similar way, we define the function $s : K \to \mathbb{R}$ by

$$s(x) := \min\{\mathcal{H}^{d-1}(\partial K \cap H) : H \text{ is a half space in } \mathbb{R}^d \text{ containing } x\}.$$

The *surface body* of K with parameter $t > 0$ is defined by

$$K(s \geq t) := \{x \in K : s(x) \geq t\}.$$

Analogously, we set

$$K(s \leq t) := \{x \in K : s(x) \leq t\}.$$

We rephrase a result of Bárány and Dalla [16], which has also been proved by Vu [84] using different techniques, see also Lemma 2.2 in [66]. Note that in the following statement, in fact, smoothness of the boundary is not needed.

Lemma 3.1. *For any $\beta \in (0, \infty)$, there exists a constant $c = c(\beta, n) \in (0, \infty)$ only depending on β and on n such that the probability of the event that P_n does not contain the $c\frac{\log n}{n}$-floating body is at most $n^{-\beta}$, whenever n is sufficiently large.*

The concept of the surface body is convenient in view of Lemma 3.2, which clarifies its connection with the random polytope K_n.

Lemma 3.2. [72, Lemma 4.2] *For all $\alpha \in (0, \infty)$, there exists a constant $c_\alpha \in (0, \infty)$ only depending on α such that*

$$\mathbf{P}(K(s \geq \tau_n) \not\subseteq K_n) \leq n^{-\alpha},$$

where

$$\tau_n := c_\alpha \frac{\log n}{n}.$$

3.1.2 Further geometric tools

A nice feature of the smoothness of K, is provided from the fact that for every point $x \in \partial K$, there exists a paraboloid Q_x, given by a quadratic form b_{Q_x}, osculating at x. The following precise description of the local behaviour of the boundary of a convex body $K \in \mathcal{K}^2_+$ is due to Reitzner [67].

Lemma 3.3. [67, Lemma 6] *Let $K \in \mathcal{K}^2_+$ and choose $\delta > 0$ sufficiently small. Then, there exists a $\lambda > 0$, only depending on δ and K, such that for each $x \in \partial K$ the following holds. Identify the hyperplane tangent to K at x with $\mathbb{R}^{d-1}$ and x with the* *origin. The λ-neighbourhood U^λ of x in ∂K defined by $\operatorname{proj}_{\mathbb{R}^{d-1}} U^\lambda = \lambda\mathbb{B}_2^{d-1}$ can be represented by a convex function $f^{(x)}(y) \in \mathcal{C}^2$, i.e. $(y, f^{(x)}(y)) \in \partial K$ for $y \in \lambda\mathbb{B}_2^{d-1}$. Denote by $f_{ij}^{(x)}(0)$ the second partial derivatives of $f^{(x)}$ at the origin. Then,*

$$b_{Q_x}(y) = \frac{1}{2}\sum_{i,j} f_{ij}^{(x)}(0) y_i y_j$$

and it holds that

$$(1+\delta)^{-1} b_{Q_x}(y) \leq f^{(x)}(y) \leq (1+\delta)\, b_{Q_x}(y)$$

for $y \in \lambda\mathbb{B}_2^{d-1}$.

Moreover, whenever $K \in \mathcal{K}_+^2$, there exists a unique unit outward normal u_x for each $x \in \partial K$. The intersection of K with $H^+(u_x, h_K(u_x) - h)$ is denoted by $C^K(x,h)$. We call $C^K(x,h)$ a cap of K at x of height h. A cap C^K is called an ε-cap if $\operatorname{vol}_d(C^K) = \varepsilon$. Analogously, a cap C^K with $\mathcal{H}^{d-1}(C^K \cap \partial K) = \varepsilon$ is called an ε-boundary cap. For the cap $C^{\mathbb{B}^d}(x,h)$, the central angle is defined as

$$\alpha(h) := \max\{\sphericalangle(x,y) : y \in C^{\mathbb{B}^d}(x,h)\}.$$

In the next Lemma we state two well-known relations regarding ε-caps and ε-boundary caps.

Lemma 3.4. [72, Lemma 6.2] *For a given $K \in \mathcal{K}_+^2$, there exist constants $\varepsilon_0, c_1, c_2 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we have that for any ε-cap C^K of K,*

$$c_1^{-1}\varepsilon^{(d-1)/(d+1)} \leq \mathcal{H}^{d-1}(C^K \cap \partial K) \leq c_1 \varepsilon^{(d-1)/(d+1)}$$

and for any ε-boundary cap $\widetilde{C}^K$ of K,

$$c_2^{-1}\varepsilon^{(d+1)/(d-1)} \leq \operatorname{vol}_d(\widetilde{C}^K) \leq c_2 \varepsilon^{(d+1)/(d-1)}.$$

This result will be used to relate Lemma 3.2 in terms of the classic floating body. For the next geometrical Lemma we assume that ε is sufficiently small.

Lemma 3.5. [83, Lemma 6.2] *Let x be a point on the boundary of K and $D(x,\varepsilon)$ the set of all points on the boundary which are of distance at most ε to x. Then, the convex hull of $D(x,\varepsilon)$ has volume at most $c_3\varepsilon^{d+1}$, where $c_3 > 0$ is a constant.*

The following result is known as the economic cap covering theorem, see [11, 13].

Proposition 3.6. [11, Theorem 4] *Assume that K is a convex body with unit volume and let $0 < t < t_0 = (2d)^{-2d}$. Then, there are caps $C_1, \dots, C_m$ and pairwise disjoint convex sets $C'_1, \dots, C'_m$ such that $C'_i \subset C_i$ for each i, and*

1. $\bigcup_{i=1}^m C'_i \subset K(t) \subset \bigcup_{i=1}^m C_i$,
2. $\mathrm{vol}_d(C'_i) \gtrsim t$ *and* $\mathrm{vol}_d(C_i) \lesssim t$ *for each* i,
3. *for each cap C with $C \cap K(v > t) = \emptyset$, there is a C_i containing C.*

We conclude this section with a statement from [11, Lemma 1], about the measure of the set of linear subspaces of $\mathbb{R}^d$ that form a small angle with a fixed vector, which will be useful later. See also [18, Lemma 10].

Lemma 3.7. *For fixed $z \in \mathbb{S}^{d-1}$ and small $a > 0$,*

$$\nu_\ell(\{L \in G(d,\ell) : \sphericalangle(z, L) \le a\}) = \Theta\big(a^{d-\ell}\big), \quad \ell \in \{1, \dots, d\}.$$

3.1.3 Bounds for normal approximation

Let X and Y be two random variables with cumulative distribution functions $F_X(u) = \mathbf{P}(X \le u)$ and $F_Y(u) = \mathbf{P}(Y \le u)$, respectively. Note that X and Y need not to be defined on a common probability space. Thus, we interpret $\mathbf{P}$ on the appropriate probability space in each case.

We define the Wasserstein distance between X and Y as

$$\mathrm{d}_W(X, Y) := \sup_{h \in \mathrm{Lip}_1} \big|\mathbf{E}\, h(X) - \mathbf{E}\, h(Y)\big| \tag{3.1}$$

where the supremum is running over all Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant less or equal than 1.

The Kolmogorov distance between the random variables X and Y is defined by

$$d_K(X, Y) = \sup_{u \in \mathbb{R}} |F_X(u) - F_Y(u)|.$$

A nice feature of the Wassertein and the Kolmogorov distance is that they metrize the convergence in distribution, i.e. given a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ and another random variable Y such that $\lim_{n\to\infty} d_W(X_n, Y) = 0$ or $\lim_{n\to\infty} d_K(X_n, Y) = 0$, then $(X_n)_{n\in\mathbb{N}}$ converges in distribution to Y.

As far as symmetric statistics of independent and identically distributed random variables are concerned, like the intrinsic volumes $V_\ell(\mathrm{conv}(X_1,\dots,X_n))$ are, quantitative bounds for normal approximation in terms of the aforementioned distances have been deeply investigated, see e.g. [90]. More recently, developments of Stein's theory were deployed to provide new bounds using the so called *difference operators*. We now introduce one of such machineries, originating from [32] and refined in [57].

Let S be a Polish space. Consider a function $f\colon \cup_{k=1}^n S^k \to \mathbb{R}$ that acts on the point configurations of at most $n \in \mathbb{N}$ points of S. Let f be measurable and symmetric, i.e. invariant under permutations of the arguments. In the setting of this chapter, S will be either a smooth convex body or its boundary, while f is an intrinsic volume of the convex hull of its arguments. Given a point $x = (x_1,\dots,x_h) \in \cup_{k=1}^n S^k$, we indicate with x^i the vector obtained from x by removing its i-th coordinate, namely $x^i \coloneqq (x_1,\dots,x_{i-1},x_{i+1},\dots,x_h)$. Analogously, we define $x^{ij} \coloneqq (x_1,\dots,x_{i-1},x_{i+1},\dots,x_{j-1},x_{j+1},\dots,x_h)$.

We now define the first- and second-order difference operators, applied to f, as

$$D_i f(x) \coloneqq f(x) - f(x^i) \quad \text{and} \quad D_{i,j} f(x) \coloneqq f(x) - f(x^i) - f(x^j) + f(x^{ij}),$$

respectively. We indicate with $X = (X_1,\dots,X_n)$ a random vector of elements of S. Let X' and $\tilde{X}$ be independent copies of X. A vector $Z = (Z_1,\dots,Z_n)$ is called a recombination of $\{X, X', \tilde{X}\}$, whenever $Z_i \in \{X_i, X_i', \tilde{X}_i\}$ for every $i \in \{1,\dots,n\}$. For a subset $A \subseteq \{1,\dots,n\}$ of the index set, we write $X^A = (X_1^A,\dots,X_n^A)$ with

$$X_i^A \coloneqq \begin{cases} X_i & : i \notin A, \\ X_i' & : i \in A. \end{cases}$$

In order to rephrase the normal approximation bound from [57], it is convenient to define the following quantities, namely,

$$\begin{aligned}
\gamma_1 &\coloneqq \sup_{(Y,Y',Z,Z')} \mathbf{E}\big[\mathbf{1}\{D_{1,2}f(Y) \neq 0\}\,\mathbf{1}\{D_{1,3}f(Y') \neq 0\}\, D_2 f(Z)^2\, D_3 f(Z')^2\big]\,, \\
\gamma_2 &\coloneqq \sup_{(Y,Z,Z')} \mathbf{E}\big[\mathbf{1}\{D_{1,2}f(Y) \neq 0\}\, D_1 f(Z)^2\, D_2 f(Z')^2\big]\,, \\
\gamma_3 &\coloneqq \mathbf{E}\big[|D_1 f(X)|^4\big]\,, \\
\gamma_4 &\coloneqq \mathbf{E}\big[|D_1 f(X)|^3\big]\,, \\
\gamma_5 &\coloneqq \sup_{A \subseteq \{1,\dots,n\}} \mathbf{E}\big[|f(X) D_1 f(X^A)^3|\big]\,,
\end{aligned}$$

where the suprema in the definitions of γ_1 and γ_2 run over all combinations of vectors (Y, Y', Z, Z') or (Y, Z, Z') that are recombinations of $\{X, X', \tilde{X}\}$.

Proposition 3.8. [57, Theorem 5.1] *Let* $W := f(X_1, \ldots, X_n)$ *and assume that* $\mathbf{E}\, W = 0$ *and* $0 < \mathbf{E}\, W^2 < \infty$. *Moreover, let* N *be a standard Gaussian random variable. Then, the following bound for the normal approximations hold:*

$$\mathrm{d}_W\left(\frac{W}{\sqrt{\mathbf{Var}\, W}}, N\right) \lesssim \frac{\sqrt{n}}{\mathbf{Var}\ W}\left(\sqrt{n^2\gamma_1} + \sqrt{n\gamma_2} + \sqrt{\gamma_3}\right) + \frac{n}{(\mathbf{Var}\ W)^{\frac{3}{2}}}\gamma_4;$$
$$d_K\left(\frac{W}{\sqrt{\mathbf{Var}\, W}}, N\right) \lesssim \frac{\sqrt{n}}{\mathbf{Var}\, W}(\sqrt{n^2\gamma_1} + \sqrt{n\gamma_2} + \sqrt{\gamma_3}) + \frac{n}{(\mathbf{Var}\, W)^{\frac{3}{2}}}\gamma_4 + \frac{n}{(\mathbf{Var}\, W)^2}\gamma_5.$$

3.2 Random points inside the convex body

We prove the following theorem on the intrinsic volume of P_n.

Theorem 3.9. *Let* $K \subset \mathbb{R}^n$ *be a convex body with twice differentiable boundary and strictly positive Gaussian curvature everywhere. Then, for all* $\ell \in \{1, \ldots, n\}$, *one has that* $(V_\ell(P_n) - \mathbf{E}\, V_\ell(P_n))/\sqrt{\mathbf{Var}\, V_\ell(P_n)}$ *converges in distribution to a standard Gaussian random variable* N, *as* $n \to \infty$. *More precisely,*

$$\mathrm{d}_W\left(\frac{V_\ell(P_n) - \mathbf{E}\, V_\ell(P_n)}{\sqrt{\mathbf{Var}\, V_\ell(P_n)}}, N\right) \lesssim n^{-\frac{1}{2}+\frac{1}{d+1}} (\log n)^{3+\frac{2}{d+1}}\,.$$

Note that the rate of convergence in Theorem 3.9 does not depend on ℓ.

In the proof of our result we will make use of the following lower and upper variance bounds, proven by Bárány, Fodor and Vígh [11], namely,

$$n^{-\frac{d+3}{d+1}} \lesssim \mathbf{Var}\, V_\ell(P_n) \lesssim n^{-\frac{d+3}{d+1}} \tag{3.2}$$

for all $\ell \in \{1, \ldots, n\}$.

According to Lemma 3.1, we see that for any $\beta \in (0, \infty)$ there exists a constant $c = c(\beta, d) \in (0, \infty)$ such that the random polytope $[X_2, \ldots, X_n]$ contains the floating body $K_{(c \log n/n)}$ with high probability. More precisely, denoting the latter event by B_1, it holds that for sufficiently large n,

$$\mathbf{P}(B_1^c) \leq (n-1)^{-\beta} \leq c_1 n^{-\beta}\,, \tag{3.3}$$

where $c_1 \in (0, \infty)$ is a constant independent of n. Note that we choose β large enough ($\beta = 5$ will be sufficient for all our purposes).

Next, we let Y, Y', Z, Z' be recombinations of our random vector $X = (X_1, \dots, X_n)$ and denote by B_2 the event that $\bigcap_{W\in\{Y,Y',Z,Z'\}}[W_4, \dots, W_n]$ contains $K_{(c\log n/n)}$. By the union bound it follows that the probability of B_2^c is also small:

$$\mathbf{P}(B_2^c) \le c_2 n^{-\beta}\,, \tag{3.4}$$

where $c_2 \in (0,\infty)$ is again a constant independent of n.

Remark 1. Let us point out that the result concerning $V_\ell(P_n)$, is not totally new. Central limit theorems for general $\ell \in \{1,\dots,d\}$ were known for a long time only for the Poisson setting in the the special case that K is the n-dimensional Euclidean unit ball, see the paper of Calka, Schreiber and Yukich [30]. We also refer to the paper of Schreiber [77] for the case $\ell = 1$. Only very recently (in parallel and independently of us) Lachièze-Rey, Schulte and Yukich [58] gave a proof for the general case by embedding the problem into the theory of so-called stabilizing functionals.

Proof of Theorem 3.9. Assume without loss of generality that K has volume one. The idea of the proof is to apply the normal approximation bound in Proposition 3.8 to the random variables

$$W = f(X_1,\dots,X_n) \coloneqq V_\ell([X_1,\dots,X_n]) - \mathbf{E}\,V_\ell(P_n).$$

To this end, we need to control, in particular, the first- and second-order difference operators $D_iW = D_iV_\ell(P_n)$ and $D_{i_1,i_2}W = D_{i_1,i_2}V_\ell(P_n)$ for $i, i_1, i_2 \in \{1,\dots,n\}$.

Conditioned on the event B_1, we use Kubota formula to estimate the first-order difference operator applied to the intrinsic volume functional $V_\ell(P_n)$ as follows:

$$\begin{aligned} D_1V_\ell(P_n) = \binom{d}{\ell}\frac{\kappa_n}{\kappa_\ell\kappa_{d-\ell}}\int_{G(d,\ell)} &\mathrm{vol}_\ell((P_n|L)\setminus([X_2,\dots,X_n]|L))\,\nu_\ell(\mathrm{d}L)\\ &\times \mathbf{1}\{X_1 \in K\setminus K_{(c\log n/n)}\}. \end{aligned} \tag{3.5}$$

For the sake of brevity we will indicate $[X_2,\dots,X_n]$ by P_{n-1}. On the event B_1 we first notice that $\mathrm{vol}_\ell((P_n|L)\setminus(P_{n-1}|L))$ is zero if $X_1 \in P_{n-1}$. So, we can restrict to the situation that $X_1 \in K \setminus P_{n-1}$, which conditioned on B_1 occurs with probability $\mathrm{vol}_d(K\setminus P_{n-1}) \lesssim \mathrm{vol}_d(K\setminus K_{(c\log n/n)}) \lesssim (\log n/n)^{\frac{2}{d+1}}$, cf. [14, Theorem 6.3].

Suppose now that the convex body K is the normalized Euclidean unit ball in $\mathbb{R}^n$. It is our aim to define a full-dimensional cap C such that $P_n \setminus P_{n-1}$ is contained in C. For this reason, we define z to be the closest point to X_1 on ∂K (we notice that z is uniquely determined if $K_{(c\log n/n)}$ is non-empty). We define the visible region of z is defined as

$$\mathrm{Vis}_z(n) \coloneqq \{x \in K \setminus K_{(c\log n/n)} : [x,z] \cap K_{(c\log n/n)} = \emptyset\}.$$

By definition of the floating body $K_{(c\log n/n)}$, the diameter of $\mathrm{Vis}_z(n)$ is equal to $c_3(\log n/n)^{\frac{1}{d+1}}$, where $c_3 \in (0,\infty)$ is a constant not depending on n. Let us denote by $D(z, c_3(\log n/n)^{\frac{1}{d+1}})$ the set of all points on the boundary of K which are of distance at most $c_3(\log n/n)^{\frac{1}{d+1}}$ to z. Then, it follows from [84, Lemma 6.2] that $C \coloneqq \mathrm{conv}(D(z, c_3(\log n/n)^{\frac{1}{d+1}}))$ has volume of order at most $\log n/n$. Moreover, C is in fact a spherical cap and the central angle of it is denoted by α. For a subspace $L \in G(d,\ell)$, one has that $(P_n|L) \setminus (P_{n-1}|L) \subseteq C|L$. The volume $\mathrm{vol}_\ell(C|L)$ of the projected cap $C|L$ is $\mathrm{vol}_\ell(C|L) \lesssim (\log n/n)^{\frac{\ell+1}{d+1}}$. Indeed, the height of $C|L$ keeps the order of the height of C, namely $(\log n/n)^{\frac{2}{d+1}}$, while the order of its base changes from $((\log n/n)^{\frac{1}{d+1}})^{d-1}$ to $((\log n/n)^{\frac{1}{d+1}})^{\ell-1}$, since L is a subspace of dimension ℓ. Note that, by construction of C, if $\sphericalangle(z,L)$, the angle between z and L, is too wide compared to α , then $C|L \subseteq P_{n-1}|L$, for sufficiently large n. In particular, $(P_n \setminus P_{n-1})|L \subseteq P_{n-1}|L$, which implies $P_n|L = P_{n-1}|L$. In fact, it is easily checked that the integrand in (3.5) can only be non-zero if $\sphericalangle(z,L) \lesssim \alpha$ (the constant can be taken to be 2 in the case of the ball). Therefore, we can restrict the integration in (3.5) to the set $\{L \in G(d,\ell) : \sphericalangle(z,L) \lesssim \alpha\}$. It is not difficult to verify that $\alpha \lesssim \mathrm{vol}_d(C)^{\frac{1}{d+1}}$, see also Equation (27) in [11].

Taken all together, this yields

$$D_1 V_\ell(P_n) \lesssim \left(\frac{\log n}{n}\right)^{\frac{\ell+1}{d+1}} \nu_\ell\Big(\Big\{L \in G(d,\ell) : \sphericalangle(z,L) \lesssim \mathrm{vol}_d(C)^{\frac{1}{d+1}}\Big\}\Big) \times \mathbf{1}\{X_1 \in K \setminus K_{(c\log n/n)}\}.$$

According to Lemma 3.7 and the fact that $\mathrm{vol}_d(C) \lesssim \log n/n$, it holds that

$$\nu_\ell\Big(\Big\{L \in G(d,\ell) : \sphericalangle(z,L) \lesssim \mathrm{vol}_d(C)^{\frac{1}{d+1}}\Big\}\Big) \lesssim \left(\frac{\log n}{n}\right)^{\frac{d-\ell}{d+1}},$$

which in turn implies

$$\begin{aligned} D_1 V_\ell(P_n) &\lesssim \left(\frac{\log n}{n}\right)^{\frac{\ell+1}{d+1}} \left(\frac{\log n}{n}\right)^{\frac{d-\ell}{d+1}} \mathbf{1}\{X_1 \in K \setminus K_{(c\log n/n)}\} \\ &= \frac{\log n}{n} \mathbf{1}\{X_1 \in K \setminus K_{(c\log n/n)}\}. \end{aligned} \tag{3.6}$$

To extend the argument for the general case, we argue as in [11, Section 6]. Namely, since K is compact, we can choose $\gamma \in (0,\infty)$ and $\Gamma \in (0,\infty)$ to be, respectively, the global lower and the global upper bound on the principal curvatures of ∂K. Remark 5 on page 126 of [74] ensures that under our assumptions on the smoothness of the convex body K all projected images of K also have a boundary with the same features as ∂K, and we choose γ and Γ such that they also bound from below and above the principal curvatures of each ℓ-dimensional projection of K. Since we can approximate ∂K locally with affine images of balls, the construction of the cap C above and the relations regarding its volume, its central angle and the subspaces L which ensure $C|L \subseteq P_{n-1}|L$ are not affected. Due to this, the relations

$$\mathrm{vol}_\ell(C|L) \lesssim (\log n/n)^{\frac{\ell+1}{d+1}}, \quad \alpha \lesssim \mathrm{vol}_d(C)^{\frac{1}{d+1}} \lesssim (\log n/n)^{\frac{1}{d+1}}$$

and

$$\sphericalangle(z, L) \lesssim \alpha$$

from the above argument still hold, but this time the implicit constants depend on γ and Γ. From here, the bound (3.6) can be obtained in the same way as for the ball.

Moreover, on the complement B_1^c of B_1, we use the trivial estimate $D_1 V_\ell(P_n) \le V_\ell(K)$ and thus conclude that

$$\begin{aligned}\mathbf{E}[(D_1V_\ell(P_n))^p] &= \mathbf{E}[(D_1V_\ell(P_n))^p\, \mathbf{1}_{B_1}] + \mathbf{E}[(D_1V_\ell(P_n))^p\, \mathbf{1}_{B_1^c}] \\ &\lesssim \left(\frac{\log n}{n}\right)^p \mathrm{vol}_d(K \setminus K_{(c\log n/n)}) \lesssim \left(\frac{\log n}{n}\right)^{p+\frac{2}{d+1}}\end{aligned}$$

for all $p \in \{1,2,3,4\}$, where we applied the probability estimate (3.3) in the second step, which ensures that the second term can be made very small for large n (the choice for p is motived by our applications below). As a consequence, we can already bound the terms appearing in the normal approximation bound in Proposition 3.8 that involve γ_3 and γ_4. Namely, using the lower variance bounds (3.2) we see that

$$\begin{aligned}\frac{\sqrt{n}}{\mathbf{Var}\, V_\ell(P_n)}\sqrt{\gamma_3} &\lesssim \frac{\sqrt{n}}{n^{-\frac{d+3}{d+1}}}\left(\frac{\log n}{n}\right)^{2+\frac{1}{d+1}} = n^{-\frac{1}{2}+\frac{1}{d+1}}\,(\log n)^{2+\frac{1}{d+1}}, \\ \frac{n}{(\mathbf{Var}\, V_\ell(P_n))^{\frac{3}{2}}}\gamma_4 &\lesssim \frac{n}{n^{-\frac{3}{2}\frac{d+3}{d+1}}}\left(\frac{\log n}{n}\right)^{3+\frac{2}{d+1}} = n^{-\frac{1}{2}+\frac{1}{d+1}}\,(\log n)^{3+\frac{2}{d+1}}.\end{aligned}$$

Next, we consider the second-order difference operator. For $z \in K \setminus K_{(c\log n/n)}$, we define the visibility region

$$\mathrm{Vis}_z(n) \coloneqq \{x \in K \setminus K_{(c\log n/n)} : [x,z] \cap K_{(c\log n/n)} = \emptyset\},$$

where $[x,z]$ denotes the closed line segment which connects x and z.

On the event B_2 it may be concluded from (3.6) that $D_i f(V)^2 \lesssim (\log n/n)^2$ for all $i \in \{1,2,3\}$ and $V \in \{Z, Z'\}$.

We note that on B_2 the following inclusion holds:

$$\begin{aligned}\{D_{1,2}f(Y) \neq 0\} &\subseteq \{Y_1 \in K \setminus K_{(c\log n/n)}\} \cap \{Y_2 \in K \setminus K_{(c\log n/n)}\} \\ &\qquad \cap \{\mathrm{Vis}_{Y_1}(n) \cap \mathrm{Vis}_{Y_2}(n) \neq \emptyset\} \\ &\subseteq \{Y_1 \in K \setminus K_{(c\log n/n)}\} \cap \Big\{Y_2 \in \bigcup_{x\in \mathrm{Vis}_{Y_1}(n)} \mathrm{Vis}_x(n)\Big\}.\end{aligned}$$

The same applies to $D_{1,3}f(Y')$ as well. We thus infer that

$$\begin{aligned}&\mathbf{E}\big[\mathbf{1}\{D_{1,2}f(Y)\neq 0\}\,\mathbf{1}_{B_2}\big] \\ &\le \mathbf{P}\big(Y_1 \in K\setminus K_{(c\log n/n)}\big)\,\mathbf{P}\Big(Y_2 \in \bigcup_{x\in \mathrm{Vis}_{Y_1}(n)} \mathrm{Vis}_x(n) \,\Big|\, Y_1 \in K\setminus K_{(c\log n/n)}\Big) \\ &\le \mathbf{P}\big(Y_1 \in K\setminus K_{(c\log n/n)}\big) \sup_{z\in K\setminus K_{(c\log n/n)}} \mathbf{P}\Big(Y_2 \in \bigcup_{x\in \mathrm{Vis}_z(n)} \mathrm{Vis}_x(n)\Big) \\ &= \mathrm{vol}_d\big(K\setminus K_{(c\log n/n)}\big) \sup_{z\in K\setminus K_{(c\log n/n)}} \mathrm{vol}_d\Big(\bigcup_{x\in \mathrm{Vis}_z(n)} \mathrm{Vis}_x(n)\Big).\end{aligned}$$

Since the diameter of the previous union is of order $(\log n/n)^{\frac{1}{d+1}}$, it follows from [83, Lemma 6.2] that

$$\Delta(n) \coloneqq \sup_{z\in K\setminus K_{(c\log n/n)}} \mathrm{vol}_d\Big(\bigcup_{x\in \mathrm{Vis}_z(n)} \mathrm{Vis}_x(n)\Big) \lesssim \frac{\log n}{n}.$$

Moreover, on the complement B_2^c of B_2 we estimate all the indicator functions by one and the value of all difference operators by the constant $V_\ell(K)$. Since $\mathbf{P}(B_2^c)$ is small in n (recall (3.4)), this readily implies

$$\gamma_2 \lesssim \left(\frac{\log n}{n}\right)^4 \mathrm{vol}_d(K\setminus K_{(c\log n/n)})\,\Delta(n) \lesssim \left(\frac{\log n}{n}\right)^{5+\frac{2}{d+1}}.$$

Analogously, we can bound γ_1. First, suppose that $Y_1 = Y_1'$. Then, conditioned on B_2,

$$\begin{aligned}&\{D_{1,2}f(Y)\neq 0\}\cap\{D_{1,3}f(Y')\neq 0\}\\&\subseteq\{\{Y_1,Y_2,Y_3'\}\subseteq K\setminus K_{(c\log n/n)}\}\cap\{\mathrm{Vis}_{Y_2}(n)\cap\mathrm{Vis}_{Y_1}(n)\neq\emptyset\}\\&\qquad\cap\{\mathrm{Vis}_{Y_3'}(n)\cap\mathrm{Vis}_{Y_1}(n)\neq\emptyset\}\\&\subseteq\{Y_1\in K\setminus K_{(c\log n/n)}\}\cap\left\{\{Y_2,Y_3'\}\subseteq\bigcup_{x\in\mathrm{Vis}_{Y_1}(n)}\mathrm{Vis}_x(n)\right\},\end{aligned}$$

and arguing as before leads to

$$\begin{aligned}&\mathbf{E}\big[\mathbf{1}\{D_{1,2}f(Y)\neq 0\}\,\mathbf{1}\{D_{1,3}f(Y')\neq 0\}\,\mathbf{1}_{B_2}\big]\\&\leq\mathbf{P}\big(Y_1\in K\setminus K_{(c\log n/n)}\big)\sup_{z\in K\setminus K_{(c\log n/n)}}\mathbf{P}\Bigg(\{Y_2,Y_3'\}\subseteq\bigcup_{x\in\mathrm{Vis}_z(n)}\mathrm{Vis}_x(n)\Bigg)\\&\leq\mathrm{vol}_d(K\setminus K_{(c\log n/n)})\,\Delta(n)^2\,.\end{aligned}$$

Note that the case $Y_1\neq Y_1'$ gives a smaller order since, by independence, it leads to an extra factor $\mathrm{vol}_d(K\setminus K_{(c\log n/n)})$. Thus, by conditioning on B_2 and its complement, we obtain

$$\gamma_1\lesssim\left(\frac{\log n}{n}\right)^4\mathrm{vol}_d(K\setminus K_{(c\log n/n)})\,\Delta(n)^2\lesssim\left(\frac{\log n}{n}\right)^{6+\frac{2}{d+1}}.$$

Now, the other terms appearing in the normal approximation bound in Lemma Proposition 3.8 can be estimated using the lower variance bounds (3.2) as follows,

$$\begin{aligned}\frac{\sqrt{n}}{\mathbf{Var}\,V_\ell(P_n)}\sqrt{n^2\gamma_1}&\lesssim\frac{\sqrt{n}}{n^{-\frac{d+3}{d+1}}}\sqrt{n^2\cdot\left(\frac{\log n}{n}\right)^{6+\frac{2}{d+1}}}=n^{-\frac{1}{2}+\frac{1}{d+1}}(\log n)^{3+\frac{1}{d+1}},\\\frac{\sqrt{n}}{\mathbf{Var}\,V_\ell(P_n)}\sqrt{n\gamma_2}&\lesssim\frac{\sqrt{n}}{n^{-\frac{d+3}{d+1}}}\sqrt{n\cdot\left(\frac{\log n}{n}\right)^{5+\frac{2}{d+1}}}=n^{-\frac{1}{2}+\frac{1}{d+1}}(\log n)^{\frac{5}{2}+\frac{1}{d+1}}.\end{aligned}$$

Putting together all estimates, we arrive at

$$\begin{aligned}\mathrm{d}_W\left(\frac{V_\ell(P_n)-\mathbf{E}\,V_\ell(P_n)}{\sqrt{\mathbf{Var}\,V_\ell(P_n)}},N\right)&\lesssim n^{-\frac{1}{2}+\frac{1}{d+1}}\Big((\log n)^{3+\frac{1}{d+1}}+(\log n)^{\frac{5}{2}+\frac{1}{d+1}}\\&\qquad+(\log n)^{2+\frac{1}{d+1}}+(\log n)^{3+\frac{2}{d+1}}\Big)\\&\lesssim n^{-\frac{1}{2}+\frac{1}{d+1}}(\log n)^{3+\frac{2}{d+1}}\end{aligned}\tag{3.7}$$

in view of the normal approximation bound in Proposition 3.8. In particular, as $n\to\infty$, this converges to zero and so the random variables

$$W_\ell(P_n) = \frac{V_\ell(P_n) - \mathbf{E}\, V_\ell(P_n)}{\sqrt{\mathbf{Var}\, V_\ell(P_n)}}$$

converge in distribution to the standard Gaussian random variable n. The proof of Theorem 3.9 is thus complete. □

3.3 Points on the surface of the convex body

We switch our interest to the random polytope K_n, whose vertices are drawn independently and uniformily on the boundary of K. Our first result concerns asymptotic lower and upper bounds, respectively, for the variances of the intrinsic volumes.

Theorem 3.10. *Let $K \in \mathcal{K}^2_+$. Choose d independent random points on ∂K according to the probability distribution $\mathcal{H}^{d-1}$ and let K_n be their convex hull. Then, for all $\ell \in \{1, \dots, d\}$,*

$$\mathbf{Var}\, V_\ell(K_n) = \Theta\big(n^{-\frac{d+3}{d-1}}\big).$$

Based on a result stated in [67, Theorem 1] concerning the behaviour of $V_\ell(K) - \mathbf{E}[V_\ell(K_n)]$, the upper variance bound of Theorem 3.10 implies a strong law of large numbers.

Theorem 3.11. *In the set-up of Theorem 3.10 and for all $\ell \in \{1, \dots, d\}$, it holds*

$$\mathbf{P}\Big(\lim_{n\to\infty} \big(V_\ell(K) - V_\ell(K_n)\big) \cdot n^{\frac{2}{d-1}} = c_{K,\ell}\Big) = 1,$$

for some constants $c_{K,\ell} \in (0, \infty)$ that depend on K and ℓ.

The constants $c_{K,\ell}$ appear in an explicit form in [67, Theorem 1] and can be expressed in form of integrals of the principal curvatures of K.

Next, we introduce the standardized intrinsic volume functionals, defined by

$$W_\ell(K_n) := \frac{V_\ell(K_n) - \mathbf{E}\, V_\ell(K_n)}{\sqrt{\mathbf{Var}\, V_\ell(K_n)}}, \quad \ell \in \{1, \dots, d\}.$$

We prove the following central limit theorem for such functionals.

Theorem 3.12. *In the set-up of Theorem 3.10 and for all $\ell \in \{1, \dots, d\}$, it holds*

$$\mathrm{d}_K\big(W_\ell(K_n), N\big) \lesssim n^{-\frac{1}{2}} (\log n)^{3+\frac{6}{d-1}},$$

where N is a standard Gaussian random variable. In particular, $W_\ell(K_n)$ converges in distribution to N, as $n \to \infty$.

Remark 2. Note that the rate of convergence in Theorem 3.12 does not depend on ℓ. Moreover, the same rate of convergence was already obtained in [82] for the case $\ell = d$.

Remark 3. Contrarily to the previous central limit theorem Theorem 3.9, when P_n was treated, using d_W this time would not improve the rate of convergence, as it will appear in the proof that the extra term from the upper bound of the Kolmogorov distance in Proposition 3.8 does not have a higher order than the other terms, which are present in both the Wassertein and the Kolmogorov distance bounds.

3.3.1 Lower variance bound

In order to prove a lower variance bound, we first introduce in Section 3.3.2 a geometrical construction taken from [72, Section 3.1]. More precisely, for $x \in \partial K$ and h sufficiently small, we define $d+1$ disjoint subsets of $C^K(x,h) \cap \partial K$ which are denoted by $D_i'(x)$, $i = 0, \dots, d$. Later, in Section 3.3.3 we fix some particular points $y_1, \dots, y_n \in \partial K$ and h_n. The event that exactly one random point is contained in each $D_i'(y_j)$, $i \in \{0, \dots, d\}$, and every other point is outside of $C^K(y_j, h_n) \cap \partial K$ is indicated by A_j, $j \in \{1, \dots, n\}$. Then, our strategy is as follows. By conditioning on the σ-field $\mathcal{F}$ generated by the positions of all $X_1, \dots, X_n$ except those which are contained in $D_0'(y_j)$ with $\mathbf{1}_{A_j} = 1$, it will turn out that

$$\mathbf{Var}[V_\ell(K_n)] \geq \mathbf{E}[\mathbf{Var}[V_\ell(K_n)|\mathcal{F}]] = \mathbf{E}\left[\sum_{j=1}^{n} \mathbf{Var}_j[V_\ell(K_n)]\mathbf{1}_{A_j}\right],$$

where the variances $\mathbf{Var}_j[\cdot]$ are taken over $X_j \in D_0'(y_j)$. Finally, it remains to determine the behaviour of $\mathbf{Var}_j[\cdot]$ and $\mathbf{P}(A_j)$, $j \in \{1, \dots, n\}$. This way we bound the variance from below by a quantity that is asymptotically of the desired order.

3.3.2 Auxiliary geometric construction

Let E be the standard paraboloid given by

$$E = \{z = (z_1, \dots, z_d) \in \mathbb{R}^d : z_d \geq z_1^2 + \dots + z_{d-1}^2\}.$$

We construct a simplex S in $C^E(0,1)$ in the following way. The base is a regular simplex whose vertices $v_1, \dots, v_d$ lie on $\partial E \cap H(e_d, 1/(3(d-1)^2))$ while $v_0 = (0, \dots, 0)$ is the

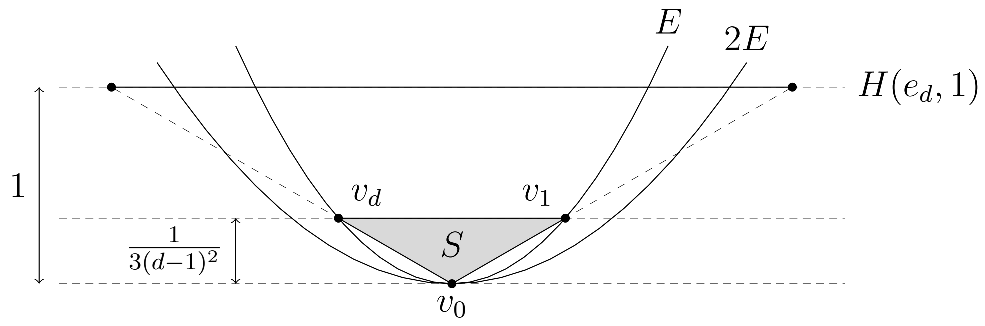

Figure 3.1: Construction of the simplex S.

apex of S. Notice that $2E \cap H(e_d, 1)$ has radius $\sqrt{2}$, while the inradius of the base of the simplex is $1/(\sqrt{3}(d-1)^2)$ and therefore, $\{\lambda z \in \mathbb{R}^d : \lambda \geq 0, z \in S\} \cap H(e_d, 1)$ has inradius $3(d-1)^2/(\sqrt{3}(d-1)^2) = \sqrt{3}$. In particular, this implies that

$$\{\lambda z \in \mathbb{R}^d : \lambda \geq 0, z \in S\} \supseteq 2E \cap H(e_d, 1),$$

see Figure 3.1 for the construction of S. For $i \in \{0, 1, \dots, d\}$, let v_i' be the orthogonal projection of v_i onto $\operatorname{span}\{e_1, \dots, e_{d-1}\}$. Consider $B_0 \coloneqq \mathbb{B}_2^{d-1}(v_0', r) \subseteq \mathbb{R}^{d-1}$ and $B_i \coloneqq \mathbb{B}_2^{d-1}(v_i', r') \subseteq \mathbb{R}^{d-1}$, $i \in \{1, \dots, d\}$, for some radii r and r' to be chosen later. Let b_E be the quadratic form associated with E, i.e. $b_E(y) = \|y\|^2$ for $y \in \mathbb{R}^{d-1}$. For $i \in \{0, \dots, d\}$, we define the lift $B_i' \coloneqq \tilde{b}(B_i)$ on ∂E of the sets B_i, where $\tilde{b}$ indicates the mapping

$$\tilde{b} \colon \mathbb{R}^{d-1} \to \partial E, \quad y \mapsto (y, b_E(y)).$$

Note that, if r and r' are small enough, then, by continuity, for any $(d+1)$-tuple of points $x_i \in B_i'$, the following still holds,

$$\{\lambda z \in \mathbb{R}^d : \lambda \geq 0, z \in [x_0, \dots, x_d]\} \supseteq 2E \cap H(e_d, 1). \tag{3.8}$$

Then, we extend the aforementioned argument to arbitrary caps of ∂K. For each point $x \in \partial K$, we consider the approximating paraboloid Q_x of K at x. Let $T_x(K)$ be the tangent space of K at the point x. The space $T_x(K)$ can be identified with $\mathbb{R}^{d-1}$ having x as its origin. Then, there exists a unique affine map A_x such that $A_x(C^E(0,1)) = C^{Q_x}(x, h)$ while mapping the coordinate axes onto the coordinate axes of $T_x(K) \times \mathbb{R}$. We define $D_i(x) \coloneqq A_x(B_i)$, $i \in \{0, \dots, d\}$. Then, it is true that $\operatorname{vol}_{d-1}(D_i(x)) = c_1 h^{\frac{d-1}{2}}$ for a constant $c_1 > 0$. We define now $D_i'(x) \coloneqq \tilde{f}^{(x)}(D_i(x))$,

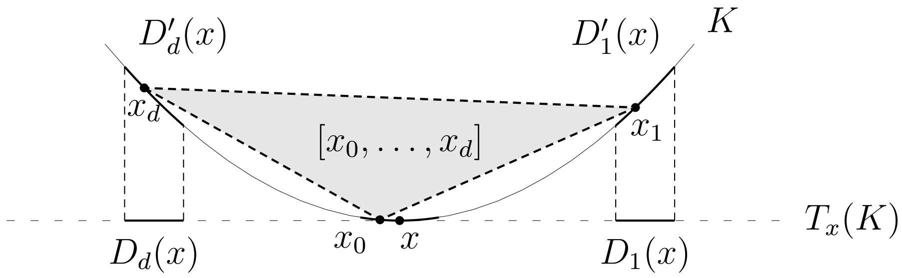

Figure 3.2: Example of a simplex $[x_0, \dots, x_d]$.

where

$$\tilde{f}^{(x)} : U \to \partial K, \quad y \mapsto (y, f^{(x)}(y))$$

for a neighbourhood $U \subseteq T_x(K)$ of x. Since $K \in \mathcal{K}^2_+$, there exist positive lower and upper bounds for the curvature. Thus, due to the curvature bounds of K, it holds that

$$c_K h^{\frac{d-1}{2}} \leq \mathcal{H}^{d-1}(D'_i(x)) \leq C_K h^{\frac{d-1}{2}}, \tag{3.9}$$

where c_K and C_K are positive constants depending only on K.

By continuity, if every x_i belongs to a ball $\mathbb{B}^d(v_i, \eta)$, (3.8) is preserved whenever $\eta > 0$ is small enough. Moreover, we can choose r and r' to be small enough such that for every $x \in \partial K$ and every $i \in \{0, \dots, d\}$, $D'_i(x) \subseteq A_x(\mathbb{B}^d(v_i, \eta))$. Indeed, define for $\varepsilon > 0$ and every $i \in \{0, \dots, d\}$, the set $U_i = \{(x, y) \in \mathbb{R}^d : x \in \mathbb{B}_2^{d-1}(\text{proj}_{\mathbb{R}^{d-1}} v_i, \eta/2), y \in [(1+\varepsilon)^{-1} b_E(x), (1+\varepsilon) b_E(x)]\}$. If ε is small enough, then $U_i \subseteq \mathbb{B}^d(v_i, \eta)$. Using Lemma 3.3, we can take h small enough such that $(1+\varepsilon)^{-1} b_{Q_x}(y) \leq f^{(x)}(y) \leq (1+\varepsilon) b_{Q_x}(y)$. In particular, if we choose $r, r' < \eta/2$, then $D'_i(x) \subseteq A_x(U_i) \subseteq A_x(\mathbb{B}^d(v_i, \eta))$. One can choose a point $x_i \in D'_i(x)$ for any $i \in \{0, \dots, d\}$, as in Figure 3.2. As a consequence of the previous inclusion, we have

$$\{\lambda z \in \mathbb{R}^d : \lambda \geq 0, z \in [x_0, \dots, x_d]\} \supseteq 2Q_x \cap H(u_x, h_K(u_x) - h) \supseteq K \cap H(u_x, h_K(u_x) - h), \tag{3.10}$$

where the last inclusion holds whenever $h \leq h_0$ for h_0 sufficiently small. Therefore, from now on r, r' and h_0 are chosen such that the previous argument holds true.

3.3.3 Proof of the lower bound

In this section we combine tools from [11, 70, 72]. Let $K \in \mathcal{K}^2_+$ and $X_1, \ldots, X_n$ be independent random points that are chosen from ∂K according to the probability distribution $\mathcal{H}^{d-1}$. Due to [70, Lemma 13], we can choose d points $y_1, \ldots, y_n \in \partial K$ and corresponding disjoint caps of K, namely, $C^K(y_j, h_n)$ for $j \in \{1, \ldots, n\}$, with $h_n = \Theta\big(n^{-\frac{2}{d-1}}\big)$. For all $i \in \{0, \ldots, d\}$ and $j \in \{1, \ldots, n\}$, we define the sets $\{D_i(y_j)\}$ and $\{D'_i(y_j)\}$ as in Section 3.3.2. Let A_j, $j \in \{1, \ldots, n\}$, be the event that exactly one random point is contained in each $D'_i(y_j)$, $i \in \{0, \ldots, d\}$, and every other point is outside of $C^K(y_j, h_n) \cap \partial K$.

Lemma 3.13. [72, Section 3.2] *For n large enough, and all $j \in \{1, \ldots, n\}$, there exists a constant $c \in (0, 1)$ such that $\mathbf{P}(A_j) \geq c$.*

Proof. The probability of the event A_j is

$$\begin{aligned}\mathbf{P}(A_j) &= n \cdot (n-1) \cdots (n-d)\, \mathbf{P}(X_{i+1} \in D'_i(y_j), i \in \{0, \ldots, d\}) \\ &\qquad \times \mathbf{P}(X_{i+1} \notin C^K(y_j, h_n) \cap \partial K, i \in \{d+1, \ldots, n-1\}) \\ &= n \cdot (n-1) \cdots (n-d) \prod_{i=0}^{d} \mathcal{H}^{d-1}(D'_i(y_j)) \prod_{k=d+1}^{n-1} (1 - \mathcal{H}^{d-1}(C^K(y_j, h_n) \cap \partial K)).\end{aligned}$$

Combining Lemma 3.4, [70, Lemma 13] and Equation (3.9), we obtain

$$\mathbf{P}(A_j) \geq c_2 n^{d+1} n^{-d-1} (1 - c_3 n^{-1})^{n-d-1} \geq c > 0,$$

where all constants are positive. □

Let $\mathcal{F}$ be the σ-field generated by the positions of all $X_1, \ldots, X_n$ except those which are contained in $D'_0(y_j)$ with $\mathbf{1}_{A_j} = 1$. Assume that $\mathbf{1}_{A_j} = \mathbf{1}_{A_k} = 1$ for some $j, k \in \{1, \ldots, n\}$ and without loss of generality that X_j and X_k are the points in $D'_0(y_j)$ and $D'_0(y_k)$. By Equation (3.10), it is not possible that there is an edge between X_j and X_k. Therefore, the change of the intrinsic volume affected by moving X_j within $D'_0(y_j)$ is independent of the change of the intrinsic volume of moving X_k within $D'_0(y_k)$. As a consequence, we obtain

$$\mathbf{Var}[V_\ell(K_n)|\mathcal{F}] = \sum_{j=1}^{n} \mathbf{Var}_j[V_\ell(K_n)] \mathbf{1}_{A_j},$$

where the variances $\mathbf{Var}_j[\cdot]$ are taken over $X_j \in D'_0(y_j)$, compare with [11].

For $j \in \{1, \ldots, n\}$ and $i \in \{0, \ldots, d\}$, let z_j^i be an arbitrary point in $D_i'(y_j)$. We indicate with N_j the normal cone of the simplex $[z_j^0, \ldots, z_j^d]$ at vertex z_j^0. Let S_j be the cone with base $H(u_{z_j^0}, h_K(u_{z_j^0}) - h_n) \cap 2Q_x$ and vertex z_j^0. Note that $u_{z_j^0}$ is the unique unit outer normal of K at z_j^0. The corresponding normal cone of S_j at z_j^0 is denoted by $\bar{n}_j$. Moreover, the angular aperture of S_j at z_j^0 is at most $c_K' \sqrt{h_n}$, where $c_K' > 0$ is a constant that depends on K. Because of this and Equation (3.10), we can find sets Σ_j such that

$$\mathbb{S}^{d-1} \cap N_j \subset \mathbb{S}^{d-1} \cap \bar{n}_j \subset \mathbb{S}^{d-1} \cap (u_{z_j^0} + c_K' \sqrt{h_n} \mathbb{B}^d) =: \Sigma_j. \tag{3.11}$$

We fix $j \in \{1, \ldots, n\}$ and $z_j^i \in D_i'(y_j)$ for all $i \in \{1, \ldots, d\}$. Let $F_j := [z_j^1, \ldots, z_j^d]$ and define

$$\widetilde{V}_\ell(z; F_j) := \binom{d}{\ell} \frac{\kappa_d}{\kappa_\ell \kappa_{d-\ell}} \int_{G(d,\ell)} \mathbf{1}_{\{L \cap \Sigma_j \neq \emptyset\}} \operatorname{vol}_\ell\big([z, F_j]|L\big) \nu_\ell(\mathrm{d}L),$$

for any $z \in D_0'(y_j)$ and any $\ell \in \{1, \ldots, d\}$.

Lemma 3.14. *Let $j \in \{1, \ldots, n\}$ and let X_j be a point chosen with respect to the normalized Hausdorff measure restricted to $D_0'(y_j)$. Then,*

$$\mathbf{Var}_j[\widetilde{V}_\ell(X_j; F_j)] = \Theta\big(n^{-2\frac{d+1}{d-1}}\big), \quad \ell \in \{1, \ldots, d\}.$$

Proof. Note that $[X_j, F_j]|L$ is a simplex in $L \in G(d, \ell)$ with base $F_j|L$ and additional point $X_j|L$. As a consequence, the height of $[X_j, F_j]|L$ is proportional to h_n and

$$\operatorname{vol}_{\ell-1}(F_j|L) = \Theta\big(h_n^{\frac{\ell-1}{2}}\big),$$

where $L \in G(d, \ell)$ with $L \cap \Sigma_j \neq \emptyset$. Thus,

$$\operatorname{vol}_\ell\big([X_j, F_j]|L\big) = \Theta\big(h_n^{\frac{\ell+1}{2}}\big).$$

Due to Lemma 3.7 and Equation (3.11), it follows

$$\int_{G(d,\ell)} \mathbf{1}_{\{L \cap \Sigma_j \neq \emptyset\}} \nu_\ell(\mathrm{d}L) = \nu_\ell(\{L \in G(d,\ell) : L \cap \Sigma_j \neq \emptyset\}) = \Theta\big(h_n^{\frac{d-\ell}{2}}\big).$$

Therefore, we obtain

$$\widetilde{V}_\ell(X_j; F_j) = \Theta\big(h_n^{\frac{d+1}{2}}\big).$$

Let X_j^1 and X_j^2 be independent copies of X_j, then

$$|\widetilde{V}_\ell(X_j^1; F_j) - \widetilde{V}_\ell(X_j^2; F_j)| = \Theta\big(h_n^{\frac{d+1}{2}}\big),$$

since the heights of $X_j^1|L$ and $X_j^2|L$ are different with probability 1. Using $h_n = \Theta\big(n^{-\frac{2}{d-1}}\big)$, we obtain

$$\begin{aligned}\mathbf{Var}_j\big[\widetilde{V}_\ell(X_j;F_j)\big] &= \frac{1}{2}\,\mathbf{E}\Big[\big|\widetilde{V}_\ell(X_j^1;F_j)-\widetilde{V}_\ell(X_j^2;F_j)\big|^2\Big]\\ &= \Theta\big(n^{-2\frac{d+1}{d-1}}\big).\end{aligned}$$ □

We can now proceed with the proof of the lower variance bound.

Proof of the lower bound of Theorem 3.10. Let $\mathcal{F}$ be the σ-field defined as above. The conditional variance formula implies that

$$\mathbf{Var}[V_\ell(K_n)] = \mathbf{E}[\mathbf{Var}[V_\ell(K_n)|\mathcal{F}]] + \mathbf{Var}[\mathbf{E}[V_\ell(K_n)|\mathcal{F}]] \geq \mathbf{E}[\mathbf{Var}[V_\ell(K_n)|\mathcal{F}]].$$

As already mentioned, $\mathcal{F}$ induces an independence property. Therefore, we obtain

$$\mathbf{Var}[V_\ell(K_n)|\mathcal{F}] = \sum_{j=1}^{n}\mathbf{Var}_j[V_\ell(K_n)]\mathbf{1}_{A_j} = \sum_{j=1}^{n}\mathbf{Var}_j[\widetilde{V}_\ell(X_j;F_j)]\mathbf{1}_{A_j}.$$

Finally, applying Lemma 3.13, Lemma 3.14 and taking expectation yields

$$\mathbf{Var}[V_\ell(K_n)] \gtrsim n^{-2\frac{d+1}{d-1}}\sum_{j=1}^{n}\mathbf{P}(A_j) \gtrsim n^{-2\frac{d+1}{d-1}}n = n^{-\frac{d+3}{d-1}}.$$ □

3.3.4 Upper variance bound

In the following, we find an upper bound for $\mathbf{Var}\,V_\ell(K_n)$, $\ell\in\{1,\dots,d\}$. The proof is based on the Efron-Stein jackknife inequality and follows the ideas of [11]. In contrast to [11], we use the concept of surface body, in particular, Lemma 3.2 about the fact that the surface body is contained in the random polytope K_n with high probability. Moreover, we make use of Lemma 3.4 for our estimates. The proof is given in full details for the case $K = \mathbb{B}_2^d$. From a geometric point of view this case is easier to handle. However, the general case is also related to this basis case. The corresponding arguments are stated at the end of the proof.

Proof of the upper bound of Theorem 3.10. First, let $K = \mathbb{B}^d$. We indicate with T_n the event that the surface body $K(s \geq \tau_n)$ is contained in K_n. Let $\ell\in\{1,\dots,d\}$. Applying the Efron-Stein jackknife inequality from [68] yields

$$\begin{aligned}\mathbf{Var}[V_\ell(K_n)] &\lesssim n\,\mathbf{E}\big[(V_\ell(K_{n+1})-V_\ell(K_n))^2\big]\\ &= n\,\mathbf{E}\big[(V_\ell(K_{n+1})-V_\ell(K_n))^2\mathbf{1}_{T_n}\big]+n\,\mathbf{E}\big[(V_\ell(K_{n+1})-V_\ell(K_n))^2\mathbf{1}_{T_n^c}\big].\end{aligned} \tag{3.12}$$

It is obvious that $(V_\ell(K_{n+1})-V_\ell(K_n))^2 \le V_\ell(K)^2$ and $\mathbf{E}[\mathbf{1}_{T_n^c}] = \mathbf{P}(T_n^c)$. Since the parameter α can be chosen arbitrarily big in Lemma 3.2, the second term in Equation (3.12) is negligible in the asymptotic analysis. By Equation (2.2), we obtain

$$\begin{aligned}\mathbf{Var}[V_\ell(K_n)] &\lesssim n\,\mathbf{E}\big[(V_\ell(K_{n+1})-V_\ell(K_n))^2\mathbf{1}_{T_n}\big]\\ &\lesssim n\,\mathbf{E}\Bigg[\int_{G(d,\ell)} \mathrm{vol}_\ell((K_{n+1}|A)\setminus(K_n|A))\nu_\ell(\mathrm{d}A)\\ &\qquad\times\int_{G(d,\ell)} \mathrm{vol}_\ell((K_{n+1}|B)\setminus(K_n|B))\,\nu_\ell(\mathrm{d}B)\mathbf{1}_{T_n}\Bigg]\\ &\lesssim n\,\mathbf{E}\Bigg[\int_{G(d,\ell)}\int_{G(d,\ell)} \mathrm{vol}_\ell((K_{n+1}|A)\setminus(K_n|A))\;\mathrm{vol}_\ell((K_{n+1}|B)\setminus(K_n|B))\\ &\qquad\times\mathbf{1}_{T_n}\nu_\ell(\mathrm{d}A)\nu_\ell(\mathrm{d}B)\Bigg].\end{aligned} \tag{3.13}$$

If $X_{n+1}|A \in K_n|A$, then the set $(K_{n+1}|A)\setminus(K_n|A)$ is clearly empty. Otherwise, $(K_{n+1}|A)\setminus(K_n|A)$ consists of several disjoint simplices which are the convex hull of $X_{n+1}|A$ and those facets of $K_n|A$ that can be "seen" from $X_{n+1}|A$. For $I=\{i_1,\ldots,i_\ell\}\subset\{1,\ldots,n\}$, we indicate with F_I the convex hull of $X_{i_1},\ldots,X_{i_\ell}$. Note that F_I and $F_I|A$ are $(\ell-1)$-dimensional simplices with probability 1. The closed half space in $\mathbb{R}^d$ which is determined by the hyperplane $A^\perp + \mathrm{aff}\,F_I$ and contains the origin is denoted by $H_0(F_I,A)$. The other half space is $H_+(F_I,A)$. The corresponding ℓ-dimensional half spaces in A are denoted by $H_0(F_I|A)$ and $H_+(F_I|A)$. Let $\tilde{\mathcal{F}}(A)$ be the set of $(\ell-1)$-dimensional facets of $K_n|A$ that can be seen from $X_{n+1}|A$. It is defined by

$$\tilde{\mathcal{F}}(A)=\{F_I|A : K_n|A\subset H_0(F_I|A),\, X_{n+1}|A\in H_+(F_I|A), I=\{i_1,\ldots,i_\ell\}\subset\{1,\ldots,n\}\}.$$

Note that $\tilde{\mathcal{F}}(A)$ is random since it depends on the points $X_1,\ldots,X_n$. In the following we use a deterministic version of it for fixed points $x_1,\ldots,x_n$. The deterministic version is denoted by $\mathcal{F}(A)$. Therefore,

$$(3.13) \lesssim n \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} \int_{G(d,\ell)} \int_{G(d,\ell)} \Big(\sum_{F \in \mathcal{F}(A)} \mathrm{vol}_\ell([x_{n+1}|A, F]) \Big) \times \Big(\sum_{F' \in \mathcal{F}(B)} \mathrm{vol}_\ell([x_{n+1}|B, F'])\mathbf{1}_{T_n} \Big) \nu_\ell(\mathrm{d}A)\nu_\ell(\mathrm{d}B)\mathcal{H}^{d-1}(\mathrm{d}x_1)\cdots\mathcal{H}^{d-1}(\mathrm{d}x_{n+1}). \tag{3.14}$$

Next, the integration is extended over all possible index sets I, J and the order of integration is changed. As a consequence, we obtain

$$(3.14) \lesssim n \int_{G(d,\ell)} \int_{G(d,\ell)} \int_{(\mathbb{S}^{d-1})^{n+1}} \Big(\sum_I \mathbf{1}\{F_I|A \in \mathcal{F}(A)\} \ \mathrm{vol}_\ell([F_I, x_{n+1}]|A) \Big) \times \Big(\sum_J \mathbf{1}\{F_J|B \in \mathcal{F}(B)\} \ \mathrm{vol}_\ell([F_J, x_{n+1}]|B)\mathbf{1}_{T_n} \Big) \times \mathcal{H}^{d-1}(\mathrm{d}x_1)\cdots\mathcal{H}^{d-1}(\mathrm{d}x_{n+1})\nu_\ell(\mathrm{d}A)\nu_\ell(\mathrm{d}B).$$

Note that $[F_I, X_{n+1}]|A$ and $[F_J, X_{n+1}]|B$ are contained in the associated caps $C_\ell(I, A) := H_+(F_I, A) \cap \mathbb{B}^d$ and $C_\ell(J, B)$. Moreover, we use the abbreviation

$$C_d(I, A) = (H_+(F_I|A) + A^\perp) \cap \mathbb{B}^d.$$

We indicate with $V_\ell(I, A) = \mathrm{vol}_\ell(C_\ell(I, A))$ and $\mathrm{vol}_d(I, A) = \mathrm{vol}_d(C_d(I, A))$ the volumes of these caps. Therefore, the variance is bounded by

$$\mathbf{Var}[V_\ell(K_n)] \lesssim n \sum_I \sum_J \int_{G(d,\ell)} \int_{G(d,\ell)} \int_{(\mathbb{S}^{d-1})^{d+1}} \mathbf{1}\{F_I|A \in \mathcal{F}(A)\} V_\ell(I, A) \mathbf{1}\{F_J|B \in \mathcal{F}(B)\} \times V_\ell(J, B)\, \mathbf{1}_{T_n}\, \mathcal{H}^{d-1}(\mathrm{d}x_1)\cdots\mathcal{H}^{d-1}(\mathrm{d}x_{n+1})\, \nu_\ell(\mathrm{d}A)\, \nu_\ell(\mathrm{d}B),$$

where the summation extends over all ℓ-tuples I and J. Of course, these tuples may have a non-empty intersection. However, if the size of $I \cap J$ is fixed to be k, then the corresponding terms in the sum are independent of the choice of $i_1, \ldots, i_\ell$ and $j_1, \ldots, j_\ell$. For any $k \in \{0, 1, \ldots, \ell\}$, we indicate with F the convex hull of $X_1, \ldots, X_\ell$ and by G

the convex hull of $X_{\ell-k+1},\ldots,X_{2\ell-k}$. As in [11], we obtain

$$\mathbf{Var}\,V_\ell(K_n) \lesssim n\sum_{k=0}^{\ell}\binom{n}{\ell}\binom{\ell}{k}\binom{n-\ell}{\ell-k}\int_{G(d,\ell)}\int_{G(d,\ell)}\int_{(\mathbb{S}^{d-1})^{n+1}}\mathbf{1}\{F_I|A\in\mathcal{F}(A)\}V_\ell(I,A)$$
$$\times\,\mathbf{1}\{F_J|B\in\mathcal{F}(B)\}V_\ell(J,B)\,\mathbf{1}_{T_n}\,\mathcal{H}^{d-1}(\mathrm{d}x_1)\cdots\mathcal{H}^{d-1}(\mathrm{d}x_{n+1})\,\nu_\ell(\mathrm{d}A)\,\nu_\ell(\mathrm{d}B). \tag{3.15}$$

We indicate with Σ_k the k-th term in the previous sum. By symmetry, we can restrict the summation to those tuples where $\mathrm{vol}_d(I,A)\geq\mathrm{vol}_d(J,B)$. In addition to that, we multiply the integrand by $\mathbf{1}\{C_d(I,A)\cap C_d(J,B)\neq\emptyset\}$. This is indeed possible because the caps have at least the point X_{n+1} in common. It follows immediately that

$$\Sigma_k\lesssim n^{2\ell-k+1}\int_{G(d,\ell)}\int_{G(d,\ell)}\int_{(\mathbb{S}^{d-1})^{n+1}}\mathbf{1}\{F|A\in\mathcal{F}(A)\}\,V_\ell(I,A)\,\mathbf{1}\{C_d(I,A)\cap C_d(J,B)\neq\emptyset\}$$
$$\times\,\mathbf{1}\{G|B\in\mathcal{F}(B)\}\,V_\ell(J,B)\,\mathbf{1}\{\mathrm{vol}_d(I,A)\geq\mathrm{vol}_d(J,B)\}$$
$$\times\,\mathbf{1}_{T_n}\,\mathcal{H}^{d-1}(\mathrm{d}x_1)\cdots\mathcal{H}^{d-1}(\mathrm{d}x_{n+1})\,\nu_\ell(\mathrm{d}A)\,\nu_\ell(\mathrm{d}B).$$

Next, we integrate with respect to $x_{2\ell-k+1},\ldots,x_n,x_{n+1}$. Due to the condition $F|A\in\mathcal{F}(A)$, the points $X_{2\ell-k+1},\ldots,X_n$ are contained in $H_0(F,A)$ and X_{n+1} is in $H_+(F,A)$. Therefore,

$$\Sigma_k\lesssim n^{2\ell-k+1}\int_{G(d,\ell)}\int_{G(d,\ell)}\int_{(\mathbb{S}^{d-1})^{2\ell-k}}(1-\mathcal{H}^{d-1}(C_d(I,A)\cap\mathbb{S}^{d-1}))^{n-2\ell+k}$$
$$\times\,\mathcal{H}^{d-1}(C_d(I,A)\cap\mathbb{S}^{d-1})\,V_\ell(I,A)\,\mathbf{1}\{C_d(I,A)\cap C_d(J,B)\neq\emptyset\,\}V_\ell(J,B)$$
$$\times\,\mathbf{1}\{\mathrm{vol}_d(I,A)\geq\mathrm{vol}_d(J,B)\}\,\mathbf{1}_{T_n}\,\mathcal{H}^{d-1}(\mathrm{d}x_1)\cdots\mathcal{H}^{d-1}(\mathrm{d}x_{2\ell-k})\,\nu_\ell(\mathrm{d}A)\,\nu_\ell(\mathrm{d}B).$$

The assumption $\mathrm{vol}_d(I,A)\geq\mathrm{vol}_d(J,B)$ implies that the height of the cap $C_d(I,A)$ is at least the height of $C_d(J,B)$. Due to $C_d(I,A)\cap C_d(J,B)\neq\emptyset$, we find a constant β such that $C_d(J,B)$ is contained in $\beta\,C_d(I,A)$. More precisely, $\beta\,C_d(I,A)$ is an enlarged homothetic copy of $C_d(I,A)$, where the center of homothety $z\in\mathbb{S}^{d-1}$ coincides with the center of the cap $C_d(I,A)$. It follows from the homogeneity that the Hausdorff measure (restricted to $\beta\,\mathbb{S}^{d-1}$) of $\beta\,C_d(I,A)$ is up to a constant $\mathcal{H}^{d-1}(C_d(I,A)\cap\mathbb{S}^{d-1})$. Therefore,

$$\int_{(\mathbb{S}^{d-1})^{\ell-k}}\mathbf{1}\{C_d(I,A)\cap C_d(J,B)\neq\emptyset\}\,\mathbf{1}\{\mathrm{vol}_d(I,A)\geq\mathrm{vol}_d(J,B)\}$$
$$\times\,V_\ell(J,B)\,\mathcal{H}^{d-1}(\mathrm{d}x_{\ell+1})\cdots\mathcal{H}^{d-1}(\mathrm{d}x_{2\ell-k})\lesssim\mathcal{H}^{d-1}(C_d(I,A)\cap\mathbb{S}^{d-1})^{\ell-k}V_\ell(I,A).$$

As in [11], the conditions $C_d(I, A) \cap C_d(J, B) \neq \emptyset$ and $\mathrm{vol}_d(I, A) \geq \mathrm{vol}_d(J, B)$ are only satisfied if the angle between z and the subspace B is not larger than twice the central angle δ of the cap $C_d(I, A)$. Moreover, δ is bounded by

$$\delta \lesssim \mathrm{vol}_d(I, A)^{1/(d+1)}. \tag{3.16}$$

Thus,

$$\begin{aligned}\Sigma_k \lesssim n^{2\ell-k+1} \int_{G(d,\ell)} \int_{G(d,\ell)} \int_{(\mathbb{S}^{d-1})^\ell} &(1 - \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}))^{n-2\ell+k} \\ &\times \mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1})^{\ell-k+1} V_\ell(I, A)^2\, \mathbf{1}\{\sphericalangle(z, B) \lesssim \mathrm{vol}_d(I, A)^{1/(d+1)}\} \\ &\times \mathbf{1}_{T_n}\, \mathcal{H}^{d-1}(\mathrm{d}x_1) \cdots \mathcal{H}^{d-1}(\mathrm{d}x_\ell)\, \nu_\ell(\mathrm{d}A)\, \nu_\ell(\mathrm{d}B).\end{aligned}$$

Due to Lemma 3.4, the condition T_n can be replaced by the condition

$$\mathrm{vol}_d(I, A) \leq c_1\, (\log n/n)^{(d+1)/(d-1)}$$

for some constant $c_1 > 0$. In the following, the economic cap covering theorem is used, recall Proposition 3.6. Let h be a positive integer such that $2^{-h} \leq \log n/n$. Note that the smallest possible value of h is $h_0 = -\lfloor \log_2(\log n/n) \rfloor$. According to the economic cap covering theorem, we find for each h a collection of caps $\{C_1, \ldots, C_{m(h)}\}$ which cover the wet part of $\mathbb{B}^d|A$ with parameter $(2^{-h})^{(\ell+1)/(d-1)}$. This collection of caps is denoted by $\mathcal{M}_h$. Each cap C_i can be viewed as a projection of a d-dimensional cap $C_i(A)$ from $\mathbb{B}^d$ to A. Now we consider an arbitrary tuple $(X_1, \ldots, X_\ell)$ which has a corresponding cap $C_d(I, A)$ having volume at most $c_1\, (\log n/n)^{(d+1)/(d-1)}$. We relate to $(X_1, \ldots, X_\ell)$ the maximal h such that $C_\ell(I, A) \subset C_i$ for some $C_i \in \mathcal{M}_h$. This is indeed possible since at least 2^{-h_0} is roughly $\log n/n$ and the volume of the caps in $\mathcal{M}_h$ tends to zero as $h \to \infty$. As a consequence, we obtain

$$V_\ell(I, A) \leq \mathrm{vol}_\ell(C_i) \lesssim 2^{-h(\ell+1)/(d-1)}$$

and

$$\mathrm{vol}_d(I, A) \leq \mathrm{vol}_d(C_i(A)) \lesssim 2^{-h(d+1)/(d-1)}.$$

According to Lemma 3.4, $\mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}) \leq \mathcal{H}^{d-1}(C_i(A) \cap \mathbb{S}^{d-1}) \lesssim 2^{-h}$. Due to the maximality of h, it holds $\mathrm{vol}_d(I, A) \geq 2^{-(h+1)(d+1)/(d-1)}$. In addition to that, it follows from Lemma 3.4 that $\mathcal{H}^{d-1}(C_d(I, A) \cap \mathbb{S}^{d-1}) \geq c_2 2^{-(h+1)}$, for some constant

$c_2 > 0$. Therefore, we obtain

$$(1-\mathcal{H}^{d-1}(C_d(I,A)\cap\mathbb{S}^{d-1}))^{n-2\ell+k}\mathcal{H}^{d-1}(C_d(I,A)\cap\mathbb{S}^{d-1})^{\ell-k+1}V_\ell(I,A)^2 \\ \lesssim (1-c_2 2^{-(h+1)})^{n-2\ell+k}2^{-h(\ell-k+1)}2^{-2h(\ell+1)/(d-1)}.$$

Then, we integrate each $(X_1,\dots,X_\ell)$ on $(C_i(A))^\ell$ and we use the fact $1-x\le\exp(-x)$ to obtain

$$\exp(-c_2(n-2\ell+k)2^{-h-1})2^{-h(\ell-k+1)}2^{-2h(\ell+1)/(d-1)}\mathcal{H}^{d-1}(C_i(A)\cap\mathbb{S}^{d-1})^\ell \\ \lesssim \exp(-c_2(n-2\ell+k)2^{-h-1})2^{-h(\ell-k+1)}2^{-2h(\ell+1)/(d-1)}2^{-h\ell}.$$

Since the volume of the wet part of $\mathbb{B}_2^\ell$ with parameter $2^{-h(\ell+1)/(d-1)}$ is $\Theta\big(2^{-2h/(d-1)}\big)$ (note that $h\to\infty$, as $n\to\infty$), we obtain

$$|\mathcal{M}_h|\lesssim\frac{2^{-2h/(d-1)}}{2^{-h(\ell+1)/(d-1)}}=2^{h(\ell-1)/(d-1)}. \tag{3.17}$$

Finally, this results in

$$\begin{aligned}\int_{G(d,\ell)}\int_{(\mathbb{S}^{d-1})^\ell}&(1-\mathcal{H}^{d-1}(C_d(I,A)\cap\mathbb{S}^{d-1}))^{n-2\ell+k}\mathcal{H}^{d-1}(C_d(I,A)\cap\mathbb{S}^{d-1})^{\ell-k+1}V_\ell(I,A)^2\\ &\times\mathbf{1}\{\sphericalangle(z,B)\lesssim\operatorname{vol}_d(I,A)^{1/(d+1)}\}\mathbf{1}_{T_n}\,\mathcal{H}^{d-1}(\mathrm{d}x_1)\cdots\mathcal{H}^{d-1}(\mathrm{d}x_\ell)\nu_\ell(\mathrm{d}B)\\ \lesssim&\sum_{h=h_0}^{\infty}\exp(-c_2(n-2\ell+k)2^{-h-1})2^{-h(\ell-k+1)}2^{-2h(\ell+1)/(d-1)}2^{-h\ell}\\ &\times|\mathcal{M}_h|\nu_\ell(\{\sphericalangle(z,B)\lesssim\operatorname{vol}_d(I,A)^{1/(d+1)}\})\\ \lesssim&\sum_{h=h_0}^{\infty}\exp(-c_2(n-2\ell+k)2^{-h-1})2^{-h[(2\ell-k+1)+(d+3)/(d-1)]}.\end{aligned}$$

Note that we used Lemma 3.7 and Equation (3.17) in the last step. As in [11], we divide the previous sum into two parts in order to see the magnitude of the variance. The integer h_1 is defined by

$$2^{-h_1}\le\frac{1}{n}<2^{-h_1+1}.$$

On the one hand, we have

$$\sum_{h=h_1}^{\infty} \exp(-c_2(n-2\ell+k)2^{-h-1})2^{-h[(2\ell-k+1)+(d+3)/(d-1)]} \leq \sum_{h=h_1}^{\infty} 2^{-h[(2\ell-k+1)+(d+3)/(d-1)]} \lesssim n^{-(2\ell-k+1)}n^{-(d+3)/(d-1)}.$$

On the other hand, let $i = h_1 - h$. Then, we can perform the following estimate, namely,

$$\begin{aligned}
&\sum_{h=h_0}^{h_1-1} \exp(-c_2(n-2\ell+k)2^{-h-1})2^{-h[(2\ell-k+1)+(d+3)/(d-1)]} \\
&\quad \leq \sum_{i=1}^{h_1-h_0} \exp(-c_2(n-2\ell+k)2^{-h_1+i-1})2^{-(h_1-i)[(2\ell-k+1)+(d+3)/(d-1)]} \\
&\quad \lesssim \sum_{i=1}^{h_1-h_0} \exp(-c_2(n-2\ell+k)2^{-h_1+i-1})n^{-(2\ell-k+1)}n^{-(d+3)/(d-1)}2^{i[(2\ell-k+1)+(d+3)/(d-1)]} \\
&\quad \lesssim n^{-(2\ell-k+1)}n^{-(d+3)/(d-1)} \sum_{i=1}^{\infty} \exp(-c_2 2^i)2^{i[(2\ell-k+1)+(d+3)/(d-1)]} \\
&\quad \lesssim n^{-(2\ell-k+1)}n^{-(d+3)/(d-1)} \sum_{j=1}^{\infty} \exp(-c_2 j)j^{5d} \\
&\quad \lesssim n^{-(2\ell-k+1)}n^{-(d+3)/(d-1)}.
\end{aligned}$$

As a consequence, it holds

$$\Sigma_k \lesssim n^{2\ell-k+1} \int_{G(d,\ell)} n^{-(2\ell-k+1)}n^{-(d+3)/(d-1)}\nu_\ell(\mathrm{d}A) \lesssim n^{-(d+3)/(d-1)}.$$

Finally, the upper bound is proven by summing up all Σ_k, $k = 0, \dots, \ell$, in Equation (3.15).

In order to extend the proof to the case of a convex body $K \in \mathcal{K}^2_+$, we follow the ideas presented in [11, Section 6]. By the compactness of ∂K there exist $\gamma > 0$ and $\Gamma > 0$, the global upper and the global lower bound on the principal curvatures of ∂K, respectively. In our setting, all projected images of ∂K also have a boundary with the same properties as ∂K, see for example [74, p. 126 Remark 5]. Without loss of generality we can choose γ and Γ to be also a bound on the principal curvatures of the boundaries of all ℓ-dimensional projections of K. Hence, one can locally approximate ∂K with affine images of balls and the volume of an ℓ-dimensional cap with small height $h > 0$ has order $h^{\frac{\ell+1}{2}}$. Note that $C^K(x, h)$ is the intersection of K with the hyperplane

$\tilde{H}(x,h) = \{y \in \mathbb{R}^d : \langle x-y, u_x \rangle = h\}$. As in [11, Equation (27)], it holds that

$$\big((x - hu_x) + \gamma_1\sqrt{h}\mathbb{B}^d\big) \cap \tilde{H}(x,h) \subset K \cap \tilde{H}(x,h) \\ \subset \big((x - hu_x) + \gamma_2\sqrt{h}\mathbb{B}^d\big) \cap \tilde{H}(x,h),$$

where the constants γ_1, γ_2 depend on γ and Γ. The last equation ensures that Equation (3.16) still holds. □

In the fashion of [11, Section 7], we derive a strong law of large numbers from the upper variance bound together with the following result of [67].

Proposition 3.15. [67, Theorem 1] *Let $K \in \mathcal{K}^2_+$ and choose d random points on ∂K independently and according to the probability distribution $\mathcal{H}^{d-1}$. Then, there exist positive constants $c_{K,\ell}$ depending on ℓ and the principal curvatures of K such that*

$$\lim_{n\to\infty} \big(V_\ell(K) - \mathbf{E}\, V_\ell(K_n)\big) \cdot n^{\frac{2}{d-1}} = c_{K,\ell}, \quad \ell \in \{1,\dots,d\}. \tag{3.18}$$

For the sake of brevity, the explicit expression of $c_{K,\ell}$ is omitted here. It can be found in [67, Equation (2)].

Proof of Theorem 3.11. Let $\ell \in \{1,\dots,d\}$. Chebyshev's inequality and the variance upper bound yield

$$\mathbf{P}\Big(\big|V_\ell(K) - V_\ell(K_n) - \mathbf{E}\big[V_\ell(K) - V_\ell(K_n)\big]\big| \cdot n^{\frac{2}{d-1}} \geq \varepsilon\Big) \leq \varepsilon^{-2} n^{\frac{4}{d-1}}\, \mathbf{Var}[V_\ell(K_n)] \lesssim n^{-1}.$$

Select now the subsequence of indices $n_k = k^2$. Then, it follows

$$\sum_{k=1}^{\infty} \mathbf{P}\Big(\big|V_\ell(K) - V_\ell(K_{n_k}) - \mathbf{E}\big[V_\ell(K) - V_\ell(K_{n_k})\big]\big| \cdot n_k^{\frac{2}{d-1}} \geq \varepsilon\Big) \lesssim \sum_{k=1}^{\infty} k^{-2} < \infty.$$

Applying the Borel-Cantelli Lemma together with Equation (3.18), we obtain that

$$\lim_{k\to\infty} \big(V_\ell(K) - V_\ell(K_{n_k})\big) \cdot n_k^{\frac{2}{d-1}} = c_{K,\ell}$$

holds with probability 1. Note that $V_\ell(K) - V_\ell(K_n)$ is a decreasing and positive sequence. Therefore, this gives

$$\big(V_\ell(K) - V_\ell(K_{n_k})\big) \cdot n_{k-1}^{\frac{2}{d-1}} \leq \big(V_\ell(K) - V_\ell(K_n)\big) \cdot n^{\frac{2}{d-1}} \leq \big(V_\ell(K) - V_\ell(K_{n_{k-1}})\big) \cdot n_k^{\frac{2}{d-1}},$$

whenever $n_{k-1} \leq n \leq n_k$. Taking the limit as $k \to \infty$, $n_{k-1}/n_k \to 1$, which allows us to conclude that the desired limit is reached by the whole sequence with probability 1. $\square$

3.3.5 Central limit theorem

In this last section, we prove the central limit theorem. In contrast with the model of uniform distribution inside the body, where floating bodies were used, here we work with surface bodies as it was already done in [82] for the case of the volume. In addition to that, we make use of the normal approximation bound of Proposition 3.8. Since the arguments are naturally easier to follow for $K = \mathbb{B}^d$, the details are given in this particular setting and the arguments for the general case are stated at the end of the proof.

Proof of Theorem 3.12. First, we prove the central limit theorem for $K = \mathbb{B}^d$. For this reason, let us introduce the two events B_1 and B_2. The event that the random polytope $[X_2, \dots, X_n]$ contains the surface body $K(s \geq \tau_n)$ is denoted by B_1. Due to the definition of B_1, it follows by Lemma 3.2 that

$$\mathbf{P}(B_1^c) \leq c_1 n^{-\alpha},$$

where $c_1 \in (0, \infty)$ is independent of d. We denote by B_2 the event that the random polytope $\bigcap_{W \in \{Y,Y',Z,Z'\}}[W_4, \dots, W_n]$ contains the surface body $K(s \geq \tau_n)$, where Y, Y', Z, Z' are recombinations of the random vector $X = (X_1, \dots, X_n)$. By taking the union bound, we obtain

$$\mathbf{P}(B_2^c) \leq c_2 n^{-\alpha},$$

where $c_2 \in (0, \infty)$ is again independent of d. Next, for any $\ell \in \{1, \dots, d\}$, we apply the bound in Proposition 3.8 to the random variables

$$W = f(X_1, \dots, X_n) \coloneqq V_\ell([X_1, \dots, X_n]) - \mathbf{E}\, V_\ell(K_n).$$

Note that $D_i W = D_i V_\ell(K_n)$ and $D_{i_1,i_2} W = D_{i_1,i_2} V_\ell(K_n)$ for $i, i_1, i_2 \in \{1, \dots, n\}$. Conditioned on the event B_1, we obtain from (2.2),

$$D_1 V_\ell(K_n) = \binom{d}{\ell} \frac{\kappa_d}{\kappa_\ell \kappa_{d-\ell}} \int_{G(d,\ell)} \mathrm{vol}_\ell\big((K_n|L) \setminus ([X_2, \dots, X_n]|L)\big)\, \nu_\ell(\mathrm{d}L). \tag{3.19}$$

We now define a full-dimensional cap C in such a way that $K_n \setminus [X_2, \dots, X_n]$ is contained in C.

We define the visibility region (with respect to the function s) of a point $z \in \partial K$ with parameter $t > 0$ as

$$\mathrm{Vis}_z(t) := \{x \in K(s \le t) : [x, z] \cap K(s \ge t) = \emptyset\},$$

where again $[x, z]$ denotes the closed line segment which connects x and z.

Consider now the visibility region $\mathrm{Vis}_{X_1}(\tau_n)$ of X_1. By definition of the surface body and by Lemma 3.4, the diameter of this visibility region is at most $c_3 \tau_n^{1/(d-1)}$, where $c_3 > 0$. We now indicate with $D(X_1, c_3\tau_n^{1/(d-1)})$ the points on ∂K with distance at most $c_3\tau_n^{1/(d-1)}$ from X_1. Then, $C := \mathrm{conv}\big(D(X_1, c_3\tau_n^{1/(d-1)})\big)$ is a spherical cap and it follows from Lemma 3.5 that C has volume of order at most $\tau_n^{(d+1)/(d-1)}$. We call α the central angle of C. For any subspace $L \in G(d, \ell)$, it holds that $(K_n|L) \setminus ([X_2, \ldots, X_n]|L) \subseteq (C|L)$. We obtain $\mathrm{vol}_\ell(C|L) \lesssim \tau_n^{(\ell+1)/(d-1)}$. Indeed, the height of $C|L$ has the same order as the height of C, namely $\tau_n^{2/(d-1)}$, while the order of its base changes from $((\tau_n)^{1/(d-1)})^{d-1}$ to $((\tau_n)^{1/(d-1)})^{\ell-1}$, since the dimension of L is ℓ. By construction of C, it now follows that if $\sphericalangle(X_1, L)$, the angle between X_1 and L, is too wide compared to α, then $C|L \subseteq K_n|L$, for sufficiently large n. Whenever this occurs, it also holds in particular that $(K_n \setminus [X_2, \ldots, X_n])|L \subseteq K_n|L$, i.e. $K_n|L = [X_2, \ldots, X_n]|L$. In fact, one can check that the integrand in (3.19) can only be non-zero if $\sphericalangle(X_1, L) \lesssim \alpha$. Therefore, we can restrict the integration to the set $\{L \in G(d, \ell) : \sphericalangle(X_1, L) \lesssim \alpha\}$. Moreover, it holds that $\alpha \lesssim \mathrm{vol}_d(C)^{1/(d+1)}$, see e.g. [11, Equation (21)]. According to Lemma 3.7, this gives

$$\nu_\ell\big(\{L \in G(d, \ell) : \sphericalangle(X_1, L) \lesssim \mathrm{vol}_d(C)^{\frac{1}{d+1}}\}\big) \lesssim \tau_n^{\frac{d-\ell}{d-1}}.$$

Putting everything together, we see that

$$D_1 V_\ell(K_n) \lesssim \tau_n^{\frac{\ell+1}{d-1}} \cdot \tau_n^{\frac{d-\ell}{d-1}} \lesssim \left(\frac{\log n}{n}\right)^{\frac{d+1}{d-1}}. \tag{3.20}$$

On the complement B_1^c of B_1 we use the trivial estimate $D_1 V_\ell(K_n) \le V_\ell(K)$. Since $\mathbf{P}(B_1^c) \lesssim n^{-\alpha}$, we obtain

$$\begin{aligned}\mathbf{E}[(D_1 V_\ell(K_n))^p] &= \mathbf{E}[(D_1 V_\ell(K_n))^p\, \mathbf{1}_{B_1}] + \mathbf{E}[(D_1 V_\ell(K_n))^p\, \mathbf{1}_{B_1^c}] \\ &\lesssim \left(\frac{\log n}{n}\right)^{p\frac{d+1}{d-1}},\end{aligned}$$

for all $p \ge 1$. As a consequence, we can bound the terms in the normal approximation bound which involve γ_3 and γ_4. Thus,

$$\frac{\sqrt{n}}{\mathbf{Var}[V_\ell(K_n)]}\sqrt{\gamma_3} \lesssim \frac{\sqrt{n}}{n^{-\frac{d+3}{d-1}}}\left(\frac{\log n}{n}\right)^{2\frac{d+1}{d-1}} = n^{-\frac{1}{2}}(\log n)^{2+\frac{4}{d-1}},$$
$$\frac{n}{(\mathbf{Var}[V_\ell(K_n)])^{\frac{3}{2}}}\gamma_4 \lesssim \frac{n}{n^{-\frac{3}{2}\frac{d+3}{d-1}}}\left(\frac{\log n}{n}\right)^{3\frac{d+1}{d-1}} = n^{-\frac{1}{2}}(\log n)^{3+\frac{6}{d-1}}.$$

By using the Cauchy-Schwarz inequality, we can estimate γ_5 as well. Namely,

$$\gamma_5 \leq \sqrt{\mathbf{Var}[V_\ell(K_n)]} \sup_{A\subseteq\{1,\dots,n\}} \sqrt{\mathbf{E}\big[|D_1 f(X^A)|\big]^6} \lesssim n^{-\frac{1}{2}\frac{d+3}{d-1}}\left(\frac{\log n}{n}\right)^{3\frac{d+1}{d-1}}.$$

Thus, we obtain

$$\frac{n}{(\mathbf{Var}[V_\ell(K_n)])^2}\gamma_5 \lesssim \frac{n}{n^{-2\frac{d+3}{d-1}}} n^{-\frac{1}{2}\frac{d+3}{d-1}}\left(\frac{\log n}{n}\right)^{3\frac{d+1}{d-1}} = n^{-\frac{1}{2}}(\log n)^{3+\frac{6}{d-1}}.$$

In the next step, we consider the terms involving the second difference operator. On the event B_2 it may be concluded from (3.20) that $D_i f(V)^2 \lesssim (\log n/n)^{2\frac{d+1}{d-1}}$ for all $i \in \{1,2,3\}$ and $V \in \{Z, Z'\}$. Moreover, we note that on B_2 the following inclusions hold

$$\{D_{1,2} f(Y) \neq 0\} \subseteq \{\mathrm{Vis}_{Y_1}(\tau_n) \cap \mathrm{Vis}_{Y_2}(\tau_n) \neq \emptyset\} \subseteq \Big\{Y_2 \in \bigcup_{x\in\mathrm{Vis}_{Y_1}(\tau_n)} \mathrm{Vis}_x(\tau_n)\Big\}.$$

The same applies to $D_{1,3} f(Y')$. Thus,

$$\mathbf{E}\big[\mathbf{1}\{D_{1,2} f(Y) \neq 0\}\mathbf{1}_{B_2}\big] \leq \sup_{z\in\partial K} \mathbf{P}\Big(Y_2 \in \bigcup_{x\in\mathrm{Vis}_z(\tau_n)} \mathrm{Vis}_x(\tau_n)\Big).$$

We note that the diameter of the previous union is at most $c_4\tau_n^{1/(d-1)}$, where $c_4 > 0$. As before, we define the spherical cap $C' := \mathrm{conv}(D(z, c_4\tau_n^{1/(d-1)}))$. It follows from Lemma 3.5 that C' has volume of order at most $\tau_n^{(d+1)/(d-1)}$. We obtain

$$\begin{aligned}\sup_{z\in\partial K} \mathbf{P}\Big(Y_2 \in \bigcup_{x\in\mathrm{Vis}_z(\tau_n)} \mathrm{Vis}_x(\tau_n)\Big) &= \sup_{z\in\partial K} \mathcal{H}^{d-1}\Big(\Big(\bigcup_{x\in\mathrm{Vis}_z(\tau_n)} \mathrm{Vis}_x(\tau_n)\Big) \cap \partial K\Big)\\ &\leq \sup_{z\in\partial K} \mathcal{H}^{d-1}\big(C' \cap \partial K\big)\\ &\lesssim \tau_n,\end{aligned}$$

where for the last inequality we have used Lemma 3.4. On the event B_2^c we use the trivial estimate $V_\ell(K)$ for all difference operators and estimate all indicators by one. Since $\mathbf{P}(B_2^c) \lesssim n^{-\alpha}$, we obtain

$$\gamma_2 \lesssim \left(\frac{\log n}{n}\right)^{1+4\frac{d+1}{d-1}}.$$

Analogously, we can bound γ_1. Indeed, suppose that $Y_1 = Y_1'$ (by independence, $Y_1 \neq Y_1'$ gives a smaller order), then

$$\{D_{1,2}f(Y) \neq 0\} \cap \{D_{1,3}f(Y') \neq 0\} \subseteq \left\{ \{Y_2, Y_3'\} \subseteq \bigcup_{x \in \mathrm{Vis}_{Y_1}(\tau_n)} \mathrm{Vis}_x(\tau_n) \right\}$$

and we obtain

$$\mathbf{E}\big[\mathbf{1}\{D_{1,2}f(Y) \neq 0\}\mathbf{1}\{D_{1,3}f(Y') \neq 0\}\big] \lesssim \left(\frac{\log n}{n}\right)^2.$$

Thus,

$$\gamma_1 \lesssim \left(\frac{\log n}{n}\right)^{2+4\frac{d+1}{d-1}}.$$

Finally,

$$\begin{aligned} \frac{\sqrt{n}}{\mathbf{Var}[V_\ell(K_n)]}\sqrt{n^2\gamma_1} &\lesssim \frac{\sqrt{n}}{n^{-\frac{d+3}{d-1}}}\sqrt{n^2\left(\frac{\log n}{n}\right)^{2+4\frac{d+1}{d-1}}} = n^{-\frac{1}{2}}(\log n)^{3+\frac{4}{d-1}}, \\ \frac{\sqrt{n}}{\mathbf{Var}[V_\ell(K_n)]}\sqrt{n\gamma_2} &\lesssim \frac{\sqrt{n}}{n^{-\frac{d+3}{d-1}}}\sqrt{n\left(\frac{\log n}{n}\right)^{1+4\frac{d+1}{d-1}}} = n^{-\frac{1}{2}}(\log n)^{\frac{5}{2}+\frac{4}{d-1}}. \end{aligned}$$

Considering all the estimates together, we obtain by Proposition 3.8

$$\begin{aligned} d_K\big(W_\ell(K_n), N\big) &\lesssim n^{-\frac{1}{2}}\big((\log n)^{3+\frac{4}{d-1}} + (\log n)^{\frac{5}{2}+\frac{4}{d-1}} \\ &\qquad + (\log n)^{2+\frac{4}{d-1}} + (\log n)^{3+\frac{6}{d-1}} + (\log n)^{3+\frac{6}{d-1}}\big) \\ &\lesssim n^{-\frac{1}{2}}(\log n)^{3+\frac{6}{d-1}}. \end{aligned}$$

For the case of a generic $K \in \mathcal{K}_+^2$ we argue as at the end of the proof of the upper bound of Theorem 3.10. Because of the global bounds on the principal curvatures and the local approximation of ∂K with affine images of balls, the construction of C and the relations regarding its volume, its central angle and the subspaces L which ensure $C|L \subseteq K_n|L$ are not afflicted. In particular, the asymptotic bounds $\mathrm{vol}_\ell(C|L) \lesssim \tau_n^{(\ell+1)/(d-1)}$, $\alpha \lesssim \mathrm{vol}_d(C)^{1/(d+1)} \lesssim \tau_n^{1/(d-1)}$ and $\sphericalangle(X_1, L) \lesssim \alpha$ stated above still hold, with the difference that the implicit constants depend on γ and Γ, the bounds on the principal curvatures of ∂K. The proof can be completed like in the case of the ball. $\square$

Chapter 4

Monotonicity of the Facets Number for Beta and Beta-prime Polytopes

In this chapter we study the expectation of the number of facets of convex hulls of independent random points distributed according to certain probability distributions. In particular, we introduce the following four classes of probability measures:

- $\mathcal{G}$ is the class of centred Gaussian distributions on $\mathbb{R}^d$ with density proportional to
$$x \mapsto \exp\Big(-\frac{\|x\|^2}{2\sigma^2}\Big), \qquad \sigma > 0,$$
- $\mathcal{H}$ is the class of heavy-tailed distributions on $\mathbb{R}^d$ with density proportional to
$$x \mapsto \Big(1 + \frac{\|x\|^2}{2\sigma^2}\Big)^{-\beta}, \qquad \beta > d/2, \sigma > 0,$$
also called beta-prime distributions.
- $\mathcal{B}$ is the class of beta-type distributions on the d-dimensional centred ball $\mathbb{B}^d_\sigma$ of radius σ with density proportional to
$$x \mapsto \Big(1 - \frac{\|x\|^2}{2\sigma^2}\Big)^{\beta}, \qquad \beta > -1, \sigma > 0,$$
also called beta distributions.
- $\mathcal{U}$ comprises the uniform distributions on the $(d-1)$-dimensional centred spheres $\mathbb{S}^{d-1}_\sigma$ with radius $\sigma > 0$.

We show the validity of following statement:

Theorem 4.1. *Let $X_1, \ldots, X_n \in \mathbb{R}^d$, $n > d$, be independent and identically distributed according to a probability measure belonging to one of the classes $\mathcal{G}$, $\mathcal{H}$, $\mathcal{B}$ or $\mathcal{U}$. Let*

$$P_{n-1} := \operatorname{conv}(X_1, \ldots, X_{n-1}) \quad \textit{and} \quad P_n := \operatorname{conv}(X_1, \ldots, X_n).$$

Then

$$\mathbf{E}\, f_{d-1}(P_n) > \mathbf{E}\, f_{d-1}(P_{n-1}).$$

It will turn out that the classes $\mathcal{G}$, $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{U}$ contain precisely the absolutely continuous rotationally symmetric probability distributions on $\mathbb{R}^d$, whose densities satisfy the natural scaling property (4.10) below, for which monotonicity of the mean facet number of the associated random convex hulls can be shown by means of arguments based on a Blaschke-Petkantschin formula, see the discussion at the end of Section 4.3 for further details. In fact, our result shows that even the stronger strict monotonicity holds.

Remark 4. It is important to note that the result is not trivial. Indeed, the fact that since the addition of a further random point can reduce the facet number implies that strict monotonicity of $n \mapsto f_{d-1}(P_n)$ cannot hold for every realization, whenever $n > d + 1$. For this reason, the expectation in Theorem 4.1 is essential.

4.1 Background results from integral geometry

We denote by $A(d, q)$ the Grassmannian of all q-dimensional affine subspaces of $\mathbb{R}^d$, where $q \in \{0, 1, \ldots, d\}$. It is a locally compact, homogeneous space with respect to the group of Euclidean motions in $\mathbb{R}^d$. The corresponding locally finite, motion invariant measure is denoted by μ_q, which is normalized in such a way that

$$\mu_q(\{E \in A(d, q) : E \cap \mathbb{B}_2^d \neq \emptyset\}) = \kappa_{d-q},$$

see [76]. For a subspace $E \in A(d, q)$, we let λ_E be the Lebesgue measure on E.

4.1.1 Blaschke-Petkantschin formulas

Our proof of Theorem 4.1 heavily relies on Blaschke-Petkantschin formulae from integral geometry. First, we rephrase a special case of the affine Blaschke-Petkantschin formula in $\mathbb{R}^d$, which appears as Theorem 7.2.7 in [76].

Proposition 4.2. *Let $f\colon (\mathbb{R}^d)^d \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int_{(\mathbb{R}^d)^d} f(x_1,\dots,x_d)\,\mathrm{d}(x_1,\dots,x_d)$$
$$= \frac{\omega_d}{2}(d-1)! \int_{A(d,d-1)} \int_{H^d} f(x_1,\dots,x_d)\Delta_{d-1}(x_1,\dots,x_d)\,\lambda_H^d(\mathrm{d}(x_1,\dots,x_d))\,\mu_{d-1}(\mathrm{d}H).$$

Besides the affine Blaschke-Petkantschin formula in $\mathbb{R}^d$ we need its spherical counterpart, which is a special case of Theorem 1 in [89] and can also be found in [59, Theorem 4].

Proposition 4.3. *Let $f\colon (\mathbb{S}^{d-1})^d \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int_{(\mathbb{S}^{d-1})^d} f(x_1,\dots,x_d)\,\mathcal{H}^{d(d-1)}_{(\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,\dots,x_d)) = (d-1)! \int_{A(d,d-1)} \int_{(H\cap\mathbb{S}^{d-1})^d} f(x_1,\dots,x_d)$$
$$\times\, \Delta_{d-1}(x_1,\dots,x_d)(1-h^2)^{-\frac{d}{2}}\,\mathcal{H}^{d(d-2)}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,\dots,x_d))\,\mu_{d-1}(\mathrm{d}H),$$

where h denotes the distance from H to the origin.

4.1.2 A slice integration formula

Finally, we will make use of the following special case of the spherical slice integration formula taken from Theorem A.4 in [9].

Proposition 4.4. *Let $f\colon \mathbb{S}^{d-1} \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int_{\mathbb{S}^{d-1}} f(x)\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}x) = \int_{-1}^{1} (1-t^2)^{\frac{d-3}{2}} \int_{\mathbb{S}^{d-2}} f(t, \sqrt{1-t^2}\,y)\,\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}y)\,\mathrm{d}t.$$

4.2 Preparatory results

4.2.1 An estimate for integrals of concave functions

The proof of the main result will make use of the next lemma, as stated in [19]. Since no proof was given in the reference, we include it here.

Lemma 4.5. *Let $h\colon (0,1) \to \mathbb{R}$ be a non-negative measurable function such that*

$$0 < \int_0^1 h(s)\,\mathrm{d}s < \infty. \tag{4.1}$$

Further, let $g\colon [0,1] \to \mathbb{R}$ be a linear function with negative slope and root $s^* \in (0,1)$. Moreover, let $L\colon \mathbb{R} \to \mathbb{R}$ be positive and strictly concave on $[0,1]$. Then,

$$\int_0^1 h(s)g(s)L(s)^{d-1}\,\mathrm{d}s > \int_0^1 h(s)g(s)\ell(s)^{d-1}\,\mathrm{d}s, \tag{4.2}$$

where $\ell(s) = \frac{L(s^*)}{s^*}s$.

Proof. We start by exploiting the positivity and strict concavity of L. For $s \in [0, s^*)$, it implies that

$$L(s) = L\left(\frac{s}{s^*}s^*\right) > \frac{s}{s^*}L(s^*), \tag{4.3}$$

while for $s \in (s^*, 1]$, it gives

$$L(s) < \frac{s}{s^*}L(s^*). \tag{4.4}$$

Since the derivative of g is negative, g is positive on $[0, s^*)$ and negative on $(s^*, 1]$. Splitting the integral on the left hand side of (4.2) at the point s^* and using (4.3) and (4.4), respectively, yields

$$\begin{aligned}
&\int_0^1 h(s)g(s)L(s)^{d-1}\,\mathrm{d}s \\
&\quad= \int_0^{s^*} h(s)g(s)L(s)^{d-1}\,\mathrm{d}s + \int_{s^*}^1 h(s)g(s)L(s)^{d-1}\,\mathrm{d}s \\
&\quad> \int_0^{s^*} h(s)g(s)\Big(\frac{s}{s^*}L(s^*)\Big)^{d-1}\,\mathrm{d}s + \int_{s^*}^1 h(s)g(s)\Big(\frac{s}{s^*}L(s^*)\Big)^{d-1}\,\mathrm{d}s \\
&\quad= \int_0^1 h(s)g(s)\ell(s)^{d-1}\,\mathrm{d}s.
\end{aligned}$$

This completes the argument. □

4.2.2 Computation of marginal densities

Recall the definitions of the distribution classes $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{U}$. As it will be clear later on, it suffices to consider the cases where the scale parameters σ, i.e. the radius of the supporting ball - are equal to 1. For this reason, from now on we restrict to these cases and denote the density of a distribution in $\mathcal{H}$ by

$$p_{\mathcal{H},\beta}(x) = \pi^{-d/2}\frac{\Gamma(\beta)}{\Gamma(\beta - \frac{d}{2})}(1 + \|x\|^2)^{-\beta}\,, \qquad x \in \mathbb{R}^d\,, \beta > d/2\,, \tag{4.5}$$

that of a distribution in $\mathcal{B}$ by

$$p_{\mathcal{B},\beta}(x) = \pi^{-d/2}\frac{\Gamma(\frac{d}{2}+\beta+1)}{\Gamma(\beta+1)}(1-\|x\|^2)^\beta\,, \qquad x\in\mathbb{B}_2^d\,,\beta>-1\,,$$

and note that the uniform distribution on $\mathbb{S}^{d-1}$ has density

$$p_{\mathcal{U}}(x) = \frac{1}{\omega_d}\,, \qquad x\in\mathbb{S}^{d-1}\,,$$

with respect to the spherical Lebesgue measure. The next lemma provides formulas for the densities of the one-dimensional marginals of these distributions and shows, that the classes $\mathcal{B}$ and $\mathcal{H}$ are in some sense closed under one-dimensional projections. Since all distributions we consider are rotationally symmetric, it is sufficient to consider projections onto the first coordinate. We would like to emphasize that the proof of Lemma 4.6 uses in an essential way the scaling property (4.10) below of the involved densities.

Lemma 4.6. *Let $\Pi\colon\mathbb{R}^d\to\mathbb{R}$ be the projection onto the first coordinate, namely $\Pi(x_1,\ldots,x_d)=x_1$ for any $(x_1,\ldots,x_d)\in\mathbb{R}^d$.*

(i) Let $\mathbf{P}\in\mathcal{H}$ be a distribution with density $p_{\mathcal{H},\beta}$ for some $\beta>d/2$. Then, the image measure of $\mathbf{P}$ under Π has density

$$f_{\mathcal{H},\beta}(x) = \pi^{-1/2}\frac{\Gamma\left(\beta-\frac{d-1}{2}\right)}{\Gamma\left(\beta-\frac{d}{2}\right)}(1+x^2)^{\frac{d-1}{2}-\beta}, \qquad x\in\mathbb{R}.$$

(ii) Let $\mathbf{P}\in\mathcal{B}$ be a distribution with density $p_{\mathcal{B},\beta}$ for some $\beta>-1$. Then, the image measure of $\mathbf{P}$ under Π has density

$$f_{\mathcal{B},\beta}(x) = \pi^{-1/2}\frac{\Gamma\big(\beta+1+\frac{d}{2}\big)}{\Gamma\big(\beta+\frac{d+1}{2}\big)}(1-x^2)^{\frac{d-1}{2}+\beta}, \qquad x\in[-1,1].$$

(iii) Let $\mathbf{P}\in\mathcal{U}$ be the uniform distribution on $\mathbb{S}^{d-1}$. Then, the image measure of $\mathbf{P}$ under Π has density

$$f_{\mathcal{U}}(x) = \pi^{-1/2}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}(1-x^2)^{\frac{d-3}{2}}, \qquad x\in[-1,1].$$

Proof. To prove (i) we put $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$, $y:=(x_2,\ldots,x_d)$ and also define

$c_{\mathcal{H},d,\beta} \coloneqq \pi^{-d/2}\frac{\Gamma(\beta)}{\Gamma(\beta-d/2)}$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} c_{\mathcal{H},d,\beta}\left(1+\|x\|^2\right)^{-\beta} \mathrm{d}(x_2,\ldots,x_d) \\
&\quad= \int_{\mathbb{R}^{d-1}} c_{\mathcal{H},d,\beta}(1+x_1^2)^{-\beta}\left(1+\frac{\|y\|^2}{1+x_1^2}\right)^{-\beta} \mathrm{d}y \\
&\quad= (1+x_1^2)^{-\beta}\int_{\mathbb{R}^{d-1}} c_{\mathcal{H},d,\beta}\left(1+\|z\|^2\right)^{-\beta}(1+x_1^2)^{\frac{d-1}{2}} \mathrm{d}z \\
&\quad= (1+x_1^2)^{\frac{d-1}{2}-\beta}\frac{c_{\mathcal{H},d,\beta}}{c_{\mathcal{H},d-1,\beta}}\int_{\mathbb{R}^{d-1}} c_{\mathcal{H},d-1,\beta}\left(1+\|z\|^2\right)^{-\beta} \mathrm{d}z \\
&\quad= \frac{c_{\mathcal{H},d,\beta}}{c_{\mathcal{H},d-1,\beta}}(1+x_1^2)^{\frac{d-1}{2}-\beta},
\end{aligned}$$

where we used the substitution $z = y/\sqrt{1+x_1^2}$. Plugging in the constants yields the desired result.

Next, we consider the distribution with density $p_{\mathcal{B},\beta}$. For $x = (x_1,\ldots,x_d) \in \mathbb{B}^d$, we put again $y \coloneqq (x_2,\ldots,x_d)$ and abbreviate $c_{\mathcal{B},d,\beta} \coloneqq \pi^{-d/2}\frac{\Gamma(\frac{d}{2}+\beta+1)}{\Gamma(\beta+1)}$. Then, similarly as above, we compute

$$\begin{aligned}
&\int_{\mathbb{B}^{d-1}} c_{\mathcal{B},d,\beta}\left(1-\|x\|^2\right)^{\beta} \mathrm{d}(x_2,\ldots,x_d) \\
&= \int_{\mathbb{B}^{d-1}} c_{\mathcal{B},d,\beta}(1-x_1^2)^{\beta}\left(1-\frac{\|y\|^2}{1-x_1^2}\right)^{\beta} \mathrm{d}y \\
&= (1-x_1^2)^{\beta}\int_{\mathbb{B}^{d-1}} c_{\mathcal{B},d,\beta}\left(1-\|z\|^2\right)^{\beta}(1-x_1^2)^{\frac{d-1}{2}} \mathrm{d}z \\
&= (1-x_1^2)^{\frac{d-1}{2}+\beta}\frac{c_{\mathcal{B},d,\beta}}{c_{\mathcal{B},d-1,\beta}}\int_{\mathbb{B}^{d-1}} c_{\mathcal{B},d-1,\beta}\left(1-\|z\|^2\right)^{\beta} \mathrm{d}z \\
&= \frac{c_{\mathcal{B},d,\beta}}{c_{\mathcal{B},d-1,\beta}}(1-x_1^2)^{\frac{d-1}{2}+\beta},
\end{aligned}$$

where we used the substitution $z = y/\sqrt{1-x_1^2}$. Again, simplification of the constants yields the desired result.

Finally, we consider the case of the uniform distribution on $\mathbb{S}^{d-1}$. We denote by F the distribution function of the image measure of $\mathbf{P} = \omega_d^{-1}\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}$ under the orthogonal projection map Π and let $x_1 \in [-1,1]$. Using the slice integration formula from

Proposition 4.4, we obtain

$$\begin{aligned}
\mathrm{F}(x_1) &= \frac{1}{\omega_d}\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}\left(\left\{u \in \mathbb{S}^{d-1} : \Pi(u) \in [-1, x_1]\right\}\right) \\
&= \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} \mathbf{1}\{\Pi(u) \in [-1, x_1]\}\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&= \frac{1}{\omega_d}\int_{-1}^{x_1} (1-t^2)^{\frac{d-3}{2}} \int_{\mathbb{S}^{d-2}} \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}y)\,\mathrm{d}t \\
&= \frac{\omega_{d-1}}{\omega_d}\int_{-1}^{x_1} (1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t.
\end{aligned}$$

Differentiation with respect to x_1, together with the definitions of ω_d and ω_{d-1}, complete the proof. □

In what follows, we shall denote by $F_{\mathcal{H},\beta}$, $F_{\mathcal{B},\beta}$ and $F_{\mathcal{U}}$ the distribution functions corresponding to the densities $f_{\mathcal{H},\beta}$, $f_{\mathcal{B},\beta}$ and $f_{\mathcal{U}}$ computed in Lemma 4.6, respectively.

Remark 5. The marginal densities of the Gaussian distributions $\mathcal{G}$ can also be computed along the lines of the proof of Lemma 4.6. This yields one-dimensional Gaussian marginals. Since random convex hulls of Gaussian points have already been treated in [19], we decided to concentrate on the classes $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{U}$.

4.3 Proof of the main result

Based on the results from the two previous sections we are now able to present the proof of our main result.

Proof of Theorem 4.1. For the classes $\mathcal{G}$, $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{U}$ it is sufficient to consider the case that the scale parameter σ is equal to 1, since the mean facet number is invariant under rescalings.

The case of the class $\mathcal{G}$ has already been treated in [19], so we refer to Theorem 5.3.1 there.

Next, we consider the heavy-tailed distribution on $\mathbb{R}^d$ with density $p_{\mathcal{H},\beta}(x) = c_{\mathcal{H},d,\beta}(1+\|x\|^2)^{-\beta}$, where $\beta > d/2$ and $c_{\mathcal{H},d,\beta} = \pi^{-d/2}\frac{\Gamma(\beta)}{\Gamma(\beta-d/2)}$. Following the ideas of [19], we start with the equality

$$\begin{aligned}
\mathbf{E}\, f_{d-1}(P_n) &= \mathbf{E} \sum_{1\le i_1<\ldots<i_d\le n} \mathbf{1}\{\mathrm{conv}(X_{i_1},\ldots,X_{i_d}) \text{ is a facet of } P_n\} \\
&= \binom{n}{d}\mathbf{P}(\mathrm{conv}(X_1,\ldots,X_d) \text{ is a facet of } P_n),
\end{aligned} \tag{4.6}$$

which holds due to the fact that the random points $X_1, \ldots, X_n$ are independent and identically distributed. Let us denote by $H \in A(d, d-1)$ the affine hull of the $(d-1)$-dimensional simplex P_d spanned by $X_1, \ldots, X_d$. In the case that P_d is a facet of P_n, all the remaining points $X_{d+1}, \ldots, X_n$ have to lie in one of the (open) half-spaces determined by H. If we denote by Π_H the orthogonal projection onto $H^\perp$, the orthogonal complement of H, we observe that P_d is a facet of P_n if and only if the point $\Pi_H(P_d)$ is not contained in the interior of the interval $\Pi_H(P_n)$ on $H^\perp$. Therefore, using Lemma 4.6, the affine Blaschke-Petkantschin formula from Proposition 4.2 and the abbreviation $F^* = F_{\mathcal{H},\beta}(\Pi_H(P_d))$, we get for the probability that P_d is a facet of P_n,

$$\begin{aligned}
&\mathbf{P}(\mathrm{conv}(X_1, \ldots, X_d) \text{ is a facet of } P_n) \\
&= \int_{(\mathbb{R}^d)^d} \big((1-F^*)^{n-d} + (F^*)^{n-d}\big) \prod_{i=1}^d c_{\mathcal{H},d,\beta}(1+\|x_i\|^2)^{-\beta} \, \mathrm{d}(x_1, \ldots, x_d) \\
&= c \int_{A(d,d-1)} \int_{H^d} \big((1-F^*)^{n-d} + (F^*)^{n-d}\big) \, \Delta_{d-1}(x_1, \ldots, x_d) \prod_{i=1}^d c_{\mathcal{H},d,\beta}(1+\|x_i\|^2)^{-\beta} \\
&\qquad \times \lambda_H^d(\mathrm{d}(x_1, \ldots, x_d)) \, \mu_{d-1}(\mathrm{d}H).
\end{aligned}$$

Next, we use the theorem of Pythagoras to decompose, for each $i \in \{1, \ldots, d\}$, the norm $\|x_i\|$. Namely, writing $\|\cdot\|_H$ for the Euclidean norm in $H \in A(d, d-1)$ and h for the distance from H to the origin in $\mathbb{R}^d$, we have that

$$\|x_i\|^2 = \|x_i\|_H^2 + h^2.$$

Therefore and as already used in the proof of Lemma 4.6, the last term of the integrand can be rewritten as

$$(1+\|x_i\|^2)^{-\beta} = (1+h^2+\|x_i\|_H^2)^{-\beta} = (1+h^2)^{-\beta}\Big(1+\frac{\|x_i\|_H^2}{1+h^2}\Big)^{-\beta}. \tag{4.7}$$

Moreover, since each hyperplane $H = H(u, h)$ is uniquely determined by its unit normal vector $u \in \mathbb{S}^{d-1}$ and its distance $h \in [0, \infty)$ to the origin, the integration over $A(d, d-1)$ can be replaced by a twofold integral over $\mathbb{S}^{d-1}$ and $[0, \infty)$. Using the substitutions $y_i = x_i/\sqrt{1+h^2}$ with $\lambda_H(\mathrm{d}x_i) = (1+h^2)^{(d-1)/2}\lambda_H(\mathrm{d}y_i)$, the rotational invariance of the underlying probability measure, and writing $F(h)$ for $F_{\mathcal{H},\beta}(h)$ as well as $f(h)$ for

$f_{\mathcal{H},\beta}(h)$, gives in view of Lemma 4.6 that

$$\begin{aligned}
&\mathbf{P}(\mathrm{conv}(X_1,\dots,X_d) \text{ is a facet of } P_n)\\
&= c\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_{H^d}\left((1-F(h))^{n-d}+F(h)^{n-d}\right)\Delta_{d-1}(x_1,\dots,x_d)\\
&\qquad\times(1+h^2)^{-d\beta}\prod_{i=1}^d c_{\mathcal{H},d,\beta}\left(1+\frac{\|x_i\|_H^2}{1+h^2}\right)^{-\beta}\lambda_H^d(\mathrm{d}(x_1,\dots,x_d))\,\mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)\\
&= c\int_{\mathbb{S}^{d-1}}\int_0^\infty\left((1-F(h))^{n-d}+F(h)^{n-d}\right)(1+h^2)^{-d\left(\beta-\frac{d-1}{2}\right)+\frac{d-1}{2}}\,\mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)\\
&\qquad\times\int_{H^d}\Delta_{d-1}(y_1,\dots,y_d)\prod_{i=1}^d c_{\mathcal{H},d-1,\beta}\left(1+\|y_i\|_H^2\right)^{-\beta}\lambda_H^d(\mathrm{d}(y_1,\dots,y_d))\\
&= c\int_0^\infty\left((1-F(h))^{n-d}+F(h)^{n-d}\right)(1+h^2)^{-d\left(\beta-\frac{d-1}{2}\right)+\frac{d-1}{2}}\,\mathrm{d}h\\
&= c\int_{-\infty}^\infty(1-F(h))^{n-d}f(h)^d(1+h^2)^{\frac{d-1}{2}}\,\mathrm{d}h,
\end{aligned}$$

where we also used the fact that the integral over H^d is a finite constant given by Equation (72) in [59] and which only depends on the space dimension d and on β.

Write now $s = F(h)$ and $L(s) = f\left(F^{-1}(s)\right)\sqrt{1+(F^{-1}(s))^2}$ to obtain

$$\mathbf{P}(\mathrm{conv}(X_1,\dots,X_d) \text{ is a facet of } P_n) = c\int_0^1(1-s)^{n-d}L(s)^{d-1}\,\mathrm{d}s.$$

Thus, combination of the above computation with (4.6) yields the representation

$$\begin{aligned}
&\mathbf{E}\,f_{d-1}(P_n)-\mathbf{E}\,f_{d-1}(P_{n-1})\\
&\qquad= c\int_0^1\left[\binom{n}{d}(1-s)-\binom{n-1}{d}\right](1-s)^{n-d-1}L(s)^{d-1}\,\mathrm{d}s.
\end{aligned}\tag{4.8}$$

In order to apply Lemma 4.5, we have to verify that $L(s)$ is strictly concave on $(0,1)$. We prove this by showing that the second derivative of $L(s)$ is negative. So, let $c_{\mathcal{H},1,\beta} := \pi^{-1/2}\frac{\Gamma(\beta)}{\Gamma(\beta-1/2)}$ and recall that $f(x) = c_{\mathcal{H},1,\beta}(1+x^2)^{\frac{d-1}{2}-\beta}$ from Lemma 4.6. Furthermore, from the definition of F it follows that

$$\left(F^{-1}(s)\right)' = \frac{1}{f\left(F^{-1}(s)\right)} = \frac{1}{c_{\mathcal{H},1,\beta}\left(1+(F^{-1}(s))^2\right)^{\frac{d-1}{2}-\beta}}.\tag{4.9}$$

We recall that

$$L(s) = f\big(F^{-1}(s)\big)\sqrt{1+(F^{-1}(s))^2} = c_{\mathcal{H},1,\beta}\big(1+(F^{-1}(s))^2\big)^{\frac{d}{2}-\beta}.$$

Hence, using (4.9), the first derivative of $L(s)$ is

$$\begin{aligned} L'(s) &= c_{\mathcal{H},1,\beta}\Big(\frac{d}{2}-\beta\Big)\big(1+(F^{-1}(s))^2\big)^{\frac{d-2}{2}-\beta} 2F^{-1}(s)\big(F^{-1}(s)\big)' \\ &= 2\Big(\frac{d}{2}-\beta\Big)\big(1+(F^{-1}(s))^2\big)^{-\frac{1}{2}} F^{-1}(s) \end{aligned}$$

and, thus, for the second derivative we find that

$$\begin{aligned} L''(s) &= 2\left(\frac{d}{2}-\beta\right)\Big[\big(1+(F^{-1}(s))^2\big)^{-\frac{1}{2}}\left(F^{-1}(s)\right)' \\ &\qquad -\frac{1}{2}\left(1+(F^{-1}(s))^2\right)^{-\frac{3}{2}} 2\left(F^{-1}(s)\right)^2\left(F^{-1}(s)\right)'\Big] \\ &= 2\left(\frac{d}{2}-\beta\right)\left(F^{-1}(s)\right)'\Big[\big(1+(F^{-1}(s))^2\big)^{-\frac{1}{2}} - \big(1+(F^{-1}(s))^2\big)^{-\frac{3}{2}}\left(F^{-1}(s)\right)^2\Big] \\ &= \frac{2}{c_{\mathcal{H},1,\beta}}\left(\frac{d}{2}-\beta\right)\left(1+(F^{-1}(s))^2\right)^{\beta-1-\frac{d}{2}}\left[1+(F^{-1}(s))^2-(F^{-1}(s))^2\right] \\ &= -\frac{2}{c_{\mathcal{H},1,\beta}}\left(\beta-\frac{d}{2}\right)\left(1+(F^{-1}(s))^2\right)^{\beta-1-\frac{d}{2}} \\ &< 0, \end{aligned}$$

where the last inequality follows from the fact that $\beta > d/2$. As a consequence, we can apply Lemma 4.5 to deduce that

$$\begin{aligned} &\mathbf{E}\, f_{d-1}(P_n) - \mathbf{E}\, f_{d-1}(P_{n-1}) \\ &= c\int_0^1 \left[\binom{n}{d}(1-s) - \binom{n-1}{d}\right](1-s)^{n-d-1}L(s)^{d-1}\,\mathrm{d}s \\ &> \left(\frac{L(d/n)}{d/n}\right)^{d-1}\binom{n}{d}\int_0^1 (1-s)^{n-d-1}s^{d-1}\Big((1-s)-\frac{n-d}{n}\Big)\,\mathrm{d}s \\ &= \left(\frac{L(d/n)}{d/n}\right)^{d-1}\binom{n}{d}\Big(\mathrm{B}(d,n-d+1)-\frac{n-d}{n}\mathrm{B}(d,n-d)\Big) \\ &= 0, \end{aligned}$$

where we used the well-known relation $\mathrm{B}(d,n-d+1) = \frac{n-d}{n}\mathrm{B}(d,n-d)$ for the beta function.

As the next case we consider the class $\mathcal{B}$ of beta-type distribution on the unit ball $\mathbb{B}_2^d$ with density $f_{\mathcal{B},\beta}$ for some $\beta > -1$. In this case the proof follows almost line by line the proof for $\mathcal{H}$, up to some minor modifications. In particular, (4.8) stays the same except that now $L(s) = f\left(F^{-1}(s)\right)\sqrt{1-(F^{-1}(s))^2}$, where $F(h) = F_{\mathcal{B},\beta}(h)$ and $f(h) = f_{\mathcal{B},\beta}(h)$. Therefore, it follows that

$$L''(s) = -\frac{2}{c_{\mathcal{B},1,\beta}}\Big(\beta + \frac{d}{2}\Big)\big(1-(F^{-1}(s))^2\big)^{-\beta-1-\frac{d}{2}},$$

where the constant $c_{\mathcal{B},1,\beta}$ is $c_{\mathcal{B},1,\beta} \coloneqq \pi^{-1/2}\Gamma(\beta+\frac{3}{2})\Gamma(\beta+1)^{-1}$. Since $F^{-1}(s) \in (-1,1)$, we obtain $L''(s) < 0$ and can conclude as in the proof for the class $\mathcal{H}$ presented above.

Finally, we consider the case of the uniform distribution on $\mathbb{S}^{d-1}$. Here we get by applying the spherical Blaschke-Petkantschin formula from Proposition 4.3 and using the abbreviations $F(h) = F_{\mathcal{U}}(h)$ and $f(h) = f_{\mathcal{U}}(h)$,

$$\begin{aligned}
&\mathbf{P}(\mathrm{conv}(X_1,\dots,X_d) \text{ is a facet of } P_n)\\
&= c\int_{A(d,d-1)}\int_{(H\cap\mathbb{S}^{d-1})^d}\big((1-F(h))^{n-d}+F(h)^{n-d}\big)\,\Delta_{d-1}(x_1,\dots,x_d)(1-h^2)^{-\frac{d}{2}}\\
&\qquad\times\mathcal{H}^{d(d-2)}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,\dots,x_d))\,\mu_{d-1}(\mathrm{d}H)\\
&= c\int_{\mathbb{S}^{d-1}}\int_0^1\int_{(H\cap\mathbb{S}^{d-1})^d}\big((1-F(h))^{n-d}+F(h)^{n-d}\big)\,\Delta_{d-1}(x_1,\dots,x_d)(1-h^2)^{-\frac{d}{2}}\\
&\qquad\times\mathcal{H}^{d(d-2)}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,\dots,x_d))\,\mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)\\
&= c\int_{\mathbb{S}^{d-1}}\int_0^1\big((1-F(h))^{n-d}+F(h)^{n-d}\big)\,(1-h^2)^{d\frac{d-2}{2}+\frac{d-1}{2}-\frac{d}{2}}\,\mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)\\
&\qquad\times\int_{(\mathbb{S}^{d-2})^d}\Delta_{d-1}(y_1,\dots,y_d)\,\mathcal{H}^{d(d-2)}_{(\mathbb{S}^{d-2})^d}(\mathrm{d}(y_1,\dots,y_d)),
\end{aligned}$$

where the substitution $x_i = y_i\sqrt{1-h^2}$ with $\mathcal{H}^{d-2}_{H\cap\mathbb{S}^{d-1}}(\mathrm{d}x_i) = (1-h^2)^{\frac{d-2}{2}}\mathcal{H}^{d-2}_{H\cap\mathbb{S}^{d-1}}(\mathrm{d}y_i)$ was used. In particular, this transforms the integration over $(H\cap\mathbb{S}^{d-1})^d$ into a d-fold integral over the unit sphere in H, which in turn has been identified with $\mathbb{S}^{d-2}$ due to rotational invariance. Since the integral over $(\mathbb{S}^{d-2})^d$ is a known positive constant only depending on d (the precise value can be deduced from [76, Theorem 8.2.3], for example), we get by rotational invariance of the underlying distribution that

$$
\begin{aligned}
&\mathbf{P}(\mathrm{conv}(X_1,\dots,X_d) \text{ is a facet of } P_n)\\
&= c\int_0^1 \left((1-F(h))^{n-d}+F(h)^{n-d}\right)(1-h^2)^{d\frac{d-3}{2}+\frac{d-1}{2}}\,\mathrm{d}h\\
&= c\int_{-1}^1 (1-F(h))^{n-d} f(h)^d (1-h^2)^{\frac{d-1}{2}}\,\mathrm{d}h.
\end{aligned}
$$

As a consequence, also for the uniform distribution on $\mathbb{S}^{d-1}$ we arrive at an expression of the form (4.8), this time with $L(s) = f(F^{-1}(s))\sqrt{1-(F^{-1}(s))^2}$. From this point on, the proof can be completed as in the case of the distribution class $\mathcal{H}$ or $\mathcal{B}$. This completes the argument. □

Remark 6. Let $p\colon \mathbb{R}^d \to [0,\infty)$ denote a probability density. By a careful inspection of the proof of Theorem 4.1 we see that the following properties of the density p have been used there. First of all, we used that p is spherically symmetric, that is, $p(x)$ only depends on $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ via $\|x\|$. By abuse of notation, we shall write $p(r) : (0,\infty) \to [0,\infty)$ with $r^2 = x_1^2 + \ldots + x_d^2$ for the radial part of the density p.

This was essential to apply the Blaschke-Petkantschin formulas, which use the invariant hyperplane measure μ_{d-1}. Moreover, given $H \in A(d, d-1)$ with distance h to the origin, we have used that we can find $\varphi(h), \psi(h) > 0$ such that

$$
p(\sqrt{r^2+h^2}) = \varphi(h)\, p\Big(\frac{r}{\psi(h)}\Big) \tag{4.10}
$$

for all $r > 0$. For example, for the density $p_{\mathcal{H},\beta}$, $\beta > d/2$, the scaling property (4.10) is satisfied with $\varphi(h) = (1+h^2)^{-\beta}$ and $\psi(h) = \sqrt{1+h^2}$, see (4.7). This scaling property has been used when we separated what happens within H from the contribution that arises from the distance of H to the origin. However, all rotationally symmetric densities with (almost everywhere differentiable) radial part satisfying the scaling property (4.10) with an (almost everywhere differentiable) function ψ have been classified by Miles [59] (see p. 376 there) and Ruben and Miles [73]. They precisely correspond to the distributions in the classes $\mathcal{G}$, $\mathcal{H}$, $\mathcal{B}$ as well as to the exceptional distributions in $\mathcal{U}$, for which Theorem 4.1 is formulated.

On the other hand, this does not mean that $\mathcal{G}$, $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{U}$ contain the only rotationally symmetric distributions on $\mathbb{R}^d$ for which such computations are possible. For example, the density with radial part $p_{\beta,j}(r) = c_{\beta,d,j}\, r^{2j}/(1+r^2)^{\beta}$, $r > 0$, $j \in \{0,1,2,\dots\}$ and $\beta > j + d/2$, which does not belong to the class $\mathcal{H}$ whenever $j > 0$, satisfies the following generalization of the scaling property (4.10):

$$p_{\beta,j}\big(\sqrt{r^2+h^2}\big) = \sum_{k=0}^{j} \varphi_k(h) p_{\beta,k}\Big(\frac{r}{\psi(h)}\Big),$$

with

$$\varphi_k(h) = \binom{j}{k} h^{2(k-j)}(1+h^2)^{-\beta}, \quad k \in \{0,\dots,j\}$$

and

$$\psi(h) = \sqrt{1+h^2}.$$

One can check that the 1-dimensional marginal density of $p_{\beta,j}$ equals

$$f_{\beta,j}(x_1) = \sum_{k=0}^{j} \binom{j}{k} \frac{c_{\beta,d,j}}{c_{\beta,d-1,k}} x_1^{2(k-j)}(1+x_1^2)^{k+\frac{d-1}{2}-\beta},$$

and that from here on the argument based on the affine Blaschke-Petkantschin formula can be applied term-by-term. Unfortunately, the computations in such and similar situations become quite involved. Moreover, to classify *all* rotationally symmetric densities on $\mathbb{R}^d$ for which these computations can be performed seems to be out of reach.

One might also ask whether the method based on Blaschke-Petkantschin formulas yields monotonicity of the mean facet number in such situations where the random points $X_1,\dots,X_n$ are independent with distributions belonging to one of the classes $\mathcal{G}$, $\mathcal{H}$, $\mathcal{B}$ and $\mathcal{U}$, but not necessarily the same (a so-called mixed case). That is, some of the X_i's are Gaussian, some distributed according to a distribution in $\mathcal{H}$ etc. (but within each class we choose every time the same scale parameter σ). Unfortunately, this does not work and, in fact, the method breaks down. The reason is that each distribution class requires its individual substitution, which is adapted to its respective scaling property (4.10). The resulting different rescalings in the hyperplane H distort the relationship between the $(d-1)$-volume in H before and after the transformation, cf. [73].

4.4 Random convex hulls on a half-sphere

In this section we consider an application of Theorem 4.1 to convex hulls generated by random points on a half-sphere. We fix $d \geq 2$, denote by $\mathbb{S}^{d-1}$ the d-dimensional unit sphere in $\mathbb{R}^{d+1}$ and define the half-sphere

$$\mathbb{S}_+^{d-1} = \{y = (y_1, \dots, y_{d+1}) \in \mathbb{S}^{d-1} : y_{d+1} > 0\}\,.$$

Furthermore, we let $\mathcal{S}$ be the class of probability distributions on $\mathbb{S}_+^d$ that have density

$$p_{\mathcal{S},\alpha}(y) = c_{\mathcal{S},\alpha}\, y_{d+1}^{\alpha}, \qquad y = (y_1, \dots, y_{d+1}) \in \mathbb{S}_+^d, \quad \alpha > -1,$$

with respect to the spherical Lebesgue measure on $\mathbb{S}_+^{d-1}$. Here, $c_{\mathcal{S},\alpha} > 0$ is a suitable normalization constant. In particular, choosing $\alpha = 0$ shows that the uniform distribution on $\mathbb{S}_+^{d-1}$ belongs to the class $\mathcal{S}$.

For fixed $\alpha > -1$ and $n \geq d+1$ we let $X_1, \dots, X_n$ be independent random points that are distributed on $\mathbb{S}_+^d$ according to the density $p_{\mathcal{S},\alpha}$. By S_n we denote the *spherical* convex hull of $X_1, \dots, X_n$, that is, the smallest spherically convex set in $\mathbb{S}_+^{d-1}$ containing the points $X_1, \dots, X_n$. For the special choice $\alpha = 0$, this model has recently been studied in [12]. In particular, it is shown in [12] that for this choice of α the mean number of facets $\mathbf{E}\, f_{d-1}(S_n)$ of the spherical random polytope S_n converges to a finite constant only depending on d, as $n \to \infty$ (a similar result is in fact valid for all distributions in $\mathcal{S}$, see [2, 31, 38]). As a special case, our next result shows the somewhat surprising fact that this limit is approached in a strictly monotone way.

Theorem 4.7. *Let $X_1, \dots, X_n$, $n \geq d+1$, be independent and identically distributed according to a probability measure belonging to the class $\mathcal{S}$. Then,*

$$\mathbf{E}\, f_{d-1}(S_n) > \mathbf{E}\, f_{d-1}(S_{n-1}).$$

Proof. Let $g\colon \mathbb{R}^d \to \mathbb{S}_+^{d-1}$ be the mapping defined as

$$g(x) = \Big(\frac{x_1}{\sqrt{1+\|x\|^2}}, \dots, \frac{x_d}{\sqrt{1+\|x\|^2}}, \frac{1}{\sqrt{1+\|x\|^2}}\Big),$$

with inverse given by

$$g^{-1}(y) = \Big(\frac{y_1}{y_{d+1}}, \dots, \frac{y_d}{y_{d+1}}\Big)$$

(this is known as the gnomonic projection). Let $\mathrm{D}g$ be the Jacobian matrix of g and put $J_g(x) := \sqrt{\det \mathrm{D}g(x)^{\intercal} \mathrm{D}g(x)}$. Then, it holds that

$$J_g(x) = (1 + \|x\|^2)^{-\frac{d+1}{2}},$$

see [20, Proposition 4.2]. Moreover, for a measurable subset $A \subset \mathbb{R}^d$ and a measurable function $f\colon A \to \mathbb{R}$ the area formula [40, Theorem 3.2.3] says that

$$\int_A f(x)\,\mathrm{d}x = \int_{g(A)} f \circ g^{-1}(y)(J_g \circ g^{-1}(y))^{-1}\,\mathcal{H}^d_{\mathbb{S}^{d-1}_+}(\mathrm{d}y).$$

Next, we notice that $1 + \|g^{-1}(y)\|^2 = y_{d+1}^{-2}$ and apply the formula with $f(x) = p_{\mathcal{H},\beta}(x)$ for some $\beta > d/2$:

$$\int_A c_{\mathcal{H},d,\beta}\,(1 + \|x\|^2)^{-\beta}\,\mathrm{d}x = \int_{g(A)} c_{\mathcal{H},d,\beta}\,y_{d+1}^{2\beta-d-1}\,\mathcal{H}^d_{\mathbb{S}^{d-1}_+}(\mathrm{d}y),$$

where $c_{\mathcal{H},d,\beta} = \pi^{-d/2}\Gamma(\beta)/\Gamma(\beta - \frac{d}{2})$ is the normalization constant of the density $p_{\mathcal{H},\beta}$ defined in (4.5). As a result, we see that the density $p_{\mathbb{S},2\beta-d-1}$ on $\mathbb{S}^d_+$ is the push-forward of the density $p_{\mathcal{H},\beta}$ on $\mathbb{R}^d$ under g. Note also that $2\beta - d - 1 > -1$ since $\beta > d/2$ and that the uniform measure on the half-sphere corresponds to the choice $\beta = (d+1)/2$.

The above discussion shows the following. Let P_n be the random convex hull in $\mathbb{R}^d$ generated by n independent points with density $p_{\mathcal{H},\beta}$. Then, the push-forward of P_n has the same distribution as the spherical random polytope S_n with $\alpha = 2\beta - d - 1$. Moreover, the facets of P_n are in one-to-one correspondence with those of S_n. As a consequence, the mean facet number of the spherical random polytope S_n is the same as the mean facet number of the random convex hull P_n, i.e.

$$\mathbf{E}\,f_{d-1}(S_n) = \mathbf{E}\,f_{d-1}(P_n).$$

Thus, the monotonicity follows from Theorem 4.1. □

Chapter 5

Threshold Phenomena for the Volume of Random Polytopes

Let N and n be natural numbers, $N > n$, and $X_1, X_2, \dots, X_N$ be independent and identically distributed random points in $\mathbb{R}^n$. As in the previous chapter, we consider two different probability distribution models:

(a) The *Beta model*, with parameter $\beta > -1$: X_1 has density proportional to

$$(1 - \|x\|_2^2)^\beta, \quad x \in \mathbb{B}_2^n.$$

We are interested in the random polytope given by

$$P_{N,n}^\beta := \operatorname{conv}(X_1, \dots, X_N).$$

(b) The *Beta-prime model*, with parameters $\beta > n/2$ and $\sigma > 0$: X_1 has density proportional to

$$\left(1 + \frac{\|x\|_2^2}{\sigma^2}\right)^{-\beta}, \quad x \in \mathbb{R}^n.$$

As before, we consider the random polytope

$$\tilde{P}_{N,n}^{\beta,\sigma} := \operatorname{conv}(X_1, \dots, X_N).$$

In this chapter, we prove threshold results for the volumes and intrinsic volumes of $P_{N,n}^\beta$ and the content of $\tilde{P}_{N,n}^{\beta,\sigma}$ with respect to log-concave isotropic measures, as the space dimension tends to infinity. In particular, it turns out that the polytope $P_{N,n}^\beta$ tends to capture the whole volume of $\mathbb{B}_2^n$ only if the number of points N is superexponential in

n. We illustrate in Figure 5.1 some 2-dimensional simulations of beta polytopes. The case $\beta = 0$ corresponds to the uniform distribution on the unit ball.

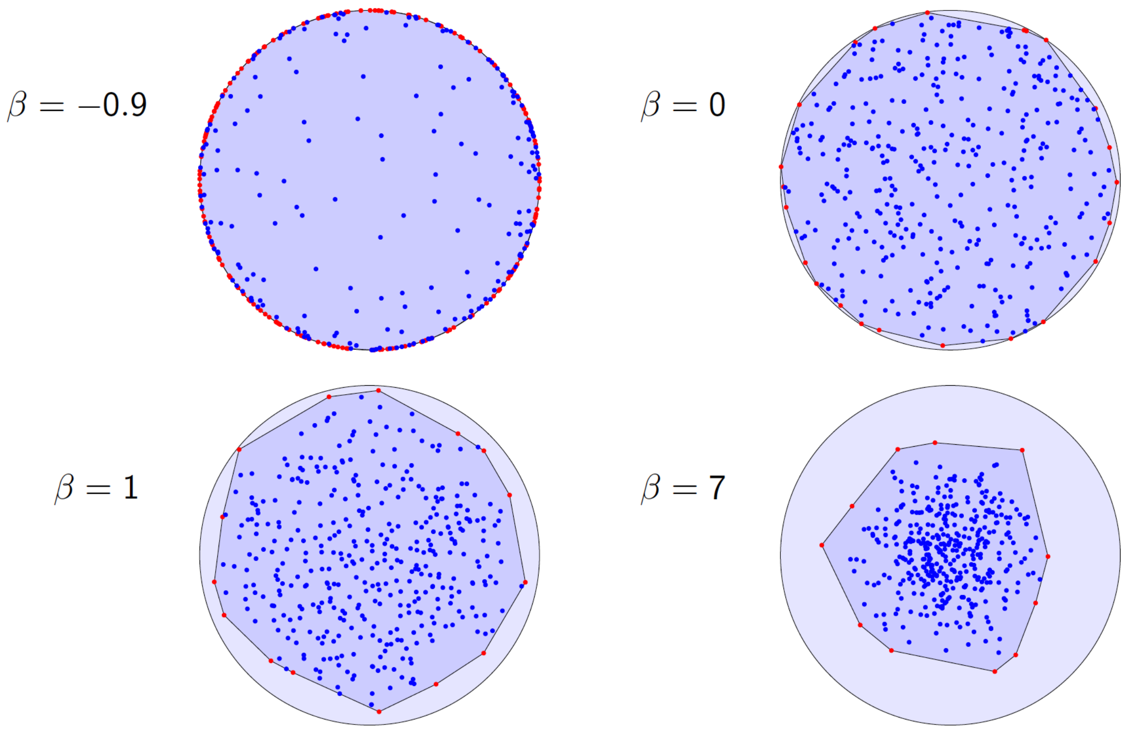

Figure 5.1: some examples of typical 2-dimensional beta polytopes $P^{\beta}_{400,2}$ inside the unit circle, according to different values of the parameter β. Note how the bigger is beta, the less spread is the polytope. In red are highlighted the vertices, whose number decreases while β increases.

Theorem 5.1 (Threshold for beta polytopes)**.** *Fix $\varepsilon \in (0, 1)$ and let $-1 < \beta = \beta(n)$ and $N = N(n)$ be sequences. Then,*

$$\lim_{n\to\infty} \frac{\mathbf{E}\,\mathrm{vol}_n(P^{\beta}_{N,n})}{\mathrm{vol}_n(\mathbb{B}^n_2)} = \begin{cases} 0 & \text{if } N \le \exp\Big((1-\varepsilon)\Big(\beta + \dfrac{n+1}{2}\Big)\log n\Big), \\ 1 & \text{if } N \ge \exp\Big((1+\varepsilon)\Big(\beta + \dfrac{n+1}{2}\Big)\log n\Big). \end{cases}$$

A special case of Theorem 5.1 is of particular interest. By its very definition (see Section 5.1.1 below), the beta distribution for $\beta = 0$ coincides with the uniform probability measure on the Euclidean ball $\mathbb{B}^n_2$. The following is thus an immediate corollary of Theorem 5.1.

Corollary 5.2. *Fix $\varepsilon \in (0, 1)$ and let $N = N(n)$ be a sequence of positive integers. Let $X_1, \ldots, X_N$ be independent random points uniformly distributed on $\mathbb{B}^n_2$ and set*

$\mathbb{B}_{N,n} := \mathrm{conv}(X_1, \dots, X_N)$. *Then,*

$$\lim_{n\to\infty} \frac{\mathbf{E}\,\mathrm{vol}_n(\mathbb{B}_{N,n})}{\mathrm{vol}_n(\mathbb{B}_2^n)} = \begin{cases} 0 & \text{if } N \le \exp\Big((1-\varepsilon)\dfrac{n+1}{2}\log n\Big), \\ 1 & \text{if } N \ge \exp\Big((1+\varepsilon)\dfrac{n+1}{2}\log n\Big). \end{cases}$$

Moreover, since the uniform distribution on the unit sphere $\mathbb{S}^{n-1}$ arises as the weak limit of the beta distribution, as $\beta \to -1$ (see for example the proof of Theorem 2.7 in [45]), the result of Theorem 2.4 in [63] can be recovered by Theorem 5.1.

Corollary 5.3. *Fix $\varepsilon \in (0,1)$ and let $N = N(n)$ be a sequence of positive integers. Let $X_1, \dots, X_N$ be independent random points uniformly distributed on $\mathbb{S}^{n-1}$ and set $S_{N,n} := \mathrm{conv}(X_1, \dots, X_N)$. Then,*

$$\lim_{n\to\infty} \frac{\mathbf{E}\,\mathrm{vol}_n(S_{N,n})}{\mathrm{vol}_n(\mathbb{B}_2^n)} = \begin{cases} 0 & \text{if } N \le \exp\big((1-\varepsilon)\dfrac{n-1}{2}\log n\big), \\ 1 & \text{if } N \ge \exp\big((1+\varepsilon)\dfrac{n-1}{2}\log n\big). \end{cases}$$

Similar threshold statements hold also for the intrinsic volumes of $P_{N,n}^{\beta}$. Lemma 4.2.6 in [75].

As pointed out in [51], the expected k-th intrinsic volume of $P_{N,n}^{\beta}$ is directly connected to the expected k-dimensional volume of $P_{N,k}^{\alpha}$ for some different parameter α depending on β, k and n. Because of this, Theorem 5.1 can be applied to establish threshold results for the intrinsic volumes $V_k(P_{N,n}^{\beta})$, $k \in \{1, \dots, n\}$, for different regimes of $k = k(n)$.

On the other hand, the case that k is a fixed integer is of independent interest, since it amounts to studying the threshold behaviour of $\mathrm{vol}_n(P_{N,n}^{\beta})$ as $\beta \to \infty$ while the dimension n stays fixed. We prove the following.

Theorem 5.4 (Threshold for intrinsic volumes of beta polytopes)**.** *Fix $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$, and let $-1 < \beta = \beta(n)$ and $N = N(n)$ be arbitrary sequences of real and natural numbers, respectively. Then*

$$\lim_{n\to\infty} \frac{\mathbf{E}\,V_k(P_{N,n}^{\beta})}{V_k(\mathbb{B}_2^n)} = \begin{cases} 1 & \text{if } N \ge \exp\Big(\exp\Big((1+\varepsilon)\log\Big(\beta + \dfrac{n-k}{2}\Big)\Big)\Big), \\ 0 & \text{if } N \le \exp\Big(\exp\Big((1-\varepsilon)\log\Big(\beta + \dfrac{n-k}{2}\Big)\Big)\Big). \end{cases}$$

The proof of Theorem 5.4, as well as a general discussion on threshold phenomena for the intrinsic volumes of $P_{N,n}^{\beta}$ is the content of Section 5.2.3.

Next, we treat the case of the beta-prime distribution. Since the underlying measure is not compactly supported, in the spirit of [64], we replace the role of the normalized volume on the ball by an arbitrary isotropic log-concave probability measure μ on $\mathbb{R}^n$, see Subsection 2.3.3 for the definition.

Theorem 5.5 (Threshold for beta-prime polytopes)**.** *Fix $\varepsilon \in (0,1)$. Let $\mu = \mu_n$ denote a sequence of isotropic log-concave measures on $\mathbb{R}^n$, let $\sigma = \sigma(n) > 0$ and $\beta = \beta(n)$ be sequences of real numbers, and let $N = N(n)$ be a sequence of natural numbers. Let $\beta - \frac{n}{2} \gg \log n$.*

(a) *If $\frac{n}{\sigma^2} \ll \frac{1}{\beta - \frac{n}{2}}$ and $N \geq 3n \log n$, then*

$$\lim_{n\to\infty} \mathbf{E}\, \mu(\tilde{P}^{\beta,\sigma}_{N,n}) = 1.$$

(b) *If $\frac{1}{\beta - \frac{n}{2}} \ll \frac{n}{\sigma^2} \ll \frac{1}{\sqrt{\beta - \frac{n}{2}}}$, then,*

$$\lim_{n\to\infty} \mathbf{E}\, \mu(\tilde{P}^{\beta,\sigma}_{N,n}) = \begin{cases} 0 & \text{if } N \leq \exp\left((1-\varepsilon)\frac{n}{\sigma^2}\left(\beta - \frac{n}{2}\right)\right), \\ 1 & \text{if } N \geq \exp\left((1+\varepsilon)\frac{n}{\sigma^2}\left(\beta - \frac{n}{2}\right)\right). \end{cases}$$

(c) *If $\frac{n}{\sigma^2} \to \infty$ and $\sigma > e^{-\frac{n}{3}}$ (in particular this holds for $\sigma \equiv 1$), then,*

$$\lim_{n\to\infty} \mathbf{E}\, \mu(\tilde{P}^{\beta,\sigma}_{N,n}) = \begin{cases} 0 & \text{if } N \leq \exp\left((\beta - \frac{n}{2}) \log\left((1-\varepsilon)\frac{n}{\sigma^2}\right)\right), \\ 1 & \text{if } N \geq \exp\left((\beta - \frac{n}{2}) \log\left((1+\varepsilon)\frac{n}{\sigma^2}\right)\right). \end{cases}$$

Since the densities of a sequence of beta-prime distributions with parameters $\sigma^2 = 2\beta \to \infty$ converge to the density of the standard multivariate Gaussian distribution, we also recover Pivovarov's threshold for Gaussian polytopes. We state it here in a slightly more explicit form than in Theorem 2.2.1 from [64]. For a related result where the log concave isotropic measures are replaced by the volume ratios of the intersection of Gaussian polytopes with balls of arbitrary radii, see Theorem 2.1 from [63].

Corollary 5.6. *Fix $\varepsilon \in (0, 1/2)$. Let $\mu = \mu_n$ denote a sequence of isotropic log-concave measures on $\mathbb{R}^n$ and let $N = N(n)$ be a sequence of natural numbers. Let $X_1, \ldots, X_N$ be independent random points distributed according to the standard Gaussian distribution* *on* $\mathbb{R}^n$ *and let* $G_{N,n} := \operatorname{conv}(X_1, \ldots, X_N)$*. Then,*

$$\lim_{n\to\infty} \mathbf{E}\,\mu(G_{N,n}) = \begin{cases} 0 & \text{if } N \le \exp\big((\frac{1}{2} - \varepsilon)n\big), \\ 1 & \text{if } N \ge \exp\big((\frac{1}{2} + \varepsilon)n\big). \end{cases}$$

The proofs of the above statements can be found in Section 5.2. We stress that in all Theorems 5.1, 5.4 and 5.5, the parameter β is actually allowed to vary with the dimension n.

5.1 Auxiliary estimates

5.1.1 The beta and beta-prime distributions

As aforementioned, our focus in this chaper is on two specific classes of probability distributions on $\mathbb{R}^n$, namely, the beta and beta-prime distributions. To introduce the beta distribution, we set

$$c_{n,\beta} := \pi^{-n/2} \frac{\Gamma\left(\beta + \frac{n}{2} + 1\right)}{\Gamma(\beta + 1)}, \quad \beta > -1,\, n \in \mathbb{N},$$

and define ν_β to be the probability measure on $\mathbb{B}_2^n$ with density function

$$p_{n,\beta}(x) := c_{n,\beta}(1 - \|x\|_2^2)^\beta, \quad x \in \mathbb{B}_2^n.$$

The corresponding one-dimensional marginal density function of ν_β is

$$f_\beta(t) := \alpha_{n,\beta}(1 - t^2)^{\beta + \frac{n-1}{2}}, \qquad t \in [-1, 1],$$

where

$$\alpha_{n,\beta} := \frac{c_{n,\beta}}{c_{n-1,\beta}} = \pi^{-1/2} \frac{\Gamma\left(\beta + \frac{n}{2} + 1\right)}{\Gamma\left(\beta + \frac{n+1}{2}\right)}.$$

Finally, for $d \in [0, 1]$, we abbreviate

$$\mathrm{F}(d) := \int_d^1 f_\beta(t)\, \mathrm{d}t.$$

To introduce the beta-prime distribution, we define

$$\tilde{c}_{n,\beta,\sigma} := \sigma^{-n}\pi^{-n/2}\frac{\Gamma(\beta)}{\Gamma(\beta-\frac{n}{2})}, \qquad \beta > \frac{n}{2}, \quad \sigma > 0, \quad n \in \mathbb{N},$$

and let $\tilde{\nu}_{\beta,\sigma}$ be the probability measure on $\mathbb{R}^n$ with density function

$$\tilde{p}_{n,\beta,\sigma}(x) := \tilde{c}_{n,\beta,\sigma}\Big(1+\frac{\|x\|_2^2}{\sigma^2}\Big)^{-\beta}, \qquad x \in \mathbb{R}^n.$$

Moreover, let

$$\tilde{\alpha}_{n,\beta,\sigma} := \frac{\tilde{c}_{n,\beta,\sigma}}{\tilde{c}_{n-1,\beta,\sigma}} = \sigma^{-1}\pi^{-1/2}\frac{\Gamma(\beta-\frac{n-1}{2})}{\Gamma(\beta-\frac{n}{2})},$$

so that

$$\tilde{f}_{\beta,\sigma}(t) := \tilde{\alpha}_{n,\beta,\sigma}(1+t^2)^{-\beta+\frac{n-1}{2}}, \qquad t \in \mathbb{R},$$

is the one-dimensional marginal density function of $\tilde{\nu}_{\beta,\sigma}$. Analogously to the beta case, for $d \in [0,\infty)$, we denote

$$\tilde{\mathrm{F}}(d) := \int_d^\infty \tilde{f}_{\beta,\sigma}(t)\,\mathrm{d}t.$$

Estimates on the asymptotic behavior of the distribution functions of ν_β and $\tilde{\nu}_{\beta,\sigma}$, in particular for the functions F and $\tilde{\mathrm{F}}$ defined above, play a central role in our work. The previous inequalities are used in the proof of the following bounds for the distribution function F.

Lemma 5.7. *Let $d \in (0,1)$. Then,*

$$\frac{1}{2\sqrt{\pi}}\frac{(1-d^2)^{\beta+\frac{n+1}{2}}}{\sqrt{\beta+\frac{n}{2}+1}} < \mathrm{F}(d) < \frac{1}{2d\sqrt{\pi}}\frac{(1-d^2)^{\beta+\frac{n+1}{2}}}{\sqrt{\beta+\frac{n}{2}}}.$$

Proof. Using the change of variable $s = 1-t^2$, we write

$$\mathrm{F}(d) = \alpha_{n,\beta}\int_d^1 (1-t^2)^{\beta+\frac{n-1}{2}}\,\mathrm{d}t = \frac{1}{2}\alpha_{n,\beta}\int_0^{1-d^2} s^{\beta+\frac{n-1}{2}}(1-s)^{-\frac{1}{2}}\,\mathrm{d}s.$$

Note that since $s \in (0, 1-d^2)$, we have $(1-s)^{-1/2} \in (1, d^{-1})$, so

$$\frac{\alpha_{n,\beta}}{2}\int_0^{1-d^2} s^{\beta+\frac{n-1}{2}}\,\mathrm{d}s < \mathrm{F}(d) < \frac{\alpha_{n,\beta}}{2d}\int_0^{1-d^2} s^{\beta+\frac{n-1}{2}}\,\mathrm{d}s.$$

The fact that

$$\frac{\alpha_{n,\beta}}{\beta+\frac{n+1}{2}} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\beta+\frac{n}{2}+1)}{\Gamma(\beta+\frac{n}{2}+\frac{3}{2})},$$

together with Lemma 2.2, completes the proof. □

Remark 7. Note that an adaptation of the above proof yields similar estimates on the growth of $\tilde{\mathrm{F}}$ if the parameter σ is an absolute constant. For instance if $\sigma = 1$, one has that

$$\frac{1}{2\sqrt{\pi}} \frac{(1+d^2)^{-\beta+\frac{n}{2}}}{\sqrt{\beta - \frac{n-1}{2}}} < \tilde{\mathrm{F}}(d) < \frac{1}{\sqrt{2\pi}} \frac{(1+d^2)^{-\beta+\frac{n}{2}}}{\sqrt{\beta - \frac{n+1}{2}}} \tag{5.1}$$

for every $d > 1$. Yet, in the general case where the parameter σ could vary with β or n we will show that the asymptotic behaviour of $\tilde{\mathrm{F}}$ in terms of σ, β and n actually depends on the growth rate of the quantity n/σ^2. This will result to the different threshold results in the statement of Theorem 5.5.

To deal with the distribution function $\tilde{\mathrm{F}}$ for an arbitrary $\sigma > 0$, we will use a different argument. Note first that a suitable substitution provides

$$\tilde{\mathrm{F}}(d) = \frac{\tilde{\alpha}_{n,\beta,\sigma}}{\sqrt{2b_n}} \int_{a_n}^{\infty} \left(1 + \frac{s^2}{2b_n}\right)^{-b_n} \mathrm{d}s, \quad b_n = \beta - \frac{n-1}{2} \text{ and } a_n = d\frac{\sqrt{2b_n}}{\sigma}. \tag{5.2}$$

It is easy to see that $\frac{\tilde{\alpha}_{n,\beta,\sigma}}{\sqrt{2b_n}} \to \frac{1}{\sqrt{2\pi}}$ whenever $b_n \to \infty$. The estimates of $\tilde{\mathrm{F}}(d)$ which will appear in the proof of Theorem 5.5 are based on (5.2) and the following lemma.

Lemma 5.8. *Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be two sequences with $a_n \geq 0$ and $\frac{1}{2} < b_n \to \infty$.*

(a) *If $\frac{a_n^4}{b_n} \to 0$, then,*

$$\int_{a_n}^{\infty} \left(1 + \frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t \sim \int_{a_n}^{\infty} e^{-\frac{t^2}{2}} \mathrm{d}t.$$

If additionally $a_n \to \infty$, then,

$$\int_{a_n}^{\infty} \left(1 + \frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t \sim \frac{e^{-\frac{a_n^2}{2}}}{a_n}.$$

(b) *If $\frac{a_n^2}{b_n} \to \infty$, then,*

$$\int_{a_n}^{\infty} \left(1 + \frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t \sim \frac{1}{\sqrt{2b_n}} \left(1 + \frac{a_n^2}{2b_n}\right)^{-(b_n - \frac{1}{2})}.$$

To prove Lemma 5.8 we use need a special version of the Laplace's method. We refer the reader to Theorem 1.1 of [87] for a more general statement than the one we present

Lemma 5.9. *Let $h\colon [a,\infty) \to \mathbb{R}$ be a strictly increasing and differentiable function. Then, as $\lambda \to \infty$,*

$$\int_a^\infty e^{-\lambda h(t)}\,\mathrm{d}t \sim \frac{e^{-\lambda h(a)}}{\lambda h'(a)}.$$

Proof of Lemma 5.8. We first show a pair of auxiliary estimates. The inequality $x - \frac{x^2}{2} \leq \log(1+x) \leq x$ gives that

$$1 \leq \frac{\left(1+\frac{t^2}{2b_n}\right)^{-b_n}}{e^{-\frac{t^2}{2}}} = \exp\Big(-b_n \log\Big(1+\frac{t^2}{2b_n}\Big) + \frac{t^2}{2}\Big) \leq e^{\frac{t^4}{8b_n}}.$$

Therefore, for any couple of sequences $0 \leq c_n < d_n$, we have

$$\int_{c_n}^{d_n} e^{-\frac{t^2}{2}}\,\mathrm{d}t \leq \int_{c_n}^{d_n} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t \leq e^{\frac{d_n^4}{8b_n}} \int_{c_n}^{d_n} e^{-\frac{t^2}{2}}\,\mathrm{d}t,$$

and in particular

$$\frac{d_n^4}{b_n} \to 0 \;\Rightarrow\; \int_{c_n}^{d_n} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t \sim \int_{c_n}^{d_n} e^{-\frac{t^2}{2}}\,\mathrm{d}t. \tag{5.3}$$

If additionally $c_n \to \infty$ we can get a more explicit approximation by using a substitution and the Laplace's method. The new estimate is

$$\frac{d_n^4}{b_n} \to 0 \text{ and } c_n \to \infty \;\Rightarrow\; \int_{c_n}^{d_n} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t \sim \frac{e^{-\frac{c_n^2}{2}}}{c_n} - \frac{e^{-\frac{d_n^2}{2}}}{d_n}. \tag{5.4}$$

Since for any t, the map $(\frac{1}{2},\infty) \ni b \mapsto \left(1+\frac{t^2}{2b}\right)^{-b}$ is decreasing, we have that for any sequence $(c_n)_{n\in\mathbb{N}}$ with $\frac{1}{2} < c_n^2 < b_n$,

$$\begin{aligned}
\int_{c_n}^\infty \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t &\leq \int_{c_n}^\infty \left(1+\frac{t^2}{2c_n^2}\right)^{-c_n^2} \mathrm{d}t \\
&= \sqrt{2}c_n \int_{\frac{1}{\sqrt{2}}}^\infty \left(1+s^2\right)^{-c_n^2} \mathrm{d}s \\
&= \sqrt{2}c_n \int_{\frac{1}{\sqrt{2}}}^\infty \exp\left(-c_n^2 \log(1+s^2)\right) \mathrm{d}s.
\end{aligned}$$

Laplace's method, see Lemma 5.9, now implies that for $c_n \to \infty$,

$$\int_{\frac{1}{\sqrt{2}}}^{\infty} \exp\left(-c_n^2 \log(1+s^2)\right) \mathrm{d}s \sim \frac{\exp\left(-c_n^2 \log(\frac{5}{4})\right)}{c_n^2}.$$

In particular,

$$b_n > c_n^2 \text{ and } c_n \to \infty \Rightarrow \int_{c_n}^{\infty} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t = o\left(e^{-c_n^2 \log(\frac{5}{4})}\right). \tag{5.5}$$

Now we have all the ingredients to show Lemma 5.8 (a), but we need to distinguish the case where $a_n \to \infty$ from the case where a_n is bounded.

First, we assume that a_n is bounded. Let $c_n > a_n$ be a sequence such that $\frac{c_n^4}{b_n} \to 0$ and $c_n \to \infty$. Splitting the integral in two parts and applying (5.3) with a_n and c_n, gives

$$\begin{aligned}\int_{a_n}^{\infty} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t &= \int_{a_n}^{c_n} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t + \int_{c_n}^{\infty} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t \\ &\sim \int_{a_n}^{c_n} e^{-\frac{t^2}{2}} \mathrm{d}t + o(1) \\ &\sim \int_{a_n}^{\infty} e^{-\frac{t^2}{2}} \mathrm{d}t.\end{aligned}$$

Second, we assume that $a_n \to \infty$. We split the integral in two parts and use the estimates (5.4) with $c_n = a_n$ and $d_n = 2a_n$, and (5.5) with $c_n = 2a_n$. This gives

$$\begin{aligned}\int_{a_n}^{\infty} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t &= \int_{a_n}^{2a_n} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t + \int_{2a_n}^{\infty} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t \\ &\sim \frac{e^{-\frac{a_n^2}{2}}}{a_n} - \frac{e^{-2a_n^2}}{2a_n} + o\left(e^{-4a_n^2 \log(\frac{5}{4})}\right) \\ &\sim \frac{e^{-\frac{a_n^2}{2}}}{a_n}.\end{aligned}$$

To show part (b) of Lemma 5.8, note that the substitution $s = (1+\frac{t^2}{2b_n})^{-1}$ gives

$$\int_{a_n}^{\infty} \left(1+\frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t = \sqrt{\frac{b_n}{2}} \int_0^{\left(1+\frac{a_n^2}{2b_n}\right)^{-1}} s^{b_n - \frac{3}{2}} (1-s)^{-\frac{1}{2}} \mathrm{d}s.$$

But, since $(0,1) \ni s \mapsto (1-s)^{-\frac{1}{2}}$ is increasing, we have

$$1 \leq \frac{\sqrt{\frac{b_n}{2}} \int_0^{\left(1+\frac{a_n^2}{2b_n}\right)^{-1}} s^{b_n-\frac{3}{2}}(1-s)^{-\frac{1}{2}} \, \mathrm{d}s}{\sqrt{\frac{b_n}{2}} \int_0^{\left(1+\frac{a_n^2}{2b_n}\right)^{-1}} s^{b_n-\frac{3}{2}} \, \mathrm{d}s} \leq \left(1 - \left(1 + \frac{a_n^2}{2b_n}\right)^{-1}\right)^{-\frac{1}{2}}.$$

Observe that the right hand side of the last equation tends to 1 because $\frac{a_n^2}{b_n} \to \infty$. Thus, the two last equations provide the equivalence

$$\begin{aligned} \int_{a_n}^{\infty} \left(1 + \frac{t^2}{2b_n}\right)^{-b_n} \mathrm{d}t &\sim \sqrt{\frac{b_n}{2}} \int_0^{\left(1+\frac{a_n^2}{2b_n}\right)^{-1}} s^{b_n-\frac{3}{2}} \, \mathrm{d}s \\ &\sim \frac{\sqrt{b_n}}{\sqrt{2}(b_n - \frac{1}{2})} \left(1 + \frac{a_n^2}{2b_n}\right)^{-(b_n-\frac{1}{2})} \\ &\sim \frac{1}{\sqrt{2b_n}} \left(1 + \frac{a_n^2}{2b_n}\right)^{-(b_n-\frac{1}{2})}, \end{aligned}$$

which completes the proof. □

5.2 Convex hulls of random points

Recall that by $P_{N,n}^{\beta}$ and $\tilde{P}_{N,n}^{\beta,\sigma}$ we denote the convex hulls arising from $N > n$ independent random points in $\mathbb{R}^n$, distributed according to the beta distribution with parameter β and the beta-prime distribution with parameters β and σ, respectively.

5.2.1 Preparatory lemmas

The proofs of Theorem 5.1 and Theorem 5.5 follow the method introduced in [37] and exploited in [63]. We thus define, for every $x \in \mathbb{R}^n$, the functions

$$q(x) := \inf\{\mathbf{P}(X \in H) : H \text{ is a halfspace containing } x\},$$

when $X \sim \nu_\beta$, and

$$\tilde{q}(x) := \inf\{\mathbf{P}(X \in H) : H \text{ is a halfspace containing } x\},$$

when $X \sim \tilde{\nu}_{\beta,\sigma}$. The following lemma implies a way to compute $q(x)$ and $\tilde{q}(x)$ in terms of the Euclidean norm of the point $x \in \mathbb{R}^n$.

Lemma 5.10. *Let H be a halfspace at distance $d \geq 0$ from the origin. Then,*

(a) $\mathbf{P}(X \in H) = \mathrm{F}(d)$, *when* $X \sim \nu_\beta$,

(b) $\mathbf{P}(X \in H) = \tilde{\mathrm{F}}(d)$, *when* $X \sim \tilde{\nu}_{\beta,\sigma}$.

Proof. We prove the lemma only for the case (a), since (b) is analogous. By rotational invariance of the measure ν_β, we may assume that $H = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq d\}$. We write

$$\begin{aligned}
\mathbf{P}(X \in H) = \nu_\beta(H) &= \int_H p_{n,\beta}(x)\,\mathrm{d}x = c_{n,\beta} \int_H (1 - \|x\|_2^2)^\beta\,\mathrm{d}x \\
&= c_{n,\beta} \int_d^1 \int_{\mathbb{B}_2^{n-1}} (1 - \|x\|_2^2)^\beta\,\mathrm{d}(x_2, \dots, x_n)\,\mathrm{d}x_1 \\
&= c_{n,\beta} \int_d^1 \int_{\mathbb{B}_2^{n-1}} (1 - t^2)^\beta \left(1 - \frac{\|y\|_2^2}{1 - t^2}\right)^\beta \mathrm{d}y\,\mathrm{d}t \\
&= c_{n,\beta} \int_d^1 (1 - t^2)^\beta \int_{\mathbb{B}_2^{n-1}} (1 - \|z\|_2^2)^\beta (1 - t^2)^{\frac{n-1}{2}}\,\mathrm{d}z\,\mathrm{d}t \\
&= \alpha_{n,\beta} \int_d^1 (1 - t^2)^{\beta + \frac{n-1}{2}} \int_{\mathbb{B}_2^{n-1}} p_{n-1,\beta}(z)\,\mathrm{d}z\,\mathrm{d}t \\
&= \int_d^1 f_\beta(t)\,\mathrm{d}t = \mathrm{F}(d),
\end{aligned}$$

which concludes the proof. □

Corollary 5.11. *For every $x \in \mathbb{R}^n$,*

(a) $q(x) = \mathrm{F}(\|x\|_2)$,

(b) $\tilde{q}(x) = \tilde{\mathrm{F}}(\|x\|_2)$.

Proof. As before, we discuss only the case (a). Note that $q(0) = 1/2 = \mathrm{F}(0)$. If $x \neq 0$, let $H(x)$ be the halfspace bounded by the tangent hyperplane to $\|x\|\mathbb{B}_2^n$ at x, not containing 0. Then, by Lemma 5.10 (a), we have

$$\mathrm{F}(\|x\|_2) = \mathbf{P}(X \in H(x)) \geq q(x).$$

Conversely, let H be a halfspace at distance d from the origin, such that $x \in H$. If $d = 0$, then, $\mathbf{P}(X \in H) \geq 1/2 \geq \mathrm{F}(\|x\|_2)$. If $d > 0$, then, again by Lemma 5.10 (a), we have $\mathbf{P}(X \in H) = \mathrm{F}(d) \geq \mathrm{F}(\|x\|_2)$, since $d \leq \|x\|$. It follows that $q(x) \geq \mathrm{F}(\|x\|_2)$. □

Using Corollary 5.11, we can relate the probability content of the random polytopes $P_{N,n}^{\beta}$ and $\tilde{P}_{N,n}^{\beta,\sigma}$ to the distribution functions F and $\tilde{\mathrm{F}}$, respectively. In particular, we upper bound the expected volume of $P_{N,n}^{\beta}$ in terms of F (and similarly for $\tilde{P}_{N,n}^{\beta,\sigma}$).

Lemma 5.12. *Let A be a bounded, measurable subset of $\mathbb{R}^n$.*

(a) *In the beta model,*

$$\mathbf{P}(A \subseteq P_{N,n}^{\beta}) \leq \frac{\mathbf{E}\,\mathrm{vol}(P_{N,n}^{\beta} \cap A)}{\mathrm{vol}(A)} \leq N \sup_{x \in A} \mathrm{F}(\|x\|_2).$$

(b) *In the beta-prime model,*

$$\mu(A)\mathbf{P}(A \subseteq \tilde{P}_{N,n}^{\beta,\sigma}) \leq \mathbf{E}\,\mu(\tilde{P}_{N,n}^{\beta,\sigma} \cap A) \leq N\mu(A) \sup_{x \in A} \tilde{\mathrm{F}}(\|x\|_2),$$

where μ is any isotropic log-concave probability measure on $\mathbb{R}^n$.

Proof. (a) First, note that, for any $x \in P_{N,n}^{\beta} = \mathrm{conv}(X_1, \ldots, X_N)$ and any halfspace H containing x, there must be some $X_i \in H$. This implies that

$$\{x \in P_{N,n}^{\beta}\} \subseteq \bigcup_{i=1}^{N} \{X_i \in H\}.$$

Since the previous inclusion holds for any halfspace H containing x, by a union bound and Corollary 5.11 (a), we get that

$$\mathbf{P}(x \in P_{N,n}^{\beta}) \leq Nq(x) = N\mathrm{F}(\|x\|_2).$$

Now, using the latter estimate,

$$\mathbf{E}\,\mathrm{vol}(P_{N,n}^{\beta} \cap A) = \mathbf{E} \int_A \mathbf{1}P_{N,n}^{\beta}(x)\,\mathrm{d}x = \int_A \mathbf{P}(x \in P_{N,n}^{\beta})\,\mathrm{d}x \leq N\,\mathrm{vol}(A) \sup_{x \in A} \mathrm{F}(\|x\|_2).$$

This proves the upper bound. On the other hand, since the event $\{A \subseteq P_{N,n}^{\beta}\}$ implies $\{\mathrm{vol}(A) \leq \mathrm{vol}_n(P_{N,n}^{\beta} \cap A)\}$, Markov's inequality gives

$$\mathrm{vol}_n(A)\mathbf{P}(A \subseteq P_{N,n}^{\beta}) \leq \mathbf{E}\,\mathrm{vol}_n(P_{N,n}^{\beta} \cap A),$$

completing the proof.

(b) The proof follows along the same line as (a), using now Corollary 5.11 (b) instead of (a). As above, for any halfspace H and any point $x \in H$, we have

$$\{x \in \tilde{P}_{N,n}^{\beta,\sigma}\} \subseteq \bigcup_{i=1}^{N} \{X_i \in H\}.$$

Again, by a union bound and Corollary 5.11 (b), we get that

$$\mathbf{P}(x \in \tilde{P}_{N,n}^{\beta,\sigma}) \leq N\tilde{q}(x) = N\tilde{\mathrm{F}}(\|x\|_2).$$

Using this estimate, we have

$$\mathbf{E}\,\mu(\tilde{P}_{N,n}^{\beta,\sigma} \cap A) = \mathbf{E}\int_A \mathbf{1}\tilde{P}_{N,n}^{\beta,\sigma}(x)\,\mu(\mathrm{d}x) = \int_A \mathbf{P}(x \in \tilde{P}_{N,n}^{\beta,\sigma})\,\mu(\mathrm{d}x) \leq N\mu(A)\sup_{x\in A}\tilde{\mathrm{F}}(\|x\|_2),$$

which proves the upper bound. Finally, by Markov's inequality, we get the lower bound

$$\mu(A)\mathbf{P}(A \subseteq \tilde{P}_{N,n}^{\beta,\sigma}) \leq \mathbf{E}\,\mu(\tilde{P}_{N,n}^{\beta,\sigma} \cap A),$$

finishing the proof. □

Next, we reproduce an analogous "ball inclusion" argument as in [37] in our setting.

Lemma 5.13. (a) *For any $R \in (0,1)$, the inclusion $R\mathbb{B}_2^n \subseteq P_{N,n}^{\beta}$ holds with probability greater than $1 - 2\binom{N}{n}(1-\mathrm{F}(R))^{N-n}$.*

(b) *For any $R > 0$, the inclusion $R\mathbb{B}_2^n \subseteq \tilde{P}_{N,n}^{\beta,\sigma}$ holds with probability greater than $1 - 2\binom{N}{n}(1-\tilde{\mathrm{F}}(R))^{N-n}$.*

Proof. Let us start with part (a). Let $J \subseteq \{1,\dots,N\}$ with $|J| = n$. With probability equal to one, the set $\{X_j\}_{j\in J}$ is affinely independent. Let H_J be the affine hyperplane defined by the affine hull of $\{X_j\}_{j\in J}$ and H_J^+, H_J^- be the corresponding closed halfspaces, determined by H_J. Moreover, let X be an additional independent beta-distributed random point and let E_J be the event, that, either $P_{N,n}^{\beta} \subseteq H_J^+$ and $\mathbf{P}(X \notin H)|_{H=H_J^+} \geq \mathrm{F}(R)$, or $P_{N,n}^{\beta} \subseteq H_J^-$ and $\mathbf{P}(X \notin H)|_{H=H_J^-} \geq \mathrm{F}(R)$. Note that here, and in the following, $\mathbf{P}(X \notin H)|_{H=G}$ denotes the evaluation of the map $H \mapsto \mathbf{P}(X \notin H)$ for the halfspace $G \subset \mathbb{R}^n$.

Suppose that $R\mathbb{B}_2^n \nsubseteq P_{N,n}^{\beta}$, so there exists some $x_0 \in R\mathbb{B}_2^n \setminus P_{N,n}^{\beta}$. Then, there exists some $J \subseteq \{1,\dots,N\}$ with $|J| = n$ such that either $P_{N,n}^{\beta} \subseteq H_J^+$ and $x_0 \in H_J^-$ or $P_{N,n}^{\beta} \subseteq H_J^-$ and $x_0 \in H_J^+$. Note that we have $\mathbf{P}(X \notin H)|_{H=H_J^+} \geq q(x_0) \geq \mathrm{F}(R)$, or

$\mathbf{P}(X \notin H)|_{H=H_J^-} \geq q(x_0) \geq \mathrm{F}(R)$ respectively, since $\|x_0\|_2 \leq R$. It follows that

$$\{R\mathbb{B}_2^n \not\subseteq P_{N,n}^\beta\} \subseteq \bigcup_{\substack{J\subseteq[N]\\|J|=n}} E_J.$$

Clearly, using the union bound,

$$\mathbf{P}(R\mathbb{B}_2^n \not\subseteq P_{N,n}^\beta) \leq \binom{N}{n}\mathbf{P}(E_{[n]}).$$

Next, note that $\mathbf{P}(X \notin H|_{H=H_{[n]}^+}) \geq \mathrm{F}(R)$ implies $\mathbf{P}(X \in H|_{H=H_{[n]}^+}) \leq 1 - \mathrm{F}(R)$, and similarly for $H_{[n]}^-$. It follows that $\mathbf{P}(E_{[n]} \mid X_1, \dots, X_n) \leq 2(1-\mathrm{F}(R))^{N-n}$. Finally, we get that $\mathbf{P}(E_{[n]}) = \mathbf{E}\big(\mathbf{P}(E_{[n]} \mid X_1, \dots, X_n)\big) \leq 2(1-\mathrm{F}(R))^{N-n}$ and, hence,

$$\mathbf{P}(R\mathbb{B}_2^n \not\subseteq P_{N,n}^\beta) \leq 2\binom{N}{n}(1-\mathrm{F}(R))^{N-n},$$

proving the statement of the lemma. The proof of part (b) is a word-by-word repetition of the proof of (a), where now $\tilde{\mathrm{F}}$ plays the role of F. □

Finally, we provide an essential lemma for the proofs of Theorem 5.1 and Theorem 5.5.

Lemma 5.14. *Let $\varepsilon > 0$ be fixed.*

(a) *In the beta model,*

$$\lim_{n\to\infty} \frac{\mathbf{E}\,\mathrm{vol}_n(P_{N,n}^\beta)}{\kappa_n} = \begin{cases} 0 & \text{if } N\mathrm{F}\big(\sqrt{1-n^{-(1-\varepsilon)}}\big) \to 0, \\ 1 & \text{if } N\mathrm{F}\big(\sqrt{1-n^{-(1+\varepsilon)}}\big) - n\log N \to \infty. \end{cases}$$

(b) *In the beta-prime model,*

$$\lim_{n\to\infty} \mathbf{E}\,\mu_n(\tilde{P}_{N,n}^{\beta,\sigma}) = \begin{cases} 0 & \text{if } N\tilde{\mathrm{F}}\big((1-\varepsilon)\sqrt{n}\big) \to 0, \\ 1 & \text{if } N\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) - n\log N \to \infty. \end{cases}$$

Proof. (a) Set $r_n = \sqrt{1-n^{-(1-\varepsilon)}}$, $A_n = \mathbb{B}_2^n \setminus r_n\mathbb{B}_2^n$ and assume that $N\mathrm{F}(r_n) \to 0$. We have that $\sup_{x\in A_n} \mathrm{F}(\|x\|_2) \leq \mathrm{F}(r_n)$, since $\|x\| \geq r_n$ for every $x \in A_n$. By using this, in conjunction with Lemma 5.12 (a),

$$\frac{\mathbf{E}\operatorname{vol}_n(P^{\beta}_{N,n}\cap A_n)}{\kappa_n} \leq \frac{\mathbf{E}\operatorname{vol}_n(P^{\beta}_{N,n}\cap A_n)}{\operatorname{vol}_n(A_n)} \leq N\mathrm{F}(r_n) \to 0.$$

Note also that $\frac{\operatorname{vol}_n(r_n\mathbb{B}_2^n)}{\operatorname{vol}_n(\mathbb{B}_2^n)} = r_n^n \to 0$. Thus

$$\frac{\mathbf{E}\operatorname{vol}_n(P^{\beta}_{N,n})}{\kappa_n} \leq \frac{\operatorname{vol}_n(r_n\mathbb{B}_2^n)}{\operatorname{vol}_n(\mathbb{B}_2^n)} + \frac{\mathbf{E}\operatorname{vol}_n(P^{\beta}_{N,n}\cap A_n)}{\kappa_n} \to 0.$$

Now, we set $s_n = \sqrt{1-n^{-(1+\varepsilon)}}$ and assume that $N\mathrm{F}(s_n) - n\log N \to \infty$. From the lower bound in Lemma 5.12 (a) with $A = s_n\mathbb{B}_2^n$ we get that

$$\frac{\mathbf{E}\operatorname{vol}_n(P^{\beta}_{N,n})}{\kappa_n} \geq s_n^n\,\mathbf{P}(s_n\mathbb{B}_2^n \subseteq P^{\beta}_{N,n}) \sim \mathbf{P}(s_n\mathbb{B}_2^n \subseteq P^{\beta}_{N,n}).$$

Hence, it suffices to show that

$$\lim_{n\to\infty}\mathbf{P}(s_n\mathbb{B}_2^n \nsubseteq P^{\beta}_{N,n}) = 0. \tag{5.6}$$

By Lemma 5.13 (a), we have, using $\binom{N}{n} \leq (eN/n)^n$,

$$\begin{aligned}\mathbf{P}(s_n\mathbb{B}_2^n \nsubseteq P^{\beta}_{N,n}) &\leq 2\binom{N}{n}(1-\mathrm{F}(s_n))^{N-n}\\ &\leq 2(eN/n)^n\exp((N-n)\log(1-\mathrm{F}(s_n)))\\ &= 2\exp\big(n\log(eN/n) + (N-n)\log(1-\mathrm{F}(s_n))\big).\end{aligned}$$

Since $\log(1-x) \leq -x$, we have

$$\begin{aligned}\mathbf{P}(s_n\mathbb{B}_2^n \nsubseteq P^{\beta}_{N,n}) &\leq 2\exp\big(n\log(eN/n) - (N-n)\mathrm{F}(s_n)\big)\\ &= 2\exp\big(n\log(N) - N\mathrm{F}(s_n)\big)\exp\Big(n\Big(\log\Big(\frac{e}{n}\Big) + \mathrm{F}(s_n)\Big)\Big).\end{aligned}$$

Since, for $n \geq e^2$, we have $\log\left(\frac{e}{n}\right) + \mathrm{F}(s_n) \leq 0$, we get

$$\mathbf{P}(s_n\mathbb{B}_2^n \nsubseteq P^{\beta}_{N,n}) \leq 2\exp\big(n\log(N) - N\mathrm{F}(s_n)\big) \to 0.$$

(b) Set $r_n = (1-\varepsilon)\sqrt{n}$, $A_n = \mathbb{R}^n \setminus r_n\mathbb{B}_2^n$ and assume $N\tilde{\mathrm{F}}(r_n) \to 0$. By the thin shell property of μ, see Theorem 2.1, we have that $\mathbf{E}\,\mu(\tilde{P}^{\beta,\sigma}_{N,n}\cap r_n\mathbb{B}_2^n) \to 0$. On the other hand Lemma 5.12 (b) gives that

$$\mathbf{E}\,\mu(\tilde{P}^{\beta,\sigma}_{N,n}\cap A_n)\leq N\sup_{x\in A_n}\tilde{\mathrm{F}}(\|x\|_2)=N\tilde{\mathrm{F}}(r_n)\to 0.$$

Therefore

$$\mathbf{E}\,\mu(\tilde{P}^{\beta,\sigma}_{N,n})=\mathbf{E}\,\mu(\tilde{P}^{\beta,\sigma}_{N,n}\cap r_n\mathbb{B}^n_2)+\mathbf{E}\,\mu(\tilde{P}^{\beta,\sigma}_{N,n}\cap A_n)\to 0.$$

Now, set $s_n=(1+\varepsilon)\sqrt{n}$ and assume that $N\tilde{\mathrm{F}}(s_n)-n\log N\to\infty$. From the lower bound in Lemma 5.12 (b) we get that

$$\mathbf{E}\,\mu(\tilde{P}^{\beta,\sigma}_{N,n})\geq\mu(s_n\mathbb{B}^n_2)\mathbf{P}(s_n\mathbb{B}^n_2\subseteq\tilde{P}^{\beta,\sigma}_{N,n}).$$

On one hand the thin shell property of μ, see Theorem 2.1, gives that $\mathbf{E}\,\mu(s_n\mathbb{B}^n_2)\to 1$. On the other hand, arguing exactly as in the proof of case (a), we can use the bound

$$\mathbf{P}(R\mathbb{B}^n_2\not\subseteq\tilde{P}^{\beta,\sigma}_{N,n})\leq 2\exp\Big(n\log N-N\tilde{\mathrm{F}}(R)\Big)\to 0.$$

Therefore

$$1\geq\mathbf{E}\,\mu(\tilde{P}^{\beta,\sigma}_{N,n})\geq\mu(s_n\mathbb{B}^n_2)(1-\mathbf{P}(s_n\mathbb{B}^n_2\not\subseteq\tilde{P}^{\beta,\sigma}_{N,n}))\to 1,$$

which completes the proof. □

5.2.2 Proofs regarding the beta model

Based on these preparations, we proceed to the proof of Theorem 5.1 on the volume of beta polytopes.

Proof of Theorem 5.1: Set $r_n=\sqrt{1-n^{-(1-\frac{\varepsilon}{2})}}$. From Lemma 5.7 we get

$$\begin{aligned}\mathrm{F}(r_n)&\leq\frac{n^{-(1-\frac{\varepsilon}{2})(\beta+\frac{n+1}{2})}}{\sqrt{\beta+\frac{n}{2}}}\\&=\exp\Big(-\Big(1-\frac{\varepsilon}{2}\Big)\Big(\beta+\frac{n+1}{2}\Big)\log n-\frac{1}{2}\log\Big(\beta+\frac{n}{2}\Big)\Big).\end{aligned}$$

The choice $N\leq\exp\big((1-\varepsilon)\big(\beta+\frac{n+1}{2}\big)\log n\big)$ implies that

$$N\mathrm{F}(r_n)\leq\exp\Big(-\frac{\varepsilon}{2}\Big(\beta+\frac{n+1}{2}\Big)\log n-\frac{1}{2}\log\Big(\beta+\frac{n}{2}\Big)\Big)\to 0,$$

as $n\to\infty$. Combined with Lemma 5.14, this yields the proof of the first part of the theorem.

Set $R_n = \sqrt{1 - n^{-(1+\frac{\varepsilon}{2})}}$. From Lemma 5.7 we get

$$\begin{aligned}\mathrm{F}(R_n) &\geq \frac{1}{2\sqrt{\pi}} \frac{n^{-(1+\frac{\varepsilon}{2})(\beta+\frac{n+1}{2})}}{\sqrt{\beta+\frac{n}{2}+1}} \\ &= \exp\Big(-\Big(1+\frac{\varepsilon}{2}\Big)\Big(\beta+\frac{n+1}{2}\Big)\log n - \frac{1}{2}\log\Big(4\pi\Big(\beta+\frac{n}{2}+1\Big)\Big)\Big).\end{aligned}$$

The choice $N = \exp\big((1+\varepsilon)(\beta+\frac{n+1}{2})\log n\big)$ implies that

$$N\mathrm{F}(R_n) \geq \exp\Big(\frac{\varepsilon}{2}\Big(\beta+\frac{n+1}{2}\Big)\log n - \frac{1}{2}\log\Big(4\pi\Big(\beta+\frac{n}{2}+1\Big)\Big)\Big),$$

and thus

$$\lim_{n\to\infty} NF(R_n) - n\log N = \infty.$$

Combined with Lemma 5.14, this yields the proof. □

Remark 8. As anticipated in Section 1, Theorem 5.1 can be formulated in a stronger way, as follows.

Consider any function $f = f(n)$ such that $f(n) \to \infty$ and $f(n) - \log n \to -\infty$ as $n \to \infty$. If $N \leq \exp\big((\beta + \frac{n+1}{2})f(n)\big)$, then, $\mathbf{E}\,\mathrm{vol}_n(P_{N,n}^\beta)/\kappa_n \to 0$ as $n \to \infty$. Analogously, for any function $g = g(n)$ such that $g(n) - \log n \to +\infty$ as $n \to \infty$, if $N \geq \exp\big((\beta + \frac{n+1}{2})g(n)\big)$, then, $\mathbf{E}\,\mathrm{vol}_n(P_{N,n}^\beta)/\kappa_n \to 1$ as $n \to \infty$. This is proved in the same way as Theorem 5.1 using $r_n^2 = 1 - \exp(-f(n)/2)$ for the upper bound and $R_n^2 = 1 - \exp(-g(n)/2)$ for the lower bound, respectively.

Notice that this is equivalent to replacing ε constant in the statement by $\varepsilon = \varepsilon(n)$ with $\varepsilon(n) \gg 1/\log n$.

Proof of Corollary 5.3: We start with the first case. Let $\varepsilon \in (0,1)$ and fix a sequence $N(n) \leq \exp((1-\varepsilon)(\frac{n-1}{2})\log n)$. As elaborated in [45] the weak limit of a sequence of beta distributions on $\mathbb{R}^n$ for $\beta \to -1$ is the unique rotational invariant probability measure on the sphere $\mathbb{S}^{n-1}$, for any fixed n. Since the map $(x_1,\dots,x_N) \mapsto \mathrm{vol}_n(\mathrm{conv}(x_1,\dots,x_N))/\mathrm{vol}_n(\mathbb{B}_2^n)$ is bounded and continuous, there exists a sequence β_n such that $|\mathbf{E}\,\mathrm{vol}_n(P_{N,n}^{\beta_n}) - \mathbf{E}\,\mathrm{vol}_n(S_{N,n})| < \varepsilon'\,\mathrm{vol}_n(\mathbb{B}_2^n)$, for any $\varepsilon' > 0$. By Theorem 5.1 we have $\mathbf{E}\,\mathrm{vol}_n(P_{N,n}^\beta) \leq \varepsilon'\,\mathrm{vol}_n(\mathbb{B}_2^n)$, and thus can conclude that $\mathbf{E}\,\mathrm{vol}_n(S_{N,n}) \leq 2\varepsilon'\,\mathrm{vol}_n(\mathbb{B}_2^n)$. The statement of the second case can be shown analogously. □

5.2.3 Intrinsic volumes of the beta polytopes

For the beta distribution, there is a known formula that relates the expected k-th intrinsic volume of $P_{N,n}^\beta$ to the expected volume of the respective k-dimensional polytope up to a different parameter β'. In particular, Proposition 2.3 in [51] states that

$$\mathbf{E}\, V_k(P^{\beta}_{N,n}) = \binom{n}{k} \frac{\kappa_n}{\kappa_k \kappa_{n-k}} \mathbf{E}\, V_k\big(P^{\beta+\frac{n-k}{2}}_{N,k}\big).$$

Since

$$V_k(\mathbb{B}^n_2) = \binom{n}{k} \frac{\kappa_n}{\kappa_k \kappa_{n-k}} V_k(\mathbb{B}^k_2),$$

see, e.g., Equation (4.8) from [75], we have

$$\frac{\mathbf{E}\, V_k(P^{\beta}_{N,n})}{V_k(\mathbb{B}^n_2)} = \frac{\mathbf{E}\, V_k\big(P^{\beta+\frac{n-k}{2}}_{N,k}\big)}{V_k(\mathbb{B}^k_2)}. \tag{5.7}$$

The above relation indicates that for any $k = k(n)$ such that $\lim_{n\to\infty} k(n) = \infty$, a threshold behavior similar to that of Theorem 5.1 holds for the intrinsic volumes of $P^{\beta}_{N,n}$, namely, as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\mathbf{E}\, V_k(P^{\beta}_{N,n})}{V_k(\mathbb{B}^n_2)} = \begin{cases} 0 & \text{if } N \le \exp\Big((1-\varepsilon)(\beta + \frac{n+1}{2}) \log k\Big), \\ 1 & \text{if } N \ge \exp\Big((1+\varepsilon)(\beta + \frac{n+1}{2}) \log k\Big). \end{cases}$$

Moreover, if $k = n - m$ for any fixed $m \in \mathbb{N}$, the ratio on the left hand side will exhibit a threshold behavior similar to that of the case $k = n$. As a special case, for $m = 1$, one can deduce by Theorem 5.1 the following threshold phenomenon for the surface area S_{n-1} of $P^{\beta}_{N,n}$.

Proposition 5.15. *Let $\varepsilon \in (0,1)$. Then, as $n \to \infty$,*

$$\lim_{n\to\infty} \frac{\mathbf{E}(S_{n-1}(P^{\beta}_{N,n}))}{S_{n-1}(\mathbb{B}^n_2)} = \begin{cases} 0 & \textit{if } N \le \exp\Big((1-\varepsilon)(\beta + \frac{n+1}{2}) \log n\Big), \\ 1 & \textit{if } N \ge \exp\Big((1+\varepsilon)(\beta + \frac{n+1}{2}) \log n\Big). \end{cases}$$

Still, by (5.7), determining the threshold behaviour of the k-th intrinsic volume when k is a fixed integer would require looking into the case that the space dimension stays fixed, while the parameter β grows to infinity. This is done in the next subsection.

5.2.4 Volume thresholds for beta polytopes in fixed dimension

Here we present the proof of Theorem 5.4. This will come as a corollary of the following general statement.

Theorem 5.16. *Let $n \in \mathbb{N}$ be a fixed integer, $\delta > 1$ and $N = \delta^\beta$.*

(a) *For any $R \in \left(0, \sqrt{\frac{\delta-1}{\delta}}\right)$, we have that $\mathbf{P}(R\mathbb{B}_2^n \subset P_N^\beta) \to 1$ as $\beta \to \infty$.*

(b) *For any $R \in \left(\sqrt{\frac{\delta-1}{\delta}}, 1\right)$, we have that $\mathbf{P}(P_N^\beta \subset R\mathbb{B}_2^n) \to 1$ as $\beta \to \infty$.*

Given Theorem 5.16, note that if $N = \delta^\beta = \exp(\beta \log \delta)$ and R_1, R_2 are such that $0 < R_1 < \sqrt{\frac{\delta-1}{\delta}} < R_2 < 1$, then

$$\lim_{\beta\to\infty} \mathbf{P}(R_1\mathbb{B}_2^n \subseteq P_{N,n}^\beta \subseteq R_2\mathbb{B}_2^n) = 1.$$

In particular

$$\lim_{\beta\to\infty} \mathbf{P}\left(R_1^n \leq \frac{\mathrm{vol}_n(P_{N,n}^\beta)}{\mathrm{vol}_n(\mathbb{B}_2^n)} \leq R_2^n\right) = 1,$$

and since this holds for any $0 < R_1 < \sqrt{\frac{\delta-1}{\delta}} < R_2 < 1$, we get that

$$\lim_{\beta\to\infty} \frac{\mathbf{E}\,\mathrm{vol}_n(P_{N,n}^\beta)}{\mathrm{vol}_n(\mathbb{B}_2^n)} = \left(\frac{\delta-1}{\delta}\right)^{\frac{n}{2}}.$$

Now since

$$N \mapsto \frac{\mathbf{E}\,\mathrm{vol}_n(P_{N,n}^\beta)}{\mathrm{vol}_n(\mathbb{B}_2^n)}$$

is an increasing function, then

$$\lim_{\delta\to\infty} \left(\frac{\delta-1}{\delta}\right)^{\frac{n}{2}} = 1$$

and

$$\lim_{\delta\to 1} \left(\frac{\delta-1}{\delta}\right)^{\frac{n}{2}} = 0,$$

we have just proved the following.

Corollary 5.17. *Let $n \in \mathbb{N}$ be a fixed integer. Let $f, g\colon (-1, \infty) \to \mathbb{R}_+$ be functions* *with $f(\beta)\to\infty$ and $g(\beta)\to 0$ as $\beta\to\infty$, and let $\delta\in(1,\infty)$. Then, as $\beta\to\infty$,*

$$\lim_{n\to\infty}\frac{\mathbf{E}\,\mathrm{vol}_n(P_{N,n}^{\beta})}{\mathrm{vol}_n(\mathbb{B}_2^n)}=\begin{cases}1 & \text{if } N\geq\exp(\beta f(\beta)),\\ 0 & \text{if } N\leq\exp(\beta g(\beta)),\\ \left(\frac{\delta-1}{\delta}\right)^{\frac{n}{2}} & \text{if } N=\exp(\beta\log(\delta)).\end{cases}$$

Proof of Theorem 5.4: The result is an immediate consequence of (5.7) and Corollary 5.17, with $f\left(\beta+\frac{n-k}{2}\right)=(\beta+\frac{n-k}{2})^{\varepsilon}$ and $g\left(\beta+\frac{n-k}{2}\right)=(\beta+\frac{n-k}{2})^{-\varepsilon}$. □

It remains to prove Theorem 5.16.

Proof of Theorem 5.16. (a) By Lemma 5.7 we have that

$$F(R)\geq\frac{1}{2\sqrt{\pi}}\frac{(1-R^2)^{\beta+\frac{n+1}{2}}}{\sqrt{\beta+\frac{n}{2}+1}},$$

thus

$$NF(R)\geq\frac{(1-R^2)^{\frac{n+1}{2}}}{2\sqrt{\pi}}\frac{1}{\sqrt{\beta+\frac{n}{2}+1}}(\delta(1-R^2))^{\beta}.$$

Observe that $\varepsilon:=\delta(1-R^2)-1>0$ because $R<\sqrt{\frac{\delta-1}{\delta}}$. It is then easy to see that

$$\lim_{\beta\to\infty}\frac{(1-R^2)^{\frac{n+1}{2}}}{2\sqrt{\pi}}\frac{1}{\sqrt{\beta+\frac{n}{2}+1}}(1+\varepsilon)^{\frac{\beta}{2}}=+\infty,$$

in particular $NF(R)\geq(1+\varepsilon)^{\beta/2}$ for large enough β. On the other hand, by Lemma 5.13 (a),

$$\begin{aligned}1-\mathbf{P}(R\mathbb{B}_2^n\subset P_N^{\beta})&\leq 2\binom{N}{n}(1-F(R))^{N-n}\\&\leq 2N^n(1-F(R))^{N-n}\\&=\exp\left(\log(2)+n\log(N)+(N-n)\log(1-F(R))\right)\\&\leq\exp\left(\log(2)+n\log(N)-(N-n)F(R)\right),\end{aligned}$$

and since $\log N=\beta\log\delta$ and n is fixed, we have that the last expression tends to 0 as $\beta\to\infty$. Thus,

$$\lim_{\beta\to\infty}\mathbf{P}(R\mathbb{B}_2^n\subseteq P_{N,n}^{\beta})=1.$$

(b) Using integration in polar coordinates and the change of variables $s=t^2$, we can see that if x is distributed according to ν_β one has

$$\begin{aligned}\mathbf{P}(\|x\| \geq R) &= c_{n,\beta} \int_{(R\mathbb{B}_2^n)^c} (1 - \|x\|_2^2)^\beta \,\mathrm{d}x \\ &= n c_{n,\beta} \kappa_n \int_R^1 (1 - t^2) t^{n-1} \,\mathrm{d}t \\ &= \frac{1}{B\left(\beta + 1, \frac{n}{2}\right)} \int_{R^2}^1 (1 - s)^\beta s^{\frac{n}{2} - 1} \,\mathrm{d}s \\ &\leq \frac{1}{B\left(\beta + 1, \frac{n}{2}\right)} \int_{R^2}^1 (1 - s)^\beta \,\mathrm{d}s = \frac{(1 - R^2)^{\beta+1}}{B\left(\beta + 1, \frac{n}{2}\right)(\beta + 1)}.\end{aligned}$$

Letting $N = \delta^\beta$ and $\varepsilon := 1 - \delta(1 - R^2)$, the above inequality implies that

$$N\mathbf{P}(\|x\| \geq R) \leq (1 - \varepsilon)^\beta \frac{1 - R^2}{B\left(\beta + 1, \frac{n}{2}\right)(\beta + 1)}.$$

Note that $\varepsilon \in (0, 1)$, since $R \in \left(\sqrt{\frac{\delta - 1}{\delta}}, 1\right)$, so using the fact that $B\left(\beta + 1, \frac{n}{2}\right) \sim \Gamma(n/2)/(\beta + 1)^{n/2}$ we can easily see that

$$\lim_{\beta \to \infty} (1 - \varepsilon)^{\frac{\beta}{2}} \frac{1 - R^2}{B\left(\beta + 1, \frac{n}{2}\right)(\beta + 1)} = 0.$$

In particular, $N\mathbf{P}(\|x\| \geq R) \leq (1 - \varepsilon)^{\frac{\beta}{2}}$ if β is sufficiently large. Combining this with the inequality $\log x \geq 1 - \frac{1}{x}$, which holds for every $x > 0$, we get

$$\begin{aligned}0 \geq N \log \mathbf{P}(\|x\| \leq R) &\geq N\Big(1 - \frac{1}{\mathbf{P}(\|x\| \leq R)}\Big) \\ &\geq N\left(1 - \frac{1}{1 - \frac{(1-\varepsilon)^{\beta/2}}{N}}\right) = -\frac{N(1 - \varepsilon)^{\beta/2}}{N - (1 - \varepsilon)^{\beta/2}}.\end{aligned}$$

It follows that $\lim_{\beta \to \infty} N \log \mathbf{P}(\|x\| \leq R) = 0$. By independence, we have that

$$\mathbf{P}(P_{N,n}^\beta \subseteq R\mathbb{B}_2^n) = \mathbf{P}(\|x\| \leq R)^N = \exp(N \log \mathbf{P}(\|x\| \leq R)),$$

which gives that $\lim_{\beta \to \infty} \mathbf{P}(P_{N,n}^\beta \subseteq R\mathbb{B}_2^n) = 1$, completing the proof. □

5.2.5 Proofs regarding the beta-prime model

Using Lemma 5.8 and the machinery developed in Section 5.2.1, we now proceed to the proof of Theorem 5.5. Set

$$b_n = \beta - \frac{n-1}{2}.$$

Under the assumptions of Theorem 5.5, $b_n \to \infty$. Thus (5.2) becomes

$$\tilde{\mathrm{F}}(d) \sim \frac{1}{\sqrt{2\pi}} \int_{a_n}^{\infty} \left(1 + \frac{s^2}{2b_n}\right)^{-b_n} \mathrm{d}s, \quad a_n = d\frac{\sqrt{2b_n}}{\sigma}. \tag{5.8}$$

Proof of Theorem 5.5 (a): Let $\varepsilon > 0$. Equation (5.8) gives that

$$\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) \sim \frac{1}{\sqrt{2\pi}} \int_{a_n}^{\infty} \left(1 + \frac{s^2}{2b_n}\right)^{-b_n} \mathrm{d}s,$$

with $\frac{a_n^4}{b_n} = 4(1+\varepsilon)^4 \frac{n^2(\beta - \frac{n-1}{2})}{\sigma^4} \to 0$ because of the assumptions. Thus, by Lemma 5.8,

$$\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) \sim \frac{1}{\sqrt{2\pi}} \int_{a_n}^{\infty} e^{-\frac{t^2}{2}} \,\mathrm{d}t.$$

Since $a_n = (1+\varepsilon)\frac{\sqrt{2b_n n}}{\sigma} \to 0$, it follows that

$$\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) \to \frac{1}{2}.$$

Therefore, for $N = 3\lceil n \log n \rceil$ and n big enough, we have

$$\begin{aligned} N\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) - n\log N &\geq \frac{2}{5}N - n\log N \\ &= \frac{6}{5}\lceil n\log n\rceil - n\log(3\lceil n\log n\rceil) \to \infty. \end{aligned}$$

Again, Lemma 5.14 yields the proof. □

Proof of Theorem 5.5 (b): From (5.8), we have

$$\tilde{\mathrm{F}}\big((1-\varepsilon)\sqrt{n}\big) \sim \frac{1}{\sqrt{2\pi}} \int_{a_n}^{\infty} \left(1 + \frac{s^2}{2b_n}\right)^{-b_n} \mathrm{d}s.$$

Due to the assumptions, $a_n = (1-\varepsilon)\frac{\sqrt{2b_n n}}{\sigma} \to \infty$ and $\frac{a_n^4}{b_n} = 4(1-\varepsilon)^4\frac{n^2(\beta - \frac{n-1}{2})}{\sigma^4} \to 0$.

Thus, by Lemma 5.8,

$$\tilde{\mathrm{F}}\big((1-\varepsilon)\sqrt{n}\big) \sim \frac{e^{-\frac{a_n^2}{2}}}{\sqrt{2\pi}a_n} = \frac{1}{\sqrt{2\pi}a_n}\exp\Big(-(1-\varepsilon)^2\frac{b_n n}{\sigma^2}\Big).$$

In particular, for $N \leq \exp\big((1-\varepsilon)^2\frac{nb_n}{\sigma^2}\big)$ and n big enough,

$$N\tilde{\mathrm{F}}\big((1-\varepsilon)\sqrt{n}\big) \leq \frac{1}{a_n} \to 0,$$

which implies

$$\mathbf{E}\,\mu(\tilde{P}_{N,n}^{\beta,\sigma}) \to 0,$$

because of Lemma 5.14.

Similarly as above, we have

$$\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) \sim \frac{1}{\sqrt{2\pi}a_n}\exp\Big(-(1+\varepsilon)^2\frac{b_n n}{\sigma^2}\Big),$$

where $a_n = (1+\varepsilon)\frac{\sqrt{2b_n n}}{\sigma}$. Because of the condition $\frac{b_n n}{\sigma^2} \to \infty$, we have that, for n big enough,

$$\begin{aligned}\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) &\sim \exp\Big(-(1+\varepsilon)^2\frac{b_n n}{\sigma^2} - \frac{1}{2}\log\frac{b_n n}{\sigma^2} - \log\big(2(1+\varepsilon)\sqrt{\pi}\big)\Big)\\ &\geq \exp\Big(-(1+3\varepsilon)\frac{b_n n}{\sigma^2}\Big),\end{aligned}$$

where the inequality holds because $(1+\varepsilon)^2 < 1+3\varepsilon$. Hence, for $N = \exp\big((1+4\varepsilon)\frac{b_n n}{\sigma^2}\big)$ and n big enough, we have

$$\begin{aligned}N\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) - n\log N &\geq \exp\left(\varepsilon\frac{b_n n}{\sigma^2}\right) - n(1+4\varepsilon)\frac{b_n n}{\sigma^2}\\ &\geq \exp\left(f(n)\right) - \frac{1+4\varepsilon}{\varepsilon}nf(n),\end{aligned} \tag{5.9}$$

where $f(n) := \varepsilon\frac{b_n n}{\sigma^2}$. The assumption on the growth of β, together with (5.9), give that $N\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) - n\log N \to \infty$, and Lemma 5.14 yields the proof. □

Proof of Theorem 5.5 (c): From (5.8) we have

$$\tilde{\mathrm{F}}\big((1-\varepsilon)\sqrt{n}\big) \sim \frac{\tilde{\alpha}_{n,\beta}}{\sqrt{2b_n}}\int_{a_n}^{\infty}\left(1+\frac{s^2}{2b_n}\right)^{-b_n}\mathrm{d}s,$$

where $a_n = (1-\varepsilon)\frac{\sqrt{2b_n n}}{\sigma}$. Note that $\frac{a_n^2}{b_n} = (1-\varepsilon)^2\frac{2n}{\sigma^2} \to \infty$ because of the assumption $\frac{n}{\sigma^2} \to \infty$. Consequently, by Lemma 5.8

$$\begin{aligned}\tilde{\mathrm{F}}\big((1-\varepsilon)\sqrt{n}\big) &\sim \frac{1}{\sqrt{2\pi}}\frac{\sqrt{b_n}}{\sqrt{2}(b_n-\frac{1}{2})}\left(1+\frac{a_n^2}{2b_n}\right)^{-(b_n-\frac{1}{2})} \\ &\sim \frac{1}{2\sqrt{b_n\pi}}\left(1+(1-\varepsilon)^2\frac{n}{\sigma^2}\right)^{-(b_n-\frac{1}{2})}.\end{aligned}$$

In particular, for $N \le \exp\big((b_n-\frac{1}{2})\log\big((1-\varepsilon)^2\frac{n}{\sigma^2}\big)\big)$ and n big enough,

$$N\tilde{\mathrm{F}}\big((1-\varepsilon)\sqrt{n}\big) \le \frac{1}{\sqrt{b_n}} \to 0,$$

which implies

$$\mathbf{E}\,\mu(\tilde{P}_{N,n}^{\beta,\sigma}) \to 0,$$

because of Lemma 5.14.

Similarly as above, we have

$$\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) \sim \frac{1}{2\sqrt{b_n\pi}}\left(1+(1+\varepsilon)^2\frac{n}{\sigma^2}\right)^{-(b_n-\frac{1}{2})}.$$

Set $\varepsilon' \in \left(0, \log\frac{(1+2\varepsilon)^2}{(1+\varepsilon)^2}\right)$. From the last equation, it is easy to see that for $N = \exp\big((b_n-\frac{1}{2})\log\big((1+2\varepsilon)^2\frac{n}{\sigma^2}\big)\big)$, and n big enough,

$$\begin{aligned}N\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) &\ge \frac{1}{4\sqrt{b_n}}\exp\left(\left(b_n-\frac{1}{2}\right)\log\frac{(1+2\varepsilon)^2\frac{n}{\sigma^2}}{1+(1+\varepsilon)^2\frac{n}{\sigma^2}}\right) \\ &\ge \exp\left(\varepsilon' b_n\right).\end{aligned}$$

Observe also that $\log\big((1+2\varepsilon)^2\frac{n}{\sigma^2}\big) < \frac{n}{2}$ because of the assumption $\sigma > e^{-\frac{n}{3}}$. Combined with $\log n \ll b_n$ we get

$$N\tilde{\mathrm{F}}\big((1+\varepsilon)\sqrt{n}\big) - n\log N \ge \exp\left(\varepsilon' b_n\right) - \frac{n^2}{2}b_n \to \infty,$$

and the result follows from Lemma 5.14. □

Proof of Corollary 5.6: We will prove the corollary just for the first case, since the second is analogous. Fix $\varepsilon \in (0, 1/2)$ and a sequence $N(n) \le \exp\big((\frac{1}{2}-\varepsilon)n\big)$. For any fixed n, it holds that $\tilde{p}_{n,\beta,\sigma}(x) \to (2\pi)^{-\frac{n}{2}}\exp\big(-\frac{\|x\|_2^2}{2}\big)$ whenever $\sigma^2 = 2\beta \to \infty$. In particular, for any $\varepsilon' > 0$, one can find σ_n such that $\left|\mathbf{E}\,\mu(\tilde{P}_{N,n}^{\frac{1}{2}\sigma_n^2,\sigma_n}) - \mathbf{E}\,\mu(G_{N,n})\right| < \varepsilon'$. We can choose $\sigma_n \gg n$ and $\beta_n = \frac{1}{2}\sigma_n^2$, so that the conditions of case *(b)* in Theorem 5.5 are met. Therefore, for n large enough, $\mathbf{E}\,\mu(\tilde{P}_{N,n}^{\frac{1}{2}\sigma_n^2,\sigma_n}) < \varepsilon'$. We can conclude that $\mathbf{E}\,\mu(G_{N,n}) < 2\varepsilon'$, which ends the proof. □

Chapter 6

The Isotropic Constant of Random Polytopes with Vertices on Convex Surfaces

Let $X_1, \dots, X_N$ be independent random points, distributed on the boundary of an isotropic convex body K in $\mathbb{R}^n$, $n \geq 2$, according to its cone measure C_K. In this chapter we establish that the isotropic constant of the random symmetric convex hull of $X_1, \dots, X_N$ is bounded by an absolute constant, as n tends to infinity, with overwhelming probability, in the case where K is also unconditional, or can grow by at most a logarithmic function of N/n, in the general case. We treat these two settings separately.

6.1 The unconditional case

Let K be an isotropic convex body in $\mathbb{R}^n$. This means that $\mathrm{vol}_n(K) = 1$, its barycenter is at the origin and

$$\int_K \langle x, \theta \rangle^2 \, \mathrm{d}x = L_K^2$$

for all directions $\theta \in \mathbb{S}^{n-1}$, where L_K is a constant independent of θ, the so-called isotropic constant of K.

Additionally, K is unconditional, i.e it is symmetric with respect to all n coordinate hyperplanes. In particular, K is symmetric with respect to the origin. The cone probability measure μ_K is defined on ∂K as follows,

$$\mu_K(B) := \frac{\mathrm{vol}_n(\{rx : x \in B, 0 \leq r \leq 1\})}{\mathrm{vol}_n(K)}, \qquad B \subset \partial K \text{ a Borel set.}$$

Although both the cone measure and the surface measure are defined on the boundary of a convex body, they differ as the former is concerned with the volume of the cone spanned by a region of the boundary. However, they coincide for example on $\mathbb{S}_1^{n-1}$, $\mathbb{S}_2^{n-1}$ and $\mathbb{S}_\infty^{n-1}$.

The uniform distribution on a convex body $K \subset \mathbb{R}^n$ shall be denoted by ν_K, i.e. for a Borel set $A \subset \mathbb{R}^n$

$$\nu_K(A) := \frac{\mathrm{vol}_n(K \cap A)}{\mathrm{vol}_n(K)},$$

For $N > n$ we let $X_1, \ldots, X_N$ be independent random points distributed according to μ_K and $K_N := \mathrm{conv}(\pm X_1, \ldots, \pm X_N)$ be the random symmetric convex hull generated by $X_1, \ldots, X_N$. We prove the following.

Theorem 6.1. *Let $K \subset \mathbb{R}^n$ be an isotropic unconditional convex body, $N > n$ and K_N the symmetric convex hull of N independent random points on ∂K with distribution μ_K. Then there exist absolute constants $c_1, c_2, C \in (0, \infty)$ such that the event that*

$$L_{K_N} \leq C$$

occurs with probability at least $1 - c_1 e^{-c_2 n}$.

Remark 9. We note that the result of Theorem 6.1, in the regime where N is proportional to the space dimension n, follows directly from the existing literature. Indeed, in this situation the random polytope K_N has precisely N vertices with probability one and it is known from [4] that an n-dimensional polytope P with $v_P > n$ vertices has an isotropic constant bounded above by a constant multiple of $\sqrt{v_P/n}$. This implies absolute boundedness of the isotropic constant of K_N even with probability one.

Let us emphasize that Theorem 6.1 generalizes the main results of both [3] and [48]. Moreover, it is the clear analogue to the main result in [33], where the authors consider random polytopes generated by points $X_1, \ldots, X_N$ chosen uniformly at random from the interior of an isotropic unconditional convex body. However, the result in [33] does not imply Theorem 6.1 and vice versa. Although the tools we use and the strategy of the proof rely on similar ingredients as those employed in [33] (and also that in [3, 5, 54]) there is a significant difference. In fact, one of the main ingredients in the proof of Theorem 6.1 is a version of Bernstein's inequality (see Lemma 6.3 below). In order to be able to apply it, an upper bound on the so-called ψ_2-norm of linear functionals with respect to the cone probability measure is needed. While this is well known in the case of the uniform distribution on K, this is not the case for the cone probability measure on ∂K, the main reason for this being the fact that the cone measure does not fit into the theory developed for log-concave measures. We shall provide such an estimate in Section 6.4.

6.2 The general case

We assume the same set-up as in the previous subsection, but now we drop the assumption that the convex body K is unconditional. That is, we assume that $K \subset \mathbb{R}^n$ is an isotropic convex body with cone probability measure μ_K. The next result is the analogue to the main result in [5], where the authors prove a similar estimate in the case that the random polytope is generated by points uniformly distributed in the interior of K. However and in contrast to Theorem 6.1, for general isotropic convex bodies we are (in general) not able to bound the isotropic constant of K_N by an absolute constant with high probability. In addition this set-up requires an upper bound for the number of vertices of K_N. Again, this is in line with the results in [5].

Theorem 6.2. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body, $n < N \le e^{\sqrt{n}}$ and K_N the symmetric convex hull of N independent random points on ∂K with distribution μ_K. Then there exist absolute constants $c_1, c_2, c_3, C \in (0,\infty)$ such that the event that*

$$L_{K_N} \le C\sqrt{\log \frac{2N}{n}}$$

occurs with probability at least $1 - c_1 e^{-c_2 n} - e^{-c_3\sqrt{N}}$.

Remark 10. As in Remark 9, if $N \le cn$ for some $c \in (0,\infty)$ the conclusion of Theorem 6.2 is again trivial. More precisely, in this regime we even have that L_{K_N} is absolutely bounded with probability one.

Remark 11. Observe that in the regime where $e^{\sqrt{n}} < N \le e^n$ one can prove that the weaker estimate

$$L_{K_N} \le C L_K \sqrt{\log \frac{2N}{n}}$$

holds with probability exponentially close to 1. This follows from the fact that one can use part (a) of Lemma 6.10 instead of (b) to lower bound $\mathrm{vol}_n(K_N)^{1/n}$ in the final proof (see also the discussion after Theorem 11.3.7 in [28]). However, this estimate does not improve Klartag's general bound as the right-hand side is of order at least $n^{1/4}$.

Remark 12. It was shawn in [3, Theorem 1.2] that if P is a polytope in $\mathbb{R}^n$ with f_P facets then

$$L_P \leq C\sqrt{\log \frac{f_P}{n}} \tag{6.1}$$

for some absolute constant $C \in (0,\infty)$. Moreover, in [80] (see also [29] for the case of the unit ball) it is proved that if ∂K is twice differentiable and has positive Gaussian curvature everywhere the expected number of facets of K_N satisfies

$$\mathbf{E}\, f_{K_N} = cN(1+o(1)),$$

as $N \to \infty$, where $c \in (0,\infty)$ is some constant depending on d and on K, and $o(1)$ is some sequence that tends to zero. (The results in [29, 80] are formulated for the non-symmetric convex hull of N random points in K, but it can be checked that the order remains the same for the symmetric convex hulls.) Thus, replacing f_P by $\mathbf{E}\, f_{K_N}$ in (6.1), the result of Theorem 6.2 might be anticipated.

The proof of Theorem 6.2 is similar to the one of Theorem 6.1, but is based on another version of Bernstein's inequality. More precisely, while in the argument for Theorem 6.1 we work with the so-called ψ_2-norm (of a certain class of linear functionals), in the context of Theorem 6.2 we are able to deal only with the ψ_1-norm, which can effectively be handled for arbitrary isotropic convex bodies.

6.3 Preliminaries

6.3.1 Orlicz spaces and Bernstein's Inequality

A convex function $M\colon [0,\infty) \to [0,\infty)$ with $M(0) = 0$ is called an *Orlicz function*. We indicate by $L^M(\mathbf{P})$ the set (of equivalence classes) of random variables $X : \Omega \to \mathbb{R}$ such that $M(|X|/\lambda) \in L^1(\mathbf{P})$, for some $\lambda > 0$. We supply $L^M(\mathbf{P})$ with the Luxemburg norm

$$\|X\|_M := \inf\{\lambda > 0 : \mathbf{E}\ M(|X|/\lambda) \leq 1\},$$

(this notation should not lead to confusion with the Minkowski $\|\cdot\|_K$ functional associated with a convex body K).

We point out that $(L^M(\mathbf{P}), \|\cdot\|_M)$ is a Banach space and we refer to it as the Orlicz space associated to M. Examples of Orlicz spaces are the L^p-spaces, for every $p \in [1,\infty)$, associated to the Orlicz functions $x \mapsto x^p$, and the spaces associated to the functions $\psi_\alpha(x) = \exp(x^\alpha) - 1$, for every $\alpha \in [1,\infty)$. In the particular case $\alpha \in \{1,2\}$, the elements of the space $(L^{\psi_\alpha}(\Omega,\mathbf{P}), \|\cdot\|_{\psi_\alpha})$ are also called sub-exponential and sub-Gaussian random variables, respectively.

The following result, known as Bernstein's inequality, taken in this form from [8, Theorem 3.5.17], allows to obtain an estimate on the tail of the distribution of a sum of independent and uniformly sub-Gaussian random variables. It will be used in the proof of Theorem 6.1. In the proof of Theorem 6.2, we need another version of Bernstein's inequality, which deals with sums of independent and uniformly sub-exponential random variables. It is written here as a particular case of [8, Theorem 3.5.16].

Lemma 6.3. *Let $Y_1,\dots,Y_n$ be independent and centred random variables defined on a common probability space $(\Omega,\mathcal{F},\mathbf{P})$.*

(a) Suppose that there exists $R \in (0,\infty)$ such that $\|Y_i\|_{\psi_2} \le R$ for every $i \in \{1,\dots,n\}$. Then, for every $t > 0$,

$$\mathbf{P}\Big(\Big|\sum_{i=1}^n Y_i\Big| > tn\Big) \le 2\exp\Big(-\frac{t^2 n}{8R^2}\Big).$$

(b) Suppose that there exists $R \in (0,\infty)$ such that $\|Y_i\|_{\psi_1} \le R$ for every $i \in \{1,\dots,n\}$. Then, for every $t > 0$,

$$\mathbf{P}\Big(\Big|\sum_{i=1}^n Y_i\Big| > tn\Big) \le 2\exp\Big(-\frac{tn}{6R}\min\Big\{\frac{t}{R},1\Big\}\Big).$$

6.3.2 Auxiliary inequalities

Although this definition relies on the 2-norm, the isotropic constant of a symmetric convex body can be bounded from above using an average of the 1-norm. As in [33] this bound will turn out to be very useful for our purposes. The first of the following inequalities is taken from [28, Lemma 11.5.2], while the second is a direct consequence of the definition of isotropic constant (Equation (2.3)). We recall that a convex body $K \subset \mathbb{R}^n$ is symmetric provided that $x \in K$ implies $-x \in K$ and centred if K has its barycentre at the origin.

Lemma 6.4. *(a) Let $K \subset \mathbb{R}^n$ be a symmetric convex body. Then there exists a constant $c \in (0,\infty)$ such that*

$$L_K \leq \frac{c}{n \,\mathrm{vol}_n(K)^{1+1/n}} \int_K \|x\|_1 \,\mathrm{d}x.$$

(b) Let $K \subset \mathbb{R}^n$ be a centred convex body. Then,

$$L_K^2 \leq \frac{1}{n \,\mathrm{vol}_n(K)^{1+2/n}} \int_K \|x\|_2^2 \,\mathrm{d}x.$$

Since we will deal with symmetric polytopes, we will make use of the following lemma that, together with the previous one, allows us to connect the isotropic constant of a polytope with properties of its facets.

Lemma 6.5. *(a) Let $K_N = \mathrm{conv}\{\pm y_1, \ldots, \pm y_N\} \subset \mathbb{R}^n$ be a symmetric polytope. Then*

$$\frac{1}{|K_N|} \int_{K_N} \|x\|_1 \,\mathrm{d}x \leq \frac{1+\sqrt{2}}{n} \max_{\substack{\{y_{i_1},\ldots,y_{i_n}\} \subset \{\pm y_1,\ldots,\pm y_N\} \\ \varepsilon_1,\ldots,\varepsilon_n = \pm 1}} \|\varepsilon_1 y_{i_1} + \ldots + \varepsilon_n y_{i_n}\|_1.$$

(b) Let $K_N = \mathrm{conv}\{\pm y_1, \ldots, \pm y_N\} \subset \mathbb{R}^n$ be a symmetric polytope. Then

$$\frac{1}{|K_N|} \int_{K_N} \|x\|_2^2 \,\mathrm{d}x \leq \frac{2}{(n+1)(n+2)} \max_{\substack{\{y_{i_1},\ldots,y_{i_n}\} \subset \{\pm y_1,\ldots,\pm y_N\} \\ \varepsilon_1,\ldots,\varepsilon_n = \pm 1}} \|\varepsilon_1 y_{i_1} + \ldots + \varepsilon_n y_{i_n}\|_2^2.$$

Proof. Following the idea of [5, Lemma 3.2], we decompose K_N as union of simplices $S_1, \ldots, S_m$ having pairwise disjoint interiors. Namely, let $K_N = \bigcup_{i=1}^m S_i$ with $S_i := \mathrm{conv}\{0, y_{i_1}, \ldots, y_{i_n}\}$ for every $i \in \{1, \ldots, m\}$. Note that for each $i \in \{1, \ldots, m\}$ the set $F_i := \mathrm{conv}\{y_{i_1}, \ldots, y_{i_n}\}$ is an $(n-1)$-dimensional simplex and that $\partial K_N = \bigcup_{i=1}^m F_i$. According to [28, Lemma 11.5.4], it holds that, for each $i \in \{1, \ldots, m\}$,

$$\frac{1}{|K_N|} \int_{K_N} \|x\|_2^2 \,\mathrm{d}x \leq \frac{n}{n+2} \max_{i \in \{1,\ldots,m\}} \frac{1}{|F_i|} \int_{F_i} \|y\|_2^2 \,\mathrm{d}y.$$

In the discussion following [28, Lemma 11.5.4], it is also shown that

$$\frac{1}{|F_i|} \int_{F_i} \|y\|_2^2 \,\mathrm{d}y \leq \frac{2}{n(n+1)} \max_{\varepsilon_1,\ldots,\varepsilon_n = \pm 1} \|\varepsilon_1 y_{i_1} + \ldots + \varepsilon_n y_{i_n}\|_2^2,$$

from which the claim (a) follows. For (b), we proceed analogously. Namely, it is stated in [28, Lemma 11.4.4] that

$$\frac{1}{|K_N|}\int_{K_N}\|x\|_1\,\mathrm{d}x \le \max_{i\in\{1,\ldots,m\}}\frac{1}{|F_i|}\int_{F_i}\|y\|_1\,\mathrm{d}y.$$

Moreover, from [28, Lemma 11.4.5] follows that, for every $i \in \{1,\ldots,m\}$,

$$\frac{1}{|F_i|}\int_{F_i}\|y\|_1\,\mathrm{d}y \le \frac{1+\sqrt{2}}{n}\max_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1}\|\varepsilon_1 y_{i_1}+\ldots+\varepsilon_n y_{i_n}\|_2^2,$$

which, combined with the previous inequality, concludes the proof. □

6.4 A ψ_2-estimate for the cone measure

In order to be able to apply Bernstein's inequality for independent and uniformly sub-Gaussian random variables (see Lemma 6.3 (a)), we need an upper bound on the ψ_2-norm on linear functionals with respect to the cone probability measure on the boundary of an isotropic unconditional convex body. We emphasize that such an estimate is the key point in the proof of Theorem 6.1 and might also be of independent interest. Bounds for the ψ_2-norm of linear functionals have been subject of a number of studies, which in turn concentrate on the case of the uniform distribution on an isotropic convex body or, more generally, on an isotropic log-concave measure, see in particular the work of Bobkov and Nazarov [22]. However, the cone measure does clearly not satisfy this property and it seems that the following Bobkov-Nazarov type estimate for the cone measure result is not available in the existing literature. We also remark that the proof in [22] does not carry over to the cone measure case. Instead we use the polar integration formula to deduce the estimate from the one for the uniform distribution. In addition, this allows to identify the extremal bodies for which the estimate is sharp, see Remark 14.

Proposition 6.6. *Let $K \subset \mathbb{R}^n$ be an isotropic unconditional convex body. Then, for every $\theta \in \mathbb{R}^n$,*

$$\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_K)} \le 3\sqrt{n}\|\theta\|_\infty.$$

Let us briefly comment that the result of Proposition 6.6 might be re-phrased by saying that for an isotropic unconditional convex body K and for every $\theta \in \mathbb{R}^n$ one has that

$$\mu_K(\{x\in\partial K : |\langle x,\theta\rangle| \ge t\sqrt{n}\|\theta\|_\infty\}) \le 2e^{-\frac{t^2}{9}}$$

for all $t > 0$. Especially, taking $\theta = (1, \dots, 1)$, which satisfies $\|\theta\|_\infty = 1$, we have that

$$\mu_K\Big(\Big\{x \in \partial K : \frac{\|x\|_1}{\sqrt{n}} \geq t\Big\}\Big) \leq 2e^{-\frac{t^2}{9}}.$$

We split the proof of Proposition 6.6 into three lemmas, the first being a comparison inequality, in the spirit of [22], where we compare the absolute moments of a linear functional on a general isotropic unconditional convex body to the ones on a rescaling of the unit ball $\mathbb{B}_1^n$.

Lemma 6.7. *Let $K \subset \mathbb{R}^n$ be an isotropic unconditional convex body and $V := \frac{\sqrt{6}}{2} n \mathbb{B}_1^n$. Then, for any $\theta \in \mathbb{R}^n$ and every $q \in \mathbb{N} \cup \{0\}$,*

$$\int_{\partial K} |\langle x, \theta \rangle|^{2q} \, \mathrm{d}\mu_K(x) \leq \int_{\partial V} |\langle x, \theta \rangle|^{2q} \, \mathrm{d}\mu_V(x).$$

Proof. We know from the computations in [22] (see in particular [28, page 307]) that, for any $\theta \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^{2q} \, \mathrm{d}\nu_K(x) \leq \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^{2q} \, \mathrm{d}\nu_V(x), \tag{6.2}$$

since K is unconditional. Then the claim holds if for any symmetric convex body K_0,

$$\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^{2q} \, \mathrm{d}\nu_{K_0}(x) = c_{n,q} \int_{\partial K_0} |\langle x, \theta \rangle|^{2q} \, \mathrm{d}\mu_{K_0}(x), \tag{6.3}$$

where $c_{n,q} \in (0, \infty)$ can depend on n, q but not K_0. We can prove Equation (6.3) using a polar integration formula for the cone measure. It says that, for every integrable function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} f(x) \, \mathrm{d}x = n \, \mathrm{vol}_n(K_0) \int_0^\infty r^{n-1} \int_{\partial K_0} f(rx) \, \mathrm{d}\mu_{K_0}(x) \, \mathrm{d}r,$$

see [60, Proposition 1]. We apply this transformation formula to $f(x) = \mathbf{1}_{K_0}(x) |\langle x, \theta \rangle|^{2q}$.

Then, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\langle x,\theta\rangle|^{2q}\,\mathrm{d}\nu_{K_0}(x) &= \int_{\mathbb{R}^n} |\langle x,\theta\rangle|^{2q} \frac{\mathbf{1}_{K_0}(x)}{\mathrm{vol}_n(K_0)}\,\mathrm{d}x \\ &= n\int_0^\infty r^{n-1}\int_{\partial K_0} |\langle rx,\theta\rangle|^{2q}\mathbf{1}_{K_0}(rx)\,\mathrm{d}\mu_{K_0}(x)\,\mathrm{d}r \\ &= n\int_0^\infty r^{n-1+2q}\int_{\partial K_0} |\langle x,\theta\rangle|^{2q}\mathbf{1}_{[0,1]}(r)\,\mathrm{d}\mu_{K_0}(x)\,\mathrm{d}r \\ &= n\int_0^1 r^{n-1+2q}\int_{\partial K_0} |\langle x,\theta\rangle|^{2q}\,\mathrm{d}\mu_{K_0}(x)\,\mathrm{d}r \\ &= \frac{n}{n+2q}\int_{\partial K_0} |\langle x,\theta\rangle|^{2q}\,\mathrm{d}\mu_{K_0}(x),\end{aligned} \tag{6.4}$$

which completes the argument. □

Remark 13. The quantitative dependence of the constant $c_{n,q} = n/(n+2q)$ on n and q is of importance on its own. This will become clear in the proofs of Lemma 6.9 and Lemma 6.12.

Lemma 6.8. *For every $c \in (0,\infty)$,*

$$\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_{c\mathbb{B}_1^n})} = c\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_{\mathbb{B}_1^n})}. \tag{6.5}$$

Proof. It is well known that $\mu_{c\mathbb{B}_1^n}$ coincides with the normalization of the Hausdorff measure on $c\,\mathbb{S}_1^{n-1}$, see [65]. Then, for $t>0$ large enough,

$$\begin{aligned}\int_{c\,\mathbb{S}_1^{n-1}} \exp\big((\langle x,\theta\rangle/t)^2\big)\,\mathrm{d}\mu_{c\mathbb{B}_1^n}(x) &= \frac{1}{\mathcal{H}^{n-1}_{c\mathbb{S}_1^n}(c\,\mathbb{S}_1^{n-1})}\int_{c\,\mathbb{S}_1^{n-1}} \exp\big((\langle x,\theta\rangle/t)^2\big)\,\mathrm{d}\mathcal{H}^{n-1}_{c\mathbb{S}_1^n}(x) \\ &= \frac{c^{n-1}}{\mathcal{H}^{n-1}_{c\mathbb{S}_1^n}(c\,\mathbb{S}_1^{n-1})}\int_{\mathbb{S}_1^{n-1}} \exp\big((c\langle x',\theta\rangle/t)^2\big)\,\mathrm{d}\mathcal{H}^{n-1}_{\mathbb{S}_1^n}(x') \\ &= \int_{\mathbb{S}_1^{n-1}} \exp\big((c\langle x',\theta\rangle/t)^2\big)\,\mathrm{d}\mu_{\mathbb{B}_1^n}(x')\,,\end{aligned}$$

where we used the homogeneous of degree $n-1$ of the Hausdorff measure in the last step. This implies the claim by definition of the ψ_2-norm. □

Lemma 6.9. *For every $\theta \in \mathbb{R}^n$,*

$$\sqrt{n}\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_{\mathbb{B}_1^n})} \le \sqrt{6}\|\theta\|_\infty.$$

Proof. The proof follows the idea of the one in [22] (see also page 305 in [28]). Let

$q \in \mathbb{N} \cup \{0\}$. Using the unconditionality of $\mathbb{B}_1^n$, expanding the power of the scalar product yields

$$\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^{2q} \,\mathrm{d}\nu_{\mathbb{B}_1^n}(x) = \sum_{q_i \in \mathbb{N}\cup\{0\}, \sum_{i=1}^n q_i = q} \binom{2q}{2q_1, \dots, 2q_n} \prod_{i=1}^n \theta_i^{2q_i} \int_{\mathbb{R}^n} \prod_{i=1}^n x_i^{2q_i} \,\mathrm{d}\nu_{\mathbb{B}_1^n}(x),$$

where we used the standard notation for multinomial coefficients. Moreover, whenever we have $q_1 + \ldots + q_n = q$, it holds

$$\int_{\mathbb{R}^n} \prod_{i=1}^n x_i^{2q_i} \,\mathrm{d}\nu_{\mathbb{B}_1^n}(x) = \frac{n!}{(n+2q)!} \prod_{i=1}^n (2q_i)!\,.$$

For the sake of completeness, we prove this claim by induction. Note that it is equivalent to

$$\int_{\mathbb{B}_1^n} \prod_{i=1}^n x_i^{2q_i} \,\mathrm{d}x = \frac{2^n}{(n+2q)!} \prod_{i=1}^n (2q_i)!\,.$$

The equality holds for $n = 1$, indeed both sides are equal to $2/(1+2q_1)$. Suppose that it holds in dimension n and for exponents $2q_1, \dots, 2q_n$ whose sum is equal to $2q$. We want to prove it in dimension $n+1$ adding a new exponent $2q_{n+1}$:

$$\begin{aligned}
\int_{\mathbb{B}_1^{n+1}} \prod_{i=1}^{n+1} x_i^{2q_i} \,\mathrm{d}x_1 \dots \mathrm{d}x_{n+1} &= \int_{-1}^{1} \int_{(1-|x_{n+1}|)\mathbb{B}_1^n} \prod_{i=1}^n x_i^{2q_i} \,\mathrm{d}x_1 \dots \mathrm{d}x_n \, x_{n+1}^{2q_{n+1}} \,\mathrm{d}x_{n+1} \\
&= \int_{-1}^{1} \int_{\mathbb{B}_1^n} (1-|x_{n+1}|)^{n+2q} \prod_{i=1}^n y_i^{2q_i} \,\mathrm{d}y_1 \dots \mathrm{d}y_n \, x_{n+1}^{2q_{n+1}} \,\mathrm{d}x_{n+1} \\
&= \frac{2^n}{(n+2q)!} \prod_{i=1}^n (2q_i)! \int_{1}^{1} (1-|x_{n+1}|)^{n+2q} x_{n+1}^{2q_{n+1}} \,\mathrm{d}x_{n+1} \\
&= \frac{2^{n+1}}{(n+2q)!} \prod_{i=1}^n (2q_i)! \int_{0}^{1} (1-z)^{n+2q} z^{2q_{n+1}} \,\mathrm{d}z \\
&= \frac{2^{n+1}}{(n+2q)!} \prod_{i=1}^n (2q_i)! \frac{(n+2q)!(2q_{n+1})!}{(n+2q+2q_{n+1}+1)!} \\
&= \frac{2^{n+1}}{(n+1+2\sum_{i=1}^{n+1} q_i)!} \prod_{i=1}^{n+1} (2q_i)!,
\end{aligned}$$

which proves the claim.

Now, if we set $\alpha := \sqrt{n}\|\theta\|_\infty$, then it holds that $\prod_{i=1}^n \theta_i^{2q_i} \le \alpha^{2q} n^{-q}$. This yields

$$\begin{aligned}\int_{\mathbb{R}^n} |\langle x,\theta\rangle|^{2q}\, \mathrm{d}\nu_{\mathbb{B}_1^n}(x) &\le \sum_{q_i\in\mathbb{N}\cup\{0\},\sum_{i=1}^n q_i = q} \binom{2q}{2q_1,\dots,2q_n} \frac{\alpha^{2q}}{n^q} \frac{n!}{(n+2q)!} \prod_{i=1}^n (2q_i)! \\ &= \sum_{q_i\in\mathbb{N}\cup\{0\},\sum_{i=1}^n q_i = q} \frac{n!(2q)!\,\alpha^{2q}}{(n+2q)!\,n^q} \\ &= \binom{n+q-1}{n-1} \frac{n!(2q)!\,\alpha^{2q}}{(n+2q)!\,n^q},\end{aligned}$$

where in the last equality we used that the cardinality of the set of indices in the sum is precisely $\binom{n+q-1}{n-1}$. Using Equation (6.4), we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\langle x,\theta\rangle|^{2q}\, \mathrm{d}\mu_{\mathbb{B}_1^n}(x) &= \frac{n+2q}{n} \int_{\mathbb{R}^n} |\langle x,\theta\rangle|^{2q}\, \mathrm{d}\nu_{\mathbb{B}_1^n}(x) \\ &\le \binom{n+q-1}{n-1} \frac{(n-1)!(2q)!\,\alpha^{2q}}{(n+2q-1)!\,n^q} \\ &= \frac{1}{(n+q)\cdots(n+2q-1)} \frac{(2q)!\,\alpha^{2q}}{q!\,n^q} \\ &\le \frac{q!}{2}\Big(\frac{2\alpha}{n}\Big)^{2q},\end{aligned}$$

where for the last step we used the inequality $2(2q)! \le (2^q q!)^2$, which can be checked by induction on $q \in \mathbb{N}$, and the fact that $n^q \le (n+q)\cdots(n+2q-1)$. When $|t| < 1/(2\alpha)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \exp\big((tn\langle x,\theta\rangle)^2\big)\, \mathrm{d}\mu_{\mathbb{B}_1^n}(x) &= 1 + \sum_{q=1}^\infty \frac{t^{2q} n^{2q}}{q!} \int_{\mathbb{R}^n} |\langle x,\theta\rangle|^{2q}\, \mathrm{d}\mu_{\mathbb{B}_1^n}(x) \\ &\le 1 + \frac{1}{2}\sum_{q=1}^\infty (2t\alpha)^{2q} \\ &= 1 + \frac{1}{2}\Big(\frac{1}{1-4t^2\alpha^2} - 1\Big).\end{aligned}$$

If t_0 is such that the last expression equals 2 when evaluated in $t = t_0$, we get that

$$n\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_{\mathbb{B}_1^n})} \le 1/t_0 = \sqrt{6}\alpha = \sqrt{6}\sqrt{n}\|\theta\|_\infty,$$

which completes the proof. $\square$

Proof of Proposition 6.6. From Lemma 6.7 and the definition of $\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_V)}$, we get

$$\begin{aligned}\int_{\partial K}\exp\Big(\langle x,\theta\rangle^2/\|\langle\cdot,\theta\rangle\|^2_{L^{\psi_2}(\mu_V)}\Big)\,\mathrm{d}\mu_K(x)&\\ &= 1+\sum_{q=1}^{\infty}\frac{1}{q!\|\langle\cdot,\theta\rangle\|^{2q}_{L^{\psi_2}(\mu_V)}}\int_{\partial K}|\langle x,\theta\rangle|^{2q}\,\mathrm{d}\mu_K(x)\\ &\leq 1+\sum_{q=1}^{\infty}\frac{1}{q!\|\langle\cdot,\theta\rangle\|^{2q}_{L^{\psi_2}(\mu_V)}}\int_{\partial V}|\langle x,\theta\rangle|^{2q}\,\mathrm{d}\mu_V(x)\\ &= 2.\end{aligned}$$

In particular, we have

$$\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_K)}\leq\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_V)}.$$

Moreover, from Lemma 6.8 and Lemma 6.9, we obtain

$$\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_V)}=\frac{\sqrt{6}}{2}n\|\langle\cdot,\theta\rangle\|_{L^{\psi_2}(\mu_{\mathbb{B}_1^n})}\leq 3\sqrt{n}\|\theta\|_\infty.$$

The proof is thus complete. □

Remark 14. Let us emphasize that we decided to use a different approach than the one used in Lemma 6.12 below in order to gain a comparison between the cone measure of an isotropic unconditional convex body and the cone measure of a suitably rescaled ball with respect to the 1-norm. Also, we have made explicit every constant in our computations.

6.5 Proof of the unconditional case

Recalling the bound for the isotropic constant presented in Lemma 6.4 (a), our proof is naturally divided into two parts. The first is concerned with a lower bound on the volume radius of our random polytope.

In the following Lemma, part (*a*) will be applied to the case of an isotropic unconditional convex body, while part (*b*) will be used for the general case of an isotropic convex body.

Lemma 6.10. *Let $K\subset\mathbb{R}^n$ be a convex body with $\mathrm{vol}_n(K)=1$ and K_N the symmetric convex hull of N independent random points on ∂K with distribution μ_K.*

(a) There exist constants $c_1 \in (1,\infty)$ and $c_2 \in (0,\infty)$ such that the event that

$$\mathrm{vol}_n(K_N)^{1/n} \geq c_2 \min\left\{\sqrt{\frac{\log(2N/n)}{n}}, 1\right\}$$

has probability greater than $1-\exp(-n)$ when $N \geq c_1 n$.

(b) There exist constants $c_1 \in (1,\infty)$ and $c_2 \in (0,\infty)$ such that the event that

$$\mathrm{vol}_n(K_N)^{1/n} \geq c_2\, L_K \sqrt{\frac{\log(2N/n)}{n}}$$

has probability greater than $1-\exp(-c_1\sqrt{N})$ when $n \leq N \leq e^{\sqrt{n}}$.

Proof. Let us start with (a). We use a coupling argument that was introduced in [48]. Let $Y_1,\dots,Y_N$ be independent random points distributed according to the uniform distribution on K, and define the symmetric random polytope

$$\widetilde{K}_N := \mathrm{conv}(\pm Y_1,\dots,\pm Y_N).$$

It is proven in [33, Proposition 2.2] that if $N \geq c_1 n$, then

$$\mathrm{vol}_n(\widetilde{K}_N)^{1/n} \geq c_2 \min\left\{\sqrt{\frac{\log(2N/n)}{n}}, 1\right\}$$

with probability greater than $1-\exp(-n)$. For $i \in \{1,\dots,N\}$, consider the random variables

$$X_i := \begin{cases} \dfrac{Y_i}{\|Y_i\|_K} & \text{if } \|Y_i\|_K \neq 0, \\ y \in \partial K & \text{if } \|Y_i\|_K = 0\,, \end{cases}$$

where y is a fixed but arbitrary point on ∂K. By definition, the points $X_1,\dots,X_N$ are independent and belong to ∂K. Moreover, the push-forward probability measure of the uniform distribution ν_K under the map $K \ni y \mapsto y/\|y\|_K \in \partial K$ is exactly the cone probability measure μ_K on K. Indeed, for any Borel set $B \subset \partial K$,

$$\mathbf{P}(X_i \in B) = \mathbf{P}(Y_i \in (0,1]B) = \frac{\mathrm{vol}_n((0,1]B)}{\mathrm{vol}_n(K)} = \mu_K(B).$$

Note also that it follows from the symmetry of K that if $X \in \partial K$, then also $-X \in \partial K$.

In particular, the symmetric random polytope

$$K_N \coloneqq \mathrm{conv}(\pm X_1, \dots, \pm X_N)$$

has the desired distribution. Moreover, by construction $K_N(\omega) \supseteq \widetilde{K}_N(\omega)$ for every realization $\omega \in \Omega$, so that

$$\mathrm{vol}_n(K_N)^{1/n} \geq \mathrm{vol}_n(\widetilde{K}_N)^{1/n} \geq c_2 \min\left\{\sqrt{\frac{\log(2N/n)}{n}}, 1\right\}$$

with probability greater than $1 - \exp(-n)$.

The proof of part (b) is similar. The only change is that for the lower bound for $\mathrm{vol}_n(\widetilde{K}_N)^{1/n}$ instead of [33, Proposition 2.2] we now use [34, Theorem 4.1] in the form of [28, Theorem 11.3.7]. □

Now that we have established the ψ_2-estimate in the previous section, we can proceed to bound the second quantity that we need in view of Lemma 6.3 (a).

Lemma 6.11. *Let $K \subset \mathbb{R}^n$ be an isotropic unconditional convex body. For $N > n$ let $X_1, \dots, X_N$ be independent random points distributed according to the cone measure on ∂K. Then there exist constants $c, C \in (0, \infty)$ such that with probability greater than $1 - \exp(-cn\log(2N/n))$ it holds*

$$\max_{\varepsilon_1,\dots,\varepsilon_n = \pm 1} \|\varepsilon_1 X_{i_1} + \ldots + \varepsilon_n X_{i_n}\|_1 \leq C\, n^{3/2}\sqrt{\log(2N/n)}$$

for all subsets of vertices $\{X_{i_1}, \dots, X_{i_n}\} \subset \{\pm X_1, \dots, \pm X_N\}$.

Proof. We start considering the points $X_1, \dots, X_n$. Fix a direction $\theta \in \mathbb{S}_\infty^{n-1}$ and an n-tuple of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, +1\}^n$. For every $i \in \{1, \dots, n\}$, we define the random variables $Y_i \coloneqq \langle \varepsilon_i X_i, \theta \rangle$. Note that by Proposition 6.6, $\|Y_i\|_{L^{\psi_2}(\mu_K)} \leq 3\sqrt{n}$ so that we can apply linearity and the ψ_2-version of Bernstein's inequality (see Lemma 6.3 (a)) in order to get

$$\mathbf{P}\big(|\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta \rangle| > tn\big) \leq 2\exp(-t^2/72), \tag{6.6}$$

for every $t > 0$. Now we notice that

$$\|\varepsilon_1 X_1 + \ldots + \varepsilon_n X_n\|_1 = \sup_{\theta \in \mathbb{S}_\infty^{n-1}} |\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta \rangle| = \max_{\theta \in \{-1,1\}^n} |\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta \rangle|.$$

Hence, we obtain

$$\begin{aligned}
&\mathbf{P}\Big(\max_{\varepsilon\in\{-1,+1\}^n}\|\varepsilon_1X_1+\ldots+\varepsilon_nX_n\|_1>tn\Big)\\
&\qquad=\mathbf{P}\Big(\max_{\varepsilon,\theta\in\{-1,1\}^n}|\langle\varepsilon_1X_1+\ldots+\varepsilon_nX_n,\theta\rangle|>tn\Big)\\
&\qquad\le 4^n\mathbf{P}\big(|\langle\varepsilon_1X_1+\ldots+\varepsilon_nX_n,\theta\rangle|>tn\big)\\
&\qquad\le\exp\big((2n+1)\log 2-t^2/72\big),
\end{aligned}\tag{6.7}$$

where we used the union bound together with the fact that, due to the unconditionality of the X_i's, $|\langle\varepsilon_1X_1+\ldots+\varepsilon_nX_n,\theta\rangle|$ has the same distribution for every choice of sign vectors ε and θ. We now consider all the subsets $\{X_{i_1},\ldots,X_{i_n}\}\subset\{\pm X_1,\ldots,\pm X_n\}$ of cardinality n. Since there are $\binom{2N}{n}\le(2eN/n)^n=\exp(n\log(2N/n))$ of such subsets, we can set $t:=C\sqrt{n\log(2N/n)}$, with $C\in(0,\infty)$ sufficiently large, and use again the union bound to get

$$\begin{aligned}
&\mathbf{P}\Big(\max_{\{X_{i_1},\ldots,X_{i_n}\}\subset\{\pm X_1,\ldots,\pm X_N\}}\max_{\varepsilon\in\{-1,+1\}^n}\|\varepsilon_1X_{i_1}+\ldots+\varepsilon_nX_{i_n}\|_1>Cn^{3/2}\sqrt{\log(2N/n)}\Big)\\
&\qquad\le\exp\big((2n+1)\log 2-(C^2/72-1)n\log(2N/n)\big)\\
&\qquad\le\exp\big(-cn\log(2N/n)\big),
\end{aligned}\tag{6.8}$$

which implies the statement by taking the complementary event. □

We are now prepared to complete the proof of Theorem 6.1.

Proof of Theorem 6.1. By Remark 9 the conclusion is clear if $N\le c_1n$ for some constant $c_1\in(0,\infty)$.

Let us next assume that there are constants $c_1,q\in(0,\infty)$ such that $c_1n\le N\le e^{an}$. Since every facet of K_N is obtained as the convex hull of a subset (of cardinality n with probability one) of all the vertices, Lemma 6.11 together with Lemma 6.5 (a) immediately gives that

$$\frac{1}{\mathrm{vol}_n(K_N)}\int_{K_N}\|x\|_1\,\mathrm{d}x\le(1+\sqrt{2})C\sqrt{n\log(2N/n)}\tag{6.9}$$

with probability greater than $1-\exp(-cn)$, where $c,C\in(0,\infty)$ are the same constants as in Lemma 6.11. Combining this with Lemma 6.4 (a) and Lemma 6.10 (a), we get that

$$\begin{aligned} L_{K_N} &\le \frac{c_3}{n}\frac{1}{\mathrm{vol}_n(K_N)^{1/n}}\frac{1}{\mathrm{vol}_n(K_N)}\int_{K_N}\|x\|_1\,\mathrm{d}x \\ &\le c_3\cdot c_2^{-1}\cdot\frac{1}{n}\sqrt{\frac{n}{\log(2N/n)}}\cdot(1+\sqrt{2})\,C\sqrt{n\log(2N/n)} \\ &= (1+\sqrt{2})\,c_3\cdot c_2^{-1}\cdot C \end{aligned} \tag{6.10}$$

with probability greater than $1-c_4\exp(-c_5 n)$.

Finally, we treat the case where $N \ge e^{an}$ for some constant $a \in (0,\infty)$. In this case Lemma 6.10 (a) yields that $\mathrm{vol}_n(K_N)^{1/n} \ge c_2$ for some constant $c_2 \in (0,\infty)$ with probability at least $1-e^{-n}$. In addition, by unconditionality of K it holds that $K \subset (\sqrt{6}/2)n\mathbb{B}_1^n$ (see, e.g., [28, p. 306]), hence

$$\frac{1}{\mathrm{vol}_n(K_N)}\int_{K_N}\|x\|_1\,\mathrm{d}x \le \frac{\sqrt{6}/2}{\mathrm{vol}_n(K_N)}\int_{K_N} n\|x\|_{K_N}\,\mathrm{d}x \le \frac{\sqrt{6}}{2}n.$$

Thus, Lemma 6.4 (a) yields the bound

$$L_{K_N} \le \frac{c}{n}\frac{1}{\mathrm{vol}_n(K_N)^{1/n}}\frac{1}{\mathrm{vol}_n(K_N)}\int_{K_N}\|x\|_1\,\mathrm{d}x \le \frac{c}{n}\frac{1}{c_2}\frac{\sqrt{6}}{2}n = \frac{\sqrt{6}}{2}\frac{c}{c_2}$$

with probability at least $1-e^{-n}$. The proof is thus complete. □

6.6 Proof of the general case

In this section we give a proof of Theorem 6.2. We start with the following ψ_1-estimate.

Lemma 6.12. *Fix an isotropic convex body $K \subset \mathbb{R}^n$ and $\theta \in \mathbb{S}^{n-1}$. Then there exists an absolute constant $c \in (0,\infty)$ such that $\|\langle\cdot,\theta\rangle\|_{L^{\psi_1}(\mu_K)} \le cL_K$.*

Proof. We recall that [28, Lemma 2.4.2] implies that

$$\|\langle\cdot,\theta\rangle\|_{L^{\psi_1}(\mu_K)} \le c\sup_{p\ge 1}\frac{\|\langle\cdot,\theta\rangle\|_{L^p(\mu_K)}}{p}$$

for some absolute constant $c \in (0,\infty)$. Moreover, from (6.4) we deduce that

$$\frac{\|\langle\cdot,\theta\rangle\|_{L^p(\mu_K)}}{p} = \Big(\frac{n+p}{n}\Big)^{1/p}\frac{\|\langle\cdot,\theta\rangle\|_{L^p(\nu_K)}}{p},$$

where ν_K is the uniform distribution on K. This implies

$$\|\langle\cdot,\theta\rangle\|_{L^{\psi_1}(\mu_K)} \leq c \sup_{p\geq 1}\Big(\frac{n+p}{n}\Big)^{1/p} \sup_{p\geq 1}\frac{\|\langle\cdot,\theta\rangle\|_{L^p(\nu_K)}}{p} \leq C\|\langle\cdot,\theta\rangle\|_{L^{\psi_1}(\nu_K)}$$

for another constant $C \in (0,\infty)$, since the first supremum is bounded by 2. However, $\|\langle\cdot,\theta\rangle\|_{L^{\psi_1}(\nu_K)}$ is bounded by a constant multiple of L_K, since every isotropic log-concave measure is known to be a so-called ψ_1-measure (this is essentially a consequence of Borell's lemma, see [28, page 81]). □

In a next step we observe that Lemma 6.10 (b) yields a lower bound for $\mathrm{vol}_n(K_N)^{1/n}$, which depends on the isotropic constant L_K of K whenever $N \leq e^{\sqrt{n}}$. In addition, Lemma 6.11 needs an adaptation. Especially, while in the unconditional case we could work with the 1-norm, here we have to deal with the 2-norm instead. Such a change causes the use of a ψ_1-estimate, which leads to a different kind of Bernstein inequality from the one used with the ψ_2-estimate. Eventually, this leads to the appearance of an additional logarithmic factor in our final result. Moreover, we have to make explicit now the dependence on L_K, since for a general isotropic convex body we do not know whether or not this quantity is bounded by an absolute constant, as explained in the introduction. In the end this will allow us to bound L_{K_N} independently of L_K if $N \leq e^{\sqrt{n}}$.

Lemma 6.13. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body. For $N > n$ let $X_1,\dots,X_N$ be independent random points distributed according to the cone measure on ∂K. Then there exist constants $c, C > 0$ such that with probability greater than $1-\exp(-cn\log(2N/n))$ it holds*

$$\max_{\varepsilon_1,\dots,\varepsilon_n=\pm 1}\|\varepsilon_1 X_{i_1}+\dots+\varepsilon_n X_{i_n}\|_2 \leq CL_K n\log(2N/n)$$

for all subsets of vertices $\{X_{i_1},\dots,X_{i_n}\}\subset\{\pm X_1,\dots,\pm X_N\}$.

Proof. The proof follows the one of Lemma 6.11 and we shall indicate the necessary modifications.

Let $X_1,\dots,X_n$ be independent random points with distribution μ_K and, for $\theta\in\mathbb{S}^{n-1}$ and $\varepsilon_1,\dots,\varepsilon_n\in\{-1,+1\}$, put $Y_i := \langle\varepsilon_i X_i,\theta\rangle$ for any $i\in\{1,\dots,n\}$. We start by noticing that Lemma 6.12 implies that if $K\subset\mathbb{R}^n$ is an arbitrary isotropic convex body, we have that $\|Y_i\|_{L^{\psi_1}(\mu_K)} \leq cL_K$ for some absolute constant $c\in(0,\infty)$ and any $i\in\{1,\dots,n\}$.

Thus, we can apply the ψ_1-version of Bernstein's inequality (Lemma 6.3 (b)), which implies that (6.6) needs to be replaced by

$$\mathbf{P}\big(|\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta\rangle| > scL_K n\big) \le 2\exp(-sn/6),$$

for some parameter $s > 1$ to be chosen later. Taking the union bound, we get

$$\mathbf{P}\Big(\max_{\varepsilon\in\{-1,+1\}^n} |\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta\rangle| > scL_K n\Big) \le \exp\big((n+1)\log 2 - sn/6\big).$$

Consider now a $\frac{1}{2}$-net $\mathcal{N}$ of $\mathbb{S}^{n-1}$ with cardinality at most 5^n (the existence of such a net is ensured by [8, Lemma 5.2.5], for example). Applying the union bound once more leads to

$$\mathbf{P}\Big(\max_{\theta\in\mathcal{N}} \max_{\varepsilon\in\{-1,+1\}^n} |\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta\rangle| > scL_K n\Big) \le \exp\big((n+1)\log 2 + n\log 5 - sn/6\big).$$

For any $\theta \in \mathbb{S}^{n-1}$ there exist a sequence $(\theta_j)_{j\in\mathbb{N}} \in \mathcal{N}^{\mathbb{N}}$ and coefficients $\delta_j \in [0, 2^{1-j}]$ such that $\theta = \sum_{j=1}^{\infty} \delta_j\theta_j$ (see [3]). In particular, this implies

$$\begin{aligned}
&\mathbf{P}\Big(\max_{\theta\in\mathbb{S}^{n-1}} \max_{\varepsilon\in\{-1,+1\}^n} |\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta\rangle| > 2scL_K n\Big)\\
&\qquad \le \mathbf{P}\Big(\max_{\theta\in\mathbb{S}^{n-1}} \max_{\varepsilon\in\{-1,+1\}^n} \sum_{j=1}^{\infty} \delta_j |\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta_j\rangle| > 2scL_K n\Big)\\
&\qquad \le \mathbf{P}\Big(\max_{\theta\in\mathcal{N}} \max_{\varepsilon\in\{-1,+1\}^n} |\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta_j\rangle| > scL_K n\Big)\\
&\qquad \le \exp\big((n+1)\log 2 + n\log 5 - sn/6\big).
\end{aligned}$$

Notice that

$$\max_{\theta\in\mathbb{S}^{n-1}} |\langle \varepsilon_1 X_1 + \ldots + \varepsilon_n X_n, \theta\rangle| = \|\varepsilon_1 X_1 + \ldots + \varepsilon_n X_n\|_2.$$

Hence, applying a union bound and taking $s := 42\log(2N/n)$, (6.8) gets replaced by

$$\begin{aligned}
\mathbf{P}\Bigg(\max_{\{X_{i_1},\ldots,X_{i_n}\}\subset\{\pm X_1,\ldots,\pm X_N\}} \max_{\varepsilon\in\{-1,+1\}^n} \|\varepsilon_1 X_{i_1} + \ldots + \varepsilon_n X_{i_n}\|_2 > 84cL_K n\,\log(2N/n)\Bigg)&\\
\le \exp\big(-n\log(2N/n)\big).&
\end{aligned}$$

This completes the proof. □

Proof of Theorem 6.2. Again, the proof follows closely the one of Theorem 6.1 and we shall indicate the necessary modifications.

The regime where $N \leq cn$ is trivial by Remark 9. Next, as long as $N \leq e^{\sqrt{n}}$ we combine this time Lemma 6.13 with Lemma 6.5 (b) to see that (6.9) gets replaced by

$$\frac{1}{\mathrm{vol}_n(K_N)} \int_{K_N} \|x\|_2^2 \, \mathrm{d}x \leq 2C^2 L_K^2 \log(2N/n)^2,$$

which holds with probability greater than $1 - \exp(-c_1 n)$, where $C \in (0, \infty)$ is an absolute constant. Combining this with Lemma 6.4 (b) and Lemma 6.10 (b), we deduce that (6.10) has to be replaced by

$$\begin{aligned} L_{K_N}^2 &\leq \frac{1}{n \, \mathrm{vol}_n(K_N)^{2/n}} \frac{1}{\mathrm{vol}_n(K_N)} \int_{K_N} \|x\|_2^2 \, \mathrm{d}x \\ &\leq \frac{1}{n} \frac{n}{c_2^2 \log(2N/n) L_K^2} C^2 L_K^2 \log(2N/n)^2 \leq \frac{C^2}{c_2^2} \log(2N/n), \end{aligned}$$

which holds with probability greater than $1 - c_3 \exp(-c_4 n) - \exp(-c_5 \sqrt{N})$. The proof is thus complete. □

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