On the Completeness of Some First-order Extensions of C

Grigory K. Olkhovikov*
Department of Philosophy I
Ruhr University Bochum
grigory.olkhovikov@{rub.de, gmail.com}

Abstract

We show the completeness of several Hilbert-style systems resulting from extending the propositional connexive logics C and C3 by the set of Nelsonian quantifiers, both in the varying domain and in the constant domain setting. In doing so, we focus on countable signatures and proceed by variations of the Henkin construction. We compare our work on the first-order extensions of C3 with the results of [10] and answer several open questions naturally arising in this respect. In addition, we consider possible extensions of C and C3 with a non-Nelsonian universal quantifier preserving a specific rapport between the interpretation of conditionals and the interpretation of the universal quantification which is visible in both intuitionistic logic and Nelson's logic but is lost if one adds the Nelsonian quantifiers on top of the propositional basis provided by C and C3. We briefly explore the completeness of systems resulting from adding this non-Nelsonian quantifier either together with the Nelsonian ones or separately to the two propositional connexive logics.

First-order logic, completeness, Nelson's logic, paraconsistent logic

1 Introduction

The present paper contains some completeness results concerning a family of first-order extensions of two propositional connexive logics, C and C3.[1] Among

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[1] More sources on connexivity in logic and on different systems of connexive logics can be found in [9] and [15].

connexive logics, C holds a special place in being, on the one hand, "one of the simplest and most natural"[12, p.178] connexive systems, and, on the other hand, being negation-inconsistent.

The propositional logic C was introduced in [14] as a connexive modification of the paraconsistent version N4 of Nelson's constructive logic of strong negation.[2] The logic C3, a variant of C which excludes the truth-value gaps, was then introduced in [10]. The connectives of C are defined in such a way that, as long as truth-value gaps are eradicated at the level of atoms, they also cannot occur at the level of compound sentences. As a result, the only difference between Hilbert-style axiomatizations of C and C3, respectively, is the presence in the latter of the axiom $\phi \vee \sim \phi$, corresponding to the law of excluded middle for the strong negation.

The quantified version of C, which we will call QC in this paper, was obtained by borrowing the semantics of $\forall$ and $\exists$ from QN4 (in the present paper, we will call them the Nelsonian quantifiers) and adding them on top of the propositional basis of C. A Hilbert-style axiomatization of QC was also proposed in [14] and was immediately shown to be complete via an embedding of the set of formulas of QC into positive intuitionistic logic. However, this work has not been extended yet to the extension of C3 with the Nelsonian quantifiers, although some proof-theoretic results about some extensions of this kind were reported already in [10]. A peculiar complication that arose relative to this type of extensions, consisted in the fact that the simple addition of Nelsonian quantifiers to C3 led to the reinstatement of truth-value gaps. This problem afflicts one of the first-order extensions of C3 introduced (in a purely proof-theoretic manner) in [10], namely, $\mathsf{QC3}_{At}$. The other system introduced in [10], QC3 eliminates them in a somewhat too direct manner. As a result, the set of admissible models is no longer closed for the models based on the same underlying Kripke frame so that the Kripke semantics of QC3 assumes a decidedly non-standard flavor.

One natural remedy to this adverse effect would have been to require the constancy of object domains associated to the nodes in Kripke models; but, in case this

[2] Nelson's original logic QN3 was introduced in [7]. It was from the very beginning a first-order logic, a first-order arithmetic even, with a semantics inspired by the Kleene's realizability semantics. However, the guiding idea behind Nelson's realizability clauses was clear enough so that their translation into Kripke semantics was completely unproblematic. See one of the early examples of such a translation — however, assuming the constancy of domains, – in [13]. QN4, on the other hand, was only introduced explicitly in a relatively recent [8, Section 4.1]; its only difference from QN3 is that the gluts, that is to say, the sentences that are both true and false at the same node of a Kripke model, are now allowed. The propositonal fragments of these logics, which we will denote by N3 and N4, respectively, also have been objects of separate study for many years now; in particular, N4 was introduced for the first time, to the best of our knowledge, in [6], and, independently, in [1].

move is taken, and the system $\mathsf{QC3}_{CD}$ is understood as the extension of C3 with the Nelsonian quantifiers under the assumption of constant domains, our attention is also inevitably drawn to the system which is now seen as a natural intermediary between QC and $\mathsf{QC3}_{CD}$. This third system, which we will denote by QC_{CD}, results from the addition of the Nelsonian $\forall$ and $\exists$ to C under the same assumption of constant domains which we had to impose on $\mathsf{QC3}_{CD}$.

The main goal of the present paper is then to spell out what happens with the completeness proofs in the family of the logics outlined in the previous paragraph, namely $\{\mathsf{QC}, \mathsf{QC}_{CD}, \mathsf{QC3}_{At}, \mathsf{QC3}, \mathsf{QC3}_{CD}\}$. Given that the first-order extensions of the propositional connexive logics remain largely unexplored, our plan for the paper is to provide a firm basis for further advancement by showing how fairly standard Henkin-style constructions can be produced for these logics, rather than to surprise the reader with new findings. That is why we also treat the completeness of QC even though it was already proven in [14] by an indirect argument; our aim is to spell out a direct proof by the usual Henkin technique that allows for further modifications aimed at getting the completeness results also for the other systems.

In achieving this goal, we adapt a mix of traditional techniques for proving completeness of intuitionistic and intermediate first-order logics; a knowledgeable reader will not fail to notice that we are influenced by the presentation of the completeness proofs given in [3, Ch. 4–5] and [5, Ch. 6–7].

However, given that C departs from N3 and N4 in its understanding of the propositonal connectives, the extension of C with the Nelsonian quantifiers cannot be viewed as the only acceptable choice, not without an additional argument that takes into account the range of other objectively existing options for such an extension. Although in this paper we mainly confine ourselves to preparing the ground for a comprehensive discussion of relative pros and cons of adopting the Nelsonian quantifiers in C, we also find it important to define and motivate, already at this point, at least one non-Nelsonian version of the universal quantifier. It turns out that it is relatively easy to take this new quantifier on board, both as an addition to the set of Nelsonian quantifiers and as the only quantifier extending the connexive propositional base — as long as one does not insist on eradicating the truth-value gaps in the style of C3. On the other hand, for the first-order extensions of C3 the non-Nelsonian universal quantifier exacerbates the problem of truth-value gaps to the point where even the assumption of constant domains is now no longer sufficient to eliminate them.

The corresponding completeness results for the extensions of C featuring the non-Nelsonian universal quantifier are then obtainable by repeating, with some minimal variations, the respective completeness proofs for the Nelsonian extensions of C and C3, which constitutes another reason for the inclusion of this whole discussion into the current preparatory work on first-order connexive logics.

The layout of the remaining part of present paper is then as follows. In Section 2, we define our notation and introduce the Kripke semantics for the first-order connexive logics with the Nelsonian quantifiers. In Section 3, we recall the axiomatization of QC given in [14], develop the basics of the Hilbertian proof theory for this system, and then prove both the general soundness theorem and its converse for the case of countable signatures. In Section 4, we introduce, for the first time in the existing literature, the axiomatizations for the Nelsonian systems, QC_{CD}, $\mathsf{QC3}_{CD}$, and show how to modify the proofs given in Section 3, so that they extend to these logics. In Section 5, we extend our completeness proofs to $\mathsf{QC3}_{At}$ and $\mathsf{QC3}$ and look at the results reported about these systems in [10] in the light of the notions and techniques developed in the previous sections. In particular, we address the question of Existence Property in the first-order extensions of $\mathsf{C3}$. Section 6 is devoted to the discussion of one possible definition of a non-Nelsonian universal quantifier which we denote by Æ, and of the axiomatizations of some logics featuring this quantifier.

Finally, in Section 7, we draw conclusions and try to map out some of the avenues for the future research.

2 Preliminaries and Notations

2.1 The First-order Language

We start by fixing some general notational conventions. In this paper, we identify the natural numbers with finite ordinals. We denote by ω the smallest infinite ordinal. For any $n \in \omega$, we will denote by $\bar{o}_n$ the sequence $(o_1, \ldots, o_n)$ of objects of any kind; moreover, somewhat abusing the notation, we will sometimes denote $\{o_1, \ldots, o_n\}$ by $\{\bar{o}_n\}$. The ordered 1-tuple will be identified with its only member. For any given $m, n \in \omega$, the notation $(\bar{p}_m)^\frown(\bar{q}_n)$ denotes the concatenation of $\bar{p}_m$ and $\bar{q}_n$.

Given a set X and a $k \in \omega$, the notation X^k (resp. $X^{\neq k}$) will denote the k-th Cartesian power of X (resp. the set of all k-tuples from X^k such that their elements are pairwise distinct). We also define that $X^\infty := \bigcup_{n \geq 0} X^n$. The *powerset* of X, that is to say, the set of its subsets, will be denoted by $\mathcal{P}(X)$; on the other hand, the *power of* X will be referred to by $|X|$, so that, for example, $|X| = \omega$ will mean that X is countably infinite. Finally, if X, Y are sets, then we will write $X \Subset Y$, if $X \subseteq Y$ and X is finite.

Given any relation R and a set X, we will denote by $R[X]$ the set $\{b \mid (\exists a \in X)(R(a, b))\}$; this notation naturally extends to cases when R is a function f or its inverse f^{-1}. In case $X = \{a\}$, we will also write $R[a]$ (resp. $f^{-1}[a]$) instead of $R[\{a\}]$ (resp. $f^{-1}[\{a\}]$). In case $\bar{a}_n \in X^n$, we will denote by $R\langle\bar{a}_n\rangle$ the set $\{\bar{b}_n \mid (\forall 1 \leq i \leq n)(b_i \in R[a_i])\}$. Similarly, by $f\langle\bar{a}_n\rangle$ we will denote the tuple $(f(a_1), \ldots, f(a_n))$.

Given any function f, we will denote its domain by $dom(f)$; the *range* of f, denoted by $rang(f)$ is just $f[dom(f)]$. In case $rang(f) \subseteq X$, we will also write $f : dom(f) \to X$. Finally, for any set X, we will denote by id_X the identity function on X.

The notations introduced above for sets and functions are also freely applied in this paper to proper classes and class functions.

In this paper, we consider the first-order language without equality based on any set of predicate letters of any arity $k \in \omega$. In particular, 0-ary predicates, or propositional letters, are allowed in our language. We do not allow functions and constants[3], though.

We fix a proper class $Pred$ of possible predicate letters. Elements of $Pred$ will be normally denoted by capital Latin letters like P and Q. If $\Omega \subseteq Pred$ is a set, then any function $\Sigma : \Omega \to \omega$ is called a *signature*. Signatures will be denoted by letters Σ and Θ; moreover, we set that $|\Sigma| = |dom(\Sigma)|$. If $n \in \omega$ and $P \in \Sigma^{-1}[n]$, then we will also write $P^n \in \Sigma$.

Since signatures are functions, we can take their unions and intersections, in case the former make sense according to the general restrictions existing for such operations.

All these notations and all of the other notations introduced in this section can be decorated by all types of sub- and superscripts.

We are going to allow parameters in our formulas, therefore, we also fix a proper set Par which we assume to be disjoint from $Pred$. The elements of Par will be denoted by small Latin letters like a, b, c, and d.

Having fixed a signature Σ, and a set $\Pi \subseteq Par$ we generate a language out of it in the following way. We use $Log := \{\sim, \wedge, \vee, \to, \forall, \exists\}$ as the set of logical symbols and $Var := \{v_i \mid i < \omega\}$ as the set of (individual) variables. Both of these sets are assumed to be disjoint from $Pred \cup Par$. The set $L(\Sigma, \Pi)$ of Σ-formulas with parameters in Π can be then defined by the usual induction on the construction of a formula; in other words, $L(\Sigma, \Pi)$ is the smallest set such that:

1. $P(\bar{\alpha}_n) \in L(\Sigma, \Pi)$ for any $n \in \omega$, $P^n \in \Sigma$, and $\bar{\alpha}_n \in (Var \cup \Pi)^n$.

2. $\{\sim \phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \to \psi), \forall x\phi, \exists x\phi\} \subseteq L(\Sigma, \Pi)$ for all $\phi, \psi \in L(\Sigma, \Pi)$ and $x \in Var$.

[3]The parameters that we speak about are not proper constants since they are not required to be defined at every node of an appropriate model.

As per usual, we get that $|L(\Sigma, \Pi)| = \max(|\Sigma|, |\Pi|, \omega)$.

The elements of Var will be also denoted by x, y, z, w, and the elements of $L(\Sigma, \Pi)$ by Greek letters like ϕ, ψ and θ. In what follows, we will also freely use $\leftrightarrow$, understanding $\phi \leftrightarrow \psi$ as an abbreviation for $(\phi \to \psi) \wedge (\psi \to \phi)$. Moreover, given an $n \in \omega$ and a $\bar{\phi}_n \in L(\Sigma, \Pi)^n$ we define that $\bigwedge \bar{\phi}_n := \phi_1 \wedge \ldots \wedge \phi_n$ with the parentheses grouped to the left, and similarly for $\bigvee \bar{\phi}_n$.

Now we can also define the (always finite) sets of parameters, free and bound variables occurring in a given $\phi \in L(\Sigma, \Pi)$ as well as the smallest (finite) signature associated with ϕ (to be denoted by $Par(\phi)$, $FV(\phi)$, $BV(\phi)$, and $Sign(\phi)$, respectively). The definition is by induction on the construction of ϕ:

1. $Par(P(\bar{\alpha}_n)) = \{\bar{\alpha}_n\} \cap Par$, $FV(P(\bar{\alpha}_n)) = \{\bar{\alpha}_n\} \cap Var$, $BV(P(\bar{\alpha}_n)) = \emptyset$, and $Sign(\phi) = \{(P, \Sigma(P))\}$.

2. $\alpha(\sim \phi) = \alpha(\phi)$ and $\alpha(\phi \circ \psi) = \alpha(\phi) \cup \alpha(\psi)$ for all $\alpha \in \{Par, FV, BV, Sign\}$ and $\circ \in \{\wedge, \vee, \to\}$.

3. $Par(Qx\phi) = Par(\phi)$, $FV(Qx\phi) = FV(\phi) \setminus \{x\}$, $BV(Qx\phi) = BV(\phi) \cup \{x\}$, and $Sign(Qx\phi) = Sign(\phi)$ for all $Q \in \{\forall, \exists\}$.

For any $\Gamma \subseteq L(\Sigma, \Pi)$ and any $\alpha \in \{Par, FV, BV, Sign\}$, we define that $\alpha(\Gamma) := \bigcup\{\alpha(\phi) \mid \phi \in \Gamma\}$. Note that, for an infinite Γ, $FV(\Gamma)$ and $BV(\Gamma)$ can be countably infinte; as for the parameter sets and the associated signatures, we clearly have $|Par(\Gamma)|, |Sign(\Gamma)| \leq \max(|\Gamma|, \omega)$ for all $\Gamma \subseteq L(\Sigma, \Pi)$.

It is also clear that for any given $\phi \in L(\Sigma, \Pi)$, any signature Θ, and any set $\Xi \subseteq Par$, we have $\phi \in L(\Theta, \Xi)$ iff $Par(\phi) \subseteq \Xi$ and $Sign(\phi) \subseteq \Theta$.

We will denote the set of $L(\Sigma, \Pi)$-formulas with free variables among the elements of $\bar{x}_n \in Var^{\neq n}$ by $L_{\bar{x}_n}(\Sigma, \Pi)$; in case $\Pi = \emptyset$, we simply write $L_{\bar{x}_n}(\Sigma)$ instead of $L_{\bar{x}_n}(\Sigma, \emptyset)$. In particular, $L_{\emptyset}(\Sigma, \Pi)$ will stand for the set of Σ-sentences with parameters in Π. If $\phi \in L_{\bar{x}_n}(\Sigma, \Pi)$ (resp. $\Gamma \subseteq L_{\bar{x}_n}(\Sigma, \Pi)$), then we will also express this by writing $\phi(\bar{x}_n)$ (resp. $\Gamma(\bar{x}_n)$).

The formulas in $L(\Sigma)$ (resp. sentences in $L_{\emptyset}(\Sigma)$) will be called *pure* Σ-formulas (resp. *pure* Σ-sentences). It is $L_{\emptyset}(\Sigma)$ that can be called a language (over Σ, which in this case serves as a vocabulary) in the most direct and complete sense: every pure Σ-sentence says something in every possible Σ-model. Pure formulas with free variables are mainly of interest as possible constituent parts of pure sentences. Parametrized formulas, including parametrized sentences, are strange hybrid entities arising from pure sentences and formulas after these latter get (partially) interpreted in some particular model, which leads to a replacement of some variables in a formula by their denotations. The parametrized formulas are, therefore, always a mixture of linguistic entities like logical symbols or variables, and the objects in the world referred to by these linguistic entities *in a given interpretation attempt*; as such they are "neither here nor there".

However, the admission of these logical chimeras turns out to be very helpful both in defining the semantics and in formulating the calculi which are also complete for the sets of pure sentences over a given vocabulary, which is the reason for their introduction in this paper.

Given any $\phi \in L(\Sigma, \Pi)$, $\alpha \in Var \cup \Pi$, and $\beta \in \Pi \cup (Var \setminus BV(\phi))$, we denote by $\phi[\beta/\alpha] \in L(\Sigma, \Pi)$ the result of simultaneously replacing every occurrence of α by β (resp. every free occurrence in case $\alpha \in Var$). The precise definition of this operation proceeds by induction on the construction of $\phi \in L(\Sigma, \Pi)$ and runs as follows:

- $P(\bar{t}_n)[\beta/\alpha] := P(\bar{s}_n)$, where $P^n \in \Sigma$, and $\bar{t}_n, \bar{s}_n \in (Var \cup \Pi)^n$ are such that, for all $1 \leq i \leq n$ we have:

$$s_i := \begin{cases} \beta, & \text{if } t_i = \alpha; \\ t_i, & \text{otherwise.} \end{cases}$$

- $(\sim \phi)[\beta/\alpha] :=\sim (\phi[\beta/\alpha])$.
- $(\phi \circ \psi)[\beta/\alpha] := \phi[\beta/\alpha] \circ \psi[\beta/\alpha]$, for $\circ \in \{\wedge, \vee, \to\}$.
- For every $x \in Var$ and $Q \in \{\forall, \exists\}$, we set:

$$(Qx\phi)[\beta/\alpha] := \begin{cases} Qx\phi, & \text{if } x = \alpha; \\ Qx(\phi[\beta/\alpha]), & \text{otherwise.} \end{cases}$$

The following lemma states that our substitution operations work as expected. We (mostly) omit the straightforward but tedious inductive proof.

Lemma 1. *Let Σ be a signature, let Π be a set of parameters, let $\phi \in L(\Sigma, \Pi)$, let $s, s' \in (Var \cup Par)$, and let $t, t' \in Par \cup (Var \setminus BV(\phi))$. Then the following statements hold:*

1. $BV(\phi[t/s]) = BV(\phi)$, $FV(\phi[t/s]) \subseteq (FV(\phi) \setminus \{s\}) \cup \{t\}$, *and* $Par(\phi[t/s]) \subseteq (Par(\phi) \setminus \{s\}) \cup \{t\}$.
2. *If* $s \notin FV(\phi) \cup Par(\phi)$, *then* $\phi[t/s] = \phi$.
3. $\phi[t/t] = \phi$.

4. We have

$$\phi[t/s][t'/s'] := \begin{cases} \phi[t'/s'], \text{ if } s' = s \text{ and } s' = t; \\ \phi[t/s], \text{ if } s' = s \text{ and } s' \neq t; \\ \phi[t'/s'][t'/s], \text{ if } s' \neq s \text{ and } s' = t; \\ \phi[t'/s'][t/s], \text{ if } s' \neq s, s \neq t', \text{ and } s' \neq t \end{cases}$$

Proof. We only sketch the proof for Part 4. If both $s' = s$ and $s' = t$, then also $s = t$. Thus we have $\phi[t/s][t'/s'] = \phi[s'/s'][t'/s'] = \phi[t'/s']$ by Part 3. Next, if both $s' = s$ and $s' \neq t$, then we have $s \neq t$. By Part 1, $FV(\phi[t/s]) \subseteq (FV(\phi) \setminus \{s\}) \cup \{t\}$ and $Par(\phi[t/s]) \subseteq (Par(\phi) \setminus \{s\}) \cup \{t\}$, therefore, $s \notin Par(\phi[t/s]) \cup FV(\phi[t/s])$. But then, Part 2 implies that $\phi[t/s][t'/s'] = \phi[t/s][t'/s] = \phi[t/s]$.

The proof for the remaining two cases proceeds by induction on the construction of ϕ. The basis and the induction step for the connectives are straightforward. As for the quantifiers, let $x \in Var$ and $Q \in \{\forall, \exists\}$ be such that $\phi = Qx\psi$. We may also assume that $s \neq s'$. The following cases arise:

Case 1. Assume that $s' = t$. Then, since $t \notin BV(\phi)$, we must also have $s' \neq x$. We have to consider the following subcases:

Case 1.1. $x = s$. By definition of substitution, we get that:

$$(Qx\psi)[t/s][t'/s'] = (Qx\psi)[t'/s'] = Qx(\psi[t'/s']) = Qx(\psi[t'/s'])[t'/s] = \\ = (Qx\psi)[t'/s'][t'/s].$$

Case 1.2. $x \neq s$. Then we argue by the Induction Hypothesis:

$$(Qx\psi)[t/s][t'/s'] = Qx(\psi[t/s])[t'/s'] = Qx(\psi[t/s][t'/s']) = \\ = Qx(\psi[t'/s'][t'/s]) = (Qx\psi)[t'/s'][t'/s].$$

Case 2. Assume that $s' \neq t$ and $t' \neq s$. The following subcases are possible:

Case 2.1. $x = s$. It follows then from $s \neq s'$ that also $s' \neq x$. The rest of the argument is as in Case 1.1.

Case 2.2. $x \neq s$. Again, two further subcases are possible. If $x = s'$ then we argue similarly to Cases 1.1 and 2.1. Otherwise, we argue by the Induction Hypothesis. □

The cases given in Lemma 1.4 are not exhaustive in that the case when $s \neq s'$, $s' \neq t$, and $s = t'$ is not solved. The following example shows that this is not a coincidence since under these assumptions one cannot, in general, push $[t'/s']$ inside the substitution cascade:

Example 1. *Let* $\Sigma = \{(P, 2)\}$*, let* $a, b, c \in Par$ *be pairwise distinct. Then we have* $P(a, c)[b/a][a/c] = P(b, c)[a/c] = P(b, a)$*, but* $P(a, c)[a/c] = P(a, a)$*, and any further substitutions can only lead to formulas of the form* $P(d, d)$*. Therefore,* $P(a, c)[b/a][a/c] \neq P(a, c)[a/c][t_1/s_1] \ldots [t_n/s_n]$ *for any* $n \in \omega$ *and any* $\bar{s}_n, \bar{t}_n \in (Var \cup Par)^n$*.*

The final case in Lemma 1.4 is sufficiently well-behaved to allow for a (restricted) introduction of the operation of simultaneous substitution of variables/parameters by parameters. We formulate this fact as a separate corollary:

Corollary 1. *Let* Σ *be a signature, let* Π *be a set of parameters, let* $\phi \in L(\Sigma, \Pi)$*, let* $n \in \omega$*, let* $\bar{s}_n \in (Var \cup Par)^{\neq n}$ *and* $\bar{t}_n \in (Par \setminus \{\bar{s}_n\})^n$*. We let* $\phi[\bar{t}_n/\bar{s}_n]$ *denote* $\phi[t_1/s_1] \ldots [t_n/s_n]$*. Then the following statements hold:*

1. *If* $(i_1, \ldots, i_n)$ *is a permutation of* $(1, \ldots, n)$*, then we have*

$$\phi[\bar{t}_n/\bar{s}_n] = \phi[t_{i_1}/s_{i_1}, \ldots, t_{i_n}/s_{i_n}].$$

2. *If* $\{s_{i_1}, \ldots, s_{i_k}\} = \{\bar{s}_n\} \cap (FV(\phi) \cup Par(\phi))$*, then*

$$\phi[\bar{t}_n/\bar{s}_n] = \phi[t_{i_1}/s_{i_1}, \ldots, t_{i_k}/s_{i_k}].$$

In the special case when $t_1 = \ldots = t_n = a \in Par$, we will write $\phi[a/\bar{s}_n]$ instead of $\phi[\bar{t}_n/\bar{s}_n]$.

The notion of substitution is necessary for the right inductive definition of a sentence that is independent from the inductive definition of an arbitrary formula. More precisely, let Σ be a signature, let Π be a subset of Par and let $c \in Par$, perhaps outside Π. Then $L_\emptyset(\Sigma, \Pi)$ is the smallest subset of $L(\Sigma, \Pi)$ satisfying the following conditions:

- $P(\bar{c}_n) \in L_\emptyset(\Sigma, \Pi)$ for all $n \geq 1$, $P^n \in \Sigma$, and $\bar{c}_n \in \Pi^n$.
- If $\phi, \psi \in L_\emptyset(\Sigma, \Pi)$, then $\sim \phi \in L_\emptyset(\Sigma, \Pi)$ and $(\phi \circ \psi) \in L_\emptyset(\Sigma, \Pi)$ for all $\circ \in \{\wedge, \vee, \to\}$.
- If $x \in Var$ and $\phi[c/x] \in L_\emptyset(\Sigma, \Pi \cup \{c\})$, then $\forall x\phi, \exists x\phi \in L_\emptyset(\Sigma, \Pi)$.

2.2 Semantics

In order to define our semantics we first fix yet another proper class $State$ which is disjoint from $Log \cup Var \cup Pred \cup Par$.

For any given signature Σ, a Σ-model is a structure of the form $\mathcal{M} = (W, \leq, U, D, V^+, V^-)$ such that:

- $\emptyset \neq W \subseteq State$ is a non-empty set of *states*, or *nodes*.
- $\leq \subseteq W \times W$ is reflexive and transitive (i.e. a *pre-order*).
- $\emptyset \neq U \subseteq Par$ is a non-empty set of parameters serving, in this context, as the *universe of objects*.
- $D : W \to (\mathcal{P}(U) \setminus \{\emptyset\})$ is such that, for all $w, v \in W$ we have:

$$w \leq v \Rightarrow D(w) \subseteq D(v).$$

  Given a $w \in W$, we will sometimes write D_w to denote $D(w)$.

- For all $\circ \in \{+, -\}$, we have that $V^\circ : dom(\Sigma) \times W \to \mathcal{P}(U^\infty)$ such that, for every $P^n \in \Sigma$, and all $w, v \in W$, it is true that:
  - $V^\circ(P, w) \subseteq (D_w)^n$.
  - $w \leq v \Rightarrow V^\circ(P, w) \subseteq V^\circ(P, v)$.

  Given a $w \in W$ and a $P^n \in \Sigma$, we will often write $V^\circ_w(P)$ in place of $V^\circ(P, w)$.

When we use subscripts and other decorated model notations, we strive for consistency in this respect. Some examples of this notational principle are given below:

$$\mathcal{M} = (W, \leq, U, D, V^+, V^-),\ \mathcal{M}' = (W', \leq', U', D', (V')^+, (V')^-),$$
$$\mathcal{M}_n = (W_n, \leq_n, U_n, D_n, (V_n)^+, (V_n)^-).$$

For a given model $\mathcal{M}$, its substructure $(W, \leq, U, D)$ is called the *underlying frame* of $\mathcal{M}$, and $\mathcal{M}$ is said to be based on $(W, \leq, U, D)$.

A model $\mathcal{M}$ is called a constant-domain model iff for all $w \in W$ we have $D_w = U$. A model $\mathcal{M}$ is called a C3-model iff for all $w \in W$ and for every $P^n \in \Sigma$, we have $V^+_w(P) \cup V^-_w(P) = (D_w)^n$. We will denote the classes of constant domain and C3-models by $\mathbb{CD}$ and $\mathbb{C}3$, respectively. In particular, if $\mathcal{M} \in \mathbb{CD} \cap \mathbb{C}3$, then we get that $V^+_w(P) \cup V^-_w(P) = U^n$ for all $w \in W$ and all $P^n \in \Sigma$.

We would like to say that a class $\mathbb{K}$ of models is *good* (that is to say, as a basis for a possible first-order extension of C) iff it is closed for the models based on the same underlying frame. Similarly, we will say that a class $\mathbb{K} \subseteq \mathbb{C}3$ is C3-*good* (that is to say, as a basis for a possible first-order extension of C3), iff whenever a Σ-model $\mathcal{M}$ is in $\mathbb{K}$ and a Σ-model $\mathcal{N} \in \mathbb{C}3$ is based on $(W, \leq, U, D)$, then $\mathcal{N} \in \mathbb{K}$. The goodness here is supposed to mean, somewhat loosely, a naturality of the resulting Kripke semantics, including (but not necessarily limited to) the possibility of a standard-looking frame correspondence theory.

It is easy to see that the class of all models and $\mathbb{CD}$ are good, whereas $\mathbb{C}3$ and $\mathbb{C}3 \cap \mathbb{CD}$ are C3-good. Interestingly enough, $\mathbb{C}3$ itself is not good, which raises the question whether any first-order extension of C3 can be also seen as a natural extension of C. We will not attempt to answer it in this paper. But, even if the question is to be answered negatively, the connection of the first-order extensions of C3 with their propositional base is already sufficient to make them interesting to look at.

The semantics of QC, our main system, is given by the pair of ternary (class-)relations, $\models^+$ and $\models^-$ which are only defined for a triple (α, β, γ) in case α is a Σ-model $\mathcal{M}$ for some signature Σ, $\beta = w \in W$, and $\gamma \in L_\emptyset(\Sigma, D_w)$. The definition of these relations is then given by the following induction on the construction of γ for any Σ-model $\mathcal{M}$ and any $w \in W$:

$$\begin{aligned}
\mathcal{M}, w \models^\circ P(\bar{c}_n) &\Leftrightarrow \bar{c}_n \in V^\circ_w(P) \qquad\qquad \circ \in \{+,-\},\ P \in \Sigma_n,\ \bar{c}_n \in (D_w)^n\\
\mathcal{M}, w \models^+ \sim \phi &\Leftrightarrow \mathcal{M}, w \models^- \phi\\
\mathcal{M}, w \models^- \sim \phi &\Leftrightarrow \mathcal{M}, w \models^+ \phi\\
\mathcal{M}, w \models^+ \phi \wedge \psi &\Leftrightarrow \mathcal{M}, w \models^+ \phi \text{ and } \mathcal{M}, w \models^+ \psi\\
\mathcal{M}, w \models^- \phi \wedge \psi &\Leftrightarrow \mathcal{M}, w \models^- \phi \text{ or } \mathcal{M}, w \models^- \psi\\
\mathcal{M}, w \models^+ \phi \vee \psi &\Leftrightarrow \mathcal{M}, w \models^+ \phi \text{ or } \mathcal{M}, w \models^+ \psi\\
\mathcal{M}, w \models^- \phi \vee \psi &\Leftrightarrow \mathcal{M}, w \models^- \phi \text{ and } \mathcal{M}, w \models^- \psi\\
\mathcal{M}, w \models^+ \phi \to \psi &\Leftrightarrow (\forall v \geq w)(\mathcal{M}, v \not\models^+ \phi \text{ or } \mathcal{M}, w \models^+ \psi)\\
\mathcal{M}, w \models^- \phi \to \psi &\Leftrightarrow (\forall v \geq w)(\mathcal{M}, v \not\models^+ \phi \text{ or } \mathcal{M}, w \models^- \psi)\\
\mathcal{M}, w \models^+ \forall x \phi &\Leftrightarrow (\forall v \geq w)(\forall a \in D_v)(\mathcal{M}, v \models^+ \phi[a/x])\\
\mathcal{M}, w \models^- \forall x \phi &\Leftrightarrow (\exists a \in D_w)(\mathcal{M}, w \models^- \phi[a/x])\\
\mathcal{M}, w \models^+ \exists x \phi &\Leftrightarrow (\exists a \in D_w)(\mathcal{M}, w \models^+ \phi[a/x])\\
\mathcal{M}, w \models^- \exists x \phi &\Leftrightarrow (\forall v \geq w)(\forall a \in D_v)(\mathcal{M}, v \models^- \phi[a/x])
\end{aligned}$$

Given a pair $(\Gamma, \Delta) \subseteq \mathcal{P}(L_\emptyset(\Sigma, \Pi)) \times \mathcal{P}(L_\emptyset(\Sigma, \Pi))$, a Σ-model $\mathcal{M}$, and a $w \in W$, we say that $(\mathcal{M}, w)$ satisfies (Γ, Δ), and write $\mathcal{M}, w \models^+ (\Gamma, \Delta)$ iff $Par(\Gamma) \cup Par(\Delta) \subseteq D_w$, and we have $\mathcal{M}, w \models^+ \phi$ for every $\phi \in \Gamma$ and $\mathcal{M}, w \not\models^+ \psi$ for every $\psi \in \Delta$.[4] In case $\Delta = \emptyset$, we simply write $\mathcal{M}, w \models^+ \Gamma$. We say that (Γ, Δ) is satisfiable iff $\mathcal{M}, w \models^+ (\Gamma, \Delta)$ for some Σ-model $\mathcal{M}$, and some $w \in W$. Otherwise we say that

[4]The provision requiring inclusion of parameter sets into D_w is necessary to exclude the cases where the parametrized sentences from Δ fail to hold due to the absence of the corresponding parameters in the domain of the respective node.

Δ follows from Γ and write $\Gamma \models \Delta$; in other words, Δ follows from Γ iff for every Σ-model $\mathcal{M}$, and every $w \in W$ such that $Par(\Gamma) \cup Par(\Delta) \subseteq D_w$, $\mathcal{M}, w \models^+ \Gamma$ implies $\mathcal{M}, w \models^+ \phi$ for some $\phi \in \Delta$. As usual, we will suppress the brackets when Γ is a singleton. Given a $\psi \in L_\emptyset(\Sigma, \Pi)$, we say that ψ is satisfiable iff $\{\psi\}$ is, and that ψ is valid iff $\emptyset \models \psi$.

These notions can be easily relativized to any given subclass $\mathbb{K}$ of the class of models. Thus, we will say that Δ follows from Γ *over* $\mathbb{K}$ (and write $\Gamma \models_{\mathbb{K}} \Delta$) iff for every Σ-model $\mathcal{M} \in \mathbb{K}$, and every $w \in W$ such that $Par(\Gamma) \cup Par(\Delta) \subseteq D_w$, $\mathcal{M}, w \models^+ \Gamma$ implies $\mathcal{M}, w \models^+ \phi$ for some $\phi \in \Delta$; and similarly for the other notions introduced in the previous paragraph. In this sense we will speak of, e.g., $\mathbb{C}3$-consequence, $\mathbb{CD}$-consequence, and so on, and will write $\Gamma \models_{\mathbb{C}3} \Delta$, $\Gamma \models_{\mathbb{CD}} \Delta$, etc.

When handling the pairs of parametrized sentences (we will often call pairs of sets also bi-sets), we will assume that the usual set-theoretic relations and operations on them are defined componentwise. Thus, for example, we will write $(\Gamma, \Delta) \subseteq (\Gamma', \Delta')$ iff both $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$; we will understand $(\Gamma, \Delta) \Subset (\Gamma', \Delta')$, $(\Gamma, \Delta) \cup (\Gamma', \Delta')$ and so forth in a similar way.

The following lemma is a standard consequence of the definitions given in this subsection

Lemma 2. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $w, v \in W$ be such that $w \leq v$, and let $\phi \in L_\emptyset(\Sigma, D_w)$. Then we have $\mathcal{M}, w \models^\circ \phi \Rightarrow \mathcal{M}, v \models^\circ \phi$ for all $\circ \in \{+, -\}$.*

Proof (a sketch). The proof proceeds by induction on the construction of a parametrized sentence. We look into the following two cases:

Case 1. $\phi = \psi \to \theta$. If $\circ \in \{+, -\}$ and $w, v \in W$ are such that $w \leq v$ and $\mathcal{M}, w \models^\circ \psi \to \theta$, then let $v' \in W$ be such that $v' \geq v$. By transitivity, $v' \geq w$. Therefore, if $\mathcal{M}, v' \models^+ \psi$, then, by $\mathcal{M}, w \models^\circ \psi \to \theta$, we also have $\mathcal{M}, v' \models^\circ \theta$. But then, since $v' \in W$ was chosen arbitrarily under the condition that $v' \geq v$, we must also have $\mathcal{M}, v \models^\circ \psi \to \theta$.

Case 2. $\phi = \forall x\psi$. If $w, v \in W$ are such that $w \leq v$ and $\mathcal{M}, w \models^+ \forall x\psi$, then let $v' \in W$ and $a \in D_{v'}$ be such that $v' \geq v$. By transitivity, $v' \geq w$, therefore, $\mathcal{M}, w \models^+ \forall x\psi$ implies that $\mathcal{M}, v' \models^+ \psi[a/x]$. But then, since $v' \in W$ and $a \in D_{v'}$ were chosen arbitrarily under the condition that $v' \geq v$, we must also have $\mathcal{M}, v \models^+ \forall x\psi$.

On the other hand, assume that $\mathcal{M}, w \models^- \forall x\psi$, and choose an $a \in D_w$ such that $\mathcal{M}, w \models^- \psi[a/x]$. By definition, $a \in D_v$, and, by the Induction Hypothesis, $\mathcal{M}, v \models^- \psi[a/x]$, whence $\mathcal{M}, v \models^- \forall x\psi$ follows. □

We observe that it follows from Lemma 2, that if $\mathcal{M}$ happens to be a constant-domain model, the quantifier clauses can be simplified:

Corollary 2. *Let Σ be a signature, let $\mathcal{M} \in \mathbb{CD}$ be a Σ-model, let $w \in W$, and let $\phi \in L_\emptyset(\Sigma, D_w)$. Then we have:*

$$\mathcal{M}, w \models^+ \forall x\phi \Leftrightarrow (\forall a \in U)(\mathcal{M}, w \models^+ \phi[a/x])$$
$$\mathcal{M}, w \models^- \forall x\phi \Leftrightarrow (\exists a \in U)(\mathcal{M}, w \models^- \phi[a/x])$$
$$\mathcal{M}, w \models^+ \exists x\phi \Leftrightarrow (\exists a \in U)(\mathcal{M}, w \models^+ \phi[a/x])$$
$$\mathcal{M}, w \models^- \exists x\phi \Leftrightarrow (\forall a \in U)(\mathcal{M}, w \models^- \phi[a/x])$$

Turning now to relativizations of the consequence relation w.r.t. the model subclasses introduced in this subsection, we observe that on the first-order level, in contrast with the propositional C3, restricting the class of admissible models to $\mathbb{C}3$ does not ensure the absence of truth-value gaps for arbitrary (pure) sentences. Indeed, consider the following example.

Example 2. *Let $\Sigma := \{(P, 1)\}$ and let the Σ-model $\mathcal{M}$ be defined as follows:*

$$W := \{w, v\},\ \leq := \{(w, v)\} \cup id_{\{w,v\}},\ U := \{a, b\},\ D := \{(w, \{a\}), (v, \{a, b\})\}.$$

Finally, set $V^+_\alpha(P) := \{a\}$ for all $\alpha \in W$, $V^-_w(P) := \emptyset$, and $V^-_v(P) := \{b\}$. Then $\mathcal{M} \in \mathbb{C}3$, but we have both $\mathcal{M}, w \not\models^+ \forall xP(x)$ and $\mathcal{M}, w \not\models^- \forall xP(x)$

However, the phenomena, illustrated by the above example, do not arise for the models in $\mathbb{CD} \cap \mathbb{C}3$, as the following lemma shows:

Lemma 3. *Let Σ be a signature, let $\mathcal{M} \in \mathbb{CD} \cap \mathbb{C}3$ be a Σ-model, and let $w \in W$. Then for any $\phi \in L_\emptyset(\Sigma, D_w)$ we have $\mathcal{M}, w \models^\circ \phi$ for some $\circ \in \{+, -\}$.*

Proof. By induction on the construction of ϕ. The atomic case, providing the basis for our induction, is obvious. We consider the induction steps where we have to deal with the following cases:

Case 1. $\phi = \psi \wedge \chi$. Then, by the Induction Hypothesis, we either have both $\mathcal{M}, w \models^+ \psi$ and $\mathcal{M}, w \models^+ \chi$, and thus also $\mathcal{M}, w \models^+ \psi \wedge \chi$, or else at least one of $\mathcal{M}, w \models^- \psi$, $\mathcal{M}, w \models^- \chi$ holds, implying that $\mathcal{M}, w \models^- \psi \wedge \chi$.

Case 2. $\phi = \psi \vee \chi$. Similar to Case 1.

Case 3. $\phi = \sim \psi$. Straightforward.

Case 4. $\phi = \psi \to \chi$. Note that the Induction Hypothesis implies that we have $\mathcal{M}, w \models^\circ \chi$ for some $\circ \in \{+, -\}$. If now $v \in W$ is such that both $v \geq w$ and $\mathcal{M}, v \models^+ \psi$, then we will also have $\mathcal{M}, v \models^\circ \chi$ by Lemma 2. But, since v was chosen arbitrarily, this also means that $\mathcal{M}, w \models^\circ \psi \to \chi$.

Case 5. $\phi = \forall x\psi$. Then, by the Induction Hypothesis, two subcases are possible: either we have $\mathcal{M}, w \models^- \psi[a/x]$ for some $a \in U$, and hence also $\mathcal{M}, w \models^- \forall x\psi$, or we have $\mathcal{M}, w \models^+ \psi[a/x]$ for all $a \in U$, and hence also $\mathcal{M}, w \models^+ \forall x\psi$ by the fact that $\mathcal{M}$ is a constant-domain model.

Case 6. $\phi = \exists x\psi$. Similar to Case 4. □

Lemma 3, together with Example 2, jointly explain why we consider $\mathbb{CD} \cap \mathbb{C}3$ a better setting for the first-order version of $\mathsf{C3}$ than the wider class $\mathbb{C}3$, even though this choice leads to a system which is different from both $\mathsf{C3}$-based first-order systems introduced in [10]. This discussion is taken up in more detail in Section 5 below. Of course, alternative natural settings for the first-order $\mathsf{C3}$ appear to be possible as well, but we leave their consideration to a future research.

We may understand a logic as a class-function, that, for any given signature Σ, returns the set of all pairs (Γ, ϕ) such that $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma)$ and ϕ is a consequence of Γ. If we use the (Nelsonian) semantics of quantifiers given in this section and interpret ϕ being a consequence of Γ by $\Gamma \models \phi$ (resp. $\Gamma \models_{\mathbb{CD}} \phi$, $\Gamma \models_{\mathbb{CD}\cap\mathbb{C}3} \phi$), then we get the definition of QC (resp. QC_{CD}, $\mathsf{QC3}_{CD}$).

Before we move on to the next section, we need to consider several important operations on models, that will be used later in the paper. The first one is a parameter substitution operation, very similar to the one we used for the formulas. More precisely, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. Consider the function $f_{[b/a]} : U \to (U \setminus \{a\}) \cup \{b\}$ such that, for every $c \in U$ we have:

$$f_{[b/a]}(c) := \begin{cases} b, & \text{if } c = a; \\ c, & \text{otherwise.} \end{cases}$$

Then we can define the model $\mathcal{M}_{[b/a]}$ resulting from the substitution of b for a as the tuple $(W, \leq, U_{[b/a]}, D_{[b/a]}, (V_{[b/a]})^+, (V_{[b/a]})^-)$, where:

- $U_{[b/a]} := f_{[b/a]}[U] = (U \setminus \{a\}) \cup \{b\}$.
- For every $w \in W$:

$$D_{[b/a]}(w) := f_{[b/a]}[D_w] = \begin{cases} (D_w \setminus \{a\}) \cup \{b\}, & \text{if } a \in D_w; \\ D_w, & \text{otherwise.} \end{cases}$$

- $(V_{[b/a]})^\circ(P, w) := \{f_{[b/a]}\langle \bar{a}_n \rangle \mid \bar{a}_n \in V^\circ(P, w)\}$ for all $\circ \in \{+, -\}$, $P^n \in \Sigma$, and $w \in W$.

The parameter substitutions in models are closely related to the parameter substitutions in formulas, so that the following lemma holds:

Lemma 4. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. For every $\circ \in \{+, -\}$, every $w \in W$, and every $\phi \in L_\emptyset(\Sigma, D_w)$, it is true that:*

$$\mathcal{M}, w \models^\circ \phi \Leftrightarrow \mathcal{M}_{[b/a]}, w \models^\circ \phi[b/a].$$

Proof. See Appendix A for details. □

We immediately state a useful corollary to Lemma 4, namely that model substitutions do not affect the satisfaction of certain formulas:

Corollary 3. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. For every $\circ \in \{+, -\}$, every $w \in W$, and every $\phi \in L_\emptyset(\Sigma, D_w \setminus \{a\})$, it is true that:*

$$\mathcal{M}, w \models^\circ \phi \Leftrightarrow \mathcal{M}_{[b/a]}, w \models^\circ \phi.$$

Proof. Note that, by Lemma 1.2, we must have $\phi[b/a] = \phi$. The corollary now follows from Lemma 4. □

Another useful operation on models allows us to add a new object to the domain of a model, as long as we make it indistinguishable from some already existing object. More precisely, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. Consider the relation $\rho_{[b:=a]} := id_{(U \cup \{b\})} \cup \{(a, b)\}$. Then we can define the model $\mathcal{M}_{[b:=a]}$ resulting from the addition of b as a copy a, setting it to the following tuple $(W, \leq, U_{[b:=a]}, D_{[b:=a]}, (V_{[b:=a]})^+, (V_{[b:=a]})^-)$, where:

- $U_{[b:=a]} := \rho_{[b:=a]}[U] = U \cup \{b\}$.
- For every $w \in W$:

$$D_{[b:=a]}(w) := \rho_{[b:=a]}[D_w] = \begin{cases} D_w \cup \{b\}, & \text{if } a \in D_w; \\ D_w, & \text{otherwise.} \end{cases}$$

- $(V_{[b:=a]})^\circ(P, w) := \bigcup\{\rho_{[b:=a]}\langle \bar{a}_n \rangle \mid \bar{a}_n \in V^\circ(P, w)\}$ for all $\circ \in \{+, -\}$, $P^n \in \Sigma$, and $w \in W$.

Just as in the case of model substitution, the operation of adding a new copy of an existing object displays a close relation to a certain kind of parameter substitutions in formulas. As a result, the following lemma holds:

Lemma 5. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. For every $n \in \omega$, every tuple $\bar{x}_n \in Var^{\neq n}$, every $\circ \in \{+, -\}$, every $w \in W$, and every $\phi \in L_{\bar{x}_n}(\Sigma, D_w)$, it is true that:*

$$\mathcal{M}, w \models^\circ \phi[a/\bar{x}_n] \Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^\circ \phi[b/\bar{x}_n].$$

Proof. See Appendix B for details. □

Lemma 5 also implies a useful corollary which we would like to state before we move on to axiomatizations of our logics.

Corollary 4. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. For every $\circ \in \{+, -\}$, every $w \in W$, and every $\phi \in L_{\emptyset}(\Sigma, D_w)$, it is true that:*

$$\mathcal{M}, w \models^{\circ} \phi \Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^{\circ} \phi.$$

Proof. Note that, by Lemma 1.2, we must have $\phi[a/\bar{x}_n] = \phi = \phi[b/\bar{x}_n]$ for every $n \in \omega$ and every tuple $\bar{x}_n \in Var^{\neq n}$. The corollary now follows from Lemma 5. □

We note, in passing, that the subclasses of models that we have considered so far, like $\mathbb{CD}$, $\mathbb{C}3$, and $\mathbb{CD} \cap \mathbb{C}3$, are clearly closed for both operations on models.

3 A Hilbert-style Axiomatization of QC

We now start with the axiomatization work for QC, the first of the three logics introduced in the previous section. We will give a direct argument showing that the axiomatization of QC, as it is given in [14], is in general sound relative to the semantics defined above; in case the signature is assumed to be at most countable, we will also show completeness.

In this way, we will show that, for a countable signature Σ, the set of all pairs (Γ, ϕ) such that $\Gamma \cup \{\phi\} \subseteq L_{\emptyset}(\Sigma)$, and ϕ follows from Γ, is recursively enumerable; in fact, our results will show that, even if we allow $\Gamma \cup \{\phi\} \subseteq L_{\emptyset}(\Sigma, \Pi)$ for an at most countable $\Pi \subseteq Par$, the respective set of pairs of the form (Γ, ϕ) remains enumerable. This is due to the fact that our axiomatization is given in a form that makes a generation of parametrized sentences from other sentences (parametrized or pure) an indispensable by-product in the process of the generation of pure sentences following from other pure sentences. The readers can easily convince themselves of this indispensability by paying attention to the form of axioms like (A15) below, as well as to their possible interaction with the rules like (MP).

Similar remarks apply to the axiomatizations of the other logical systems considered in this paper.

Given a signature Σ and an infinite set Π of parameters, the (Σ, Π)-instantiation of Hilbert-style axiomatization presented in [14] includes all parametrized sentences that are instances of the following schemes (for all $\phi, \psi, \chi \in L_{\emptyset}(\Sigma, \Pi)$, all $c \in \Pi$, all

$x \in Var$, and all $\theta \in L_x(\Sigma, \Pi)$):

$$\phi \to (\psi \to \phi) \tag{A1}$$
$$(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)) \tag{A2}$$
$$(\phi \wedge \psi) \to \phi \tag{A3}$$
$$(\phi \wedge \psi) \to \psi \tag{A4}$$
$$(\chi \to \phi) \to ((\chi \to \psi) \to (\chi \to (\phi \wedge \psi))) \tag{A5}$$
$$\phi \to (\phi \vee \psi) \tag{A6}$$
$$\psi \to (\phi \vee \psi) \tag{A7}$$
$$(\phi \to \chi) \to ((\psi \to \chi) \to ((\phi \vee \psi) \to \chi)) \tag{A8}$$
$$\sim\sim \phi \leftrightarrow \phi \tag{A9}$$
$$\sim (\phi \wedge \psi) \leftrightarrow (\sim \phi \vee \sim \psi) \tag{A10}$$
$$\sim (\phi \vee \psi) \leftrightarrow (\sim \phi \wedge \sim \psi) \tag{A11}$$
$$\sim (\phi \to \psi) \leftrightarrow (\phi \to \sim \psi) \tag{A12}$$
$$\sim \exists x\theta \leftrightarrow \forall x \sim \theta \tag{A13}$$
$$\sim \forall x\theta \leftrightarrow \exists x \sim \theta \tag{A14}$$
$$\forall x\theta \to \theta[c/x] \tag{A15}$$
$$\theta[c/x] \to \exists x\theta \tag{A16}$$

The rules of inference are then as follows:

$$\text{From } \phi, \phi \to \psi \text{ infer } \psi \tag{MP}$$
$$\text{From } \phi \to \theta[c/x] \text{ infer } \phi \to \forall x\theta \tag{R$\forall$}$$
$$\text{From } \theta[c/x] \to \psi \text{ infer } \exists x\theta \to \psi \tag{R$\exists$}$$

Given any particular application of the rules (R∀) and (R∃) the parameter c is called *the main parameter of the rule application* and must have no occurrences in $\phi \to \psi$.[5]

For any $\Delta \Subset L_\emptyset(\Sigma, \Pi)$ and any $\bar{\phi}_n \in L_\emptyset(\Sigma, \Pi)^n$, such that $\Delta \subseteq \{\bar{\phi}_n\}$, we say that $\bar{\phi}_n$ is a (Σ, Π)-deduction in QC of ϕ_n from the premises Δ iff, for every $1 \leq i \leq n$, ϕ_i is either (1) an instance of (A1)–(A16), or (2) $\phi_i \in \Delta$, or (3) ϕ_i is obtained from some ϕ_j, ϕ_k such that $1 \leq j, k < i$ by an application of (MP), or else (4) is obtained from some ϕ_j such that $1 \leq j < i$ by an application of either (R∀) or (R∃) and

[5]We do not need to require that $x \notin FV(\phi)$ since we assume that our deductions consist of parametrized sentences. Moreover, note that a parameter is always substitutable for a variable, hence the usual provisions associated to axiom schemas like (A15) and (A16) can be omitted in our case.

the main parameter of this application is outside $Par(\Delta)$. Moreover, $\bar{\phi}_n$ is called a proof iff it is a deduction from the empty set of premises. For any $\Gamma \subseteq L_\emptyset(\Sigma, \Pi)$, we say that $\phi \in L_\emptyset(\Sigma, \Pi)$ is (Σ, Π)-*deducible* from Γ (and write $\Gamma \vdash_{(\Sigma,\Pi)} \phi$) iff there exists a (Σ, Π)-deduction $\bar{\phi}_n$ from the premises Δ for some $\Delta \Subset \Gamma$ such that $\phi_n = \phi$. We say that $\phi \in L_\emptyset(\Sigma, \Pi)$ is *deducible* from Γ (and write $\Gamma \vdash \phi$) iff for every infinite set $Par(\Gamma \cup \{\phi\}) \subseteq \Xi \subseteq Par$, we have $\Gamma \vdash_{(Sign(\Gamma \cup \{\phi\}), \Xi)} \phi$.

We now take a brief look at some properties of deducibility. We establish, first, that certain renamings of parameters in deductions by “fresh” parameters are always possible:

Lemma 6. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $\Delta \cup \{\phi\} \Subset L_\emptyset(\Sigma, \Pi)$, let $\bar{\phi}_n$ be a (Σ, Π)-deduction of ϕ from the premises in Δ, and let $\bar{a}_m \in Par^{\neq m}$ be a non-repeating listing of $Par(\{\bar{\phi}_n\}) \setminus Par(\Delta \cup \{\phi\})$. Assume, moreover, that $\bar{b}_m \in (Par \setminus Par(\{\bar{\phi}_n\}))^{\neq m}$. Then $\phi_1[\bar{b}_m/\bar{a}_m], \ldots, \phi_n[\bar{b}_m/\bar{a}_m]$ is a $(\Sigma, Par(\Delta \cup \{\phi\}) \cup \{\bar{b}_m\})$-deduction of $\phi = \phi_n$ from the premises in Δ.*

Proof. By Lemma 1.2 and the choice of $\bar{a}_m$, we know that the formulas from $\Delta \cup \{\phi\}$ are not affected by the substitution of $\bar{b}_m$ for $\bar{a}_m$; on the other hand, for every $1 \leq i \leq n$ we have that:

$$\begin{aligned} Par(\phi_i[\bar{b}_m/\bar{a}_m]) &\subseteq (Par(\phi_i) \setminus \{\bar{a}_m\}) \cup \{\bar{b}_m\} && \text{(by Lemma 1.1)} \\ &\subseteq (Par(\{\bar{\phi}_n\}) \setminus \{\bar{a}_m\}) \cup \{\bar{b}_m\} \\ &= (Par(\{\bar{\phi}_n\}) \setminus (Par(\{\bar{\phi}_n\}) \setminus Par(\Delta \cup \{\phi\}))) \cup \{\bar{b}_m\} \\ &= Par(\Delta \cup \{\phi\}) \cup \{\bar{b}_m\} && \text{(by } (\Delta \cup \{\phi\}) \subseteq \{\bar{\phi}_n\}) \end{aligned}$$

It remains to show that $\phi_1[\bar{b}_m/\bar{a}_m], \ldots, \phi_n[\bar{b}_m/\bar{a}_m]$ is indeed a deduction; in doing so, we proceed by induction on $r \leq n$. More precisely, we show that, for every such r, $\phi_1[\bar{b}_m/\bar{a}_m], \ldots, \phi_r[\bar{b}_m/\bar{a}_m]$ is a $(\Sigma, Par(\Delta \cup \{\phi\}) \cup \{\bar{b}_m\})$-deduction of $\phi_r[\bar{b}_m/\bar{a}_m]$ from the premises in $\Delta \cap \{\bar{\phi}_k\}$.

Basis. $r = 1$. The following cases are then possible:

Case 1. $\phi_1 \in \Delta$. Then, by Lemma 1.2 and the fact that $\{\bar{a}_m\} \cap Par(\Delta) = \emptyset$, we must have $\phi_1[\bar{b}_m/\bar{a}_m] = \phi_1 \in \Delta$.

Case 2. ϕ_1 is an instance of an axiom schema. Then $\phi_1[\bar{b}_m/\bar{a}_m]$ is clearly an instance of the same axiom schema.

Step. $r = k + 1$. Then, by IH, $\phi_1[\bar{b}_m/\bar{a}_m], \ldots, \phi_k[\bar{b}_m/\bar{a}_m]$ is a $(\Sigma, Par(\Delta \cup \{\phi\}) \cup \{\bar{b}_m\})$-deduction of $\phi_k[\bar{b}_m/\bar{a}_m]$ from the premises in $\Delta \cap \{\bar{\phi}_k\}$. If now ϕ_r is in Δ or an instance of an axiom schema, then we reason as in the Basis. Otherwise, the following cases are possible:

Case 1. For some i, j such that $1 \leq i, j \leq k$ it is true that $\phi_j = \phi_i \to \phi_r$. But then $\phi_j[\bar{b}_m/\bar{a}_m] = \phi_i[\bar{b}_m/\bar{a}_m] \to \phi_r[\bar{b}_m/\bar{a}_m]$ so that $\phi_r[\bar{b}_m/\bar{a}_m]$ is obtained by an application of (MP) from $\phi_j[\bar{b}_m/\bar{a}_m]$ and $\phi_i[\bar{b}_m/\bar{a}_m]$.

Case 2. For some $1 \leq i \leq k$ we have $\phi_i = \psi \to \chi[c/x]$ for corresponding x, c, ψ, and χ, such that $c \notin Par(\Delta) \cup Par(\psi \to \chi)$, whereas $\phi_r = \psi \to \forall x \chi$.

Two subcases are then possible.

Case 2.1. $c \in Par(\phi)$. Then we get that $c \notin \{\bar{a}_m\} \cup \{\bar{b}_m\}$, and we reason as follows:

$$\begin{aligned}\phi_i[\bar{b}_m/\bar{a}_m] &= (\psi \to \chi[c/x])[\bar{b}_m/\bar{a}_m] \\ &= \psi[\bar{b}_m/\bar{a}_m] \to \chi[c/x][\bar{b}_m/\bar{a}_m] \\ &= \psi[\bar{b}_m/\bar{a}_m] \to \chi[\bar{b}_m/\bar{a}_m][c/x] \qquad \text{(by Lemma 1.4)}\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\phi_n[\bar{b}_m/\bar{a}_m] &= \psi[\bar{b}_m/\bar{a}_m] \to \forall x(\chi[\bar{b}_m/\bar{a}_m]) \\ &= \psi[\bar{b}_m/\bar{a}_m] \to \forall x(\chi[\bar{b}_m/\bar{a}_m])\end{aligned}$$

It remains to notice that $c \notin Par(\Delta)$, and that:

$$Par(\psi[\bar{b}_m/\bar{a}_m] \to \chi[\bar{b}_m/\bar{a}_m]) \subseteq Par(\psi \to \chi) \cup \{\bar{b}_m\},$$

therefore, by the choice of $\bar{b}_m$, we must have

$$c \notin Par(\psi[\bar{b}_m/\bar{a}_m] \to \chi[\bar{b}_m/\bar{a}_m]).$$

Case 2.2. $c \notin Par(\phi)$. Then $c \notin Par(\Delta) \cup Par(\phi)$ and yet c occurs in our deduction, therefore, $c = a_j$ for some $1 \leq j \leq m$. We may assume, wlog, that $j = m$ (otherwise, we can just re-shuffle our listing $\bar{a}_m$).

But then we get that:

$$\begin{aligned}\phi_i[\bar{b}_m/\bar{a}_m] &= (\psi \to \chi[a_m/x])[\bar{b}_m/\bar{a}_m] \\ &= \psi[\bar{b}_m/\bar{a}_m] \to \chi[a_m/x][\bar{b}_m/\bar{a}_m] \\ &= \psi[\bar{b}_m/\bar{a}_m] \to \chi[\bar{b}_m/\bar{a}_m][b_m/x] \qquad \text{(by Lemma 1.4)} \\ &= \psi[\bar{b}_{m-1}/\bar{a}_{m-1}] \to \chi[\bar{b}_{m-1}/\bar{a}_{m-1}][b_m/x] \\ &\qquad \text{(by Corollary 1.2 and } a_m = c \notin Par(\psi \to \chi))\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\phi_n[\bar{b}_m/\bar{a}_m] &= \psi[\bar{b}_m/\bar{a}_m] \to \forall x(\chi[\bar{b}_m/\bar{a}_m]) \\ &= \psi[\bar{b}_{m-1}/\bar{a}_{m-1}] \to \forall x(\chi[\bar{b}_{m-1}/\bar{a}_{m-1}])\end{aligned}$$

again, by Corollary 1.2 and $a_m = c \notin Par(\psi \to \chi)$. It remains to notice that $b_m \notin Par(\Delta)$, and that:

$$\begin{aligned} Par(\psi[\bar{b}_{m-1}/\bar{a}_{m-1}] \to \chi[\bar{b}_{m-1}/\bar{a}_{m-1}]) &\subseteq Par(\psi \to \chi) \cup \{\bar{b}_{m-1}\} \\ &\subseteq Par(\{\bar{\phi}_n\}) \cup \{\bar{b}_{m-1}\}, \end{aligned}$$

therefore, by the choice of $\bar{b}_m$, we must have

$$b_m \notin Par(\psi[\bar{b}_{m-1}/\bar{a}_{m-1}] \to \chi[\bar{b}_{m-1}/\bar{a}_{m-1}]).$$

Therefore, $\phi_n[\bar{b}_m/\bar{a}_m]$ is obtained from $\phi_i[\bar{b}_m/\bar{a}_m]$ by a correct application of (R$\forall$).

Case 3. For some $1 \leq i \leq k$ we have $\phi_i = \psi[c/x] \to \chi$ for corresponding x, c, ψ, and χ, such that $c \notin Par(\Delta) \cup Par(\psi \to \chi)$, whereas $\phi_r = \exists\psi \to \chi$. This case is dual to Case 2. □

Lemma 7. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. Then $\Gamma \vdash \phi$ iff $\Gamma \vdash_{(Sign(\Gamma\cup\{\phi\}),\Xi)} \phi$ for some infinite set $Par(\Gamma \cup \{\phi\}) \subseteq \Xi \subseteq Par$.*

Proof. The left-to-right direction is trivial. As for the right-to-left direction, assume that the set $Par(\Gamma\cup\{\phi\}) \subseteq \Xi \subseteq Par$ is infinite, and that we have $\Gamma \vdash_{(Sign(\Gamma\cup\{\phi\}),\Xi)} \phi$. Then for some $\Delta \Subset \Gamma$ and for some $\bar{\phi}_n \in L_\emptyset(Sign(\Gamma \cup \{\phi\}), \Xi)^n$ it is true that $\bar{\phi}_n$ is a $(Sign(\Gamma \cup \{\phi\}), \Xi)$-deduction of $\phi = \phi_n$ from the premises in Δ. Let $m \in \omega$ and let $\bar{a}_m \in \Xi^{\neq m}$ be a non-repeating listing of $Par(\{\bar{\phi}_n\}) \setminus Par(\Delta \cup \phi)$.

Choose any infinite parameter set $\Xi' \supseteq Par(\Gamma \cup \{\phi\})$. Since $Par(\{\bar{\phi}_n\})$ is finite, we can choose a non-repeating tuple $\bar{b}_m \in Par^{\neq m}$ such that $\{\bar{b}_m\} \subseteq \Xi' \setminus Par(\{\bar{\phi}_n\})$. Now Lemma 6 implies that $\phi_1[\bar{b}_m/\bar{a}_m], \ldots, \phi_n[\bar{b}_m/\bar{a}_m]$ is a $(Sign(\Gamma \cup \{\phi\}), Par(\Delta \cup \phi) \cup \{\bar{b}_m\})$-deduction (and hence also a $(Sign(\Gamma \cup \{\phi\}), \Xi')$-deduction) of $\phi = \phi_n$ from the premises in Δ. □

Next, we need to establish some particular deducibility relations to be used later:

Lemma 8. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, let $\phi, \psi, \chi \in L_\emptyset(\Sigma, \Pi)$, let $\Gamma \subseteq L_\emptyset(\Sigma, \Pi)$, and let $a \in Par \setminus Par(\Gamma)$. Next, let $x \in Var$, and let $\theta \in L_x(\Sigma, \Pi)$. Moreover, let $m, n \in \omega$, $\bar{\phi}_n \in L_\emptyset(\Sigma, \Pi)^n$, and $\bar{\psi}_m \in L_\emptyset(\Sigma, \Pi)^m$ be such that $\{\bar{\phi}_n\} \subseteq$*

$\{\bar{\psi}_m\}$. *Then the following deducibility relations hold:*

$$\vdash \phi \to \phi \tag{T1}$$

$$\vdash (\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \wedge \psi) \to \chi) \leftrightarrow (\psi \to (\phi \to \chi)) \tag{T2}$$

$$\vdash ((\phi \to \psi) \wedge (\phi \vee \psi \vee \chi)) \to (\psi \vee \chi) \tag{T3}$$

$$\vdash ((\phi \to \psi) \wedge (\phi \wedge \chi)) \to (\phi \wedge \psi \wedge \chi) \tag{T4}$$

$$\vdash \bigwedge \bar{\psi}_m \to \bigwedge \bar{\phi}_n \tag{T5}$$

$$\vdash \bigvee \bar{\phi}_n \to \bigvee \bar{\psi}_m \tag{T6}$$

$$\vdash \forall x(\theta \to \phi) \leftrightarrow (\exists x\theta \to \phi) \tag{T7}$$

$$\vdash \forall x(\phi \to \theta) \leftrightarrow (\phi \to \forall x\theta) \tag{T8}$$

$$(\Gamma \vdash \phi \to \psi \,\&\, \Gamma \vdash \psi \to \chi) \Rightarrow \Gamma \vdash \phi \to \chi \tag{DR1}$$

The proof is as in the intuitionistic (and classical) case. Using (T5) and (T6), we may extend our notational conventions and write $\bigwedge \Gamma$ and $\bigvee \Gamma$ for an arbitrary $\Gamma \Subset L_\emptyset(\Sigma, \Pi)$.

Lemma 9 (Deduction Theorem)**.** *Let* Σ *be a signature, let* $\Pi \subseteq Par$ *be a set, and let* $\Gamma \cup \{\phi, \psi\} \subseteq L_\emptyset(\Sigma, \Pi)$*. Then* $\Gamma, \phi \vdash \psi$ *iff* $\Gamma \vdash \phi \to \psi$*.*

Proof. The right-to-left direction is straightforward due to the presence of (MP) in our system. As for the other direction, assume that, for some infinite $\Xi \subseteq Par$ such that $Par(\Gamma \cup \{\phi, \psi\}) \subseteq \Xi$, and for some $n \in \omega$, the sequence $\bar{\phi}_n \in L_\emptyset(Sign(\Gamma \cup \{\phi, \psi\}), \Xi)^n$ is a $(Sign(\Gamma \cup \{\phi, \psi\}), \Xi)$-deduction of $\psi = \phi_n$ from the premises in $\Delta \Subset \Gamma \cup \{\phi\}$.

Now, if $\phi \notin \Delta$, then we must also have $\Gamma \vdash \psi$. But then we can append to $\bar{\phi}_n$ the sentence $\psi \to (\phi \to \psi)$ as an instance of (A1) followed by $\phi \to \psi$ as the result of applying (MP) to the previous sentence and ψ. The resulting sequence is clearly a deduction of $\phi \to \psi$ from the premises in $\Delta \subseteq \Gamma$ so that $\Gamma \vdash \phi \to \psi$.

On the other hand, if $\phi \in \Delta$, then consider the sequence $\phi \to \phi_1, \ldots, \phi \to \phi_n$, and show, by induction on n, that, for every $1 \leq k \leq n$, we can add enough elements to it so that its initial fragment $\phi \to \phi_1, \ldots, \phi \to \phi_k$ turns into a deduction of $\phi \to \phi_k$ from the premises in $(\Delta \setminus \{\phi\}) \cap \{\bar{\phi}_k\}$.

Basis. $k = 1$. We reason as in the intuitionistic (and classical) case.

Step. $k = r + 1$ for some $r \geq 1$. In case ϕ_k is in $\Delta \cup \{\phi\}$, or is an instance of an axiom schema, or is obtained from earlier formulas by an application of (MP), we again reason as in the intuitionistic (and classical) case. There remain two cases connected with the use of the quantifier rules:

Case 1. For some $1 \leq i \leq r$ we have $\phi_i = \theta \to \chi[c/x]$ for corresponding x, c, θ, and χ, such that $c \notin Par(\Delta \cup \{\phi\}) \cup Par(\theta \to \chi)$, whereas $\phi_k = \theta \to \forall x\chi$. The Induction Hypothesis then implies that for some $s \in \omega$ we have transformed the sequence $\phi \to \phi_1, \ldots, \phi \to \phi_r$ into some $(Sign(\Gamma \cup \{\phi, \psi\}), \Xi)$-deduction $\chi_1, \ldots, \chi_s$ of $\phi \to \phi_r = \chi_s$ from the premises in $(\Delta \setminus \{\phi\}) \cap \{\bar{\phi}_r\}$. We now extend $\chi_1, \ldots, \chi_s$ by adding the proof of $(\phi \to (\theta \to \chi[c/x])) \to ((\phi \wedge \theta) \to \chi[c/x])$ as an instance of (T2) followed by an occurrence of $(\phi \wedge \theta) \to \chi[c/x]$ resulting from an application of (MP) to this instance of (T2) and the formula $\phi \to (\theta \to \chi[c/x]) = \phi \to \phi_i = \chi_j$ for some $1 \leq j \leq s$. Immediately after that, we add the formula $(\phi \wedge \theta) \to \forall x\chi$. Since $c \notin Par(\Delta \cup \{\theta, \phi, \chi\})$, the latter formula is obtained from $(\phi \wedge \theta) \to \chi[c/x]$ by an application of (R$\forall$). We insert, next, the proof of $((\phi \wedge \theta) \to \forall x\chi) \to (\phi \to (\theta \to \forall x\chi))$ as an instance of (T2). The sentence $\phi \to (\theta \to \forall x\chi) = \phi \to \phi_k$ now follows from the latter sentence and from $(\phi \wedge \theta) \to \forall x\chi$ by an application of (MP).

Case 2. For some $1 \leq i \leq r$ we have $\phi_i = \theta[c/x] \to \chi$ for corresponding x, c, θ, and χ, such that $c \notin Par(\Delta \cup \{\phi\}) \cup Par(\theta \to \chi)$, whereas $\phi_k = \exists x\theta \to \chi$. The reasoning here is parallel to the argument for Case 1. ☐

We immediately state a useful corollary:

Corollary 5. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$, let $\Delta \Subset L_\emptyset(\Sigma, \Pi)$, let $x \in Var$, let $\psi \in L_x(\Sigma, \Pi)$, and let $a \in Par \setminus Par(\Gamma \cup \{\psi\})$. Then the following statements hold:*

1. $\Gamma \cup \Delta \vdash \phi \Leftrightarrow \Gamma \vdash \bigwedge \Delta \to \phi$.

2. $\Gamma \vdash \psi[a/x] \Leftrightarrow \Gamma \vdash \forall x\psi$.

Proof. (Part 1). By Lemma 9 and (T2).

(Part 2). The right-to-left direction follows by (A15). For the left-to-right direction, note that $Sign(\psi)$ must be non-empty, so choose any $k \in \omega$ and any P such that $P^k \in Sign(\psi)$. By (T1) and (A16), we must have $\vdash \chi$ for $\chi := \exists \bar{x}_k(P(\bar{x}_k) \to P(\bar{x}_k))$, hence also $\Gamma \vdash \chi$. On the other hand, since $Par(\chi) = \emptyset$, we must have $\Gamma \cup \{\chi\} \vdash \psi[a/x]$, so that, by Lemma 9, also $\Gamma \vdash \chi \to \psi[a/x]$. Since $a \notin Par(\Gamma \cup \{\chi \to \psi\})$, the rule (R$\forall$) is applicable, and we get that $\Gamma \vdash \chi \to \forall x\psi$. One further application of (MP) gives us that $\Gamma \vdash \forall x\psi$. ☐

Our proof system is sound relative to the semantics of QC introduced in the previous section; more precisely, the following theorem holds:

Theorem 1. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \vdash \phi$, then $\Gamma \models \phi$.*

Proof. Assume that $\Gamma \vdash \phi$. Fix any infinite parameter set $\Xi \subseteq Par(\Gamma \cup \{\phi\})$ and any $(Sign(\Gamma \cup \{\phi\}), \Xi)$-deduction $\bar{\phi}_n$ of $\phi = \phi_n$ from the premises in $\Delta \Subset \Gamma$. We will show that, for every $r \leq n$, we have $\Delta \models \phi_r$ by induction on r, whence $\Gamma \models \phi = \phi_n$ obviously follows.

Basis. $r = 1$. Let $\Sigma \supseteq Sign(\Gamma \cup \{\phi_1\})$, let $\mathcal{M}$ be a Σ-model and let $w \in W$ be such that $D_w \supseteq Par(\Delta \cup \{\phi_1\})$ and $\mathcal{M}, w \models^+ \Delta$. Two cases are possible. If ϕ_1 is an instance of an axiom schema, then clearly $\mathcal{M}, w \models^+ \phi_1$. Otherwise, we must have $\phi_1 \in \Delta$, and then $\mathcal{M}, w \models^+ \phi_1$ follows from $\mathcal{M}, w \models^+ \Delta$.

Step. $r = k + 1$. Then the Induction Hypothesis implies that $\Delta \models^+ \phi_i$ for any $1 \leq i \leq k$. Again, let $\Sigma \supseteq Sign(\Delta \cup \{\phi_r\})$, let $\mathcal{M}$ be a Σ-model and let $w \in W$ be such that $D_w \supseteq Par(\Delta \cup \{\phi_r\})$ and $\mathcal{M}, w \models^+ \Delta$. If ϕ_r is an instance of an axiom schema or a premise, then we reason as in the Basis. Otherwise the following cases are possible:

Case 1. For some i, j such that $1 \leq i, j \leq k$ it is true that $\phi_j = \phi_i \to \phi_r$. Again, let $\Sigma \supseteq Sign(\Delta \cup \{\phi_r\})$, let $\mathcal{M}$ be a Σ-model and let $w \in W$ be such that $D_w \supseteq Par(\Delta \cup \{\phi_r\})$ and $\mathcal{M}, w \models^+ \Delta$. Assume, for contradiction, that $\mathcal{M}, w \not\models^+ \phi_r$. Then we choose a tuple $\bar{a}_m \in Par^{\neq m}$ such that $\{\bar{a}_m\} = Par(\phi_i) \setminus D_w$ and choose any $b \in D_w$. Next, we set $\mathcal{M}' := \mathcal{M}_{[a_1:=b]\ldots[a_m:=b]}$. By Corollary 4, we have both $\mathcal{M}', w \models^+ \Delta$ and $\mathcal{M}', w \not\models^+ \phi_r$. On the other hand, $Par(\phi_i) \cup Par(\phi_r) \subseteq D'_w$ so that the Induction Hypothesis implies that $\mathcal{M}', w \models^+ \phi_i \wedge (\phi_i \to \phi_r)$, which is in contradiction with $\mathcal{M}', w \not\models^+ \phi_r$.

Case 2. For some $1 \leq i \leq k$ we have $\phi_i = \psi \to \chi[c/x]$ for corresponding x, c, ψ, and χ, such that $c \notin Par(\Delta) \cup Par(\psi \to \chi)$, whereas $\phi_r = \psi \to \forall x \chi$. Again, let $\Sigma \supseteq Sign(\Delta \cup \{\phi_r\})$, let $\mathcal{M}$ be a Σ-model and let $w \in W$ be such that $D_w \supseteq Par(\Delta \cup \{\phi_r\})$ and $\mathcal{M}, w \models^+ \Delta$. Observe that we have then that $\Delta \vdash \psi \to \chi[c/x]$. Assume, for contradiction, that $\mathcal{M}, w \not\models^+ \psi \to \forall x \chi$. Then there must be a $v \in W$ such that $w \leq v$ and we have both $\mathcal{M}, v \models^+ \psi$ and $\mathcal{M}, v \not\models^+ \forall x \chi$; the latter means that, for some $u \in W$ such that $v \leq u$ and for some $a \in D_u$ we must have $\mathcal{M}, u \not\models^+ \chi[a/x]$. By transitivity of $\leq$ and Lemma 2, we get that $\mathcal{M}, u \models^+ (\Delta \cup \{\psi\}, \{\chi[a/x]\})$. If $a = c$, then we are done. Otherwise, we choose any $d \in Par \setminus U$ and set $\mathcal{M}' := \mathcal{M}_{[d/c][c:=a]}$. By Corollary 3 and Corollary 4, we get that $\mathcal{M}', u \models^+ \Delta \cup \{\psi\}$; on the other hand, we get, by Corollary 3, that $\mathcal{M}_{[d/c]}, u \not\models^+ \chi[a/x][d/c]$. However, by the choice of a, c we have $c \notin Par(\chi) \cup \{a\}$ and so Lemma 1.2 implies that $\chi[a/x][d/c] = \chi[a/x]$. Therefore, $\mathcal{M}_{[d/c]}, u \not\models^+ \chi[a/x]$, whence, by Lemma 5, it follows that $\mathcal{M}', u \not\models^+ \chi[c/x]$. Now, since $c \in D'_u$, and the Induction Hypothesis implies that $\Delta \models^+ \psi \to \chi[c/x]$, we must also have $\mathcal{M}', u \models^+ \psi \to \chi[c/x]$. The obtained contradiction shows that, in fact, we must have had $\mathcal{M}, w \models^+ \psi \to \forall x \chi$ all along.

Case 3. For some $1 \leq i \leq k$ we have $\phi_i = \psi[c/x] \to \chi$ for corresponding x, c, ψ, and χ, such that $c \notin Par(\Delta) \cup Par(\psi \to \chi)$, whereas $\phi_r = \exists x\psi \to \chi$. This case is dual to Case 2. ☐

We now proceed to show the converse of Theorem 1. We will only show it for countable signatures and countable parameter sets. Again, we start with some definitions. A given bi-set $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$, is called:

- *Non-trivial*, if $\Delta \neq \emptyset$ and $\Gamma \nvdash \bigvee \Delta'$ for every $\emptyset \neq \Delta' \Subset \Delta$.
- *Complete*, if $\Gamma \cup \Delta = L_\emptyset(Sign(\Gamma \cup \Delta), Par(\Gamma \cup \Delta))$.
- $\exists$-*complete*, if for every $\exists x\phi \in L_\emptyset(Sign(\Gamma\cup\Delta), Par(\Gamma\cup\Delta))$ such that $\exists x\phi \in \Gamma$, there exists an $a \in Par(\Gamma \cup \Delta)$ such that $\phi[a/x] \in \Gamma$.

A given $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ is called (Σ, Π)-appropriate iff $Sign(\Gamma\cup\Delta) = \Sigma$, $Par(\Gamma \cup \Delta) = \Pi$, and (Γ, Δ) is non-trivial, complete, and $\exists$-complete. In the lemmas that follow below, we list some properties of non-trivial bi-sets and then, more specifically, some properties of the non-trivial bi-sets that also happen to be appropriate.

Lemma 10. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ be non-trivial. Then the following statements hold:*

1. *If $(\Gamma', \Delta') \subseteq (\Gamma, \Delta)$ and $\Delta' \neq \emptyset$, then (Γ', Δ') is non-trivial.*
2. *If $\phi \in L_\emptyset(\Sigma, \Pi)$, then one of $(\Gamma \cup \{\phi\}, \Delta)$, $(\Gamma, \Delta \cup \{\phi\})$ is non-trivial.*
3. *If $\exists x\phi \in L_\emptyset(\Sigma, \Pi)$, and $a \in Par \setminus \Pi$, then one of $(\Gamma \cup \{\exists x\phi, \phi[a/x]\}, \Delta)$, $(\Gamma, \Delta \cup \{\exists x\phi\})$ is non-trivial.*
4. *If $\phi \to \psi \in \Delta$, then $(\Gamma \cup \{\phi\}, \{\psi\})$ is non-trivial.*
5. *If $\sim (\phi \to \psi) \in \Delta$, then $(\Gamma \cup \{\phi\}, \{\sim \psi\})$ is non-trivial.*
6. *If $\forall x\phi \in \Delta$, and $a \in Par \setminus \Pi$, then $(\Gamma, \{\phi[a/x]\})$ is non-trivial.*
7. *If $\sim \exists x\phi \in \Delta$, and $a \in Par \setminus \Pi$, then $(\Gamma, \{\sim \phi[a/x]\})$ is non-trivial.*

Proof. Part 1 is straightforward. As for Part 2, assume that both $(\Gamma \cup \{\phi\}, \Delta)$ and $(\Gamma, \Delta \cup \{\phi\})$ are trivial. Then there must be $\Delta', \Delta'' \Subset \Delta$ such that (wlog, due to (A6) and (MP)), both $\Gamma \cup \{\phi\} \vdash \bigvee \Delta'$ and $\Gamma \vdash \phi \vee \bigvee \Delta''$. By Lemma 9, the former deducibility relation implies that also $\Gamma \vdash \phi \to \bigvee \Delta'$. Applying (A6) and (T3), we infer that $\Gamma \vdash \bigvee \Delta' \vee \bigvee \Delta''$. Applying (T6) next, we can show that also

$\Gamma \vdash \bigvee(\Delta' \cup \Delta'')$ (we basically need to erase the repetitions in $\bigvee \Delta' \vee \bigvee \Delta''$). Since $\Delta' \cup \Delta'' \Subset \Delta$, this contradicts the non-triviality of (Γ, Δ).

(Part 3). Again, assume that both $(\Gamma \cup \{\exists x\phi, \phi[a/x]\}, \Delta)$ and $(\Gamma, \Delta \cup \{\exists x\phi\})$ are trivial. Then, by Part 2, $(\Gamma \cup \{\exists x\phi\}, \Delta)$ must be non-trivial. Let $\Delta' \Subset \Delta$ be such that $\Gamma \cup \{\exists x\phi, \phi[a/x]\} \vdash \bigvee \Delta'$. By Lemma 9, we must have then $\Gamma \cup \{\exists x\phi\} \vdash \phi[a/x] \to \bigvee \Delta'$. Since $a \in Par$, by its choice, is outside $Par(\Gamma \cup \Delta \cup \{\phi\})$, we get that:

$$\Gamma \cup \{\exists x\phi\} \vdash \exists x\phi \to \bigvee \Delta' \qquad \text{(by (R}\exists\text{))}$$

$$\Gamma \cup \{\exists x\phi\} \vdash \bigvee \Delta' \qquad \text{(by (MP))}$$

The latter deducibility clearly contradicts the non-triviality of $(\Gamma \cup \{\exists x\phi\}, \Delta)$.

(Part 4). Assume that $\phi \to \psi \in \Delta$, but $(\Gamma \cup \{\phi\}, \{\psi\})$ is trivial, that is to say, that we have $\Gamma \cup \{\phi\} \vdash \psi$. By Lemma 9, we have $\Gamma \vdash \phi \to \psi$, which contradicts the non-triviality of (Γ, Δ).

(Part 5). Assume that $\sim (\phi \to \psi) \in \Delta$. By Part 2, either $(\Gamma \cup \{\phi \to\sim \psi\}, \Delta)$ or $(\Gamma, \Delta \cup \{\phi \to\sim \psi\})$ must be non-trivial. The former case is in contradiction with (A12), therefore, $(\Gamma, \Delta \cup \{\phi \to\sim \psi\})$ must be non-trivial, and, by Part 4, $(\Gamma \cup \{\phi\}, \{\sim \psi\})$ must be non-trivial as well.

(Part 6). Assume that $\forall x\phi \in \Delta$, and that $a \in Par \setminus \Pi$, but $(\Gamma, \{\phi[a/x]\})$ is trivial, that is to say, that we have $\Gamma \vdash \phi[a/x]$. By Corollary 5.2, we must have then that $\Gamma \vdash \forall x\phi$ which is in contradiction with the non-triviality of (Γ, Δ).

(Part 7). Assume that $\sim \exists x\phi \in \Delta$, and that $a \in Par \setminus \Pi$. By Part 2, either $(\Gamma \cup \{\forall x \sim \phi\}, \Delta)$ or $(\Gamma, \Delta \cup \{\forall x \sim \phi\})$ must be non-trivial. The former case is in contradiction with (A13), therefore, $(\Gamma, \Delta \cup \{\forall x \sim \phi\})$ must be non-trivial, and, by Part 6, $(\Gamma, \{\sim \phi[a/x]\})$ must be non-trivial as well. □

Lemma 11. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ be (Σ, Π)-appropriate. Let $\phi, \psi \in L_\emptyset(\Sigma, \Pi)$, let $x \in Var$, and let $\chi \in L_x(\Sigma, \Pi)$. Then the following statements hold:*

1. *If $\Gamma \vdash \phi$, then $\phi \in \Gamma$.*
2. *$\phi \wedge \psi \in \Gamma$ iff $\phi, \psi \in \Gamma$.*
3. *$\sim (\phi \wedge \psi) \in \Gamma$ iff $\sim \phi \in \Gamma$ or $\sim \psi \in \Gamma$.*
4. *$\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$.*
5. *$\sim (\phi \vee \psi) \in \Gamma$ iff $\sim \phi, \sim \psi \in \Gamma$.*
6. *$\sim\sim \phi \in \Gamma$ iff $\phi \in \Gamma$.*

7. *$\exists x\chi \in \Gamma$ iff $\chi[a/x] \in \Gamma$ for some $a \in Par(\Gamma \cup \Delta)$.*

8. *$\sim \forall x\chi \in \Gamma, \Delta$ iff $\sim \chi[a/x] \in \Gamma$ for some $a \in Par(\Gamma \cup \Delta)$.*

9. *If $\phi \to \psi \in \Gamma$ and $\phi \in \Gamma$, then $\psi \in \Gamma$.*

10. *If $\sim (\phi \to \psi) \in \Gamma$ and $\phi \in \Gamma$, then $\sim \psi \in \Gamma$.*

11. *If $\forall x\chi \in \Gamma$ and $a \in Par(\Gamma \cup \Delta)$, then $\chi[a/x] \in \Gamma$.*

12. *If $\sim \exists x\chi \in \Gamma$ and $a \in Par(\Gamma \cup \Delta)$, then $\sim \chi[a/x] \in \Gamma$.*

Proof. Part 1 follows from the non-triviality and completeness of (Γ, Δ).

Most of the remaining parts are proven by a straightforward reference to Part 1 plus the corresponding part of our axiomatization sometimes combined with the reference to the earlier dual parts of the Lemma. Exceptions are Part 4 (reference to (T1)) and Part 7, where one must use the $\exists$-completeness of (Γ, Δ). ☐

Certain types of non-trivial bi-sets are in general extendable to certain types of appropriate bi-sets, which is the subject of the next lemma:

Lemma 12. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ be non-trivial. Then, for every $\Xi \subseteq Par$ disjoint from Π and such that $|\Xi| = \omega$, there exists a $(\Sigma, \Pi \cup \Xi)$-appropriate bi-set (Γ', Δ') such that $(\Gamma', \Delta') \supseteq (\Gamma, \Delta)$.*

Proof. Let $\{a_n \mid n \in \omega\}$ be an enumeration of Ξ, and let $\{\psi_n \mid n \in \omega\}$ be an enumeration of $L_\emptyset(\Sigma, \Pi \cup \Xi)$. We now define a countably infinite increasing chain of non-trivial bi-sets

$$(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_n, \Delta_n) \subseteq \ldots$$

by setting $(\Gamma, \Delta) := (\Gamma_0, \Delta_0)$, and for any $k \in \omega$, if ψ_k is not of the form $\exists x\phi$ we set:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\psi_k\}, \Delta_k), \text{ if } (\Gamma_k \cup \{\psi_k\}, \Delta_k) \text{ is non-trivial} \\ (\Gamma_k, \Delta_k \cup \{\psi_k\}), \text{ otherwise.} \end{cases}$$

In case ψ_k has the form $\exists x\phi$, we set

$$\nu[\Gamma_k, \Delta_k, \psi_k] := \{n \in \omega \mid a_n \in \Xi \setminus Par(\Gamma_k \cup \Delta_k \cup \{\psi_k\})\}$$

and define:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k), \text{ if } m = \min \nu[\Gamma_k, \Delta_k, \exists x\phi] \\ \qquad\qquad \text{and } (\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k) \text{ is non-trivial} \\ (\Gamma_k, \Delta_k \cup \{\exists x\phi\}), \text{ otherwise.} \end{cases}$$

We show that the chain $(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_n, \Delta_n) \subseteq \ldots$ is well-defined and that, for every $k \in \omega$ the bi-set (Γ_k, Δ_k) is non-trivial and we have $|\nu[\Gamma_k, \Delta_k, \psi_k]| = \omega$.

This claim is obviously true when $k = 0$. If $k = r + 1$, and the claim is true for (Γ_r, Δ_r), then $(\Gamma_{r+1}, \Delta_{r+1})$ is well-defined by the Induction Hypothesis and is non-trivial by Lemma 10.2–3. Finally, we have $Par(\Gamma_{r+1} \cup \Delta_{r+1} \cup \{\psi_{r+1}\}) = Par(\Gamma_r \cup \Delta_r \cup \{\psi_r, \psi_{r+1}\})$ in case ψ_r is not of the form $\exists x\phi$ and $Par(\Gamma_{r+1} \cup \Delta_{r+1} \cup \{\psi_{r+1}\}) \subseteq Par(\Gamma_r \cup \Delta_r \cup \{\exists x\phi, \phi[a_m/x], \psi_{k+1}\})$ for certain fresh $a_m \in Par$ when $\psi_r = \exists x\phi$. In both cases the difference with $Par(\Gamma_r \cup \Delta_r \cup \{\psi_r\})$ is clearly finite so that $|\nu[\Gamma_{r+1}, \Delta_{r+1}, \psi_{r+1}]| = \omega$ obviously holds.

We now set $(\Gamma', \Delta') := (\bigcup_{n\in\omega} \Gamma_n, \bigcup_{n\in\omega} \Delta_n)$ and show that this bi-set satisfies the requirements of the Lemma. It is clear that $(\Gamma', \Delta') \supseteq (\Gamma, \Delta)$. Moreover, for every $k \in \omega$, we have $\psi_k \in \Gamma_{k+1} \cup \Delta_{k+1}$, therefore it is also clear that $Sign(\Gamma' \cup \Delta') = \Sigma$, that $Par(\Gamma' \cup \Delta') = \Pi \cup \Xi$, and that (Γ', Δ') is complete.

It remains to show non-triviality and $\exists$-completeness of (Γ', Δ'). If $\emptyset \neq \Delta^* \Subset \Delta'$ is such that $\Gamma' \vdash \bigvee \Delta^*$, then consider any deduction of $\bigvee \Delta^*$ from the premises in $\Gamma^* \Subset \Gamma$ and choose any $k \in \omega$ such that $(\Gamma^*, \Delta^*) \subseteq (\Gamma_k, \Delta_k)$. Then $\Gamma_k \vdash \bigvee \Delta^*$, which contradicts the non-triviality of (Γ_k, Δ_k).

As for the $\exists$-completeness, if $\exists x\phi \in \Gamma'$, then $\exists x\phi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$, therefore, for some $k \in \omega$, we must have $\exists x\phi = \psi_k$. Clearly, $\psi_k \in \Delta_{k+1} \subseteq \Delta'$ would contradict the non-triviality of (Γ', Δ'). Therefore, we must have $\exists x\phi, \phi[a_m/x] \in \Gamma_{k+1} \subseteq \Gamma'$ for an appropriate $a_m \in \Xi$. □

We are now in a position to prove the completeness of our axiomatization in the countable case. Given a signature Σ and a parameter set Π, we will call a bi-set (Γ, Δ) (Σ, Π)-nice iff (Γ, Δ) is (Σ, Ξ)-appropriate, for some $\Xi \subseteq \Pi$ such that $|\Pi \setminus \Xi| = \omega$. Given a (Σ, Π)-appropriate bi-set (Γ, Δ), and a countably infinite $\Xi \subseteq Par$ which is disjoint from Π, we can define the following Σ-model $\mathcal{M}_{(\Gamma,\Delta,\Xi)}$ by setting:

- $W_{(\Gamma,\Delta,\Xi)} := \{(\Gamma', \Delta') \mid (\Gamma', \Delta') \text{ is } (\Sigma, \Pi \cup \Xi)\text{-nice}\}$.
- For $(\Gamma_0, \Delta_0), (\Gamma_1, \Delta_1) \in W_{(\Gamma,\Delta,\Xi)}$, we have $(\Gamma_0, \Delta_0) \leq_{(\Gamma,\Delta,\Xi)} (\Gamma_1, \Delta_1)$ iff $\Gamma_0 \subseteq \Gamma_1$.
- $U_{(\Gamma,\Delta,\Xi)} := \Pi \cup \Xi$.
- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta,\Xi)}$, we have $D_{(\Gamma,\Delta,\Xi)}(\Gamma_0, \Delta_0) := Par(\Gamma_0 \cup \Delta_0)$.
- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta,\Xi)}$, $n \in \omega$, $P^n \in Sign(\Gamma_0 \cup \Delta_0)$, and $\bar{a}_n \in Par(\Gamma_0 \cup \Delta_0)^n$ we have $\bar{a}_n \in V^+_{(\Gamma,\Delta,\Xi)}(P, (\Gamma_0, \Delta_0))$ iff $P(\bar{a}_n) \in \Gamma_0$.

- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta,\Xi)}$, $n \in \omega$, $P^n \in Sign(\Gamma_0 \cup \Delta_0)$, and $\bar{a}_n \in Par(\Gamma_0 \cup \Delta_0)^n$ we have $\bar{a}_n \in V^-_{(\Gamma,\Delta,\Xi)}(P, (\Gamma_0, \Delta_0))$ iff $\sim P(\bar{a}_n) \in \Gamma_0$.

It is straightforward to show that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}$ is indeed a model of QC, and using the usual methods, a truth lemma can be shown for this model:

Lemma 13. *Let (Γ, Δ) be a (Σ, Π)-appropriate bi-set and let $\Xi \subseteq Par$ be countably infinite and disjoint from Π. Then, for every $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta,\Xi)}$ and every $\phi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$ it is true that:*

1. *$\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \phi$ iff $\phi \in \Gamma_0$.*
2. *$\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \phi$ iff $\sim \phi \in \Gamma_0$.*

Proof. We prove both parts of the Lemma simultaneously by induction on the construction of $\phi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$.

Basis. ϕ is atomic. Both parts of the Lemma hold by the definition of $V^+_{(\Gamma,\Delta,\Xi)}$ and $V^-_{(\Gamma,\Delta,\Xi)}$

Step. The following cases are possible.

Case 1. $\phi = \psi \wedge \chi$ for some $\psi, \chi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$. Then, for Part 1 of the Lemma we reason as follows:

$$\begin{aligned}
\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \psi \wedge \chi &\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \psi \\
&\qquad \text{and } \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \chi \\
&\Leftrightarrow \psi \in \Gamma_0 \text{ and } \chi \in \Gamma_0 && \text{(by IH)} \\
&\Leftrightarrow \psi \wedge \chi \in \Gamma_0 && \text{(by Lemma 11.2)}
\end{aligned}$$

For Part 2, the reasoning is similar:

$$\begin{aligned}
\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \psi \wedge \chi &\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \psi \\
&\qquad \text{or } \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \chi \\
&\Leftrightarrow \sim \psi \in \Gamma_0 \text{ or } \sim \chi \in \Gamma_0 && \text{(by IH)} \\
&\Leftrightarrow \sim (\psi \wedge \chi) \in \Gamma_0 && \text{(by Lemma 11.3)}
\end{aligned}$$

Case 2. $\phi = \psi \vee \chi$ for some $\psi, \chi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$. Similar to Case 1.

Case 3. $\phi = \sim \psi$ or some $\psi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$. Then, for Part 1 of the Lemma we reason as follows:

$$\begin{aligned}
\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \sim \psi &\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \psi \\
&\Leftrightarrow \sim \psi \in \Gamma_0 && \text{(by IH)}
\end{aligned}$$

As for Part 2, the argument is as follows:

$$\begin{aligned}
\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \sim \psi &\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \psi \\
&\Leftrightarrow \psi \in \Gamma_0 && \text{(by IH)} \\
&\Leftrightarrow \sim\sim \psi \in \Gamma_0 && \text{(by Lemma 11.6)}
\end{aligned}$$

Case 4. $\phi = \psi \to \chi$ for some $\psi, \chi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$. Again, we consider Part 1 first:

($\Leftarrow$). If $\psi \to \chi \in \Gamma_0$, and $(\Gamma_0, \Delta_0) \leq_{(\Gamma,\Delta,\Xi)} (\Gamma_1, \Delta_1)$, then $\Gamma_0 \subseteq \Gamma_1$ so that $\psi \to \chi \in \Gamma_1$. Now, if $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \models^+ \psi$, then, by the Induction Hypothesis, $\psi \in \Gamma_1$, and, by Lemma 11.9, $\chi \in \Gamma_1$. Applying the Induction Hypothesis one more time, we get that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \models^+ \chi$. Since $(\Gamma_1, \Delta_1) \geq_{(\Gamma,\Delta,\Xi)} (\Gamma_0, \Delta_0)$ was chosen arbitrarily, we get that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \psi \to \chi$.

($\Rightarrow$). If $\psi \to \chi \notin \Gamma_0$, then, by the completeness of (Γ_0, Δ_0), we get that $\psi \to \chi \in \Delta_0$. But then, by Lemma 10.4, we get that $(\Gamma_0 \cup \{\psi\}, \{\chi\})$ must be non-trivial, and, clearly $Par(\Gamma_0 \cup \{\psi, \chi\}) \subseteq Par(\Gamma_0 \cup \Delta_0)$. Therefore, the parameter set $\Pi' := (\Pi \cup \Xi) \setminus Par(\Gamma_0 \cup \{\psi, \chi\})$ must be countably infinite. Now, we partition Π' into two further countably infinite sets, Π_0 and Π_1. By Lemma 12, we can find a $(\Sigma, Par(\Gamma_0 \cup \{\psi, \chi\}) \cup \Pi_0)$-appropriate bi-set $(\Gamma_1, \Delta_1) \supseteq (\Gamma_0 \cup \{\psi\}, \{\chi\})$. For this latter bi-set, we have that $(\Pi \cup \Xi) \setminus Par(\Gamma_1 \cup \Delta_1) = \Pi_1$, so that (Γ_1, Δ_1) is also $(\Sigma, \Pi \cup \Xi)$-nice and thus in $W_{(\Gamma,\Delta,\Xi)}$. Moreover, we must have $\Gamma_1 \supseteq \Gamma_0$ so that $(\Gamma_1, \Delta_1) \geq_{(\Gamma,\Delta,\Xi)} (\Gamma_0, \Delta_0)$. Next, we have $\psi \in \Gamma_1$ so that the Induction Hypothesis implies that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \models^+ \psi$. Finally, we have $\chi \in \Delta_1$, hence also $\chi \notin \Gamma_1$ by the non-triviality of (Γ_1, Δ_1), whence further $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \not\models^+ \chi$ by the Induction Hypothesis. But then $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \not\models^+ \psi \to \chi$.

Part 2 of the Lemma in this Case is similar to Part 1.

Case 5. $\phi = \forall x \psi$ for some $\psi \in L_x(\Sigma, \Pi \cup \Xi)$. We consider Part 1 first:

($\Leftarrow$). If $\forall x \psi \in \Gamma_0$, and $(\Gamma_0, \Delta_0) \leq_{(\Gamma,\Delta,\Xi)} (\Gamma_1, \Delta_1)$, then $\Gamma_0 \subseteq \Gamma_1$ so that $\forall x \psi \in \Gamma_1$. Now, if $a \in D_{(\Gamma,\Delta,\Xi)}(\Gamma_1, \Delta_1) = Par(\Gamma_1 \cup \Delta_1)$, then, by Lemma 11.11, $\psi[a/x] \in \Gamma_1$, and further, by the Induction Hypothesis, $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \models^+ \psi[a/x]$. Since $(\Gamma_1, \Delta_1) \geq_{(\Gamma,\Delta,\Xi)} (\Gamma_0, \Delta_0)$ and $a \in D_{(\Gamma,\Delta,\Xi)}(\Gamma_1, \Delta_1)$ were chosen arbitrarily, we get that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \forall x \psi$.

($\Rightarrow$). If $\forall x \psi \notin \Gamma_0$, then, by the completeness of (Γ_0, Δ_0), we get that $\forall x \psi \in \Delta_0$. We know that $(\Pi \cup \Xi) \setminus Par(\Gamma_0 \cup \Delta_0)$ is infinite, therefore, we can choose any parameter a in this set. Now Lemma 10.6 tells us that $(\Gamma_0, \{\psi[a/x]\})$ must be non-trivial, and, clearly $Par(\Gamma_0 \cup \{\psi[a/x]\}) \subseteq Par(\Gamma_0 \cup \Delta_0) \cup \{a\}$. Therefore, the parameter set $\Pi' := (\Pi \cup \Xi) \setminus Par(\Gamma_0 \cup \{\psi[a/x]\})$ must be countably infinite. We partition Π' into two further countably infinite sets, Π_0 and Π_1. By Lemma 12, we can find a $(\Sigma, Par(\Gamma_0 \cup \{\psi[a/x]\}) \cup \Pi_0)$-appropriate bi-set $(\Gamma_1, \Delta_1) \supseteq (\Gamma_0, \{\psi[a/x]\})$.

For this latter bi-set, we have that $(\Pi \cup \Xi) \setminus Par(\Gamma_1 \cup \Delta_1) = \Pi_1$, so that (Γ_1, Δ_1) is also $(\Sigma, \Pi \cup \Xi)$-nice and thus in $W_{(\Gamma,\Delta,\Xi)}$. Moreover, we must have $\Gamma_1 \supseteq \Gamma_0$ so that $(\Gamma_1, \Delta_1) \geq_{(\Gamma,\Delta,\Xi)} (\Gamma_0, \Delta_0)$. Next, we have $\psi[a/x] \in \Delta_1$, hence also $\psi[a/x] \notin \Gamma_1$ by the non-triviality of (Γ_1, Δ_1), whence further $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \not\models^+ \psi[a/x]$ by the Induction Hypothesis. But then $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \not\models^+ \forall x\psi$.

We know turn to Part 2 of the Lemma, and reason as follows:

$$\begin{aligned} \sim \forall x\psi \in \Gamma_0 &\Leftrightarrow (\exists a \in Par(\Gamma_0 \cup \Delta_0))(\sim \psi[a/x] \in \Gamma_0) && \text{(by Lemma 11.8)}\\ &\Leftrightarrow (\exists a \in Par(\Gamma_0 \cup \Delta_0)(\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \psi[a/x]) && \text{(by IH)}\\ &\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \forall x\psi \end{aligned}$$

Case 6. $\phi = \exists x\psi$ for some $\psi \in L_x(\Sigma, \Pi \cup \Xi)$. Similar to Case 5. ☐

Theorem 2. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.*

Proof. We argue by contraposition. If $\Gamma \not\vdash \phi$, then the bi-set $(\Gamma, \{\phi\})$ must be non-trivial. But then, choose two infinitely countable parameter sets Ξ_0 and Ξ_1 such that $\{\Pi, \Xi_0, \Xi_1\}$ forms a pairwise disjoint family of sets. Then we can find, by Lemma 12, a $(\Pi \cup \Xi_0)$-appropriate bi-set $(\Gamma', \Delta') \supseteq (\Gamma, \{\phi\})$; (Γ', Δ') is also $(\Pi \cup \Xi_0 \cup \Xi_1)$-nice. We clearly have $\phi \in \Delta'$, so also $\phi \notin \Gamma'$ by the non-triviality of (Γ', Δ'). Now Lemma 13 implies that we have both $\mathcal{M}_{(\Gamma',\Delta',\Xi_1)}, (\Gamma', \Delta') \models^+ \Gamma' \supseteq \Gamma$ and $\mathcal{M}_{(\Gamma',\Delta',\Xi_1)}, (\Gamma', \Delta') \not\models^+ \phi$. Therefore, $\Gamma \not\models \phi$ as desired. ☐

4 Hilbert-style axiomatizations of QC_{CD} and $\mathsf{QC3}_{CD}$

In order to obtain the axiomatization of QC_{CD}, we extend the set of axioms with the parametrized sentences which are instances of the following scheme:

$$\forall x(\phi \vee \psi) \to (\phi \vee \forall x\psi) \tag{A17}$$

We do not need to require separately that $x \notin FV(\phi)$ since this already follows from the fact that $\forall x(\phi \vee \psi) \to (\phi \vee \forall x\psi)$ is a parametrized sentence.

We can then define the notion of $(\Sigma, \Pi)_{CD}$-deduction and the deducibility relation $\vdash_{CD}$ for this extended system. Lemmas 6–9 then extend to our amended deduction and deducibility notions and the only change in the proofs is that one needs to mention the extended set of axioms in place of the set of axioms for QC.

Similarly, we can now prove the following theorem in almost the same way as Theorem 1:

Theorem 3. *Let* Σ *be a signature, let* $\Pi \subseteq Par$ *be a set, and let* $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$*. If* $\Gamma \vdash_{CD} \phi$*, then* $\Gamma \models_{\mathbb{CD}} \phi$*.*

Turning now to the converse of Theorem 3 in the countable case, we observe, first, that we need to extend the notion of an appropriate bi-set. More precisely, given a bi-set $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$, we say that (Γ, Δ) is $\forall$-complete iff for every $\forall x\phi \in L_\emptyset(Sign(\Gamma \cup \Delta), Par(\Gamma \cup \Delta))$ such that $\forall x\phi \in \Delta$, there exists an $a \in Par(\Gamma \cup \Delta)$ such that $\phi[a/x] \in \Delta$. A bi-set (Γ, Δ) is then called $(\Sigma, \Pi)_{CD}$-appropriate iff it is (Σ, Π)-appropriate (in the sense of the previous section, except that non-triviality is understood relative to $\vdash_{CD}$) and $\forall$-complete.

Next, we need to extend the lemma on non-trivial bi-sets:

Lemma 14. *Let* Σ *be a signature, let* $\Pi \subseteq Par$ *be a set, and let* $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ *be CD-non-trivial. Then all of the statements in Lemma 10 hold, and, in addition, it is true that, if* $\forall x\phi \in L_\emptyset(\Sigma, \Pi)$*, and* $a \in Par \setminus \Pi$*, then one of* $(\Gamma \cup \{\forall x\phi\}, \Delta)$*,* $(\Gamma, \Delta \cup \{\forall x\phi, \phi[a/x]\})$ *is CD-non-trivial.*

Proof. The proof of Lemma 10 can be simply repeated replacing the non-triviality everywhere with the CD-non-triviality. As for the additional part, assume that both $(\Gamma \cup \{\forall x\phi\}, \Delta)$ and $(\Gamma, \Delta \cup \{\forall x\phi, \phi[a/x]\})$ are CD-trivial. Then, by Lemma 10.2, $(\Gamma, \Delta \cup \{\forall x\phi\})$ must be CD-non-trivial. Let $\emptyset \neq \Delta' \Subset \Delta$ be such that, wlog, $\Gamma \vdash_{CD} \phi[a/x] \vee (\forall x\phi \vee \bigvee \Delta')$. Since $x \notin FV(\forall x\phi \vee \bigvee \Delta')$, Lemma 1.2 implies that $\Gamma \vdash_{CD} (\phi \vee (\forall x\phi \vee \bigvee \Delta'))[a/x]$. By Corollary 5.2, we must have then $\Gamma \vdash_{CD} \forall x(\phi \vee (\forall x\phi \vee \bigvee \Delta'))$, whence, by (A17) and (MP), $\Gamma \vdash_{CD} \forall x\phi \vee \forall x\phi \vee \bigvee \Delta'$. By (T6), we get, next, that $\Gamma \vdash_{CD} \forall x\phi \vee \bigvee \Delta'$, which contradicts the CD-non-triviality of $(\Gamma, \Delta \cup \{\forall x\phi\})$. □

We note, furthermore, that Lemma 11 (on appropriate bi-sets) carries over to CD-appropriate bi-sets without any non-trivial change in the proof. Next, we show that in the countable case any CD-non-trivial bi-set can be extended to a CD-appropriate one over an extended set of parameters.

Lemma 15. *Let* Σ *be an at most countable signature, let* $\Pi \subseteq Par$ *be an at most countable set, and let* $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ *be CD-non-trivial. Then, for every* $\Xi \subseteq Par$ *disjoint from* Π *and such that* $|\Xi| = \omega$*, there exists a* $(\Sigma, \Pi \cup \Xi)_{CD}$*-appropriate bi-set* (Γ', Δ') *such that* $(\Gamma', \Delta') \supseteq (\Gamma, \Delta)$*.*

Proof. We adapt the proof of Lemma 12 to our current environment. Again, let $\{a_n \mid n \in \omega\}$ be an enumeration of Ξ, and let $\{\psi_n \mid n \in \omega\}$ be an enumeration of $L_\emptyset(\Sigma, \Pi \cup \Xi)$. We now define a countably infinite increasing chain of CD-non-trivial bi-sets

$$(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_n, \Delta_n) \subseteq \ldots$$

by setting $(\Gamma,\Delta) := (\Gamma_0,\Delta_0)$, and for any $k \in \omega$, if ψ_k is neither of the form $\exists x\phi$ nor of the form $\forall x\phi$, then we set:

$$(\Gamma_{k+1},\Delta_{k+1}) := \begin{cases}(\Gamma_k \cup \{\psi_k\},\Delta_k), & \text{if } (\Gamma_k \cup \{\psi_k\},\Delta_k) \text{ is } CD\text{-non-trivial}\\ (\Gamma_k,\Delta_k \cup \{\psi_k\}), & \text{otherwise.}\end{cases}$$

For the remaining cases, we will use the subsets of ω of the form $\nu[\Gamma_k,\Delta_k,\psi_k]$ as defined in the proof of Lemma 12.

Namely, in case ψ_k has the form $\exists x\phi$, we set:

$$(\Gamma_{k+1},\Delta_{k+1}) := \begin{cases}(\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\},\Delta_k) \text{ if } m = \min\nu[\Gamma_k,\Delta_k,\exists x\phi]\\ \qquad\qquad \text{and } (\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\},\Delta_k) \text{ is CD-non-trivial}\\ (\Gamma_k,\Delta_k \cup \{\exists x\phi\}), \text{ otherwise.}\end{cases}$$

Finally, in case ψ_k has the form $\forall x\phi$, we set:

$$(\Gamma_{k+1},\Delta_{k+1}) := \begin{cases}(\Gamma_k \cup \{\forall x\phi\},\Delta_k), & \text{if } (\Gamma_k \cup \{\forall x\phi\},\Delta_k) \text{ is } CD\text{-non-trivial}\\ (\Gamma_k,\Delta_k \cup \{\forall x\phi, \phi[a_m/x]\}), & \text{if } m = \min\nu[\Gamma_k,\Delta_k,\forall x\phi], \text{ otherwise.}\end{cases}$$

The rest of the argument is exactly as in the proof of Lemma 12 except that we need to add the reference to Lemma 14 in order to show that in the latter case the bi-set remains CD-non-trivial. Another addition is the argument for $\forall$-completeness of the resulting set $(\Gamma',\Delta') := (\bigcup_{n\in\omega}\Gamma_n, \bigcup_{n\in\omega}\Delta_n)$ which is similar to the one for the $\exists$-completeness given in the proof of Lemma 12. □

Before we start with the construction of the canonical model, we need one final ingredient which was not necessary in the case of QC but which is normally required as long as the domains are assumed to be constant. We formulate this additional argumentative ingredient in the following lemma:

Lemma 16. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, let $(\Gamma,\Delta) \subseteq (L_\emptyset(\Sigma,\Pi), L_\emptyset(\Sigma,\Pi))$ be $(\Sigma,\Pi)_{CD}$-appropriate, and let $(\Gamma_0,\Delta_0) \Subset (L_\emptyset(\Sigma,\Pi), L_\emptyset(\Sigma,\Pi))$ be such that $(\Gamma \cup \Gamma_0,\Delta_0)$ is CD-non-trivial. Then there exists a $(\Sigma,\Pi)_{CD}$-appropriate bi-set (Γ',Δ') such that $(\Gamma',\Delta') \supseteq (\Gamma \cup \Gamma_0,\Delta_0)$.*

Proof. Once again we re-use the construction from Lemma 15 with a further additional twist. Namely, we let $\{a_n \mid n \in \omega\}$ be an enumeration of Π and we let $\{\psi_n \mid n \in \omega\}$ be an enumeration of $L_\emptyset(\Sigma,\Pi)$. But this time we define a countably infinite increasing chain of finite bi-sets

$$(\Gamma_0,\Delta_0) \subseteq \ldots \subseteq (\Gamma_n,\Delta_n) \subseteq \ldots$$

such that, for every $k \in \omega$, the bi-set $(\Gamma \cup \Gamma_k, \Delta_k)$ is CD-non-trivial. In this chain, (Γ_0, Δ_0) is given in the formulation of the lemma and for any $k \in \omega$, if ψ_k is neither of the form $\exists x\phi$ nor of the form $\forall x\phi$, then we set:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\psi_k\}, \Delta_k), & \text{if } (\Gamma \cup \Gamma_k \cup \{\psi_k\}, \Delta_k) \text{ is } CD\text{-non-trivial} \\ (\Gamma_k, \Delta_k \cup \{\psi_k\}), & \text{otherwise.} \end{cases}$$

In case ψ_k has the form $\exists x\phi$, we set:

$$\mu_\Gamma[\Gamma_k, \Delta_k, \exists x\phi] := \{n \in \omega \mid a_n \in \Pi \mid (\Gamma \cup \Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k) \text{ is } CD\text{-non-trivial}\}$$

and we define:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k), & \\ \qquad \text{if } \mu_\Gamma[\Gamma_k, \Delta_k, \exists x\phi] \neq \emptyset \text{ and } m = \min \mu_\Gamma[\Gamma_k, \Delta_k, \exists x\phi] & \\ (\Gamma_k, \Delta_k \cup \{\exists x\phi\}), \ \text{if } \mu_\Gamma[\Gamma_k, \Delta_k, \exists x\phi] = \emptyset. & \end{cases}$$

Finally, in case ψ_k has the form $\forall x\phi$, we set

$$\mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] := \{n \in \omega \mid a_n \in \Pi \mid (\Gamma \cup \Gamma_k, \Delta_k \cup \{\forall x\phi, \phi[a_m/x]\}) \text{ is } CD\text{-non-trivial}\}$$

and we define:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\forall x\phi\}, \Delta_k), \ \text{if } \mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] = \emptyset & \\ (\Gamma_k, \Delta_k \cup \{\forall x\phi, \phi[a_m/x]\}), & \\ \qquad \text{if } \mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] \neq \emptyset \text{ and } m = \min \mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi]. & \end{cases}$$

We show that the chain $(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_n, \Delta_n) \subseteq \ldots$ is well-defined and that, for every $k \in \omega$, we have $(\Gamma_k, \Delta_k) \Subset (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ and the bi-set $(\Gamma \cup \Gamma_k, \Delta_k)$ is CD-non-trivial.

This claim is obviously true when $k = 0$. If $k = r + 1$, and the claim is true for (Γ_r, Δ_r), then $(\Gamma_{r+1}, \Delta_{r+1})$ is well-defined by the Induction Hypothesis and we clearly have $(\Gamma_{r+1}, \Delta_{r+1}) \Subset (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$. It remains to show the CD-non-triviality, and, in doing so, we have to consider the three cases in our definition:

Case 1. ψ_r is neither of the form $\exists x\phi$ nor of the form $\forall x\phi$. Then the CD-non-triviality of $(\Gamma \cup \Gamma_{r+1}, \Delta_{r+1})$ follows from (the CD-version of) Lemma 10.2.

Case 2. ψ_r has the form $\exists x\phi$. If $\mu_\Gamma[\Gamma_r, \Delta_r, \exists x\phi] \neq \emptyset$, then we are done. Otherwise, we must have $\mu_\Gamma[\Gamma_r, \Delta_r, \exists x\phi] = \emptyset$. If now $(\Gamma \cup \Gamma_r, \Delta_r \cup \{\exists x\phi\})$ is CD-trivial, then by (the CD-version of) Lemma 10.2, the bi-set $(\Gamma \cup \Gamma_r \cup \{\exists x\phi\}, \Delta_r)$ must be CD-non-trivial. On the other hand, since $\mu_\Gamma[\Gamma_r, \Delta_r, \exists x\phi] = \emptyset$, we must have, wlog,

$$(\forall m \in \omega)(\Gamma \cup \Gamma_r \cup \{\exists x\phi, \phi[a_m/x]\} \vdash_{CD} \bigvee \Delta_r) \tag{1}$$

We now reason as follows:

$$(\forall m \in \omega)(\Gamma \vdash_{CD} (\exists x\phi \wedge \phi[a_m/x] \wedge \bigwedge \Gamma_r) \to \bigvee \Delta_r) \text{ (by (1) and Cor. 5.1)} \quad (2)$$

$$(\forall m \in \omega)(\Gamma \vdash_{CD} (\phi[a_m/x] \wedge \bigwedge \Gamma_r) \to \bigvee \Delta_r) \text{ (by (2), (T4) and (DR1))} \quad (3)$$

$$(\forall m \in \omega)(\Gamma \vdash_{CD} \phi[a_m/x] \to (\bigwedge \Gamma_r \to \bigvee \Delta_r)) \text{ (by (3), (T2) and (DR1))} \quad (4)$$

Now, since $(\Gamma_r, \Delta_r) \Subset (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$, Lemma 1.2 implies that

$$(\forall m \in \omega)(\Gamma \vdash_{CD} (\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r))[a_m/x]) \quad (5)$$

Since we clearly have $(\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r))[a_m/x] \in L_\emptyset(\Sigma, \Pi)$ for every $m \in \omega$, the (CD-version of) Lemma 11.1 allows us to infer that:

$$(\forall m \in \omega)((\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r))[a_m/x] \in \Gamma) \quad (6)$$

By the CD-non-triviality of (Γ, Δ), it follows, further, that:

$$(\forall m \in \omega)((\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r))[a_m/x] \notin \Delta) \quad (7)$$

Finally, since $\{a_n \mid n \in \omega\}$ is an enumeration of Π, $\forall$-completeness of (Γ, Δ) implies that:

$$\forall x(\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r)) \notin \Delta \quad (8)$$

Applying the completeness of (Γ, Δ), we get that:

$$\Gamma \vdash_{CD} \forall x(\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r)) \quad (9)$$

Now it remains to apply (T7) and (T2) to get, successively:

$$\Gamma \vdash_{CD} \exists x\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r) \quad (10)$$

and:

$$\Gamma \vdash_{CD} (\exists x\phi \wedge \bigwedge \Gamma_r) \to \bigvee \Delta_r \quad (11)$$

but the latter equation implies, by Corollary 5.1, that $\Gamma \cup \Gamma_r \cup \{\exists x\phi\} \vdash_{CD} \bigvee \Delta_r$ which is in contradiction with the CD-non-triviality of $(\Gamma \cup \Gamma_r \cup \{\exists x\phi\}, \Delta_r)$. The obtained contradiction shows that we must have $\mu_\Gamma[\Gamma_r, \Delta_r, \exists x\phi] \neq \emptyset$, whence the CD-non-triviality of $(\Gamma \cup \Gamma_{r+1}, \Delta_{r+1})$ easily follows.

Case 3. ψ_r has the form $\forall x\phi$. If $\mu_\Gamma[\Gamma_r, \Delta_r, \forall x\phi] \neq \emptyset$, then we are done. Otherwise, we must have $\mu_\Gamma[\Gamma_r, \Delta_r, \forall x\phi] = \emptyset$. If now $(\Gamma \cup \Gamma_r \cup \{\forall x\phi\}, \Delta_r)$ is CD-trivial, then by (the CD-version of) Lemma 10.2, the bi-set $(\Gamma \cup \Gamma_r, \Delta_r \cup \{\forall x\phi\})$ must be CD-non-trivial. On the other hand, since $\mu_\Gamma[\Gamma_r, \Delta_r, \forall x\phi] = \emptyset$, we must have, wlog,

$$(\forall m \in \omega)(\Gamma \cup \Gamma_r \vdash_{CD} \forall x\phi \vee \phi[a_m/x] \vee \bigvee \Delta_r) \quad (12)$$

$$(\forall m \in \omega)(\Gamma \cup \Gamma_r \vdash_{CD} \phi[a_m/x] \vee \bigvee \Delta_r) \quad \text{(by (12), (T3), and (A15))} \quad (13)$$

$$(\forall m \in \omega)(\Gamma \vdash_{CD} \bigwedge \Gamma_r \to \phi[a_m/x] \vee \bigvee \Delta_r) \quad \text{(by (13) and Cor. 5.1)} \quad (14)$$

Now, since $(\Gamma_r, \Delta_r) \Subset (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$, Lemma 1.2 implies that

$$(\forall m \in \omega)(\Gamma \vdash_{CD} (\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r)[a_m/x]) \quad (15)$$

Since we clearly have $(\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r)[a_m/x] \in L_\emptyset(\Sigma, \Pi)$ for every $m \in \omega$, the (CD-version of) Lemma 11.1 allows us to infer that:

$$(\forall m \in \omega)((\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r)[a_m/x] \in \Gamma) \quad (16)$$

By the CD-non-triviality of (Γ, Δ), it follows, further, that:

$$(\forall m \in \omega)((\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r)[a_m/x] \notin \Delta) \quad (17)$$

Finally, since $\{a_n \mid n \in \omega\}$ is an enumeration of Π, $\forall$-completeness of (Γ, Δ) implies that:

$$\forall x(\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r) \notin \Delta \quad (18)$$

Applying again the completeness of (Γ, Δ), we get that:

$$\Gamma \vdash_{CD} \forall x(\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r) \quad (19)$$

Now it remains to apply (T8), (DR1), and (A17) to get, successively:

$$\Gamma \vdash_{CD} \bigwedge \Gamma_r \to \forall x(\phi \vee \bigvee \Delta_r) \quad (20)$$

and:

$$\Gamma \vdash_{CD} \bigwedge \Gamma_r \to \forall x\phi \vee \bigvee \Delta_r \quad (21)$$

but the latter equation implies, by Corollary 5.1, that $\Gamma \cup \Gamma_r \vdash_{CD} \forall x\phi \vee \bigvee \Delta_r$ which is in contradiction with the CD-non-triviality of $(\Gamma \cup \Gamma_r, \Delta_r \cup \{\forall x\phi\})$. The obtained contradiction shows that we must have $\mu_\Gamma[\Gamma_r, \Delta_r, \forall x\phi] \neq \emptyset$, whence the CD-non-triviality of $(\Gamma \cup \Gamma_{r+1}, \Delta_{r+1})$ easily follows.

Having defined our chain of bi-sets, we set:

$$(\Gamma', \Delta') = (\Gamma \cup \bigcup_{n\in\omega} \Gamma_n, \bigcup_{n\in\omega} \Delta_n),$$

and we show that this latter bi-set satisfies the conditions of the Lemma arguing as in the proofs of Lemmas 12 and 15. For example, to show that (Γ', Δ') is $\forall$-complete, assume that $\forall x\phi \in \Delta' \subseteq L_\emptyset(\Sigma, \Pi)$. Then, for some $k \in \omega$, we must have $\forall x\phi = \psi_k$. Consider $(\Gamma_{k+1}, \Delta_{k+1})$. If $\mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] = \emptyset$, then we must have $\forall x\phi \in \Gamma_{k+1} \subseteq \Gamma'$, which would contradict the non-triviality of (Γ', Δ'). Therefore, we must have $\mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] \neq \emptyset$, but then also $\phi[a_m/x] \in \Delta_{k+1} \subseteq \Delta'$ for $m = \min \mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi]$. □

Our canonical model construction for QC_{CD} now looks as follows. Given a signature Σ, a parameter set Π, and a $(\Sigma, \Pi)_{CD}$-appropriate bi-set (Γ, Δ), we can define the following constant domain Σ-model $\mathcal{M}_{(\Gamma,\Delta)}$ by setting:

- $W_{(\Gamma,\Delta)} := \{(\Gamma', \Delta') \mid (\Gamma', \Delta') \text{ is } (\Sigma, \Pi)_{CD}\text{-appropriate}\}$.
- For $(\Gamma_0, \Delta_0), (\Gamma_1, \Delta_1) \in W_{(\Gamma,\Delta)}$, we have $(\Gamma_0, \Delta_0) \leq_{(\Gamma,\Delta)} (\Gamma_1, \Delta_1)$ iff $\Gamma_0 \subseteq \Gamma_1$.
- $U_{(\Gamma,\Delta)} := \Pi = D_{(\Gamma,\Delta)}(\Gamma_0, \Delta_0) = Par(\Gamma_0 \cup \Delta_0)$ for every $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta)}$.
- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta)}$, $n \in \omega$, $P^n \in \Sigma = Sign(\Gamma_0 \cup \Delta_0)$, and $\bar{a}_n \in \Pi^n = Par(\Gamma_0 \cup \Delta_0)^n$ we have $\bar{a}_n \in V^+_{(\Gamma,\Delta)}(P, (\Gamma_0, \Delta_0))$ iff $P(\bar{a}_n) \in \Gamma_0$.
- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta)}$, $n \in \omega$, $P^n \in \Sigma = Sign(\Gamma_0 \cup \Delta_0)$, and $\bar{a}_n \in \Pi^n = Par(\Gamma_0 \cup \Delta_0)^n$ we have $\bar{a}_n \in V^-_{(\Gamma,\Delta)}(P, (\Gamma_0, \Delta_0))$ iff $\sim P(\bar{a}_n) \in \Gamma_0$.

It is straightforward to show that $\mathcal{M}_{(\Gamma,\Delta)}$ is indeed a constant domain model of QC, and using the usual methods, a truth lemma can be shown for this model:

Lemma 17. *Let (Γ, Δ) be a $(\Sigma, \Pi)_{CD}$-appropriate bi-set. Then, for every $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta)}$ and every $\phi \in L_\emptyset(\Sigma, \Pi)$ it is true that:*

1. *$\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \models^+ \phi$ iff $\phi \in \Gamma_0$.*
2. *$\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \models^- \phi$ iff $\sim \phi \in \Gamma_0$.*

Proof. We prove both parts of the Lemma simultaneously by induction on the construction of $\phi \in L_\emptyset(\Sigma, \Pi)$. The proof for the induction basis and for the induction steps associated with $\wedge$, $\vee$, and $\sim$ are exactly as in the proof of Lemma 13. We consider the remaining cases:

Step. The following cases are possible.

Case 4. $\phi = \psi \to \chi$ for some $\psi, \chi \in L_\emptyset(\Sigma, \Pi)$. We consider Part 1 first:

$(\Rightarrow)$. If $\psi \to \chi \notin \Gamma_0$, then, by the completeness of (Γ_0, Δ_0), we get that $\psi \to \chi \in \Delta_0$. But then, by (the CD-version of) Lemma 10.4, we get that $(\Gamma_0 \cup \{\psi\}, \{\chi\})$ must be CD-non-trivial. Next, by Lemma 16, there must be a $(\Gamma_1, \Delta_1) \in W_{(\Gamma,\Delta)}$ such that $(\Gamma_1, \Delta_1) \supseteq (\Gamma_0 \cup \{\psi\}, \{\chi\})$. Now, $\psi \in \Gamma_1$ implies, by the Induction Hypothesis, that $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_1, \Delta_1) \models^+ \psi$. On the other hand, we have $\chi \in \Delta_1$, hence also $\chi \notin \Gamma_1$ by the CD-non-triviality of (Γ_1, Δ_1), whence, further, $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_1, \Delta_1) \not\models^+ \chi$ by the Induction Hypothesis. But then $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \not\models^+ \psi \to \chi$.

The proofs for the $(\Leftarrow)$-part and for Part 2 are as in Lemma 13.

Case 5. $\phi = \forall x\psi$ for some $\psi \in L_x(\Sigma, \Pi)$. We consider Part 1 first:

$(\Rightarrow)$. If $\forall x\psi \notin \Gamma_0$, then, by the completeness of (Γ_0, Δ_0), we get that $\forall x\psi \in \Delta_0$. Therefore, by $\forall$-completeness of (Γ_0, Δ_0), we must have $\psi[a/x] \in \Delta$ for some $a \in \Pi$. By the CD-non-triviality of (Γ_0, Δ_0), it follows that $\psi[a/x] \notin \Gamma_0$, whence further $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \not\models^+ \psi[a/x]$ by the Induction Hypothesis. But then $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \not\models^+ \forall x\psi$.

Again, the proofs for the $(\Leftarrow)$-part and for Part 2 are as in Lemma 13, and the case of the existential quantifier is parallel to Case 5. ☐

We now formulate and prove the converse of Theorem 3 for the countable case:

Theorem 4. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \models_{\mathbb{CD}} \phi$ then $\Gamma \vdash_{CD} \phi$.*

Proof. Again, we argue by contraposition. If $\Gamma \nvdash_{CD} \phi$, then the bi-set $(\Gamma, \{\phi\})$ must be CD-non-trivial. But then, choose an infinitely countable parameter set Ξ disjoint from Π. We can find, by Lemma 15, a $(\Sigma, \Pi \cup \Xi)_{CD}$-appropriate bi-set $(\Gamma', \Delta') \supseteq (\Gamma, \{\phi\})$. We clearly have $\phi \in \Delta'$, so also $\phi \notin \Gamma'$ by the CD-non-triviality of (Γ', Δ'). Now Lemma 17 implies that we have both $\mathcal{M}_{(\Gamma',\Delta')}, (\Gamma', \Delta') \models^+ \Gamma' \supseteq \Gamma$ and $\mathcal{M}_{(\Gamma',\Delta')}, (\Gamma', \Delta') \not\models^+ \phi$. Therefore, $\Gamma \not\models_{\mathbb{CD}} \phi$ as desired. ☐

It is now easy to see that one can obtain a complete axiomatization for $\mathsf{QC3}_{CD}$ by extending the axiomatization for QC_{CD} with the following additional axiom schema:

$$\phi \vee \sim \phi \tag{A18}$$

Re-using, with a slight modification, the previous definitions of this sort, one can define the deducibility relation $\vdash_{C3CD}$ and prove the following theorem:

Theorem 5. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \vdash_{C3CD} \phi$, then $\Gamma \models_{\mathbb{C3} \cap \mathbb{CD}} \phi$.*

Moreover, by repeating the series of constructions leading to Theorem 4 above, it is straigtforward to check that the presence of (A18) in our axiomatization guarantees that the respective canonical model is in $\mathbb{C}3$. Proceeding in this way, one also arrives at the corresponding completeness theorem for the countable case:

Theorem 6. *Let* Σ *be an at most countable signature, let* $\Pi \subseteq Par$ *be an at most countable set, and let* $\Gamma \cup \{\phi\} \subseteq L_{\emptyset}(\Sigma, \Pi)$*. If* $\Gamma \models_{\mathbb{C}3\cap\mathbb{CD}} \phi$ *then* $\Gamma \vdash_{C3CD} \phi$*.*

5 Comparison of $\mathsf{QC3}_{CD}$ with the systems $\mathsf{QC3}$ and $\mathsf{QC3}_{At}$

The systems $\mathsf{QC3}$ and $\mathsf{QC3}_{At}$ were introduced in [10], purely proof-theoretically, as the first-order extensions of $\mathsf{C3}$. Each of these two systems was given in two forms: first, in the form of a Hilbert-style calculus and then in the form of its (unlabelled) sequent counterpart. The two forms were shown in [10] to be equivalent in the sense that the derivability relations from a finite set of premises obtained in each of the two types of proof systems were shown to coincide for both $\mathsf{QC3}$ and $\mathsf{QC3}_{At}$.

Since in the present paper we are focusing on the Hilbert-style axiomatizations of various first-order extensions of C, we will omit the discussion of sequent calculi introduced in [10]. As for the Hilbert-style calculi for $\mathsf{QC3}$ and $\mathsf{QC3}_{At}$, they are obtained by extending the axiomatization of QC by (A18) in the case of $\mathsf{QC3}$ and by the following restriction of (A18) in the case of $\mathsf{QC3}_{At}$:

$$\phi \vee \sim \phi \qquad \text{for } \phi \text{ atomic} \qquad (\text{A18}_{At})$$

It is clear that $\mathsf{QC3}_{At}$ can be shown to axiomatize the logic of $\mathsf{C3}$-models by a trivial modification of the completeness proof given for QC in Section 3 of the present paper. Denoting by $\vdash_{C3}$ the deducibility relation induced by $\mathsf{QC3}_{At}$, we get the following

Theorem 7. *Let* Σ *be an at most countable signature, let* $\Pi \subseteq Par$ *be an at most countable set, and let* $\Gamma \cup \{\phi\} \subseteq L_{\emptyset}(\Sigma, \Pi)$*. Then* $\Gamma \models_{\mathbb{C}3} \phi$ *iff* $\Gamma \vdash_{C3} \phi$*.*

Proof (a sketch). We repeat the proofs of Theorem 1 and 2, noting that the canonical Σ-model $\mathcal{M}_{(\Gamma,\Delta,\Xi)}$ constructed for a given (Σ, Π)-appropriate bi-set (Γ, Δ), and a given countably infinite $\Xi \subseteq Par$ disjoint from Π must be in $\mathbb{C}3$, due to the presence of (A18_{At}) in our system and by Lemma 11.4. □

Incidentally, the observation made in the proof of Theorem 7 also implies that one can equivalently axiomatize $\mathsf{QC3}_{CD}$ by replacing (A18) with (A18_{At}). Of course, one could also directly infer the remaining cases of (A18) in the axiomatization of $\mathsf{QC3}_{CD}$ based on (A18_{At}) arguing by induction on the construction of a parametrized sentence, but the semantic argument provides us with a shortcut to this result as well.

The question of the right semantics for $\mathsf{QC3}$ is more tricky. Example 2 shows that $\mathsf{QC3}$ is strictly stronger than the logic of $\mathsf{C3}$-models which we just recognized as $\mathsf{QC3}_{At}$. On the other hand, Theorem 6 above shows that $\mathsf{QC3}$ must be a subsystem of $\mathsf{QC3}_{CD}$. There remains the question whether this subsystem is proper.

First of all, it is clear that, seen from a semantical point of view, QC3 must be the logic of QC3-complete models, where, for any given signature Σ, a Σ-model is QC3-complete iff for every $w \in W$ and every $\phi \in L_\emptyset(\Sigma, D_w)$ it is true that $\mathcal{M}, w \models^+ \phi$ or $\mathcal{M}, w \models^- \phi$. So let us denote by $\mathbb{QC}3$ the class of QC3-complete models and by $\vdash_{QC3}$ the deducibility relation induced by QC3. The corresponding completeness proof is obtained from the completeness proof for QC by a trivial modification very similar to the one required in the case of $\mathsf{QC3}_{At}$. In this way, we get that:

Theorem 8. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. Then $\Gamma \models_{\mathbb{QC}3} \phi$ iff $\Gamma \vdash_{QC3} \phi$.*

Proof (a sketch). Similar to Theorem 7. $\square$

Now, Example 2 shows that $\mathbb{C}3 \not\subseteq \mathbb{QC}3$, and, on the other hand, Lemma 3 shows that $\mathbb{C}3 \cap \mathbb{CD} \subseteq \mathbb{QC}3$. The question is whether we also have $\mathbb{QC}3 \subseteq \mathbb{C}3 \cap \mathbb{CD}$. The following example can be used to show that this question must be answered in the negative:

Example 3. *Consider the signature $\Sigma = \{(p, 0), (Q, 1)\}$ and consider the following varying-domain Σ-model $\mathcal{M}$, where $W = \{1, 2\}$, $\leq$ is the natural order on W, $U = \{a, b\}$, $D(1) = \{a\}$, $D(2) = U$, $V^+(p, i) = 1$ iff $i = 2$, $V^-(p, i) = 1$ for all $i \in W$, and we have $V^+(Q, i) = \{a\}$ and $V^-(Q, i) = D(i)$ for all $i \in W$.*

The following lemma can then be shown to hold:

Lemma 18. *Let Σ and $\mathcal{M}$ be defined as in the Example 3. Then the following statements are true:*

1. *$\mathcal{M}$ is* QC3*-complete.*
2. *$\mathcal{M}, 1 \not\models^+ \forall x(p \vee Q(x)) \to (p \vee \forall x Q(x))$.*

Even though the model $\mathcal{M}$ of Example 3 is an obvious paraconsistent variant of a model often used to show that (A17) fails in intuitionistic logic, the proof of Lemma 18 requires a surprisingly careful and tiresome induction on the construction of the parametrized sentence. It is therefore relegated to Appendix C.

Lemma 18 shows that QC3 is a proper subsystem of $\mathsf{QC3}_{CD}$ (as long as the signature is not too small) since we must have $\mathcal{M}, 1 \models^+ \mathsf{QC3}$, yet $\mathcal{M}, 1 \not\models^+ \forall x(p \vee Q(x)) \to (p \vee \forall x Q(x)) \in \mathsf{QC3}_{CD}$.

It also shows that $\mathbb{C}3 \cap \mathbb{CD} \subsetneq \mathbb{QC}3$ (as long as the signature is not too small).

Finally, it shows that the frame correspondence theory in its usual form is not possible for QC3, since the class of QC3-complete models in the corresponding signature is neither good nor C3-good. Indeed, whereas $\mathcal{M}$ of Example 3 was shown to be $\mathsf{QC3}$-complete, this is not the case for the model $\mathcal{N} \in \mathbb{C}3$ which only differs from $\mathcal{M}$ in that $\mathcal{N}, 1 \not\models^- Q(a)$. That $\mathcal{N} \notin \mathbb{QC}3$ is evident from the fact that we have both $\mathcal{N}, 1 \not\models^+ \forall x Q(x)$ and $\mathcal{N}, 1 \not\models^- \forall x Q(x)$.

To sum up, we have shown that all the systems in the set $\{\mathsf{QC3}_{At}, \mathsf{QC3}, \mathsf{QC3}_{CD}\}$ are pairwise disjoint. Out of these three systems, $\mathsf{QC3}_{At}$ is complete relative to a $\mathsf{C3}$-good class of models, but suffers from the truth-value gap problem in that it fails to verify the general form of the law of excluded middle given by (A18); it is also inconvenient that the set of theorems of $\mathsf{QC3}_{At}$ is not closed for formula substitutions. The truth-value gap problem is avoided in $\mathsf{QC3}$, however, this system is complete relative to a class of models which, as is shown above, is not $\mathsf{C3}$-good and it is not clear how to supply $\mathsf{QC3}$ with a better semantics. Therefore, at least as long as a better candidate is not found and proposed, we are inclined to favor $\mathsf{QC3}_{CD}$ as the correct first-order version of the propositional logic $\mathsf{C3}$ since it is both complete relative to a $\mathsf{C3}$-good class of models and verifies the unrestricted version of the law of excluded middle which we take to be a distinctive mark of $\mathsf{C3}$-like systems.

We end this section with a brief discussion of constructive truth and constructible falsity properties in the first-order extensions of C, since this subject was also discussed in [10]. It is known that in the intuitionistic first-order logic, its characteristic constructive understanding of truth manifests itself in the following properties:

(DP) Disjunctive Property: for every signature Σ, if $\phi, \psi \in L_\emptyset(\Sigma)$ and $\phi \vee \psi$ is a theorem, then either ϕ or ψ is a theorem.

(EP) Existence Property: for every signature Σ, if $\exists x\phi \in L_\emptyset(\Sigma)$ and $\exists x\phi$ is a theorem then there exists an $a \in Par$ such that $\phi[a/x]$ is a theorem.

Whereas both (DP) and (EP) fail in classical logic, they are preserved in the first-order Nelson's logics, both $\mathsf{QN3}$ and $\mathsf{QN4}$; moreover, they are complemented in these logics by the following constructible falsity counterparts, showing that the treatment of falsehoods now also becomes constructive:

(DP_F) Negated Conjunction Property: for every signature Σ, if $\phi, \psi \in L_\emptyset(\Sigma)$ and $\sim (\phi \wedge \psi)$ is a theorem, then either $\sim \phi$ or $\sim \psi$ is a theorem.

(EP_F) Negated Universal Property: for every signature Σ, if $\sim \forall x\phi \in L_\emptyset(\Sigma)$ and $\sim \forall x\phi$ is a theorem then there exists an $a \in Par$ such that $\sim \phi[a/x]$ is a theorem.

The logic QC is known to have all the four properties in $\{DP, DP_F, EP, EP_F\}$.[6] It

[6] Apparently this was known at the time of writing [14], although, quite surprisingly, neither EP nor EP_F are mentioned there; on the other hand, the satisfaction of both DP and DP_F is established in [14, Proposition 2].

is easy to see that the same sort of arguments can be used to show that the said four properties of constructive truth and constructible falsity are still satisfied by QC_{CD}.

With the first-order extensions of C3 the situation is a little bit more tricky. Due to the presence of ($\mathrm{A18}_{At}$) in all such systems, it is easy to see right away that both DP and DP_F must fail. However, a proof-theoretic argument given for [10, Theorem 6.5] shows that, surprisingly, both EP and EP_F are still satisfied by $\mathsf{QC3}_{At}$.

It remains to see whether this rather peculiar (although not completely unknown: see [11]) phenomenon persists when we extend $\mathsf{QC3}_{At}$ to QC3 and then further to $\mathsf{QC3}_{CD}$. The answer is, again, in the negative:

Proposition 1. *Both* QC3 *and* $\mathsf{QC3}_{CD}$ *fail every property in* $\{EP, EP_F\}$

Proof. Indeed, consider signature $\Sigma = \{(P, 1)\}$. Then the pure Σ-sentence $\exists x(\forall x P(x) \vee \sim P(x))$ is provable in both QC3 and $\mathsf{QC3}_{CD}$, as the following derivation in QC3 shows (where $a \in Par$ is chosen arbitrarily):

$$\sim P(a) \to (\forall x P(x) \vee \sim P(a)) \quad \text{by (A7)} \quad (22)$$
$$(\forall x P(x) \vee \sim P(a)) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{by (A16)} \quad (23)$$
$$\sim P(a) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (22)–(23) by (DR1)} \quad (24)$$
$$\exists x \sim P(x) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (24) by (R}\exists\text{)} \quad (25)$$
$$\sim \forall x P(x) \to \exists x \sim P(x) \quad \text{by (A14)} \quad (26)$$
$$\sim \forall x P(x) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (25)–(26) by (DR1)} \quad (27)$$
$$\forall x P(x) \to (\forall x P(x) \vee \sim P(a)) \quad \text{by (A6)} \quad (28)$$
$$\forall x P(x) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (23),(28) by (DR1)} \quad (29)$$
$$(\forall x P(x) \vee \sim \forall x P(x)) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (27),(29) by (A8)} \quad (30)$$
$$\forall x P(x) \vee \sim \forall x P(x) \quad \text{by (A18)} \quad (31)$$
$$\exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (30),(31) by (MP)} \quad (32)$$

However, $\forall x P(x) \vee \sim P(a)$ is not a theorem of $\mathsf{QC3}_{CD}$ for any $a \in Par$ (and hence also not a theorem of its proper subsystem QC3) as the following constant domain C3-model shows.

Indeed, let $\Sigma = \{(P, 1)\}$ and let Σ-model $\mathcal{M}$ be such that with $W = \{1\}$, $\leq = \{(1, 1)\}$, $U = D(1) = \{a, b\}$, $V^+(P, 1) = \{a\}$ and $V^-(P, 1) = \{b\}$. It is easy to see that we have both $\mathcal{M}, 1 \not\models^+ \sim P(a)$ and $\mathcal{M}, 1 \not\models^+ \forall x P(x)$. The preceding argument disproves EP for both QC3 and $\mathsf{QC3}_{CD}$; as for EP_F, it is enough to notice that the formula $\sim \forall x(\sim \forall x P(x) \wedge P(x))$ is provable by applying (A14), (A11) and (A9) to $\exists x(\forall x P(x) \vee \sim P(x))$. On the other hand, $\sim (P(a) \wedge \sim \forall x P(x))$ is not a theorem for any $a \in Par$ as is witnessed by the model $\mathcal{M}$ defined above. □

6 The peculiar quantifier Æ

Both intuitionistic logic and some of the logics inspired by it, display a very close parallelism between the interpretation of the implication connective and the interpretation of the universal quantifier. For example, in one typical description of the intuitionistic meaning of logical symbols (clearly paraphrasing the so-called Brouwer-Heyting-Kolmogorov interpretation) we can read that:

> The second group is composed of $\forall$, $\to$, and $\neg$. A proof of $\forall x A(x)$ is a construction of which we can recognize that, when applied to any number n, it yields a proof of $A(\bar{n})$. Such a proof is therefore an operation that carries natural numbers into proofs. A proof of $A \to B$ is a construction of which we can recognize that, applied to any proof of A, it yields a proof of B. Such a proof is therefore an operation carrying proofs into proofs.
>
> (M. Dummett — [2, p. 8])

We see that $\to$ and $\forall$ are grouped together in that they both refer to a general construction producing proofs, the one out of (other) proofs, the other out of objects in the domain of discourse, which, in the example at hand, are natural numbers. The difference between the two constructions consists, first of all, in the input allowed by each of them. And this difference is not that big, since both natural numbers and proofs are, according to intuitionism, just two varieties of constructions, and one of this varieties can serve as a representative of the other one as the goedelization technique has taught us.

The other obvious difference is of course that the implicational construction returns a proof of one and the same sentence for every possible input, whereas the universal quantifier construction each time returns a proof of a different substitution instance based on the input. This difference is much more serious and we are not going to downplay it, although it does not cancel the objectively existing close parallelism between the two constructions.

This close parallelism is also reflected in the Kripke semantics for intuitionistic logic by the coincidence of the quantifier patterns in the corresponding clauses in the definition of the satisfaction relation. These clauses can be given, in view of the notational conventions accepted in this paper, as follows:

$$\mathcal{M}, w \models \phi \to \psi \Leftrightarrow (\forall v \geq w)(\mathcal{M}, v \models \phi \Rightarrow \mathcal{M}, v \models \psi)$$
$$\mathcal{M}, w \models \forall x \psi \Leftrightarrow (\forall v \geq w)(\forall a)(a \in D_v \Rightarrow \mathcal{M}, v \models \psi[a/x])$$

The introduction of Nelson's logic made it necessary to conceive of the falsification conditions for connectives and quantifiers as something possibly different from a mere negation of verification conditions. Thus, although the clauses above were still accepted for the definition of the verification relation $\models^+$, the conditions for falsifying the implications and universally quantified sentences had to be given independently. But also in this extension of intuitionistic logic the parallelism between the implication and the universal quantifier remained untouched, as is evident from the formulation of these conditions used in both $\mathsf{QN3}$ and $\mathsf{QN4}$ (again, adapted to our notational conventions):

$$\mathcal{M}, w \models^- \phi \to \psi \Leftrightarrow \mathcal{M}, w \models^+ \phi \text{ and } \mathcal{M}, w \models^- \psi$$
$$\mathcal{M}, w \models^- \forall x\psi \Leftrightarrow (\exists a)(a \in D_w \text{ and } \mathcal{M}, w \models^- \psi[a/x])$$

One could rephrase the idea behind these stipulations along the lines of the BHK approach to Nelson's logic by saying that a falsification of a conditional sentence consists in the fact that a proof of an antecedent has been constructed, along with a refutation of a consequent. Similarly, a falsification of a quantified sentence means that an object has been constructed, along with a refutation of a substitution instance of the quantified formula induced by this object.

Now, in QC as well as in the other Nelsonian extensions of C considered in this paper, this parallelism of the falsification conditions between $\to$ and $\forall$ appears to be lost in that we have:

$$\mathcal{M}, w \models^- \phi \to \psi \Leftrightarrow (\forall v \geq w)(\mathcal{M}, v \models^+ \phi \Rightarrow \mathcal{M}, v \models^- \psi),$$

whereas the falsification clause for the universal quantifier remains Nelsonian. Through the BHK lens, the matter looks as if we are now saying that a proper refutation of a conditional sentence must be a general construction, which, given a proof of the antecedent, spits out a refutation of the consequent (and does that recognizably, as M. Dummett would probably insist). However, were we to think of the possible refutations of the universally quantified sentences along the same lines, we would probably have to say that a proper refutation of a universally quantified sentence must be a general construction, which, given a construction of a possible object in our domain, recognizably returns a refutation of the substitution instance of the quantified formula induced by this object. It is natural to think that a formal explication of this idea may have looked as something like this:

$$\mathcal{M}, w \models^- \forall x\phi \Leftrightarrow (\forall v \geq w)(\forall a)(a \in D_v \Rightarrow \mathcal{M}, v \models^- \phi[a/x]).$$

However, it is now evident that, in doing so, we are just ascribing to the universal quantifier the falsification condition borrowed from the Nelsonian existential quantifier (also used in the semantics of QC).

We leave it to the reader to judge whether the idea of keeping the interpretations of $\forall$ and $\to$ bound together also in the first-order extensions of C has any intuitive appeal.[7] In the present paper, we confine ourselves to pointing out some of the formal consequences of realizing this idea by having a quantifier with the verification clause borrowed from the Nelsonian $\forall$ and the verification clause borrowed from the Nelsonian $\exists$. We will denote this quantifier by Æ and will assign it the following semantics:

$$\mathcal{M}, w \models^{+} \text{Æ}x\phi \Leftrightarrow (\forall v \geq w)(\forall a \in D_v)(\mathcal{M}, v \models \phi[a/x])$$
$$\mathcal{M}, w \models^{-} \text{Æ}x\phi \Leftrightarrow (\forall v \geq w)(\forall a \in D_v)(\mathcal{M}, v \models^{-} \phi[a/x]).$$

One very interesting property of Æ is that it commutes with the strong negation, that is to say, the following principle becomes valid:

$$\sim \text{Æ}x\phi \leftrightarrow \text{Æ}x \sim \phi \tag{A19}$$

One may also express this property of Æ by saying that this quantifier is "self-dual". It is also clear that if we simply want to extend with Æ the language of any system in the set $\{\mathsf{QC}, \mathsf{QC}_{CD}\}$, then we can obtain a sound and complete (in the countable case) axiomatization for such an extension by simply adding the following two schemas to the list of its axioms:

$$\text{Æ}x\phi \leftrightarrow \forall x\phi \tag{A20}$$
$$\sim \text{Æ}x\phi \leftrightarrow \sim \exists x\phi \tag{A21}$$

The situation is somewhat different if we wish to have Æ as the only quantifier in our language. In this case, given an axiomatization for any system in $\{\mathsf{QC}, \mathsf{QC}_{CD}\}$, one has to omit the axioms (A13)–(A17) together with the rules (R$\forall$) and (R$\exists$), and replace them with (A19) and the following Æ-analogues of (A15), (A17) and (R$\forall$), respectively:

$$\text{Æ}x\theta \to \theta[c/x] \tag{A15$_{\text{Æ}}$}$$
$$\text{Æ}x(\phi \vee \psi) \to (\phi \vee \text{Æ}x\psi) \tag{A17$_{\text{Æ}}$}$$
$$\text{From } \phi \to \theta[c/x] \text{ infer } \phi \to \text{Æ}x\theta \tag{R$\forall_{\text{Æ}}$}$$

In this way, we get two additional systems C(Æ) and C_{CD}(Æ).

The soundness and completeness proofs for these new systems are simpler versions of the proofs given in the earlier sections of this paper for their Nelsonian

[7]The reader may usefully compare our discourse with the attempt at "connexivization" of other propositional connectives besides $\to$ and $\sim$ in [4].

analogues. For example, in the completeness proof of $\mathsf{C}(\text{Æ})$, we no longer need to require that appropriate (and nice) bi-sets are $\exists$-complete, moreover, we no longer need several auxiliary statements like Lemma 10.8 and the second case in the main construction given in the proof of Lemma 12 is no longer relevant. These simplifications also apply to $\mathsf{C}_{CD}(\text{Æ})$. However, in the case of $\mathsf{C}_{CD}(\text{Æ})$, we will still need both the $\forall$-completeness (more precisely, its Æ-analogue) and Case 3 in the main construction given in the proofs of the statements like Lemma 15 and 16. The rest of the argument is basically the same as for the corresponding Nelsonian systems.

The introduction of Æ into first-order extensions of C3, however, can only be easily done in the case of $\mathsf{QC3}_{At}$, where Æ can function both as an addition to the set of Nelsonian quantifiers and as the only quantifier in the same fashion as for QC. In the case of QC3 one needs to further amend its already non-standard semantics and speak of the (QC3+Æ)-complete models and (C3+Æ)-complete models depending on whether we add Æ together with the set of Nelsonian quantifiers or alone. In this case (QC3+Æ)-complete (resp. (C3+Æ)-complete) models are the models that never display truth-value gaps for the parametrized sentences in the language based on $\{\wedge, \vee, \sim, \rightarrow, \forall, \exists, \text{Æ}\}$ (resp. $\{\wedge, \vee, \sim, \rightarrow, \text{Æ}\}$) as the set of logical symbols.

We have seen in Section 5, that the class of QC3-complete models is not closed for the models based on the same underlying frame; the same clearly holds for the classes of (C3+Æ)-complete models and (QC3+Æ)-complete models. Indeed, the model $\mathcal{M}$ constructed in the proof of Proposition 1 is neither (C3+Æ)-complete nor (QC3+Æ)-complete since we have both $\mathcal{M}, 1 \not\models^+ \text{Æ}xP(x)$ and $\mathcal{M}, 1 \not\models^- \text{Æ}xP(x)$. However, the model $\mathcal{M}'$ which is only different from $\mathcal{M}$ in that we have $\mathcal{M}', 1 \models^+ P(b)$ is easily shown to be both (C3+Æ)-complete and (QC3+Æ)-complete. The following lemma provides the main stepping stone to the latter claim:

Lemma 19. *Denote by $L^{\text{Æ}}$ the language based on $\{\wedge, \vee, \sim, \rightarrow, \forall, \exists, \text{Æ}\}$ as the set of logical symbols. Let $x \in Var$, let $\Sigma = \{(P, 1)\}$ and let Σ-model $\mathcal{M}'$ be such that with $W = \{1\}$, $\leq = \{(1, 1)\}$, $U = D(1) = \{a, b\}$, $V^+(P, 1) = \{a, b\}$ and $V^-(P, 1) = \emptyset$. Then, for all $\phi \in L^{\text{Æ}}_x(\Sigma, U)$ and for every $\circ \in \{+, -\}$ it is true that:*

$$\mathcal{M}', 1 \models^\circ \phi[a/x] \Leftrightarrow \mathcal{M}', 1 \models^\circ \phi[b/x].$$

Proof. By induction on the construction of $\phi[a/x]$. Both the basis and the induction step cases for the propositional connectives are straightforward (for the implication case, note that our model consists of a single state). We treat the quantifier cases.

Case 1. $\phi[a/x] = \forall y \psi[a/x]$. We may assume, wlog, that $y \neq x$, and we reason as follows:

(Part 1). We have $\mathcal{M}', 1 \models^+ \phi[a/x]$ iff $\mathcal{M}', 1 \models^+ \psi[a/x, a/y] \wedge \psi[a/x, b/y]$ iff, by Corollary 2.1, $\mathcal{M}', 1 \models^+ \psi[a/y, a/x] \wedge \psi[b/y, a/x]$, iff, by the Induction Hypothesis,

$\mathcal{M}', 1 \models^+ \psi[a/y, b/x] \wedge \psi[b/y, b/x]$, iff, by Corollary 2.1, $\mathcal{M}', 1 \models^+ \psi[b/x, a/y] \wedge \psi[b/x, b/y]$ iff $\mathcal{M}', 1 \models^+ \phi[b/x]$.

(Part 2). We have $\mathcal{M}', 1 \models^- \phi[a/x]$ iff, for some $c \in \{a, b\}$, we have $\mathcal{M}', 1 \models^- \psi[a/x, c/y]$ iff, by Corollary 2.1, $\mathcal{M}', 1 \models^- \psi[c/y, a/x]$ for this c, iff, by the Induction Hypothesis, $\mathcal{M}', 1 \models^- \psi[c/y, b/x]$ for the said c, iff, by Corollary 2.1, $\mathcal{M}', 1 \models^- \psi[b/x, c/y]$ iff $\mathcal{M}', 1 \models^- \phi[b/x]$.

Case 2. $\phi[a/x] = \exists y\psi[a/x]$. Similar to Case 1.

Case 3. $\phi[a/x] = \text{Æ}y\psi[a/x]$. We argue similarly to Part 1 of Case 1, since we know that for any $\circ \in \{+, -\}$ we have $\mathcal{M}', 1 \models^\circ \phi[a/x]$ iff $\mathcal{M}', 1 \models^\circ \psi[a/x, a/y] \wedge \psi[a/x, b/y]$ iff, by Corollary 2.1, $\mathcal{M}', 1 \models^\circ \psi[a/y, a/x] \wedge \psi[b/y, a/x]$, iff, by the Induction Hypothesis, $\mathcal{M}', 1 \models^\circ \psi[a/y, b/x] \wedge \psi[b/y, b/x]$, iff, by Corollary 2.1, $\mathcal{M}', 1 \models^\circ \psi[b/x, a/y] \wedge \psi[b/x, b/y]$ iff $\mathcal{M}', 1 \models^\circ \phi[b/x]$. ☐

The following Proposition then makes our claim about $\mathcal{M}'$ more precise:

Proposition 2. *Let Σ, $\mathcal{M}'$, and $L^{\text{Æ}}$ be defined as in Lemma 19. Then for every $\phi \in L^{\text{Æ}}_{\emptyset}(\Sigma, U)$ it is true that $\mathcal{M}', 1 \models^\circ \phi$ for some $\circ \in \{+, -\}$.*

Proof. Again, we argue by induction on the construction of $\phi \in L^{\text{Æ}}_{\emptyset}(\Sigma, U)$. The basis and most of the induction cases are as in the proof of Lemma 3 since $\mathcal{M}' \in \mathbb{CD} \cap \mathbb{C}3$. As for the only new induction case, assume that $\phi = \text{Æ}x\psi$. Then, by the Induction Hypothesis, we must have either $\mathcal{M}', 1 \models^+ \psi[a/x]$ or $\mathcal{M}', 1 \models^- \psi[a/x]$. In the former case, Lemma 19 implies that also $\mathcal{M}', 1 \models^+ \psi[b/x]$ and hence $\mathcal{M}', 1 \models^+ \phi$. In the latter case, Lemma 19 implies that also $\mathcal{M}', 1 \models^- \psi[b/x]$ and hence $\mathcal{M}', 1 \models^- \phi$. ☐

The fact that $\mathcal{M}, \mathcal{M}' \in \mathbb{CD}$ is particularly important in that it shows that, as long as Æ is present in the language, even the imposition of constant domains does not return us to a standard type of semantics and thus cannot be considered as any sort of remedy for the truth-value gap problem. In other words, not only do the classes of (QC3+Æ)-complete models and (C3+Æ)-complete models fail to be C3-good themselves, but their intersections with $\mathbb{CD}$ also fail to be C3-good.

Due to this phenomenon, also the addition of Æ to $\mathsf{QC3}_{CD}$ inevitably leads to a system with a non-standard semantics and with poor prospects for any traditional forms of frame correspondence theory.

7 Conclusion and future work

In the main part of our paper, we were focused on the completeness for the three systems QC, QC_{CD}, and $\mathsf{QC3}_{CD}$. These systems naturally arise as the result of extension of the propositional paraconsistent logics C and C3 with the Nelsonian quantifiers. We have succeeded in proving the general version of the soundness theorem for all these logics, as well as its converse in the countable case.

The Henkin technique used in these proofs proved to be easily adaptable to the treatment of the systems $\mathsf{QC3}_{At}$ and QC3, introduced in [10], even though it turned out that QC3 is a somewhat inconvenient extension of C3 since its class of intended models is not closed for the models based on the same underlying frame even if we restrict our attention to C3-models only. Finally, we have answered in the negative the question about the existence properties in QC3 and $\mathsf{QC3}_{CD}$.

Moreover, we have considered a relatively novel and peculiar quantifier Æ which combines the verification and falsification conditions of the two Nelsonian quantifiers. The intuitive motivation for the introduction of Æ in place of the Nelsonian version of $\forall$ is that such an introduction would be parallel to the amendment of the Nelsonian interpretation of the implication connective in C. We have sketched the application of the techniques developed in the main part of our paper to the systems where Æ is either added to the Nelsonian quantifiers or replaces them, and found that, in each case, a modicum of an amendment allows to obtain a Hilbert-style proof system which is sound and (in the countable case) complete for the logic at hand. We have also observed how the presence of this novel quantifier tends to exacerbate the problem of truth-value gap reinstatement in first-order extensions of C3 which appeared earlier in relation to Nelsonian quantifiers in $\mathsf{QC3}_{At}$.

However, the more general issue of the possibility of extending C with a (partially) non-Nelsonian set of quantifiers is by no means exhausted by the sketchy discourse contained in Section 6 of our paper. It is our hope that we will be able to return to this topic in our future research and to consider other well-motivated examples of non-Nelsonian quantifiers which show a certain degree of harmony with the basic motivating intuitions of C.

Turning one more time to the family of logics extending C with the Nelsonian set of quantifiers, we would like to add that one can easily see that the argument for the completeness of QC given in our paper can be straightforwardly extended to the signatures of arbitrary power by replacing every induction on ω with a suitable transfinite induction and by increasing the power of the sets of "fresh parameters" used in Lemmas 12 and 13 accordingly.

Unfortunately, such an easy extension is not possible in the case of QC_{CD}, and $\mathsf{QC3}_{CD}$, since the proof of the respective version of Lemma 16 for any of the two systems requires essentially that every bi-set in the increasing chain obtained in its main construction is finite. However, a standard workaround for this difficulty is also well-known and boils down to giving an independent proof of the compactness theorem for the system at hand. Again, in our future work, we hope to provide a satisfactorily complete version of such a proof and thus to close the issue of completeness for the axiomatizations of QC_{CD}, and $\mathsf{QC3}_{CD}$ presented in this paper.

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A Proof of Lemma 4

The proof proceeds by induction on the construction of ϕ for all $\circ \in \{+, -\}$ and all $w \in W$ simultaneously.

Basis. Let $\phi = P(\bar{a}_n)$ for some $P^n \in \Sigma$ and some $\bar{a}_n \in (D_w)^n$. Then we have:

$$\begin{aligned}\mathcal{M}, w \models^\circ P(\bar{a}_n) &\Leftrightarrow \bar{a}_n \in V^\circ(P, w)\\ &\Leftrightarrow f_{[b/a]}\langle \bar{a}_n \rangle \in (V_{[b/a]})^\circ(P, w)\\ &\Leftrightarrow \mathcal{M}_{[b/a]}, w \models^\circ P(f_{[b/a]}\langle \bar{a}_n \rangle)\\ &\Leftrightarrow \mathcal{M}_{[b/a]}, w \models^\circ P(\bar{a}_n)[b/a]\end{aligned}$$

Step. The cases for $\wedge$, $\vee$, and $\to$ are straightforward, given that the parameter substitutions in formulas commute with the connectives. We consider the quantifiers:

Case 1. We have $\circ = +$ and $\phi = \forall x \psi$ for some $\psi \in L_x(\Sigma, D_w)$. Then we have, for the ($\Rightarrow$)-part:

$$\begin{aligned}\mathcal{M}, w \models^+ \forall x \psi &\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/x])\\ &\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b/a]}, v \models^+ \psi[c/x][b/a]) \qquad \text{(by IH)}\end{aligned}$$

If now $v \geq w$ and $d \in D_{[b/a]}(v)$, then two cases are possible:

Case 1.1. $d \in D_v \setminus \{a\}$. Then we must have $\mathcal{M}_{[b/a]}, v \models^+ \psi[d/x][b/a] = \psi[b/a][d/x]$ by Lemma 1.4 and the fact that $d \neq a$ and $x \notin \{a, b\}$.

Case 1.2. $d = b$. Then $a \in D_v$ and we must have $\mathcal{M}_{[b/a]}, v \models^+ \psi[a/x][b/a] = \psi[b/a][b/x] = \psi[b/a][d/x]$ by Lemma 1.4.

Summing up, we get that $(\forall v \geq w)(\forall d \in D_{[b/a]}(v))(\mathcal{M}_{[b/a]}, v \models^+ \psi[b/a][d/x])$, so that $\mathcal{M}_{[b/a]}, w \models^+ \forall x(\psi[b/a])$ and hence also $\mathcal{M}_{[b/a]}, w \models^+ (\forall x \psi)[b/a]$.

For the ($\Leftarrow$)-part, we have:

$$\begin{aligned}\mathcal{M}_{[b/a]}, w \models^+ (\forall x \psi)[b/a] &\Leftrightarrow \mathcal{M}_{[b/a]}, w \models^+ \forall x(\psi[b/a])\\ &\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b/a]}(v))(\mathcal{M}_{[b/a]}, v \models^+ \psi[b/a][d/x])\\ &\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b/a]}(v))(\mathcal{M}_{[b/a]}, v \models^+ \psi[d/x][b/a]),\end{aligned}$$

where the latter equivalence holds by Lemma 1.4. and the fact that $d \neq a$ and $x \notin \{a, b\}$. But then the Induction Hypothesis implies that $(\forall v \geq w)(\forall d \in D_v \setminus \{a\})(\mathcal{M}, v \models^+ \psi[d/x])$. In case $a \notin D_v$, we also get that $\mathcal{M}, w \models^+ \forall x \psi$. Otherwise, we must have $a \in D_v$ and, therefore, $b \in D_{[b/a]}(v)$. Now, given any $v \geq w$, our chain of equivalences implies that $\mathcal{M}_{[b/a]}, v \models^+ \psi[b/x][b/a] = \psi[b/a][b/x] = \psi[a/x][b/a]$ by Lemma 1.4 and the fact that $a \neq b$ and $x \notin \{a, b\}$. Therefore, the Induction Hypothesis again implies that $\mathcal{M}, v \models^+ \psi[a/x]$, and we get that $\mathcal{M}, w \models^+ \forall x \psi$ also in this case.

Case 2. We have $\circ = -$ and $\phi = \forall x \psi$ for some $\psi \in L_x(\Sigma, D_w)$. Then we have, for the ($\Rightarrow$)-part:

$$\begin{aligned}\mathcal{M}, w \models^- \forall x \psi &\Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[c/x])\\ &\Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b/a]}, w \models^- \psi[c/x][b/a]) \qquad \text{(by IH)}\end{aligned}$$

We now choose the corresponding $c \in D_w$. If $c \in D_w \setminus \{a\}$, then also $c \in D_{[b/a]}(w)$, and we must have $\mathcal{M}_{[b/a]}, w \models^- \psi[c/x][b/a] = \psi[b/a][c/x]$ by Lemma 1.4. and the fact that $b \neq x$ and $a \notin \{c, x\}$, whence $\mathcal{M}_{[b/a]}, w \models^- \forall x\psi[b/a]$. Otherwise, we must have $c = a \in D_w$ so that also $b \in D_{[b/a]}(w)$. But then, $\mathcal{M}_{[b/a]}, w \models^- \psi[a/x][b/a] = \psi[b/a][b/x]$ by Lemma 1.4, and, again, $\mathcal{M}_{[b/a]}, w \models^- \forall x\psi[b/a]$ follows.

For the $(\Leftarrow)$-part, we have:

$$\begin{aligned} \mathcal{M}_{[b/a]}, w \models^- (\forall x\psi)[b/a] &\Leftrightarrow (\exists d \in D_{[b/a]}(w))(\mathcal{M}_{[b/a]}, w \models^- \psi[b/a][d/x]) \\ &\Leftrightarrow (\exists d \in D_{[b/a]}(w))(\mathcal{M}_{[b/a]}, w \models^- \psi[d/x][b/a]), \end{aligned}$$

where the latter equivalence holds by Lemma 1.4. and the fact that $d \neq a$ and $x \notin \{a, b\}$. Now, if $d \in D_w \setminus \{a\}$, then also $\mathcal{M}, w \models^- \psi[d/x]$ by the Induction Hypothesis, and thus $\mathcal{M}, w \models^- \forall x\psi$. Otherwise, we must have $d = b$, but then also $a \in D_w$, and we get that $\mathcal{M}_{[b/a]}, w \models^- \psi[d/x][b/a] = \psi[b/a][b/x] = \psi[a/x][b/a]$ by Lemma 1.4 and $a \neq x$. Therefore, $\mathcal{M}, w \models^- \psi[a/x]$ by the Induction Hypothesis, and, again, $\mathcal{M}, w \models^- \forall x\psi$.

The case of the existential quantifier is parallel to the case of the universal quantifier.

B Proof of Lemma 5

Again, the proof is by induction on the construction of $\phi[a/\bar{x}_n]$ for all $\circ \in \{+, -\}$, all $\bar{x}_n \in Var^{\neq n}$, and all $w \in W$ simultaneously.

Basis. Let $\phi[a/\bar{x}_n] = P(\bar{a}_m)$ for some $P^m \in \Sigma$ and some $\bar{a}_m \in (D_w)^m$. If now $\mathcal{M}, w \models^\circ P(\bar{a}_m)$, then $\bar{a}_m \in V^\circ(P, w)$. Let $\phi[b/\bar{x}_n] = P(\bar{b}_m)$. We want to show that $\bar{b}_m \in \rho_{[b:=a]}\langle\bar{a}_m\rangle \subseteq (V_{[b:=a]})^\circ(P, w)$. Indeed, fix an $1 \leq i \leq m$. If $a_i \neq a$, then a_i does not replace an occurrence of x_j for any $1 \leq j \leq n$, and, therefore, also $b_i = a_i \in \rho_{[b:=a]}[a_i]$. Otherwise $a_i = a$, and then, depending on whether a replaces an occurrence of x_j for some $1 \leq j \leq n$ or not, we will have $b_i = a$ or $b_i = b$, so, in any case, $b_i \in \rho_{[b:=a]}[a_i]$. But then $\bar{b}_m \in \rho_{[b:=a]}\langle\bar{a}_m\rangle \subseteq (V_{[b:=a]})^\circ(P, w)$, and we must have $\mathcal{M}_{[b:=a]}, w \models^\circ P(\bar{b}_m)$.

In the other direction, if $\phi[b/\bar{x}_n] = P(\bar{b}_m)$ and $\mathcal{M}_{[b:=a]}, w \models^\circ P(\bar{b}_m)$, then we must have $\bar{b}_m \in \rho_{[b:=a]}\langle\bar{c}_m\rangle$ for some $\bar{c}_m \in V^\circ(P, w)$. Moreover, it is easy to see that there exists a unique $\bar{c}_m \in (D_w)^m$ such that $\bar{b}_m \in \rho_{[b:=a]}\langle\bar{c}_m\rangle$ (since we have to replace all b's in $\bar{b}_m$ with a's in order for $\bar{c}_m$ to end up in $(D_w)^m$), so we must have $\bar{c}_m \in V^\circ(P, w)$ for this unique tuple. Let $\phi[a/\bar{x}_n] = P(\bar{a}_m)$. We will show that $\bar{a}_m = \bar{c}_m$. Indeed, fix an $1 \leq i \leq m$. If $b_i \notin \{a, b\}$, then $c_i \ \rho_{[b:=a]} \ b_i$ implies that $c_i = b_i$; on the other hand, if $b_i \notin \{a, b\}$, then b_i does not replace an occurrence of x_j in $\phi \in L_{\bar{x}_n}(\Sigma, D_w)$ for any $1 \leq j \leq n$, and we must have $a_i = b_i$. Therefore $a_i = c_i$. Next, if $b_i = b$, then $c_i = a$ since $\bar{c}_m \in (D_w)^m$ and $b \notin D_w$; on the other hand, if $b_i = b$ then b_i must replace an occurrence of x_j in $\phi \in L_{\bar{x}_n}(\Sigma, D_w)$ for some $1 \leq j \leq n$. This same occurrence will be replaced by a in $P(\bar{a}_m)$, hence $a_i = a$. Summing up, we get that $c_i = a = a_i$. Finally, if $b_i = a$, then, again $c_i = a$ and also the occurrence of b_i does not replace an occurrence of any x_j, so also $a_i = a$. Again we get that $c_i = a = a_i$.

In this way, we see that $\bar{a}_m = \bar{c}_m \in V^\circ(P, w)$ and thus $\mathcal{M}, w \models^\circ P(\bar{a}_m) = \phi[a/\bar{x}_n]$.

Step. The cases for $\wedge$, $\vee$, and $\rightarrow$ are straightforward, given that the parameter substitutions in formulas commute with the connectives. We consider the quantifiers:

Case 1. We have $\circ = +$ and $\phi = \forall y \psi$ for some $\psi \in L_{(\bar{x}_n)^\frown y}(\Sigma, D_w)$ and some $y \in Var \setminus \{\bar{x}_n\}$. Then we have, for the ($\Rightarrow$)-part:

$$
\begin{aligned}
\mathcal{M}, w \models^+ (\forall y \psi)[a/\bar{x}_n] &\Leftrightarrow \mathcal{M}, w \models^+ \forall y(\psi[a/\bar{x}_n]) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[a/\bar{x}_n][c/y]) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/y][a/\bar{x}_n]) && \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\}) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/y][b/\bar{x}_n]) && \text{(by IH)} \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y]) && \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\})
\end{aligned}
$$

We now fix any $v \geq w$. In case $a \notin D_v$, we have $D_v = D_{[b:=a]}(v)$ so we can already conclude that $(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y])$. Otherwise, we have $a \in D_v$ and $D_{[b:=a]}(v) = D_v \cup \{b\}$, so, in particular, we have that $\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][a/y]$. By Lemma 1.4 and $y \notin \{\bar{x}_n\}$, we conclude that $\mathcal{M}_{[b:=a]}, v \models^+ \psi[a/y][b/\bar{x}_n]$. Now, the Induction Hypothesis implies that $\mathcal{M}, v \models^+ \psi[a/y][a/\bar{x}_n] = \psi[a/\bar{x}_n][a/y]$, and, applying the Induction Hypothesis one more time, we get that $\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][b/y]$. Summing this up with the fact that

$(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y])$, we again arrive at the conclusion that

$(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y])$.

Since we thus get the latter conclusion for an arbitrary $v \geq w$, we infer that $(\forall v \geq w)(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y])$, whence it follows that $\mathcal{M}_{[b:=a]}, w \models^+ \forall y(\psi[b/\bar{x}_n]) = (\forall y \psi)[b/\bar{x}_n]$.

Turning now to the ($\Leftarrow$)-part, we reason as follows:

$$\begin{aligned}
\mathcal{M}_{[b:=a]}, w \models^+ (\forall y\psi)[b/\bar{x}_n] &\Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^+ \forall y(\psi[b/\bar{x}_n]) \\
&\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][d/y]) \\
&\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[d/y][b/\bar{x}_n]) \\
&\qquad\qquad\qquad\qquad (\text{by Lemma 1.4 and } y \notin \{\bar{x}_n\}) \\
&\Rightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/y][b/\bar{x}_n]) \\
&\qquad\qquad\qquad\qquad (\text{by } D_v \subseteq D_{[b:=a]}(v)) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/y][a/\bar{x}_n]) \qquad (\text{by IH}) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[a/\bar{x}_n][c/y]) \\
&\qquad\qquad\qquad\qquad (\text{by Lemma 1.4 and } y \notin \{\bar{x}_n\}) \\
&\Leftrightarrow \mathcal{M}, w \models^+ \forall y(\psi[a/\bar{x}_n]) = (\forall y\psi)[a/\bar{x}_n]
\end{aligned}$$

Case 2. We have $\circ = +$ and $\phi = \forall y\psi$ for some $\psi \in L_{\bar{x}_n}(\Sigma, D_w)$, where $y = x_i$ for some $1 \leq i \leq n$. Then we must have $n \geq 1$. If now $n > 1$, then we have $\forall y\psi[a/\bar{x}_n] = \forall y\psi[a/(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)]$, and similarly for $\forall y\psi[b/\bar{x}_n]$, so we can reason as in Case 1 replacing $\bar{x}_n$ everywhere with $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. In case $n = 1$, we must have $y = x_1$. Then we have, for the ($\Rightarrow$)-part:

$$\begin{aligned}
\mathcal{M}, w \models^+ (\forall x_1\psi)[a/x_1] &\Leftrightarrow \mathcal{M}, w \models^+ \forall x_1\psi \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/x_1]) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/x_1][a/x_1]) \qquad (\text{by Lemma 1.4}) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1][b/x_1]) \qquad (\text{by IH}) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1]) \qquad (\text{by Lemma 1.4})
\end{aligned}$$

We now fix any $v \geq w$. In case $a \notin D_v$, we have $D_v = D_{[b:=a]}(v)$ so we can already conclude that $(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1])$. Otherwise, we have $a \in D_v$ and $D_{[b:=a]}(v) = D_v \cup \{b\}$, so, in particular, we have that $\mathcal{M}, v \models^+ \psi[a/x_1]$. Now, the Induction Hypothesis implies that $\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/x_1]$. Summing this up with the fact that $(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1])$, we again arrive at the conclusion that $(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1])$.

Since we thus get the latter conclusion for an arbitrary $v \geq w$, we infer that $(\forall v \geq w)(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1])$, whence it follows that $\mathcal{M}_{[b:=a]}, w \models^+ \forall x_1\psi = (\forall x_1\psi)[b/x_1]$.

Turning now to the ($\Leftarrow$)-part, we reason as follows:

$$\begin{aligned}
&\mathcal{M}_{[b:=a]}, w \models^+ (\forall x_1 \psi)[b/x_1] \Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^+ \forall x_1 \psi \\
&\quad \Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[d/x_1]) \\
&\quad \Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[d/x_1][b/x_1]) && \text{(by Lemma 1.4)} \\
&\quad \Rightarrow (\forall v \geq w)(\forall d \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[d/x_1][b/x_1]) && \text{(by } D_v \subseteq D_{[b:=a]}(v)) \\
&\quad \Leftrightarrow (\forall v \geq w)(\forall d \in D_v)(\mathcal{M}, v \models^+ \psi[d/x_1][a/x_1]) && \text{(by IH)} \\
&\quad \Leftrightarrow (\forall v \geq w)(\forall d \in D_v)(\mathcal{M}, v \models^+ \psi[d/x_1]) && \text{(by Lemma 1.4)} \\
&\quad \Leftrightarrow \mathcal{M}, w \models^+ \forall x_1 \psi = (\forall x_1 \psi)[a/x_1]
\end{aligned}$$

Case 3. We have $\circ = -$ and $\phi = \forall y \psi$ for some $\psi \in L_{(\bar{x}_n)^\frown y}(\Sigma, D_w)$ and some $y \in Var \setminus \{\bar{x}_n\}$. Then we have, for the ($\Rightarrow$)-part:

$$\begin{aligned}
&\mathcal{M}, w \models^- (\forall y \psi)[a/\bar{x}_n] \Leftrightarrow \mathcal{M}, w \models^- \forall y(\psi[a/\bar{x}_n]) \\
&\quad \Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[a/\bar{x}_n][c/y]) \\
&\quad \Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[c/y][a/\bar{x}_n]) && \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\}) \\
&\quad \Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b:=a]}, w \models^- \psi[c/y][b/\bar{x}_n]) && \text{(by IH)} \\
&\quad \Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b:=a]}, w \models^- \psi[b/\bar{x}_n][c/y]) && \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\}) \\
&\quad \Rightarrow (\exists c \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^- \psi[b/\bar{x}_n][c/y]) && \text{(by } D_w \subseteq D_{[b:=a]}(w)) \\
&\quad \Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^- \forall y(\psi[b/\bar{x}_n]) = (\forall y \psi)[b/\bar{x}_n]
\end{aligned}$$

Turning now to the ($\Leftarrow$)-part, we reason as follows:

$$\begin{aligned}
&\mathcal{M}_{[b:=a]}, w \models^- (\forall y \psi)[b/\bar{x}_n] \Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^- \forall y(\psi[b/\bar{x}_n]) \\
&\quad \Leftrightarrow (\exists d \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^- \psi[b/\bar{x}_n][d/y]) \\
&\quad \Leftrightarrow (\exists d \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^- \psi[d/y][b/\bar{x}_n]) \ \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\})
\end{aligned}$$

We now choose a corresponding $d \in D_{[b:=a]}(w)$. If $d \in D_w$, then, by IH, we get that $\mathcal{M}, w \models^- \psi[d/y][a/\bar{x}_n]$ whence $\mathcal{M}, w \models^- \psi[a/\bar{x}_n][d/y]$ by Lemma 1.4 and $y \notin \{\bar{x}_n\}$. Now $\mathcal{M}, w \models^- \forall y(\psi[a/\bar{x}_n]) = (\forall y \psi)[a/\bar{x}_n]$ follows immediately.

Otherwise we have $d = b$, and we get that $\mathcal{M}_{[b:=a]}, w \models^- \psi[b/y][b/\bar{x}_n] = \psi[b/(\bar{x}_n)^\frown y]$, whence, by IH, $\mathcal{M}, w \models^- \psi[a/(\bar{x}_n)^\frown y] = \psi[a/\bar{x}_n][a/y]$. Hence also $\mathcal{M}, w \models^- \forall y(\psi[a/\bar{x}_n]) = (\forall y \psi)[a/\bar{x}_n]$ follows.

Case 4. We have $\circ = -$ and $\phi = \forall y \psi$ for some $\psi \in L_{\bar{x}_n}(\Sigma, D_w)$, where $y = x_i$ for some $1 \leq i \leq n$. Then we must have $n \geq 1$. Again, the subcase $n > 1$ can be reduced to Case 3 above. In case $n = 1$, we must have $y = x_1$. Then we have, for the (⇒)-part:

$$\begin{aligned}
\mathcal{M}, w \models^- (\forall x_1\psi)[a/x_1] &\Leftrightarrow \mathcal{M}, w \models^- \forall x_1\psi \\
&\Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[c/x_1]) \\
&\Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[c/x_1][a/x_1]) && \text{(by Lemma 1.4)} \\
&\Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b:=a]}, w \models^- \psi[c/x_1][b/x_1]) && \text{(by IH)} \\
&\Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b:=a]}, w \models^- \psi[c/x_1]) && \text{(by Lemma 1.4)} \\
&\Rightarrow (\exists c \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^{\cdot} \psi[c/x_1]) && \text{(by } D_v \subseteq D_{[b:=a]}(v)) \\
&\Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^- \forall x_1\psi = (\forall x_1\psi)[b/x_1]
\end{aligned}$$

Turning now to the (⇐)-part, we reason as follows:

$$\begin{aligned}
\mathcal{M}_{[b:=a]}, w \models^- (\forall x_1\psi)[b/x_1] &\Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^- \forall x_1\psi \\
&\Leftrightarrow (\exists d \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^- \psi[d/x_1])
\end{aligned}$$

We now choose a corresponding $d \in D_{[b:=a]}(w)$. In the subcase $d \in D_w$ we are done by the Induction Hypothesis.

In the subcase $d = b$, we must have $a \in D_w$, and we get that $\mathcal{M}_{[b:=a]}, w \models^- \psi[b/x_1]$, whence, by the Induction Hypothesis, $\mathcal{M}, w \models^- \psi[a/x_1]$.

In this way, we get that $\mathcal{M}, w \models^- \forall x_1\psi = (\forall x_1\psi)[a/x_1]$ in both subcases.

The case of the existential quantifier is parallel to the case of the universal quantifier.

C Proof of Lemma 18

We assume that the signature Σ and the Σ-model $\mathcal{M}$ are defined as in Example 3. We prove a couple of auxiliary lemmas first:

Lemma 20. *Let $\phi \in L_\emptyset(\Sigma, U)$. Then, for some $\circ \in \{+, -\}$, we have $\mathcal{M}, 2 \models^\circ \phi$.*

Proof. 2 is the maximal state in $\mathcal{M}$. □

Lemma 21. *Let $x \in Var$, let $\phi \in L_x(\Sigma, U)$. Then the following statements hold:*

1. *If both $\mathcal{M}, 2 \models^+ \phi[b/x]$ and $\mathcal{M}, 2 \not\models^- \phi[b/x]$, then $\mathcal{M}, 2 \models^+ \phi[a/x]$.*

2. *If both $\mathcal{M}, 2 \models^- \phi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \phi[b/x]$, then $\mathcal{M}, 2 \models^- \phi[a/x]$.*

Proof. By induction on the construction of $\phi[b/x]$.

Basis. If $\phi[b/x]$ is atomic, then we must have $\phi[b/x] \in \{Q(a), Q(b), p\}$.

(Part 1). The situation when both $\mathcal{M}, 2 \models^{+} \phi[b/x]$ and $\mathcal{M}, 2 \not\models^{-} \phi[b/x]$ is therefore impossible, so our statement holds vacuously.

(Part 2). If both $\mathcal{M}, 2 \models^{-} \phi[b/x]$ and $\mathcal{M}, 2 \not\models^{+} \phi[b/x]$, then we must have $\phi[b/x] = Q(b)$. Two cases are possible:

Case 1. $\phi = Q(b)$. Then $\phi[a/x] = Q(b)$, and we have $\mathcal{M}, 2 \models^{-} \phi[a/x] = Q(b) = \phi[b/x]$ by our assumption.

Case 2. $\phi = Q(x)$. Then $\phi[a/x] = Q(a)$, and we have $\mathcal{M}, 2 \models^{-} \phi[a/x] = Q(a)$ by the definition of $\mathcal{M}$.

Step. The following cases are possible:

Case 1. $\phi[b/x] = \psi[b/x] \wedge \chi[b/x]$.

(Part 1). If both $\mathcal{M}, 2 \models^{+} \phi[b/x]$ and $\mathcal{M}, 2 \not\models^{-} \phi[b/x]$, then we must have, on the one hand, that both $\mathcal{M}, 2 \models^{+} \psi[b/x]$ and $\mathcal{M}, 2 \models^{+} \chi[b/x]$. On the other hand, we must have both $\mathcal{M}, 2 \not\models^{-} \psi[b/x]$ and $\mathcal{M}, 2 \not\models^{-} \chi[b/x]$. Therefore, by IHp1, we must have also that $\mathcal{M}, 2 \models^{+} \psi[a/x] \wedge \chi[a/x]$.

(Part 2). If both $\mathcal{M}, 2 \models^{-} \phi[b/x]$ and $\mathcal{M}, 2 \not\models^{+} \phi[b/x]$, then we must have, on the one hand, that either $\mathcal{M}, 2 \models^{-} \psi[b/x]$ or $\mathcal{M}, 2 \models^{-} \chi[b/x]$. On the other hand, we must have either $\mathcal{M}, 2 \not\models^{+} \psi[b/x]$ or $\mathcal{M}, 2 \not\models^{+} \chi[b/x]$.

Assume, wlog, that $\mathcal{M}, 2 \models^{-} \psi[b/x]$. If also $\mathcal{M}, 2 \not\models^{+} \psi[b/x]$, then, by IHp2, we must have $\mathcal{M}, 2 \models^{-} \psi[a/x]$, whence $\mathcal{M}, 2 \models^{-} \psi[a/x] \wedge \chi[a/x]$. Otherwise, we must have $\mathcal{M}, 2 \models^{+} \psi[b/x]$, but then we must have $\mathcal{M}, 2 \not\models^{+} \chi[b/x]$, and, by Lemma 20, that $\mathcal{M}, 2 \models^{-} \chi[b/x]$. But now IHp2 is again applicable and yields that $\mathcal{M}, 2 \models^{-} \chi[a/x]$ whence also $\mathcal{M}, 2 \models^{-} \psi[a/x] \wedge \chi[a/x]$.

Case 2. $\phi[b/x] = \psi[b/x] \vee \chi[b/x]$. Similar to Case 1.

Case 3. $\phi[b/x] = \sim \psi[b/x]$.

(Part 1). If both $\mathcal{M}, 2 \models^{+} \phi[b/x]$ and $\mathcal{M}, 2 \not\models^{-} \phi[b/x]$, then we must have both $\mathcal{M}, 2 \models^{-} \psi[b/x]$ and $\mathcal{M}, 2 \not\models^{+} \psi[b/x]$. But then, also $\mathcal{M}, 2 \models^{-} \psi[a/x]$ follows by IHp2 and, further, $\mathcal{M}, 2 \models^{+} \sim \psi[a/x]$.

(Part 2). Parallel to Part 1.

Case 4. $\phi[b/x] = \psi[b/x] \rightarrow \chi[b/x]$.

(Part 1). If both $\mathcal{M}, 2 \models^{+} \phi[b/x]$ and $\mathcal{M}, 2 \not\models^{-} \phi[b/x]$, then we must have $\mathcal{M}, 2 \not\models^{-} \chi[b/x]$ since 2 is the maximal node; whence Lemma 20 implies that also $\mathcal{M}, 2 \models^{+} \chi[b/x]$. Now, by IHp1, we also get that $\mathcal{M}, 2 \models^{+} \chi[a/x]$, whence, further, $\mathcal{M}, 2 \models^{+} \psi[a/x] \rightarrow \chi[a/x]$.

(Part 2). If both $\mathcal{M}, 2 \models^{-} \phi[b/x]$ and $\mathcal{M}, 2 \not\models^{+} \phi[b/x]$, then we must have $\mathcal{M}, 2 \not\models^{+} \chi[b/x]$ since 2 is the maximal node; whence Lemma 20 implies that also $\mathcal{M}, 2 \models^{-} \chi[b/x]$. Now, by IHp2, we also get that $\mathcal{M}, 2 \models^{-} \chi[a/x]$, whence, further, $\mathcal{M}, 2 \models^{-} \psi[a/x] \rightarrow \chi[a/x]$.

Case 5. $\phi[b/x] = \forall y\psi[b/x]$. We may assume, wlog, that $y \neq x$. Note that we have, by Corollary 1, that $\psi[b/x][c/y] = \psi[c/y][b/x]$ for every $c \in U$ under this condition.

(Part 1). If both $\mathcal{M}, 2 \models^+ \phi[b/x]$ and $\mathcal{M}, 2 \not\models^- \phi[b/x]$, then we must have, on the one hand, both $\mathcal{M}, 2 \models^+ \psi[a/y][b/x]$ and $\mathcal{M}, 2 \models^+ \psi[b/y][b/x]$. On the other hand, we must have both $\mathcal{M}, 2 \not\models^- \psi[a/y][b/x]$ and $\mathcal{M}, 2 \not\models^- \psi[b/y][b/x]$. But then IHp1 implies that both $\mathcal{M}, 2 \models^+ \psi[a/y][a/x] = \psi[a/x][a/y]$ and $\mathcal{M}, 2 \models^+ \psi[b/y][a/x] = \psi[a/x][b/y]$, whence, given that 2 is a maximal node, it follows that $\mathcal{M}, 2 \models^+ \forall y\psi[a/x]$.

(Part 2). If both $\mathcal{M}, 2 \models^- \phi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \phi[b/x]$, then we must have, on the one hand, either $\mathcal{M}, 2 \models^- \psi[a/y][b/x]$ or $\mathcal{M}, 2 \models^- \psi[b/y][b/x]$. On the other hand, we must have either $\mathcal{M}, 2 \not\models^+ \psi[a/y][b/x]$ or $\mathcal{M}, 2 \not\models^+ \psi[b/y][b/x]$.

Assume, wlog, that $\mathcal{M}, 2 \models^- \psi[a/y][b/x]$. If also $\mathcal{M}, 2 \not\models^+ \psi[a/y][b/x]$, then, by IHp2, we must have $\mathcal{M}, 2 \models^- \psi[a/y][a/x] = \psi[a/x][a/y]$, whence $\mathcal{M}, 2 \models^- \forall y\psi[a/x]$. Otherwise, we must have $\mathcal{M}, 2 \not\models^+ \psi[b/y][b/x]$, whence, by Lemma 20, it follows that $\mathcal{M}, 2 \models^- \psi[b/y][b/x]$. But then IHp2 is applicable and yields that $\mathcal{M}, 2 \models^- \psi[b/y][a/x] = \psi[a/x][b/y]$, whence again $\mathcal{M}, 2 \models^- \forall y\psi[a/x]$.

Case 6. $\phi[b/x] = \exists y\psi[b/x]$. Similar to Case 5. $\square$

Lemma 22. *Let $x \in Var$, let $\phi \in L_x(\Sigma, \{a\})$. Then the following statements hold:*

1. *If both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$, then $\mathcal{M}, 2 \models^+ \phi[b/x]$.*
2. *If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then $\mathcal{M}, 2 \models^- \phi[b/x]$.*
3. *For some $\circ \in \{+, -\}$, we have $\mathcal{M}, 1 \models^\circ \phi[a/x]$.*

Proof. By induction on the construction of $\phi[a/x]$.

Basis. If $\phi[a/x]$ is atomic, then we must have $\phi[a/x] \in \{Q(a), p\}$.

(Part 1). The situation when both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$ is therefore impossible, so our statement holds vacuously.

(Part 2). If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then we must have $\phi[a/x] = p$. Then $\phi[b/x] = p$ as well, and we have $\mathcal{M}, 2 \models^- \phi[b/x] = p$ by the definition of $\mathcal{M}$.

(Part 3). Trivial by the definition of $\mathcal{M}$.

Step. The following cases are possible:

Case 1. $\phi[a/x] = \psi[a/x] \wedge \chi[a/x]$.

(Part 1). If both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$, then we must have, on the one hand, that both $\mathcal{M}, 1 \models^+ \psi[a/x]$ and $\mathcal{M}, 1 \models^+ \chi[a/x]$. On the other hand, we must have both $\mathcal{M}, 1 \not\models^- \psi[a/x]$ and $\mathcal{M}, 1 \not\models^- \chi[a/x]$. Therefore, by IHp1, we must have also that $\mathcal{M}, 2 \models^+ \psi[b/x] \wedge \chi[b/x]$.

(Part 2). If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then we must have, on the one hand, that either $\mathcal{M}, 1 \models^- \psi[a/x]$ or $\mathcal{M}, 1 \models^- \chi[a/x]$. On the other hand, we must have either $\mathcal{M}, 1 \not\models^+ \psi[a/x]$ or $\mathcal{M}, 1 \not\models^+ \chi[a/x]$.

Assume, wlog, that $\mathcal{M}, 1 \models^- \psi[a/x]$. If also $\mathcal{M}, 1 \not\models^+ \psi[a/x]$, then, by IHp2, we must have $\mathcal{M}, 2 \models^- \psi[b/x]$, whence $\mathcal{M}, 2 \models^- \psi[b/x] \wedge \chi[b/x]$. Otherwise, we must have $\mathcal{M}, 1 \models^+ \psi[a/x]$, but then we must have $\mathcal{M}, 1 \not\models^+ \chi[a/x]$, and, by IHp3, that $\mathcal{M}, 1 \models^- \chi[a/x]$. But now IHp2 is again applicable and yields that $\mathcal{M}, 2 \models^- \chi[b/x]$ whence also $\mathcal{M}, 2 \models^- \psi[b/x] \wedge \chi[b/x]$.

(Part 3). Trivial (by application of the corresponding truth-table).

Case 2. $\phi[a/x] = \psi[a/x] \vee \chi[a/x]$. Similar to Case 1.

Case 3. $\phi[a/x] =\sim \psi[a/x]$.

(Part 1). If both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$, then we must have both $\mathcal{M}, 1 \models^- \psi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \psi[a/x]$. But then, also $\mathcal{M}, 2 \models^- \psi[b/x]$ follows by IHp2 and, further, $\mathcal{M}, 2 \models^+ \sim \psi[b/x]$.

(Part 2). Parallel to Part 1.

(Part 3). Trivial (by application of the corresponding truth-table).

Case 4. $\phi[a/x] = \psi[a/x] \to \chi[a/x]$.

(Part 1). If both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$, then assume that $\mathcal{M}, 2 \not\models^+ \phi[b/x]$. The latter means that we have both $\mathcal{M}, 2 \models^+ \psi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \chi[b/x]$, whence it follows, by IHp1, that either $\mathcal{M}, 1 \not\models^+ \chi[a/x]$ or $\mathcal{M}, 1 \models^- \chi[a/x]$. By IHp3, we know that we must have $\mathcal{M}, 1 \models^- \chi[a/x]$ in both cases. But the latter means that we must have $\mathcal{M}, 1 \models^- \psi[a/x] \to \chi[a/x]$, which contradicts our assumption. Therefore, we must have $\mathcal{M}, 2 \models^+ \phi[b/x]$.

(Part 2). If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then assume that $\mathcal{M}, 2 \not\models^- \phi[b/x]$. The latter means that we have both $\mathcal{M}, 2 \models^+ \psi[b/x]$ and $\mathcal{M}, 2 \not\models^- \chi[b/x]$, whence it follows, by IHp2, that either $\mathcal{M}, 1 \not\models^- \chi[a/x]$ or $\mathcal{M}, 1 \models^+ \chi[a/x]$. By IHp3, we know that we must have $\mathcal{M}, 1 \models^+ \chi[a/x]$ in both cases. But the latter means that we must have $\mathcal{M}, 1 \models^+ \psi[a/x] \to \chi[a/x]$, which contradicts our assumption. Therefore, we must have $\mathcal{M}, 2 \models^- \phi[b/x]$.

(Part 3). Trivial (by application of the corresponding truth-table).

Case 5. $\phi[b/x] = \forall y \psi[b/x]$. We may assume, wlog, that $y \neq x$. Note that we have, by Corollary 1, that $\psi[b/x][c/y] = \psi[c/y][b/x]$ for every $c \in U$ under this condition.

(Part 1). Assume that both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$. Now $\mathcal{M}, 1 \models^+ \phi[a/x]$ implies that $\mathcal{M}, 1 \models^+ \psi[a/x][a/y]$, whereas $\mathcal{M}, 1 \not\models^- \phi[a/x]$, by the definition of $\mathcal{M}$, implies that $\mathcal{M}, 1 \not\models^- \psi[a/x][a/y]$. Therefore, by IHp1, we must have $\mathcal{M}, 2 \models^+ \psi[a/y][b/x]$.

Next, choose a $z \in Var$ such that $z \notin FV(\psi[a/y][a/x]) \cup BV(\psi[a/y][a/x]) \cup \{x, y\}$

and consider $\psi[z/x][z/y]$. Then Lemma 1.4 and Lemma 1.2 imply that

$$\psi[z/x][z/y][b/z] = \psi[z/x][b/z][b/y] = \psi[b/z][b/x][b/y] = \psi[b/x][b/y].$$

A parallel argument shows that also $\psi[z/x][z/y][a/z] = \psi[a/x][a/y]$. Thus we have shown that both $\mathcal{M}, 1 \models^+ \psi[z/x][z/y][a/z]$ and $\mathcal{M}, 1 \not\models^- \psi[z/x][z/y][a/z]$, whence, by IHp1, $\mathcal{M}, 2 \models^+ \psi[z/x][z/y][b/z] = \psi[b/x][b/y]$.

Thus we have shown that

$$\mathcal{M}, 2 \models^+ \psi[b/x][a/y] \wedge \psi[b/x][b/y],$$

so that also $\mathcal{M}, 2 \models^+ \phi[b/x] = \forall y\psi[b/x]$ holds.

(Part 2). If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then assume that $\mathcal{M}, 2 \not\models^- \phi[b/x]$. The latter means that we have both $\mathcal{M}, 2 \not\models^- \psi[a/y][b/x]$ and $\mathcal{M}, 2 \not\models^- \psi[b/y][b/x]$. Now Lemma 20 implies that we must also have both $\mathcal{M}, 2 \models^+ \psi[a/y][b/x]$ and $\mathcal{M}, 2 \models^+ \psi[b/y][b/x]$, whence, by Lemma 21.1, we must have both $\mathcal{M}, 2 \models^+ \psi[a/y][a/x] = \psi[a/x][a/y]$ and $\mathcal{M}, 2 \models^+ \psi[b/y][a/x] = \psi[a/x][b/y]$.

Next, since we have $\mathcal{M}, 2 \not\models^- \psi[a/y][b/x]$, it also follows by IHp2 that either $\mathcal{M}, 1 \not\models^- \psi[a/y][a/x]$ or $\mathcal{M}, 1 \models^+ \psi[a/y][a/x]$. By IHp3, $\mathcal{M}, 1 \models^+ \psi[a/y][a/x] = \psi[a/x][a/y]$ holds in both cases.

Summing up, we have shown that all of the following holds:

$$\mathcal{M}, 1 \models^+ \psi[a/x][a/y], \mathcal{M}, 2 \models^+ \psi[a/x][a/y], \mathcal{M}, 2 \models^+ \psi[a/x][b/y],$$

which, by definition of $\mathcal{M}$, implies that $\mathcal{M}, 1 \models^+ \forall y\psi[a/x]$, contrary to our assumption. The obtained contradiction shows that we must have $\mathcal{M}, 2 \models^- \phi[b/x]$.

(Part 3). Assume that $\mathcal{M}, 1 \not\models^- \phi[a/x]$. Then we must have $\mathcal{M}, 1 \not\models^- \phi[a/x][a/y]$, whence IHp3 further implies that $\mathcal{M}, 1 \models^+ \phi[a/x][a/y]$. But then, by IHp1, it follows that $\mathcal{M}, 2 \models^+ \phi[a/x][b/y]$. Moreover, we must have $\mathcal{M}, 2 \models^+ \phi[a/x][a/y]$ by monotonicity. Summing up, we get that $\mathcal{M}, 1 \models^+ \forall y\psi[a/x] = \phi[a/x]$.

Case 6. $\phi[b/x] = \exists y\psi[b/x]$. Similar to Case 5. □

Proof of Lemma 18. (Part 1). By Lemma 20 and Lemma 22.3.

(Part 2). We have $\mathcal{M}, 1 \models^+ p \vee Q(a)$ as well as $\mathcal{M}, 2 \models^+ p \vee Q(a)$ and $\mathcal{M}, 2 \models^+ p \vee Q(b)$, so that $\mathcal{M}, 1 \models^+ \forall x(p \vee Q(x))$. However, we also have $\mathcal{M}, 1 \not\models^+ p$ and $\mathcal{M}, 2 \not\models^+ Q(b)$, whence $\mathcal{M}, 1 \not\models^+ \forall xQ(x)$, so that, finally, $\mathcal{M}, 1 \not\models^+ p \vee \forall xQ(x)$. □

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